

ON THE SPITZER–HÄRM REGIME AND NON LOCAL APPROXIMATIONS: MODELING, ANALYSIS AND NUMERICAL SIMULATIONS *

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Abstract. This paper is devoted to the derivation of the Spitzer–Härm limit from the coupled system of PDEs describing the evolution of charged particles and electromagnetic fields. We identify a relevant asymptotic regime which leads to a non linear diffusion equation for the electron temperature. Then, we discuss some intermediate models, which remain of hydrodynamic nature but involve a nonlocal coupling through integral or pseudo-differential operators. In particular, we exhibit important mathematical properties of the so-called Schurtz-Nicolaï model like the well-posedness and the maximum principle. We also design numerical schemes for the non local models and analyze their consistency and stability properties.

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1. Introduction. We are concerned with the following equations

$$(1.1) \quad \partial_t f_k + \nabla_x \cdot (v f_k) \pm \frac{q Z_k}{m_k} \nabla_v \cdot ((E + v \wedge B) f_k) = \sum_l C_{kl}(f_k, f_l)$$

where the unknowns $f_k(t, x, v)$ stand for the number density in phase space of charged particles within the species labelled by k . These quantities depend on the time variable $t \geq 0$, the space variable $x \in \mathbb{R}^3$ and the velocity variable $v \in \mathbb{R}^3$. The parameters q and m_k are the electron charge and the mass of the particles, respectively. In what follows the index $k = 0$ is used for electrons, and positive indices k are used for ions. Then, the sign $+$ in front of the acceleration term corresponds to positively charged particles, the sign $-$ corresponds to electrons. By convention we set $Z_0 = 1$ and Z_k is the ionization number for the ion specie $k > 0$. The right hand side in (1.1) describes interparticles interactions. Usually, in plasma physics, it is given by the Landau-Fokker-Planck operators

$$(1.2) \quad C_{kl}(f_k, f_l) = \Gamma_{kl} \nabla_v \cdot \left(\int_{\mathbb{R}^3} S_{k,\alpha}(v - v_*) (\nabla_v - \frac{m_k}{m_l} \nabla_{v_*}) f_l(v_*) f_k(v) dv_* \right)$$

with

$$\Gamma_{kl} = \frac{4\pi Z_k^2 Z_l^2 q^4 \ln \Lambda}{\varepsilon_0^2 m_k^2},$$

$\ln \Lambda$ being the Coulomb logarithm, and ε_0 the vacuum permittivity. The kernel of the collision operator is given by

$$S_{k,\alpha}(z) = \left(\frac{k_B \Theta_k}{m_k z^2} \right)^{\frac{\alpha}{2}} \frac{\Pi_z}{|z|}, \quad \Pi_z = \mathbb{I} - \frac{z \otimes z}{|z|^2},$$

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with k_B the Boltzmann constant. The case $\alpha < -3$ is traditionally referred to as *hard potentials*, the case $\alpha = -3$ as *Maxwell molecules*, and the case $\alpha > -3$ as *soft potentials*. The most relevant case in plasma physics corresponds to the Coulombian interactions between charged particles where $\alpha = 0$. For further analysis it will be interesting to consider slightly different operators, like for example the Boltzmann and BGK models, having the same fundamental properties (conservation and dissipation). Here and below, we consider the densities, current densities and temperatures defined as velocity average of the microscopic unknowns

$$\rho_k(t, x) = \int_{\mathbb{R}^3} f_k(t, x, v) dv, \quad J_k(t, x) = Z_k q \rho_k u_k(t, x) = Z_k q \int_{\mathbb{R}^3} v f_k(t, x, v) dv,$$

$$\rho_k |u_k|^2 + 3\rho_k \frac{k_B \Theta_k}{m_k} = \int_{\mathbb{R}^3} v^2 f_k dv.$$

Finally, the particles are subject to a force field determined by the electromagnetic field (E, B) , which is self-consistently defined by the Maxwell equations :

$$(1.3) \quad \begin{cases} \partial_t E - c^2 \operatorname{curl}_x B = \frac{1}{\varepsilon_0} \left(J_0 - \sum_{k>0} J_k \right), \\ \partial_t B + \operatorname{curl}_x E = 0, \\ \operatorname{div}_x E = \frac{q}{\varepsilon_0} \left(\sum_{k>0} Z_k \rho_k - \rho_0 \right), \\ \operatorname{div}_x B = 0, \end{cases}$$

with c the speed of light.

We refer to [2, 9] for an introduction to the physics background on the model. The mathematical theory on existence–uniqueness issues for Vlasov–Maxwell equations with collisional terms is not complete, depending on the complexity of the collision operator. Concerning weak solutions we refer to [13], but considering Boltzmann or Landau operators might lead to very weak notion of solutions [20, sp. Section IV]. Impressive progress have appeared recently dealing with classical solutions close to equilibrium, with proofs based on subtle energy estimates [17, 29, 30]. Here we will be interested in asymptotic questions. Due to the multiscale nature of the problem the cost of numerical simulations of the system becomes prohibitive in many practical situations. This motivates to seek reduced models. Therefore our goal is first to identify relevant parameters and asymptotic regimes, that can be embodied into a scaling term $0 < \varepsilon \ll 1$ and second to derive the corresponding limit equations. Then, having understood this behavior we seek intermediate models, depending on ε . These models will be of hydrodynamic type, that means describing the evolution of macroscopic quantities like the charge and current densities and the temperatures. As a typical example, the description of laser plasma interactions, as in the modeling of Inertial Confinement Fusion (ICF), is highly demanding in computational resources. Simulations of the fully microscopic model is not affordable at the scales of physical interest and usually this situation is modeled by fluid codes. It turns out that electron heat flow is a crucial aspect of laser fusion and it has been observed that these codes often produce overestimated heat fluxes compared to experiments. Comparisons to kinetic codes, available in very simplified geometries, have confirmed this drawback,

which motivates that quest for more accurate macroscopic models.

Here, to start with, we adopt the following simplified framework:

- Ions reduce to one specie.
- The distribution of positive charge has already been thermalized, so that

$$f_i(t, x, v) = \frac{\rho_i}{(2\pi k_B \Theta_i / m_i)^{3/2}} \exp\left(-\frac{|v - u_i|^2}{2k_B \Theta_i / m_i}\right).$$

- The associated macroscopic quantities ρ_i, Θ_i only depend on the space variable and the current of the positive particles vanishes $u_i = 0$.

Therefore we are interested in the evolution of the distribution of electrons $f_e(t, x, v)$, driven by

$$\partial_t f_e + v \cdot \nabla_x f_e - \frac{q}{m_e} \nabla_v \cdot ((E + v \wedge B) f_e) = C_{ee}(f_e) + C_{ei}(f_e).$$

Coming back to (1.2), the collision operators read

$$\begin{aligned} C_{ee}(f) &= \Gamma_{ee} \nabla_v \cdot \left(\int S_{e,\alpha}(v - v_\star) (\nabla_v - \nabla_{v_\star}) f(v_\star) f(v) dv_\star \right), \\ C_{ei}(f) &= \frac{\rho_i}{(2\pi k_B \Theta_i / m_i)^{3/2}} \Gamma_{ei} \\ &\quad \nabla_v \cdot \left(\int S_{e,\alpha}(v - v_\star) (\nabla_v - \frac{m_e}{m_i} \nabla_{v_\star}) \exp\left(-\frac{v_\star^2}{2k_B \Theta_i / m_i}\right) f(v) dv_\star \right). \end{aligned}$$

The electromagnetic field satisfies

$$\begin{cases} \partial_t E - c^2 \operatorname{curl}_x B = \frac{J_e}{\varepsilon_0} = \frac{q}{\varepsilon_0} \int_{\mathbb{R}^3} v f_e dv, \\ \partial_t B + \operatorname{curl}_x E = 0, \\ \operatorname{div}_x E = \frac{q}{\varepsilon_0} (Z_i \rho_i - \rho_e) = \frac{q}{\varepsilon_0} \left(Z_i \rho_i - \int_{\mathbb{R}^3} f_e dv \right), \\ \operatorname{div}_x B = 0. \end{cases}$$

The paper is organized as follows. The next section is devoted to the asymptotic analysis of the problem. We start with the dimensional analysis of the equations, in order to identify a set of relevant dimensionless parameters (section 2.1). A first approximation consists in simplifying the electron/ion collision term, based on the scaling $m_e/m_i \ll 1$ (section 2.2). In particular, we bring out the fundamental properties of the approximate collision operator: charge and energy conservation, entropy dissipation. Then by using Hilbert expansions we are led to the so-called Spitzer-Härm regime where the dynamics is driven by a non linear diffusion equation for the electron temperature; we identify the diffusion coefficient which depends on the details of the collision operator (section 2.3). The intermediate model which is discussed in section 3 is derived on physical grounds and it is quite popular for the simulation of ICF experiments. There, the flux is obtained as a suitable convolution of the gradient of the temperature, which leads to a non local model. We shall establish some remarkable mathematical properties of the model. In section 4.1 we design and analyze numerical schemes for the non local models. It is completed in section 4.2 by a set of commented numerical simulations. Eventually a conclusion summarizes the main contribution of the paper.

2. The Spitzer-Härm regime.

2.1. Dimensional analysis. Let us write now the equations in dimensionless form. To this end, we introduce a particle reference density $\bar{\rho}_e$ and a reference temperature $\bar{\Theta}_e$. Then, $\sqrt{k_B \bar{\Theta}_e / m_e}$ defines the thermal velocity and $k_B \bar{\Theta}_e / q$ defines a reference potential. We also need time and length units, T and L respectively. Then we define dimensionless variables by setting

$$t = T t', \quad x = L x', \quad v = \sqrt{\frac{k_B \bar{\Theta}_e}{m_e}} v'.$$

Next, we define the dimensionless density by

$$f_e(t, x, v) = \frac{\bar{\rho}_e}{L^3 \left(\frac{k_B \bar{\Theta}_e}{m_e} \right)^{3/2}} f'(t', x', v'),$$

while the electromagnetic field scales as follows

$$E(t, x) = \frac{k_B \bar{\Theta}_e}{q} \frac{1}{L} E'(t', x'), \quad B(t, x) = \frac{k_B \bar{\Theta}_e}{q} \frac{1}{T c^2} B'(t', x').$$

Note that

$$\frac{k_B \bar{\Theta}_e}{q} \frac{1}{L} \times \frac{q}{k_B \bar{\Theta}_e} T c^2 \sqrt{\frac{m_e}{k_B \bar{\Theta}_e}} = \frac{T c^2}{L} \sqrt{\frac{m_e}{k_B \bar{\Theta}_e}}$$

measures the ratio of the electric force over the magnetic force. We also set

$$\rho_i(x) = \frac{\bar{\rho}_i}{L^3} \rho'_i(x'), \quad \Theta_i(x) = \bar{\Theta}_i \Theta'_i(x'),$$

with $\bar{\rho}_i$ and $\bar{\Theta}_i$ reference values for the ion density and temperature respectively. Finally, typical length scales are defined by

$$\begin{aligned} \text{Debye length:} \quad \lambda_D &= \sqrt{\frac{\varepsilon_0 k_B \bar{\Theta}_e L^3}{q^2 \bar{\rho}_e}}, \\ \text{electron mean free path:} \quad \ell &= \frac{\varepsilon_0^2 k_B^2 \bar{\Theta}_e^2 L^3}{4\pi \bar{\rho}_e q^4 \ln \Lambda}. \end{aligned}$$

Up to a slight change of notation, the dimensionless equations read as follows:

$$\begin{aligned} \partial_t f + \sqrt{\frac{k_B \bar{\Theta}_e}{m_e}} \frac{T}{L} v \cdot \nabla_x f - \sqrt{\frac{k_B \bar{\Theta}_e}{m_e}} \frac{T}{L} \nabla_v \cdot \left((E + \sqrt{\frac{m}{k_B \bar{\Theta}_e}} \frac{L}{T} \frac{k_B \bar{\Theta}_e}{m c^2} v \wedge B) f \right) \\ = T \Gamma_{ee} \frac{\bar{\rho}_e}{L^3} \left(\frac{m_e}{k_B \bar{\Theta}_e} \right)^{3/2} \left(C_{ee}(f) + Z_i C_{ei}(f) \right), \end{aligned}$$

with now

(2.1)

$$\begin{aligned} C_{ee}(f) &= \nabla_v \cdot \left(\int_{\mathbb{R}^3} \Theta_e^{\alpha/2} \frac{\Pi_{v-v_*}}{|v-v_*|^{\alpha+1}} (\nabla_v - \nabla_{v_*}) f(v_*) f(v) dv_* \right), \\ C_{ei}(f) &= \frac{Z_i \bar{\rho}_i}{\bar{\rho}_e} \left(\frac{m_i \bar{\Theta}_e}{m_e \bar{\Theta}_i} \frac{1}{2\pi \bar{\Theta}_i} \right)^{3/2} \rho_i \\ &\quad \nabla_v \cdot \left(\int_{\mathbb{R}^3} \Theta_e^{\alpha/2} \frac{\Pi_{v-v_*}}{|v-v_*|^{\alpha+1}} (\nabla_v - \frac{m_e}{m_i} \nabla_{v_*}) \exp\left(-\frac{m_i \bar{\Theta}_e}{m_e \bar{\Theta}_i} \frac{v_*^2}{2\bar{\Theta}_i}\right) f(v) dv_* \right), \end{aligned}$$

coupled to the Maxwell system

$$\begin{aligned}\partial_t E - \operatorname{curl}_x B &= \frac{q^2 \bar{\rho}_e T}{\varepsilon_0 L^2 \sqrt{k_B \bar{\Theta}_e} m_e} \int_{\mathbb{R}^3} v f \, dv, \\ \left(\frac{L}{Tc}\right)^2 \partial_t B + \operatorname{curl}_x E &= 0, \\ \nabla_x \cdot B &= 0, \\ \nabla_x \cdot E &= \frac{q^2 \bar{\rho}_e}{\varepsilon_0 k_B \bar{\Theta}_e L} \left(\frac{Z_i \bar{\rho}_i}{\bar{\rho}_e} \rho_i - \int_{\mathbb{R}^3} f \, dv \right).\end{aligned}$$

The dynamics is therefore governed by the dimensionless parameters

$$\frac{1}{\varepsilon} = \sqrt{\frac{k_B \bar{\Theta}_e}{m_e}} \frac{T}{L}, \quad \eta = \sqrt{\frac{k_B \bar{\Theta}_e}{m_e}} \frac{1}{c},$$

the ratio of the thermal velocity over the velocity unit defined by the time and length scales, and the ratio of the thermal velocity over the light speed, respectively;

$$\frac{L}{\lambda_D}, \quad \frac{T}{\tau},$$

where the relaxation time

$$\tau = \frac{\ell}{\sqrt{\frac{k_B \bar{\Theta}_e}{m_e}}}$$

is the time necessary for the electron moving at speed $\sqrt{\frac{k_B \bar{\Theta}_e}{m_e}}$ to travel the distance ℓ and the mass ratio m_e/m_i , the temperature ratio $\bar{\Theta}_e/\bar{\Theta}_i$ and the density ratio $\bar{Z} = Z_i \bar{\rho}_i / \bar{\rho}_e$. Indeed, we have

$$(2.2) \quad \partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f - \frac{1}{\varepsilon} \nabla_v \cdot \left((E + \varepsilon \eta^2 v \wedge B) f \right) = \frac{T}{\tau} \left(C_{ee}(f) + Z_i C_{ei}(f) \right),$$

and

$$(2.3) \quad \begin{aligned}\partial_t E - \operatorname{curl}_x B &= -\frac{1}{\varepsilon} \left(\frac{L}{\lambda_D} \right)^2 \int_{\mathbb{R}^3} v f \, dv, \\ \varepsilon^2 \eta^2 \partial_t B + \operatorname{curl}_x E &= 0, \\ \nabla_x \cdot B &= 0, \\ \nabla_x \cdot E &= \left(\frac{L}{\lambda_D} \right)^2 \left(\bar{Z} \rho_i - \int_{\mathbb{R}^3} f \, dv \right).\end{aligned}$$

REMARK 2.1. *It can be convenient to rewrite*

$$\lambda_D = \frac{1}{\omega_P} \sqrt{\frac{k_B \bar{\Theta}_e}{m_e}},$$

with

$$\omega_P = \sqrt{\frac{q^2 \bar{\rho}_e}{L^3 m_e \varepsilon_0}},$$

the plasma frequency, and

$$\ell = \lambda_D^2 \times \frac{1}{r_e} \times \frac{k_B \bar{\Theta}_e}{m_e c^2}$$

with $r_e = q^2/(\varepsilon_0 m_e c^2) \simeq 2.82 \cdot 10^{-15}$ m., the classical electron radius.

2.2. Approximation of the electron/ion collision operator; Conserved quantities and entropy dissipation. Taking into account $m_e/m_i \ll 1$ and $\bar{\Theta}_e/\bar{\Theta}_i$ fixed to a positive constant, the electron/ion collision operator simplifies to

$$(2.4) \quad \tilde{C}_{ei}(f_e)(v) = \bar{Z} \rho_i \nabla_v \cdot \left(\Theta_e^{\alpha/2} \frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v f(v) \right),$$

see e. g. [24]. This approximation is often used in practice, and we adopt from now on to replace C_{ei} by \tilde{C}_{ei} in the kinetic equation (2.2). As a matter of fact, we observe that the system (2.2)- (2.3), with the approximate operator (2.4), conserves energy and dissipates the entropy.

PROPOSITION 2.1. *Let (f, E, B) be a (smooth enough) solution of (2.2)–(2.3) on the whole space, with the approximate operator (2.4). Then, we have*

$$\frac{d}{dt} \left[\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{v^2}{2} f \, dv \, dx + \frac{1}{2} \left(\frac{\lambda_D}{L} \right)^2 \int_{\mathbb{R}^3} (E^2 + \varepsilon^2 \eta^2 B^2) \, dx \right] = 0,$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \ln(f) \, dv \, dx = \frac{T}{\tau} \int_{\mathbb{R}^3} (C_{ee}(f) + Z_i \tilde{C}_{ei}(f)) \ln(f) \, dv \, dx \leq 0.$$

This statement is a consequence of the following fundamental properties of the collision operators.

LEMMA 2.2. *Let C_{ee} and \tilde{C}_{ei} be defined by (2.1) and (2.4), respectively. We have*

$$(2.5) \quad \int_{\mathbb{R}^3} C_{ee}(f) \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \, dv = 0,$$

$$(2.6) \quad \int_{\mathbb{R}^3} \tilde{C}_{ei}(f) \begin{pmatrix} 1 \\ v^2 \end{pmatrix} \, dv = 0,$$

$$(2.7) \quad \begin{aligned} \int_{\mathbb{R}^3} C_{ee}(f) \ln(f) \, dv &= - \int_{\mathbb{R}^3} \Theta_e^{\alpha/2} \frac{\Pi_{v-v_*}}{|v-v_*|^{\alpha+1}} \left(\frac{\nabla_v f}{f}(v) - \frac{\nabla_{v_*} f}{f}(v_*) \right) \\ &\quad \cdot \left(\frac{\nabla_v f}{f}(v) - \frac{\nabla_{v_*} f}{f}(v_*) \right) f(v_*) f(v) \, dv_* \, dv \\ &= - \int_{\mathbb{R}^3} \left| \Pi_{v-v_*} (\nabla_v - \nabla_{v_*}) f(v) f(v_*) \right|^2 \frac{\Theta_e^{\alpha/2} \, dv_* \, dv}{|v-v_*|^{\alpha+1} f(v) f(v_*)} \\ &\leq 0, \end{aligned}$$

$$(2.8) \quad \begin{aligned} \int_{\mathbb{R}^3} \tilde{C}_{ei}(f) \ln(f) \, dv &= -4\bar{Z}\rho_i \int_{\mathbb{R}^3} \Theta_e^{\alpha/2} \frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v \sqrt{f} \cdot \nabla_v \sqrt{f} \, dv \\ &= -4\bar{Z}\rho_i \int_{\mathbb{R}^3} \frac{\Theta_e^{\alpha/2}}{|v|^{\alpha+1}} \left| \Pi_v \nabla_v \sqrt{f} \right|^2 \, dv \leq 0. \end{aligned}$$

More precisely the entropy dissipation vanishes in (2.7) if and only if the distribution function is a Maxwellian $f(v) = \rho(2\pi\Theta_e)^{-3/2}e^{-|v-u|^2/2\Theta_e}$ for some $\rho, \Theta_e \geq 0$ and $u \in \mathbb{R}^3$; and the entropy dissipation vanishes in (2.8) if and only if the distribution is isotropic: $f(v) = F(|v|)$. Actually, we observe that $\tilde{C}_{ei}(\varphi(|v|)f(v)) = \varphi(|v|)\tilde{C}_{ei}(f(v))$.

Proof. The proof of (2.5)–(2.8) follows by direct evaluation, using integration by parts and the fact that $(\mathbb{I} - \frac{z \otimes z}{|z|^2})z = 0$. Remark that the collision operator $C_{ee} + Z_i \tilde{C}_{ei}$ does not preserve the momentum, but it preserves only mass and energy. However we observe that

$$\int_{\mathbb{R}^3} \tilde{C}_{ei}(f)v \, dv = -\bar{Z}\rho_i \int_{\mathbb{R}^3} \Theta_e^{\alpha/2} \frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v f \, dv = -2\bar{n}\rho_i \Theta_e^{\alpha/2} \int_{\mathbb{R}^3} \frac{v}{|v|^{\alpha+3}} f \, dv$$

which of course vanishes when f is isotropic. In the Maxwellian molecules case ($\alpha = -3$), the momentum variation is proportional to the current density :

$$\int_{\mathbb{R}^3} \tilde{C}_{ei}(f)v \, dv \Big|_{\alpha=-3} = -2 \frac{\bar{Z}\rho_i}{\Theta_e^{3/2}} \int_{\mathbb{R}^3} v f \, dv.$$

Clearly the collision operator C_{ee} (resp. C_{ei}) and the entropy dissipation vanish when f is a Maxwellian (resp. isotropic function). Let us prove the reverse implication. Saying that the entropy dissipation associated to the electron–electron collisions vanishes means that $\frac{1}{f(v)f(v_*)}(\nabla_v - \nabla_{v_*})f(v)f(v_*) = \nabla_v \ln(f(v)) - \nabla_{v_*} \ln(f(v_*))$ is proportional to $v - v_*$ (which means that it is equal to $\alpha(v, v_*)(v - v_*)$ for some scalar function α). Let us set $g(v) = \ln(f(v))$. Clearly any affine function $b \cdot v + c$, with $b \in \mathbb{R}^3$, $c \in \mathbb{R}$, satisfies this relation. Possibly at the price of adding such a function we assume from now on that $\nabla_v g(0) = 0$. In particular, we obtain $\nabla_v g(v) = \tilde{\alpha}(v)v$, $\tilde{\alpha}(v) \in \mathbb{R}$. Writing $g(v) = \tilde{g}(|v|^2/2, v/|v|)$ it implies that \tilde{g} does not depend on the second argument, hence $g(v) = \tilde{g}(|v|^2/2)$ is radially symmetric. (By the way, it proves that the entropy dissipation of the electron–ion collision operator vanishes only if f is isotropic.) Therefore, we are led to $v(\alpha(v, v_*) - \tilde{g}'(|v|^2/2)) - v_*(\alpha(v, v_*) - \tilde{g}'(|v_*|^2/2)) = 0$. Up to the negligible set in $\mathbb{R}^3 \times \mathbb{R}^3$ where v and v_* are colinear, this relation implies that $\tilde{g}'(|v|^2/2) = \tilde{g}'(|v_*|^2/2)$ is constant, hence $\tilde{g}(|v|^2/2) = a|v|^2$, $a \in \mathbb{R}$. We conclude that the functions that make the electron–electron entropy dissipation vanish are given by $f(v) = \exp(a|v|^2 + b \cdot v + c)$ (with $a < 0$ to guaranty integrability), which can be recast as $\rho(2\pi\Theta_e)^{-3/2}e^{-|v-u|^2/2\Theta_e}$, with a suitable relation $(a, b, c) \mapsto (\rho, u, \Theta_e)$. ■

Based on (2.5) and (2.6), we can look at the first three moments of the equation (1.1). Using charge density, current density and temperature definition, we have :

$$(2.9) \quad \begin{cases} \partial_t \rho + \frac{1}{\varepsilon} \nabla_x \cdot (\rho u) = 0, \\ \rho \left(\partial_t u + \frac{1}{\varepsilon} (u \cdot \nabla_x) u \right) + \frac{1}{\varepsilon} \nabla_x \cdot \mathbf{P} = \frac{\mathbf{T}}{\tau} Z_i \int_{\mathbb{R}^3} v \tilde{C}_{ei}(f) \, dv - \frac{1}{\varepsilon} \rho (E + \varepsilon \eta^2 u \wedge B), \\ \partial_t (3\rho\Theta_e + \rho u^2) + 2\nabla_x \cdot \mathbf{Q} + \frac{1}{\varepsilon} \nabla_x \cdot (3\rho\Theta_e u + 2\mathbf{P}u) = -2\rho E \cdot u - \nabla_x \cdot (\rho u^2 u), \end{cases}$$

coupled with equations (2.3). These equations are not closed since the pressure tensor \mathbf{P} and the heat flux \mathbf{Q} are defined by

$$\mathbf{P} = \int_{\mathbb{R}^3} (v - u) \otimes (v - u) f \, dv, \quad \mathbf{Q} = \frac{1}{2\varepsilon} \int_{\mathbb{R}^3} |v - u|^2 (v - u) f \, dv$$

respectively. The system also involves the microscopic distribution function through the integral term $\int v \tilde{C}_{ei}(f) dv$. These terms in general cannot be expressed by means of the low order moments and the macroscopic quantities ρ, u, Θ_e .

2.3. Asymptotic regime: Hilbert expansion. The asymptotic regime we are interested in assumes

$$\varepsilon \ll 1, \quad \lambda_\varepsilon = \frac{\lambda_D}{L} \ll 1, \quad \frac{\tau}{T} \simeq \varepsilon^2 \ll 1,$$

while η, Z_i and \bar{Z} are kept fixed. We are thus concerned with the behavior of the solutions of

$$\begin{aligned} \partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f - \frac{1}{\varepsilon} \nabla_v \cdot \left((E + \varepsilon \eta^2 v \wedge B) f \right) &= \frac{1}{\varepsilon^2} \left(C_{ee}(f) + Z_i \tilde{C}_{ei}(f) \right), \\ \partial_t E - \operatorname{curl}_x B &= -\frac{1}{\varepsilon \lambda_\varepsilon^2} \int_{\mathbb{R}^3} v f dv, \\ \varepsilon^2 \eta^2 \partial_t B + \operatorname{curl}_x E &= 0, \\ \nabla_x \cdot B &= 0, \\ \nabla_x \cdot E &= \frac{1}{\lambda_\varepsilon^2} \left(\bar{Z} \rho_i - \int_{\mathbb{R}^3} f dv \right), \end{aligned}$$

with $\varepsilon \rightarrow 0$ and $\lambda_\varepsilon \rightarrow 0$, the other parameters being fixed. As it will be detailed below, in this regime the system (2.9) reduces to

$$(2.10) \quad \rho = \bar{Z} \rho_i, \quad \partial_t \Theta + \frac{2 \nabla_x \cdot Q}{3 \rho} = 0,$$

where the flux Q is proportional to the gradient of a certain power of Θ , hence a non linear diffusion equation for the electron temperature. Relevant intermediate models can be thought of as closure relations defining the flux Q by means of Θ .

Of course, we have in mind the Landau-Fokker-Planck operator (2.1) and the approximate operator (2.4) which are the most relevant for the application to plasma physics. But it is worth bringing out the key assumptions on the collisions operators which are needed to derive the asymptotic models, without specializing too much. The basic assumptions state as follows

- A1) $C_{ee}(f)$ vanishes if and only if f is a Maxwellian, $\tilde{C}_{ei}(f)$ vanishes if and only if f is isotropic, $(C_{ee} + Z_i C_{ei})(f)$ vanishes if and only if f makes the two vanish, which means that f is a centered Maxwellian,
- A2) $\int_{\mathbb{R}^3} C_{ee}(f) (1, v, v^2) dv = 0$, $\int_{\mathbb{R}^3} \tilde{C}_{ei}(f) (1, v^2) dv = 0$,
- A3) $\int_{\mathbb{R}^3} C_{ee}(f) \ln(f) dv \leq 0$, $\int_{\mathbb{R}^3} \tilde{C}_{ei}(f) \ln(f) dv \leq 0$.

Clearly, Proposition 2.1 holds for any operators satisfying (A2)–(A3). We refer to Lemma 2.2 for the case of (2.1) and (2.4). It is interesting to consider other operators like e. g.

- Boltzmann operator

$$C_{ee}^B(f) = \int_{\mathbb{R}^3 \times \mathbb{S}^2} B(|v - v_*|, (v - v_*) \cdot \omega) \left(f(v'_*) f(v') - f_*(v_*) f(v) \right) d\omega dv_*,$$

with $v' = v - \omega(v - v_*) \cdot \omega$, $v'_* = v_* + \omega(v - v_*) \cdot \omega$ and $d\omega$ is the normalized measure on \mathbb{S}^2 .

- BGK operator

$$C_{ee}^{BGK}(f) = \frac{M[f] - f}{\tau_{ee}(\rho, \Theta)}, \quad M[f](v) = \frac{\rho}{(2\pi\Theta)^{3/2}} e^{-|v-u|^2/2\Theta}$$

with $(\rho, \rho u, \rho u^2 + 3\rho\Theta) = \int_{\mathbb{R}^3} (1, v, v^2) f \, dv$.

- Fokker-Planck operator

$$C_{ee}^{FP}(f) = \nabla_v \cdot ((v-u)f + \Theta \nabla_v f).$$

Similarly, the electron-ion operator (2.4) could be replaced by the following simplified Boltzmann operator

$$\tilde{C}_{ei}^R(f)(v) = \frac{1}{\tau_{ei}(\rho, \Theta, |v|)} \left(\int_{\mathbb{S}^2} f(|v|\omega) \, d\omega - f(v) \right),$$

where τ_{ei} is a positive function. This expression can be obtained from the Boltzmann operator for electron-ion collision in the asymptotic $m_e/m_i \ll 1$, see [10, 16].

We obtain (formally) the asymptotic equation by developing the solution in power series of ε :

$$f(t, x, v) = \sum_j \varepsilon^j F_j(t, x, v)$$

and identifying terms that arise in the equation with the same power of ε . At leading order we get

$$(C_{ee} + Z_i \tilde{C}_{ei})(F_0) = 0.$$

Therefore, the leading term is a centered Maxwellian, by (A1). Moreover, by using $\lambda_\varepsilon \ll 1$, the Maxwell system leads to the quasi-neutrality relations

$$(2.11) \quad \rho = \int F_0 \, dv = \bar{Z} \rho_i, \quad J = \int v F_0 \, dv = 0.$$

We conclude that

$$F_0(t, x, v) = \frac{\rho(x)}{(2\pi\Theta(t, x))^{3/2}} \exp\left(-\frac{v^2}{2\Theta(t, x)}\right), \quad \rho = \bar{Z} \rho_i.$$

Hence the goal is to determine the evolution equation satisfied by the temperature Θ .

To this end, for ρ and Θ positive, we set $M_{\rho, \Theta}(v) = \frac{\rho}{(2\pi\Theta)^{3/2}} e^{-v^2/2\Theta}$ and we introduce the linearized operator

$$L_{\rho, \Theta} f(v) = \frac{d}{dz} (C_{ee} + Z_i \tilde{C}_{ei})(M_{\rho, \Theta} + zf)(v) \Big|_{z=0}.$$

Owing to assumptions (A1)-(A3), we observe that the linearized operator satisfies

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v^2 \end{pmatrix} L_{\rho, \Theta} f(v) \, dv = 0,$$

and

$$\int_{\mathbb{R}^3} \frac{f}{M_{\rho,\Theta}} L_{\rho,\Theta} f(v) dv = \frac{1}{2} \frac{d^2}{dz^2} \int_{\mathbb{R}^3} (C_{ee} + Z_i \tilde{C}_{ei})(M_{\rho,\Theta} + zf) \ln(M_{\rho,\Theta} + zf) dv \Big|_{z=0} \leq 0.$$

Differentiating $(C_{ee} + Z_i \tilde{C}_{ei})(M_{\rho,\Theta}) = 0$ with respect to ρ and Θ we observe that $L_{\rho,\Theta}([\frac{\rho}{\rho} + \frac{\theta}{2\Theta}(\frac{v^2}{\Theta} - \frac{3}{2})]M_{\rho,\Theta}) = 0$ for any ρ and θ . We slightly strengthen these properties that follow from (A1-A3) by requiring

$$\text{B1) } \quad \text{Ker}(L_{\rho,\Theta}) = \text{Span}(M_{\rho,\Theta}, v^2 M_{\rho,\Theta}).$$

It is satisfied by most of the classical collision operators, which furthermore are usually self-adjoint. For the Landau-Fokker-Planck operator (2.1) and (2.4) we have

$$\begin{aligned} L_{\rho,\Theta} f(v) &= \nabla_v \cdot \int \Theta^{\alpha/2} \frac{\Pi_{v-v_*}}{|v-v_*|^{\alpha+1}} M_{\rho,\Theta}(v) M_{\rho,\Theta}(v_*) \\ &\quad \times \left(\nabla_v \frac{f}{M_{\rho,\Theta}}(v) - \nabla_{v_*} \frac{f}{M_{\rho,\Theta}}(v_*) \right) dv_* \\ &\quad + Z_i \bar{\rho} \Theta^{\alpha/2} \nabla_v \cdot \left(\frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v f \right). \end{aligned}$$

Observe that

$$\begin{aligned} & - \int \frac{f}{M_{\rho,\Theta}} L_{\rho,\Theta} f dv \\ &= \frac{\Theta^{\alpha/2}}{2} \int \left| \Pi_{v-v_*} \left(\nabla_v \frac{f}{M_{\rho,\Theta}}(v) - \nabla_{v_*} \frac{f}{M_{\rho,\Theta}}(v_*) \right) \right|^2 \frac{M_{\rho,\Theta}(v) M_{\rho,\Theta}(v_*)}{|v-v_*|^{\alpha+1}} dv_* dv \\ &\quad + Z_i \bar{\rho} \Theta^{\alpha/2} \int \frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v \frac{f}{M_{\rho,\Theta}} \cdot \nabla_v \frac{f}{M_{\rho,\Theta}} dv. \end{aligned}$$

Coming back to the Hilbert expansion, we obtain

$$\begin{aligned} (2.12) \quad L_{\rho,\Theta} F_1 &= v \cdot \nabla_x F_0 + E \cdot \nabla_v F_0 \\ &= v F_0(t, x, v) \cdot \left(\frac{\nabla_x \rho}{\rho} - \frac{3}{2} \frac{\nabla_x \Theta}{\Theta} + \frac{E}{\Theta} \right) + v \frac{v^2}{2} F_0(t, x, v) \cdot \frac{\nabla_x \Theta}{\Theta^2}, \end{aligned}$$

with $F_0(t, x, v) = M_{\rho(x), \Theta(t, x)}(v)$. Bearing in mind (A2), we remark that the zeroth and second moments of $v \cdot \nabla_x F_0 + E \cdot \nabla_v F_0$ vanish so that (2.12) makes sense. We need further assumptions:

B2) For any Φ verifying $\int_{\mathbb{R}^3} \Phi dv = \int_{\mathbb{R}^3} v^2 \Phi dv = 0$, there exists a unique Γ such that $L_{\rho,\Theta} \Gamma = \Phi$ with the constraints $\int_{\mathbb{R}^3} \Gamma dv = \int_{\mathbb{R}^3} v^2 \Gamma dv = 0$.

This assumption has the flavor of the Fredholm alternative that would identify the range $\text{Ran}(L_{\rho,\Theta})$ to the orthogonal to the kernel $\text{Span}(M_{\rho,\Theta}, v^2 M_{\rho,\Theta})$. Note however that proving that $\text{Ran}(L_{\rho,\Theta})$ is closed depends on the collision kernels. It is satisfied when the collision operator satisfies the following spectral gap inequality: there exists $\lambda_{\rho,\Theta} > 0$ such that

$$(2.13) \quad - \int_{\mathbb{R}^3} L_{\rho,\Theta} F \frac{F}{M_{\rho,\Theta}} dv \geq \lambda_{\rho,\Theta} \int_{\mathbb{R}^3} \frac{|F|^2}{M_{\rho,\Theta}} dv$$

holds for any F verifying $\int_{\mathbb{R}^3} F dv = \int_{\mathbb{R}^3} v^2 F dv = 0$. Typically it holds for the BGK operator or the Boltzmann operator in the case of Maxwell molecules or hard potentials with an angular cutoff, see [7], and we refer to [11] for the Landau-Fokker-Planck operator. The case of soft potentials imposes to work with suitable weighted

spaces and the proofs involve quite intricate arguments, see [15]. In appendix A.1, we detail the arguments for the Landau-Fokker-Planck operator by using the fine estimates derived in [1, 17, 25]. For the time being, we do not detail the functional difficulties associated to (B2). The useful consequence of (B2) relies on the fact that there exist (vector valued) functions $G_{\rho,\Theta}$ and $H_{\rho,\Theta}$ verifying $L_{\rho,\Theta}G_{\rho,\Theta} = vM_{\rho,\Theta}(v)$ and $L_{\rho,\Theta}H_{\rho,\Theta} = vv^2M_{\rho,\Theta}(v)$. As a matter of fact, we observe that the matrices $\int v \otimes G_{\rho,\Theta} dv$ and $\int vv^2 \otimes H_{\rho,\Theta} dv$ are negative definite, since we can rewrite

$$\int v \otimes G_{\rho,\Theta} dv = \int L_{\rho,\Theta}(G_{\rho,\Theta}) \otimes G_{\rho,\Theta} \frac{dv}{M_{\rho,\Theta}}$$

and

$$\int vv^2 \otimes H_{\rho,\Theta} dv = \int L_{\rho,\Theta}(H_{\rho,\Theta}) \otimes H_{\rho,\Theta} \frac{dv}{M_{\rho,\Theta}}$$

which are negative matrices by virtue of the dissipation properties (2.7) and (2.8). Furthermore, using symmetry and homogeneity arguments, we can derive simpler formulae.

LEMMA 2.3. *We suppose that the collision operator satisfies the following properties:*

- i) $L_{\rho,\Theta}f = \rho L_{1,\Theta}f$,
- ii) denoting τ_Θ the application defined by $\tau_\Theta f(v) = f(\sqrt{\Theta}v)$, we have $\tau_\Theta(L_{1,\Theta}(f)) = \frac{1}{\Theta^{3/2}}L_{1,1}(\tau_\Theta f)$,
- iii) for any isometry $R \in O(\mathbb{R}^3)$, we have $L_{1,\Theta}(f \circ R) = L_{1,\Theta}(f) \circ R$.

Then, there exists functions $g, h : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$G_{\rho,\Theta}(v) = g\left(\frac{|v|}{\sqrt{\Theta}}\right) v e^{-v^2/2\Theta},$$

and

$$(2.14) \quad H_{\rho,\Theta}(v) = \Theta h(|v|/\sqrt{\Theta}) v e^{-v^2/2\Theta}.$$

Consequently, there exists some constants q_1, q_2, q_3, q_4 , with $q_1 > 0, q_4 > 0$ such that

$$\begin{aligned} \int v \otimes G_{\rho,\Theta} dv &= -q_1 \Theta^{5/2} \mathbb{I}, & \int vv^2 \otimes G_{\rho,\Theta} dv &= -q_2 \Theta^{7/2} \mathbb{I}, \\ \int v \otimes H_{\rho,\Theta} dv &= -q_3 \Theta^{7/2} \mathbb{I}, & \int vv^2 \otimes H_{\rho,\Theta} dv &= -q_4 \Theta^{9/2} \mathbb{I}. \end{aligned}$$

Proof. By i), the equation defining $G_{\rho,\Theta}$ becomes $L_{1,\Theta}G_{\rho,\Theta} = vM_{1,\Theta}(v)$. We deduce that it does not depend on ρ . Next, ii) yields $L_{1,1}\left(\frac{1}{\sqrt{\Theta}}\tau_\Theta G_{\rho,\Theta}\right)(v) = vM_{1,1}(v)$ which means $G_{\rho,\Theta}(v) = \sqrt{\Theta}G_{1,1}(v/\sqrt{\Theta})$. Eventually, repeating the arguments in [12, Lemma 3], iii) implies $G_{1,1}(v) = vg(|v|)M_{1,1}(v)$. Similar reasoning applies for $H_{\rho,\Theta}$. Now, we compute

$$\begin{aligned} \int v \otimes G_{\rho,\Theta} dv &= \int v \otimes v g\left(\frac{|v|}{\sqrt{\Theta}}\right) e^{-v^2/2\Theta} dv = \Theta^{5/2} \int v \otimes v g(|v|) e^{-v^2/2} dv, \\ \int vv^2 \otimes G_{\rho,\Theta} dv &= \Theta^{7/2} \int v \otimes v v^2 g(|v|) e^{-v^2/2} dv, \\ \int v \otimes H_{\rho,\Theta} dv &= \Theta^{7/2} \int v \otimes v h(|v|) e^{-v^2/2} dv, \\ \int vv^2 \otimes H_{\rho,\Theta} dv &= \Theta^{9/2} \int v \otimes v v^2 h(|v|) e^{-v^2/2} dv. \end{aligned}$$

The matrices in the right hand side are clearly diagonal and we set $q_1 = -\frac{1}{3} \int v^2 g(|v|) e^{-v^2/2} dv$, $q_2 = -\frac{1}{3} \int v^4 g(|v|) e^{-v^2/2} dv$, $q_3 = -\frac{1}{3} \int v^2 h(|v|) e^{-v^2/2} dv$ and $q_4 = -\frac{1}{3} \int v^4 g(|v|) e^{-v^2/2} dv$. As discussed above, q_1 and q_4 are positive as a consequence of the dissipation properties of the collision operator. ■

We postpone to appendix A.2 the proof of the following claim which makes the previous statement relevant for our purpose.

LEMMA 2.4. *The assumptions of Lemma 2.3 hold for the operators (2.1) and (2.4). Having disposed of these preliminaries, we go back to the equation (2.12). We remind that $F_0(t, x, v) = M_{\rho(x), \Theta(t, x)}$ is the Maxwellian with density $\rho(x)$ and temperature $\Theta(t, x)$. The functions G and H we consider are actually parameterized by t, x . Therefore, we get*

$$F_1(t, x, v) = G_{\rho, \Theta}(t, x, v) \cdot \left(\frac{\nabla_x \rho}{\rho} - \frac{3}{2} \frac{\nabla_x \Theta}{\Theta} + \frac{E}{\Theta} \right)(t, x) + H_{\rho, \Theta}(t, x, v) \cdot \frac{\nabla_x \Theta}{\Theta^2}(t, x).$$

Now the electric field has to be determined by the constraint of vanishing current. Indeed, the Maxwell equation tells us that

$$\begin{aligned} \lambda_\varepsilon^2 \left(\partial_t E - \operatorname{curl}_x B \right) &= \mathcal{O}(\lambda_\varepsilon^2) \\ &= - \int \frac{v}{\varepsilon} f dv \simeq - \int v F_1 dv. \end{aligned}$$

Hence, we impose

$$\int v F_1 dv = 0$$

which yields the following definition of the asymptotic electric field

$$\begin{aligned} E(t, x) &= -\Theta(t, x) \left(\left(\frac{\nabla_x \rho}{\rho} - \frac{3}{2} \frac{\nabla_x \Theta}{\Theta} \right)(t, x) \right. \\ &\quad \left. + \left(\int v \otimes G_{\rho, \Theta}(t, x, v) dv \right)^{-1} \int v \otimes H_{\rho, \Theta}(t, x, v) dv \frac{\nabla_x \Theta}{\Theta^2}(t, x) \right). \end{aligned}$$

Having obtained the expression of the corrector F_1 , we use the compatibility condition for the relation

$$L_{\rho, \Theta} F_2 + R_{ee}(F_1) = \partial_t F_0 + v \cdot \nabla_x F_1 - E \cdot \nabla_v F_1$$

with R_{ee} the second derivative of C_{ee} evaluated at $M_{\rho, \Theta}$. (The magnetic field effect disappears at this order.)

The energy conservation leads to the evolution equation for the temperature. Indeed, we have

$$\partial_t \int v^2 F_0 dv = -\nabla_x \cdot \int v v^2 F_1 dv,$$

which recasts as

$$\begin{aligned}
3\rho\partial_t\Theta &= \nabla_x \cdot \left[- \int vv^2 \otimes G_{\rho,\Theta}(t,x,v) \, dv \left(\frac{\nabla_x \rho}{\rho} - \frac{3}{2} \frac{\nabla_x \Theta}{\Theta} + \frac{E}{\Theta} \right) \right. \\
&\quad \left. - \int vv^2 \otimes H_{\rho,\Theta}(t,x,v) \, dv \frac{\nabla_x \Theta}{\Theta^2} \right] \\
&= \nabla_x \cdot \left[\int vv^2 \otimes G_{\rho,\Theta}(t,x,v) \, dv \left(\int v \otimes G_{\rho,\Theta}(t,x,v) \, dv \right)^{-1} \right. \\
&\quad \left. \times \int v \otimes H_{\rho,\Theta}(t,x,v) \, dv \frac{\nabla_x \Theta}{\Theta^2} \right. \\
&\quad \left. - \int vv^2 \otimes H_{\rho,\Theta}(t,x,v) \, dv \frac{\nabla_x \Theta}{\Theta^2} \right].
\end{aligned}$$

Owing to Lemma 2.3 and 2.4 we arrive at the following non linear diffusion equation

$$\partial_t\Theta = \frac{1}{3\rho} \nabla_x \cdot \left[\left(q_4 - \frac{q_2 q_3}{q_1} \right) \Theta^{5/2} \nabla_x \Theta \right].$$

This heat flux

$$(2.15) \quad Q_{SH} = -\frac{\bar{q}}{2} \Theta^{5/2} \nabla_x \Theta,$$

with $\bar{q} = q_4 - \frac{q_2 q_3}{q_1}$ corresponds to the so-called Spitzer-Härm heat flux, see [28].

Written in this way, it is not clear that the flux is defined with a positive coefficient. There is an alternative way to obtain the diffusion coefficient. Indeed, multiplying (2.12) by v and integrating, we obtain the following expression for the electric field

$$E = \Theta \left(\int_{\mathbb{R}^3} v \otimes v F_0 \, dv \right)^{-1} \left[\int_{\mathbb{R}^3} v L_{\rho,\Theta} F_1 \, dv - \int_{\mathbb{R}^3} v \otimes v \frac{v^2}{2} F_0 \, dv \frac{\nabla_x \Theta}{\Theta^2} \right] + \frac{3}{2} \nabla_x \Theta - \Theta \frac{\nabla_x \rho}{\rho}.$$

It recasts as

$$\begin{aligned}
E &= \frac{\Theta}{\rho\Theta} \left(\int_{\mathbb{R}^3} v L_{\rho,\Theta} F_1 \, dv - \frac{5}{2} \rho \Theta^2 \frac{\nabla_x \Theta}{\Theta^2} \right) + \frac{3}{2} \nabla_x \Theta - \Theta \frac{\nabla_x \rho}{\rho} \\
&= \frac{1}{\rho} \int_{\mathbb{R}^3} v L_{\rho,\Theta} F_1 \, dv - \nabla_x \Theta - \Theta \frac{\nabla_x \rho}{\rho}.
\end{aligned}$$

Therefore, (2.12) becomes

$$\mathcal{L}_{\rho,\Theta} F_1 = \frac{v}{2\Theta} \left(\frac{v^2}{\Theta} - 5 \right) F_0 \cdot \nabla_x \Theta,$$

where we have set

$$\mathcal{L}_{\rho,\Theta} F = L_{\rho,\Theta} F - \frac{v F_0}{\rho\Theta} \int_{\mathbb{R}^3} v_* L_{\rho,\Theta} F(v_*) \, dv_*.$$

In particular we observe that

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \mathcal{L}_{\rho,\Theta} F \, dv = 0, \quad \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \frac{v}{2\Theta} \left(\frac{v^2}{\Theta} - 5 \right) F_0 \, dv = 0.$$

We assume the following analog of (B2) for the modified operator:

B3) For any Φ verifying $\int_{\mathbb{R}^3} \Phi \, dv = \int_{\mathbb{R}^3} v^2 \Phi \, dv = 0$ and $\int_{\mathbb{R}^3} v \Phi \, dv = 0$, there exists a unique Γ such that $\mathcal{L}_{\rho, \Theta} \Gamma = \Phi$ with the constraints $\int_{\mathbb{R}^3} \Gamma \, dv = \int_{\mathbb{R}^3} v^2 \Gamma \, dv = 0$ and $\int_{\mathbb{R}^3} v \Gamma \, dv = 0$.

Clearly it holds when the estimate (2.13) is satisfied. We can define χ as the solution of

$$\mathcal{L}_{1,1} \chi = \frac{v}{2}(v^2 - 5) \frac{e^{-v^2/2}}{(2\pi)^{3/2}}, \quad \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} \chi(v) \, dv = 0$$

and, reasoning as in Lemma 2.3, we get

$$\mathcal{L}_{\rho, \Theta} \left[\frac{1}{\sqrt{\Theta(t, x)}} \chi \left(\frac{\cdot}{\sqrt{\Theta(t, x)}} \right) \right] (v) = \frac{1}{\sqrt{\Theta(t, x)}} \frac{v}{2\sqrt{\Theta}} \left(\frac{v^2}{\Theta} - 5 \right) \frac{\rho}{(2\pi\Theta)^{3/2}} e^{-v^2/(2\Theta)},$$

and accordingly

$$F_1(t, x, v) = \frac{1}{\sqrt{\Theta(t, x)}} \chi \left(\frac{v}{\sqrt{\Theta(t, x)}} \right) \cdot \nabla_x \Theta(t, x).$$

Finally, we obtain (using the fact that the flux $\int v F_1 \, dv$ vanishes)

$$\begin{aligned} \int_{\mathbb{R}^3} v v^2 F_1 \, dv &= 2\Theta^{3/2} \int_{\mathbb{R}^3} \frac{v}{2\sqrt{\Theta}} \left(\frac{v^2}{\Theta} - 5 \right) \otimes \frac{1}{\sqrt{\Theta(t, x)}} \chi \left(\frac{v}{\sqrt{\Theta(t, x)}} \right) \, dv \nabla_x \Theta \\ &= 2\Theta^{5/2} \int_{\mathbb{R}^3} \frac{v}{2} (v^2 - 5) \otimes \chi(v) \, dv \nabla_x \Theta. \end{aligned}$$

We observe that the matrix

$$\int_{\mathbb{R}^3} \frac{v}{2} (v^2 - 5) \otimes \chi(v) \, dv = \int_{\mathbb{R}^3} \mathcal{L}_{1,1} \chi \otimes \chi(v) \, dv$$

is non negative. Indeed, for any $\xi \in \mathbb{R}^3 \setminus \{0\}$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} \mathcal{L}_{1,1} \chi \otimes \chi(v) \, dv \xi \cdot \xi &= \int_{\mathbb{R}^3} \mathcal{L}_{1,1}(\chi \cdot \xi)(\chi \cdot \xi)(v) \, dv \\ &= \int_{\mathbb{R}^3} L_{1,1}(\chi \cdot \xi)(\chi \cdot \xi)(v) \, dv - \int_{\mathbb{R}^3} v \chi \cdot \xi \, dv \int_{\mathbb{R}^3} v L_{1,1}(\chi \cdot \xi) \, dv \\ &= \int_{\mathbb{R}^3} L_{1,1}(\chi \cdot \xi)(\chi \cdot \xi)(v) \, dv \geq 0. \end{aligned}$$

Actually, as observed in Lemma 2.3, see iii), it can be expressed as a mere scalar matrix since we have $\chi(v) = v \Xi(|v|) e^{-v^2/2}$, with $Z : \mathbb{R} \rightarrow \mathbb{R}$.

For some collision operators, we can explicitly solve the equations that define the corrector F_1 and in turn, we get an explicit formula for the diffusion coefficient. For example, consider the following combination :

$$\begin{aligned} C_{ee}(f)(v) &= \frac{1}{\tau_e} (M[f] - f)(v), \\ C_{ei}(f)(v) &= \frac{Z \rho_i}{\tau_i} \left(\int_{\mathbb{S}^2} f(|v|\omega) \, d\omega - f(v) \right), \end{aligned}$$

with $\tau_e \in \mathbb{R}$ and $\tau_i \in \mathbb{R}$. The linearized operator then reads

$$L_{\rho, \Theta} f = \frac{1}{\tau_e} (\mathbf{P}(f/M_{\rho, \Theta})M_{\rho, \Theta} - f) + Z_i \frac{\bar{Z}\rho_i}{\tau_i} \left(\int_{\mathbb{S}^2} f(|v|\omega) d\omega - f(v) \right)$$

where \mathbf{P} is the orthogonal projection of $L^2(M_{\rho, \Theta} dv)$ on $\text{Span}\{1, v, v^2\}$:

$$\mathbf{P}g = \frac{\tilde{\rho}}{\rho} + \frac{v \cdot \tilde{u}}{\Theta} + \frac{\tilde{\theta}}{2\Theta} \left(\frac{v^2}{\Theta} - 3 \right)$$

with

$$\begin{pmatrix} \tilde{\rho} \\ \rho \tilde{u} \\ 3\tilde{\rho}\Theta + 3\rho\tilde{\theta} \end{pmatrix} = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} g M_{\rho, \Theta} dv.$$

Then we obtain

$$\chi(v) = -\frac{\tau_e}{2} \frac{v(v^2 - 5)}{1 + Z_i \tau_e / \tau_i} \frac{e^{-v^2/2}}{(2\pi)^{3/2}},$$

and thus

$$\begin{aligned} E &= -\Theta (\nabla_x \ln \rho + \nabla_x \ln \Theta), \\ \bar{q} &= \frac{16\sqrt{2}}{3\sqrt{\pi}} \frac{\tau_e}{1 + Z_i \tau_e / \tau_i} \int_0^\infty \xi^{5/2} \left(\xi - \frac{5}{2} \right) e^{-\xi} d\xi = 80\sqrt{2} \frac{\tau_e}{1 + Z_i \tau_e / \tau_i}. \end{aligned}$$

For a general collision operator however there is no such explicit formula and the numerical evaluation of the coefficient might remain a difficulty. Continuing the Hilbert expansion at a higher order usually leads to an ill-posed problem. A rigorous proof of the asymptotic regime is certainly a tough piece of analysis; we refer for related problems to [5, 6]. We are now going to discuss an ε -dependent macroscopic model that can be found in the physics literature.

3. An asymptotic nonlocal model : the Schurtz-Nicolai model. For some applications in plasma physics, we need a more precise model of conduction, retaining more of the microscopic features. Let us describe here the derivation of an intermediate model, which is widely used in several plasma physics codes. According to experimental results, the idea consists in expressing the heat flux as a space convolution of the Spitzer-Härm heat flux, with a kernel $W_\varepsilon(x, x')$ that tends to a Dirac function as ε goes to 0:

$$Q(t, x) = \int_{\mathbb{R}^3} W_\varepsilon(x, x') Q_{SH}(t, x') dx'$$

This expression is inserted in (2.10) to close the equation, Q_{SH} being still defined by (2.15). This approach dates back to Luciani and Mora [21] and it has been revisited since then by many authors [14, 22, 23, 27]. The point is precisely to propose a relevant definition of the kernel W_ε . In what follows, we present a derivation inspired from the work of Schurtz and Nicolai [27]: the idea is to describe the heat flux as the result of the transport of the energy associated to the solution of (2.12). We set $\Omega = \frac{v}{|v|}$ and $r = |v|$ and we introduce $\mathcal{E}_{t,r}(x, \Omega)$, solution of

$$(3.1) \quad \mathcal{E} + \varepsilon \Omega \cdot \nabla_x \mathcal{E} = r^2 F_1(t, x, r\Omega).$$

Here $r^2 F_1/\varepsilon$ is seen as a source of energy and \mathcal{E} can be interpreted as the radiative intensity associated to this source, subject to a damping of order $1/\varepsilon$. As a matter of fact, when $\varepsilon = 0$, \mathcal{E} coincides to $r^2 F_1$. The nonlocal heat flux is defined by

$$Q(t, x) = \frac{1}{2} \int_{\mathbb{R}^3} v \mathcal{E} dv = \frac{1}{2} \int_0^\infty \int_{\mathbb{S}^2} r^3 \mathcal{E} dr \Omega d\Omega.$$

The solution of (3.1) can be obtained by using the method of characteristics and we get

$$\mathcal{E}_{t,r}(x, \Omega) = \int_0^\infty \frac{e^{-\frac{s}{\varepsilon}}}{\varepsilon} F_1(t, x - s\Omega, r\Omega) r^2 ds.$$

Remark that according Lemma 2.3, there exists a (isotropic) vector $\bar{F}_1(t, x, r)$ satisfying $F_1(t, x, r\Omega) = \Omega \cdot \bar{F}_1(t, x, r)$. Accordingly, we can write :

$$Q(t, x) = \int_0^\infty \int_{\mathbb{S}^2} \frac{e^{-\frac{s}{\varepsilon}}}{\varepsilon s^2} \int_0^\infty \frac{r^5}{2} \bar{F}_1(t, x - s\Omega, r) dr d\Omega s^2 ds.$$

The expression of the Spitzer-Härm heat flux is obtained from F_1 ; we remind that

$$\int_0^\infty \frac{r^5}{2} \bar{F}_1(t, x, r) dr = \frac{3}{4\pi} Q_{SH}(t, x).$$

We set $x' = s\Omega$, $dx' = s^2 ds d\Omega$. The nonlocal heat flux can be defined by the following convolution formula

$$Q(t, x) = \frac{3}{4\pi\varepsilon} \int_{\mathbb{R}^3} \frac{e^{-\frac{|x'|}{\varepsilon}}}{|x'|^2} Q_{SH}(t, x - x') dx'.$$

This expression makes a link between Schurtz-Nicolai's approach [27] and the seminal work of Luciani-Mora [21].

It turns out that a useful approximate formula can be derived for the non local heat flux. We set $q_t(x, \Omega) = \int_0^\infty \frac{r^3}{2} \mathcal{E}_{t,r}(x, \Omega) dr$. Multiply (3.1) by $\frac{r^3}{2}$ and integrate. It yields

$$q_t(x, \Omega) + \varepsilon \Omega \cdot \nabla_x q_t(x, \Omega) = \Omega \cdot \int_0^\infty \frac{r^5}{2} \bar{F}_1(t, x, r) dr = \frac{3}{4\pi} \Omega \cdot Q_{SH}(t, x).$$

Thus we can write the following moments system

$$\begin{aligned} \int_{\mathbb{S}^2} \Omega q_t(x, \Omega) d\Omega + \varepsilon \nabla_x \cdot \int_{\mathbb{S}^2} \Omega \otimes \Omega q_t(x, \Omega) d\Omega &= \frac{3}{4\pi} \int_{\mathbb{S}^2} \Omega \otimes \Omega d\Omega Q_{SH}(t, x) \\ \int_{\mathbb{S}^2} \Omega \otimes \Omega q_t(x, \Omega) d\Omega + \varepsilon \int_{\mathbb{S}^2} \Omega \otimes \Omega \nabla_x \cdot (\Omega q_t(x, \Omega)) d\Omega &= 0. \end{aligned}$$

We now make a new approximation in order to close this system. For small ε 's it is natural to expect that $\Omega q_t(x, \Omega)$ becomes isotropic which motivates the approximation $\Omega q_t(x, \Omega) \approx \frac{3}{4\pi} \Omega \otimes \Omega \int_{\mathbb{S}^2} \Omega' q_t(x, \Omega') d\Omega'$. Inserting this approximation in the moment system leads to

$$\begin{aligned} \int_{\mathbb{S}^2} \Omega q_t(x, \Omega) d\Omega - \varepsilon^2 \frac{3}{4\pi} \nabla_x \cdot \int_{\mathbb{S}^2} (\Omega \otimes \Omega) (\Omega \otimes \Omega) : \nabla_x \int_{\mathbb{S}^2} \Omega' q_t(x, \Omega') d\Omega' d\Omega \\ = \frac{3}{4\pi} \int_{\mathbb{S}^2} \Omega \otimes \Omega d\Omega Q_{SH}(t, x). \end{aligned}$$

Then, the nonlocal heat flux is defined, within this approximation, by the elliptic equation

$$(3.2) \quad Q - \frac{\varepsilon^2}{3} \Delta_x Q = Q_{SH},$$

coupled with equations (2.10) and (2.15). This simplified model, or variant of it, appears in several simulation codes. It is worth mentioning that similar equations have been derived in the modeling of electrostatic interactions in biomolecular processes, [19, 18]. We also refer to [3] for a similar derivation applied to radiative transfer problems. We can establish the following statement which provides a rigorous basis to the non local model.

THEOREM 3.1. *Let $\bar{q} > 0$ and suppose $\rho(x) \geq \underline{\rho} > 0$. Let $\Theta_0 \in L^\infty(\mathbb{R}^3)$ be a non negative function such that $\rho\Theta_0 \in L^1(\mathbb{R}^3)$. There exists a unique function such that $\rho\Theta \in C^1([0, +\infty[; L^1(\mathbb{R}^3))$ solution of the system (2.10), (2.15), (3.2) with Θ_0 as initial condition. In addition, Θ satisfies the maximum principle: for any $t \geq 0$ and a.e. $x \in \mathbb{R}^3$,*

$$(3.3) \quad 0 \leq \min_{z \in \mathbb{R}^3} \Theta_0(z) \leq \Theta(t, x) \leq \max_{z \in \mathbb{R}^3} \Theta_0(z).$$

Proof. We denote by \mathcal{F} the Fourier transform, by \mathcal{F}^{-1} the inverse Fourier transform

$$\mathcal{F}(\psi)(w) = \int_{\mathbb{R}^3} \psi(x) e^{-ix \cdot w} dx, \quad \mathcal{F}^{-1}(\phi)(x) = \left(\frac{1}{2\pi}\right)^3 \int_{\mathbb{R}^3} \phi(w) e^{ix \cdot w} dw,$$

with w the Fourier space variable. We start by rewriting the system (3.2) in the Fourier space. Since

$$\mathcal{F}Q(w) = \frac{1}{1 + \varepsilon^2 w^2 / 3} \mathcal{F}Q_{SH}(w),$$

we get

$$\frac{3}{2} \partial_t \mathcal{F}(\rho\Theta) = -\mathcal{F}(\nabla_x \cdot Q) = +iw\mathcal{F}Q = -\frac{3\bar{q}}{7\varepsilon^2} \left(1 - \frac{1}{1 + \varepsilon^2 w^2 / 3}\right) \mathcal{F}(\Theta^{7/2}).$$

Back to the real space, it becomes:

$$\rho \partial_t \Theta = -\frac{2\bar{q}}{7\varepsilon^2} \left(\Theta^{7/2} - W_\varepsilon \star \Theta^{7/2}\right),$$

with

$$W_\varepsilon = \mathcal{F}^{-1} \left(\frac{1}{1 + \varepsilon^2 w^2 / 3} \right).$$

We have the following expression for the convolution kernel, see [4, Section 6.5],

$$W_\varepsilon(x) = \frac{3}{4\pi\varepsilon^2|x|} e^{-\sqrt{3}|x|/\varepsilon}.$$

This kernel is positive and unitary in the sense that

$$W_\varepsilon \geq 0 \quad \text{and} \quad W_\varepsilon \star 1 = \frac{3}{4\pi\varepsilon^2} \int_{\mathbb{R}^3} \frac{e^{-\sqrt{3}|x|/\varepsilon}}{|x|} dx = \frac{3}{\varepsilon^2} \int_0^\infty z e^{-\sqrt{3}z/\varepsilon} dz = 1$$

holds. It is also worth pointing out that

$$\frac{1}{\varepsilon^2} \left(1 - \frac{1}{1 + \varepsilon^2 w^2 / 3} \right) \rightarrow \frac{w^2}{3}$$

as ε goes to 0, so that the operator $\frac{1}{\varepsilon^2}(I - W_\varepsilon \star)$ tends to $\frac{\Delta_x}{3}$, which is another way to check that the non local model is formally consistent with the Spitzer–Härm limit. For proving Theorem 3.1, we introduce a cut-off of the non-linearity: the L^∞ estimate will show that the cut-off is actually not relevant. We set $M = \max \Theta_0 > 0$ (bearing in mind that Θ_0 is non negative). Then we define

$$f_M(\psi) = \frac{2\bar{q}}{7} \begin{cases} \psi^{7/2}, & \text{if } \psi \leq M, \\ \frac{7}{2}M^{5/2}\psi - \frac{5}{2}M^{7/2} & \text{else.} \end{cases}$$

The function $\psi \mapsto f_M(\psi)$ is non-decreasing and ($L = \bar{q}M^{5/2}$)-Lipschitzian. Let us consider the auxiliary problem

$$(3.4) \quad \begin{cases} \rho \partial_t u + f_M(u) = g, \\ u(t=0) = \Theta_0, \end{cases}$$

with $\rho g \in C^0([0, +\infty[; L^1(\mathbb{R}^3))$. With a standard contraction argument, we show that there exists a unique solution u of (3.4) with $\rho u \in C^1([0, +\infty[; L^1(\mathbb{R}^3))$. Observe also that $u \geq 0$ when $g \geq 0$ and $\Theta_0 \geq 0$. Let $(u_n)_{n \in \mathbb{N}}$ be the sequence defined by $u_0(t, x) = \Theta_0(x)$ and

$$(3.5) \quad \begin{cases} \rho \partial_t u_{n+1} + f_M(u_{n+1}) = W_\varepsilon \star f_M(u_n), \\ u_{n+1}(t=0, x) = \Theta_0(x). \end{cases}$$

This sequence is well-defined because u_{n+1} is solution of (3.4) with $g = W_\varepsilon \star f_M(u_n)$. So, we have, for any n , $\rho u_n \in C^1([0, +\infty[; L^1(\mathbb{R}^3))$.

We have

$$\rho \partial_t (u_{n+1} - u_n) + f_M(u_{n+1}) - f_M(u_n) = W_\varepsilon \star f_M(u_n) - W_\varepsilon \star f_M(u_{n-1}).$$

We set

$$\text{sg}(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ -1 & \text{else.} \end{cases}$$

Then we get

$$\begin{aligned} \rho \partial_t (u_{n+1} - u_n) \text{sg}(u_{n+1} - u_n) + (f_M(u_{n+1}) - f_M(u_n)) \text{sg}(u_{n+1} - u_n) \\ = (W_\varepsilon \star f_M(u_n) - W_\varepsilon \star f_M(u_{n-1})) \text{sg}(u_{n+1} - u_n). \end{aligned}$$

Since f_M is non decreasing, we have $(f_M(u_{n+1}) - f_M(u_n)) \text{sg}(u_{n+1} - u_n) \geq 0$, and, since W_ε is non negative and unitary, we get

$$\begin{aligned} \int_{\mathbb{R}^3} \left| (W_\varepsilon \star f_M(u_n) - W_\varepsilon \star f_M(u_{n-1})) \text{sg}(u_{n+1} - u_n) \right| dx &\leq \int_{\mathbb{R}^3} W_\varepsilon \star (L|u_n - u_{n-1}|) dx \\ &\leq L \int_{\mathbb{R}^3} |u_n - u_{n-1}| dx. \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^3} \rho |u_{n+1} - u_n|(t, x) \, dx \leq \frac{L}{\rho} \int_0^t \int_{\mathbb{R}^3} \rho |u_n - u_{n-1}|(s, x) \, dx \, ds.$$

We deduce that

$$\|\rho(u_{n+1} - u_n)(t, \cdot)\|_{L^1(\mathbb{R}^3)} \leq \frac{(Lt/\rho)^{n+1}}{(n+1)!} \sup_{s \in [0, t]} \|\rho(u_1 - u_0)(s, \cdot)\|_{L^2(\mathbb{R}^3)}$$

holds. Since the series $\sum_n \frac{(Lt/\rho)^{n+1}}{(n+1)!}$ converges, it implies that the sequence $(u_n)_{n \in \mathbb{N}}$ satisfies the Cauchy criterion and thus it has a limit Θ with $\rho\Theta \in C^0([0, +\infty[; L^1(\mathbb{R}^3))$. Passing to the limit in (3.5) we observe that Θ satisfies

$$\rho \partial_t \Theta + f_M(\Theta) = W_\varepsilon \star f_M(\Theta),$$

with initial data Θ_0 . It remains to discuss the L^∞ estimate.

Of course, the initial guess verifies $u_0(t, x) = \Theta_0(x) \leq M$. Let us now assume that $\max_{x \in \mathbb{R}^3} u_n(t, x) \leq M$ holds. Since W_ε is unitary, we have $W_\varepsilon \star f_M(M) = f_M(M)$. Therefore, we can write

$$\rho \partial_t (u_{n+1} - M) + f_M(u_{n+1}) - f_M(M) = W_\varepsilon \star f_M(u_n) - W_\varepsilon \star f_M(M).$$

We use the function

$$(\psi)_+ = \begin{cases} \psi & \text{if } \psi > 0, \\ 0 & \text{else.} \end{cases}$$

Then we have

$$\begin{aligned} \rho \partial_t (u_{n+1} - M) (u_{n+1} - M)_+ + (f_M(u_{n+1}) - f_M(M)) (u_{n+1} - M)_+ \\ = (W_\varepsilon \star f_M(u_n) - W_\varepsilon \star f_M(M)) (u_{n+1} - M)_+. \end{aligned}$$

Since f_M is non decreasing, we observe that

$$(f_M(u_{n+1}) - f_M(M)) (u_{n+1} - M)_+ \geq 0,$$

and, since W_ε is non negative, we get

$$W_\varepsilon \star (f_M(u_n) - f_M(M)) (u_{n+1} - M)_+ \leq 0.$$

It follows that

$$\frac{d}{dt} \int_{\mathbb{R}^3} \rho (u_{n+1} - M)_+^2(t, x) \, dx \leq 0.$$

Since initially $(u_{n+1}(0, x) - M) = (\Theta_0(x) - M) \leq 0$, we conclude that $u_{n+1}(t, x) \leq M$. With similar arguments, we treat the bound from below and we finally obtain, for any $n \in \mathbb{N}$,

$$\min_{z \in \mathbb{R}^3} \Theta_0(z) \leq u_n(t, x) \leq \max_{z \in \mathbb{R}^3} \Theta_0(z).$$

In particular, we have

$$f_M(u_n(x, t)) = \frac{2\bar{q}}{7} u_n(x, t)^{7/2}.$$

Therefore, u_n converges to the solution Θ of the system (2.10), (2.15), (3.2) and it satisfies (3.3). ■

In addition, it is worth pointing conservation and dissipation properties of the model.

PROPOSITION 3.2. *The total energy is conserved*

$$(3.6) \quad \frac{d}{dt} \int_{\mathbb{R}^3} \rho \Theta \, dx = 0.$$

Suppose furthermore that $\rho = \rho_\star > 0$ is constant. Then, the following entropy dissipation holds

$$(3.7) \quad \frac{d}{dt} \left(\int_{\mathbb{R}^3} \rho_\star |\Theta|^2 \, dx + \frac{\varepsilon^2}{3} \int_{\mathbb{R}^3} \rho_\star |\nabla_x \Theta|^2 \, dx \right) \leq 0$$

Proof. The energy conservation follows directly by integration of equation (2.10). Next, we apply the operator $(Id - \frac{\varepsilon^2}{3} \Delta_x)$ to (2.10). Then we use (3.2) to make the Spitzer-Härm heat flux appear. We obtain

$$\left[Id - \frac{\varepsilon^2}{3} \Delta_x \right] \partial_t \Theta = \frac{2}{3\rho_\star} \nabla_x \cdot (\kappa \nabla_x \Theta).$$

Multiply by Θ , and integrate with respect to the space variable. Using several integrations by part we obtain

$$\frac{d}{dt} \left(\int_{\mathbb{R}^3} |\Theta|^2 \, dx + \frac{\varepsilon^2}{3} \int_{\mathbb{R}^3} |\nabla_x \Theta|^2 \, dx \right) = -\frac{4}{3\rho_\star} \int_{\mathbb{R}^3} \kappa |\nabla_x \Theta|^2 \, dx \leq 0.$$

■

In the next section, we propose a numerical scheme to solve efficiently the system (2.10), (2.15), (3.2), and preserving the properties in Proposition 3.2 and the maximum principle (3.3).

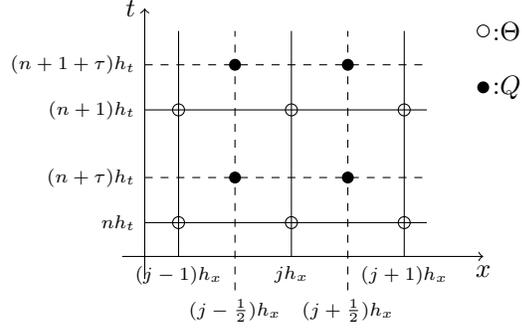
4. Numerical analysis.

4.1. A numerical scheme for the Schurtz-Nicolai model. For the sake of simplicity we discuss the numerical issues by restricting ourselves to the one-dimension framework. However, we consider models slightly more general than equation (3.2) derived above. Precisely let us consider two (smooth) positive functions $\nu : \mathbb{R} \rightarrow (0, \infty)$ and $\kappa : \mathbb{R} \rightarrow (0, \infty)$. We are concerned with the numerical approximation of the system

$$(4.1) \quad \partial_t \Theta = -\frac{2}{3\rho} \partial_x Q,$$

$$(4.2) \quad Q - \varepsilon^2 \nu(\Theta) \partial_x^2 Q = -\kappa(\Theta) \partial_x \Theta,$$

with a prescribed initial data $\Theta(t=0, x) = \Theta_{Init}(x)$. For the time being we neglect the question of the boundary condition and we consider the problem as set on the whole line $x \in \mathbb{R}$ (see Remark 4.3 below). The function κ is related to the definition of the Spitzer-Härm flux and relies on the identification of the coefficient in (2.15). The additional function ν has been introduced as a tuning parameter to fit the results

FIG. 1. *Staggered grids*

with kinetic simulations, see [21, 22, 23, 27]. Our goal is to derive and analyze a numerical scheme for (4.1)–(4.2).

Let $h_t > 0$ and $h_x > 0$ stand for time and space steps, respectively. The scheme is based on the approximation of the following integrated form of (4.1), at the point jh_x

$$\Theta((n+1)h_t, jh_x) = \Theta(nh_t, jh_x) - \frac{2}{3\rho(jh_x)} \left(\partial_x \left(\int_{nh_t}^{(n+1)h_t} Q(s, x) ds \right) \right)_{x=jh_x}$$

together with (4.2). We adopt a Finite Difference viewpoint but temperature and fluxes will be evaluated on staggered grids, see Fig. 1. The numerical unknown Θ_j^n , with $n \in \mathbb{N}$ and $j \in \mathbb{Z}$, is intended to be an approximation of $\Theta(nh_t, jh_x)$. We denote $\rho_j = \rho(jh_x)$, with $\rho : \mathbb{R} \rightarrow (0, \infty)$ a given function. We define the scheme by

$$\Theta_j^{n+1} = \Theta_j^n - h_t \frac{2}{3\rho_j} (\nabla^D \cdot \mathcal{Q}^{n, n+1})_j$$

where we are left with the task of defining the numerical flux $\mathcal{Q}_{j+1/2}^{n, n+1}$ intended to approximate the flux

$$\frac{1}{h_t} \int_{nh_t}^{(n+1)h_t} Q(s, (j+1/2)h_x) ds,$$

and the numerical operator $\nabla^D \cdot \bullet$ intended to be a discrete version of the derivative $\partial_x \bullet$.

For further purposes it is convenient to introduce the following operators

$$\begin{aligned} \nabla^D : (\Theta_j)_{j \in \mathbb{Z}} &\longmapsto \left((\nabla^D \Theta)_{j+1/2} = \frac{\Theta_{j+1} - \Theta_j}{h_x} \right)_{j \in \mathbb{Z}}, \\ \nabla^D \cdot : (Q_{j+1/2})_{j \in \mathbb{Z}} &\longmapsto \left((\nabla^D \cdot Q)_j = \frac{Q_{j+1/2} - Q_{j-1/2}}{h_x} \right)_{j \in \mathbb{Z}}, \\ \Delta^D : (\Theta_j)_{j \in \mathbb{Z}} &\longmapsto \left((\Delta^D \Theta)_j = \frac{\Theta_{j+1} - 2\Theta_j + \Theta_{j-1}}{h_x^2} \right)_{j \in \mathbb{Z}}. \end{aligned}$$

Note that $(\nabla^D \cdot) \circ \nabla^D = \Delta^D$. We shall also use the following definition, with a slight abuse of notation,

$$\Delta^D : (Q_{j+1/2})_{j \in \mathbb{Z}} \mapsto \left((\Delta^D Q)_{j+1/2} = \frac{Q_{j+3/2} - 2Q_{j+1/2} + Q_{j-1/2}}{h_x^2} \right)_{j \in \mathbb{Z}}.$$

A naive approach consists in using an explicit discretization: we set $\mathcal{Q}_{j+1/2}^{n,n+1} = Q_{j+1/2}^n$, intended to be an approximation of $Q(nh_t, (j+1/2)h_x)$, where the approximation of (4.2) yields

$$Q_{j+1/2}^n - \varepsilon^2 \nu_{j+1/2}^n \Delta^D Q_{j+1/2}^n = -\kappa_{j+1/2}^n (\nabla^D \Theta^n)_{j+1/2}.$$

Here, for the nonlinear terms we use

$$\nu_{j+1/2}^n = \nu \left(\frac{\Theta_{j+1}^n + \Theta_j^n}{2} \right), \quad \kappa_{j+1/2}^n = \kappa \left(\frac{\Theta_{j+1}^n + \Theta_j^n}{2} \right).$$

But for $\varepsilon = 0$ the problem reduces to a nonlinear heat equation and the scheme we are writing is nothing but a mere explicit scheme for the heat equation. In particular it will be certainly constrained by a parabolic CFL condition where the time step h_t should be proportional to h_x^2 . Such a condition induces a prohibitive numerical cost and in particular the computation will not be affordable for any extension to multi-dimensional problems. This remark motivates to seek an implicit definition of the numerical fluxes. The scheme we propose is based on the observation that (4.1) can be approached by $Q((n+\tau)h_t, (j+1/2)h_x)$ for $0 \leq \tau \leq 1$. Hence pick $0 \leq \tau \leq 1$; we define the numerical flux $\mathcal{Q}_{j+1/2}^{n,n+1} = Q_{j+1/2}^{n+\tau}$ which has to be thought of as an approximation of the flux $Q((n+\tau)h_t, (j+1/2)h_x)$. It leads to the following discrete version of (4.2)

$$(4.3) \quad Q_{j+1/2}^{n+\tau} - \varepsilon^2 \nu_{j+1/2}^{n+\tau} \Delta^D Q_{j+1/2}^{n+\tau} = -\kappa_{j+1/2}^{n+\tau} \left(\tau (\nabla^D \Theta^{n+1})_{j+1/2} + (1-\tau) (\nabla^D \Theta^n)_{j+1/2} \right).$$

It is coupled to

$$(4.4) \quad \Theta_j^{n+1} = \Theta_j^n - h_t \frac{2}{3\rho_j} \frac{Q_{j+1/2}^{n+\tau} - Q_{j-1/2}^{n+\tau}}{h_x} = \Theta_j^n - h_t \frac{2}{3\rho_j} (\nabla^D \cdot Q^{n+\tau})_j.$$

Inserting (4.4) into (4.3) yields

$$Q_{j+1/2}^{n+\tau} - \varepsilon^2 \nu_{j+1/2}^{n+\tau} \Delta^D Q_{j+1/2}^{n+\tau} - \frac{2\tau h_t}{3} \kappa_{j+1/2}^{n+\tau} \nabla^D \left(\frac{1}{\rho} \nabla^D \cdot Q^{n+\tau} \right)_{j+1/2} = -\kappa_{j+1/2}^{n+\tau} (\nabla^D \Theta^n)_{j+1/2}.$$

The detailed expression of the third term in the left side is

$$\nabla^D \left(\frac{1}{\rho} \nabla^D \cdot Q^{n+\tau} \right)_{j+1/2} = \frac{1}{h_x^2} \left(\frac{Q_{j+3/2}^{n+\tau} - Q_{j+1/2}^{n+\tau}}{\rho_{j+1}} - \frac{Q_{j+1/2}^{n+\tau} - Q_{j-1/2}^{n+\tau}}{\rho_j} \right).$$

Finally, the nonlinearities are defined as follows

$$\begin{aligned} \nu_{j+1/2}^{n+\tau} &= \nu \left(\tau \frac{\Theta_{j+1}^{n+1} + \Theta_j^{n+1}}{2} + (1-\tau) \frac{\Theta_{j+1}^n + \Theta_j^n}{2} \right), \\ \kappa_{j+1/2}^{n+\tau} &= \kappa \left(\tau \frac{\Theta_{j+1}^{n+1} + \Theta_j^{n+1}}{2} + (1-\tau) \frac{\Theta_{j+1}^n + \Theta_j^n}{2} \right). \end{aligned}$$

h_x in the Crank-Nicolson case ($\tau = 1/2$). Anyway, the scheme is stable under the standard parabolic CFL condition $\frac{2h_t}{h_x^2} \leq 1$.

Proof. We associate to a sequence $(u_j)_{j \in \mathbb{Z}}$ the Fourier series $\mathcal{F}_D(u)(\xi) = \sum_{j \in \mathbb{Z}} u_j e^{2i\pi j \xi}$. The amplification factor of the scheme (4.6) reads

$$\frac{\mathcal{F}_D(\tilde{\Theta}^{n,r})}{\mathcal{F}_D(\Theta^n)}(\xi) = \frac{1 - \frac{2}{h_x^2} \left(\varepsilon^2 \nu_\star + \frac{2(\tau-1)h_t \kappa_\star}{3\rho_\star} \right) (1 - \cos \xi)}{1 - \frac{2}{h_x^2} \left(\varepsilon^2 \nu_\star + \frac{2\tau h_t \kappa_\star}{3\rho_\star} \right) (1 - \cos \xi)}.$$

The amplification factor satisfies $|\frac{\mathcal{F}_D(\Theta^{n+1})}{\mathcal{F}_D(\Theta^n)}(\xi)| \leq 1$ iff the condition (4.7) holds. A similar stability analysis appeared in [3]. ■

THEOREM 4.2 (Maximum principle). *We suppose that $\rho = \rho_\star > 0$ is constant and that $0 < \underline{\nu} \leq \nu(z) \leq \bar{\nu}$, $0 < \underline{\kappa} \leq \kappa(z) \leq \bar{\kappa}$ holds for any $z \geq 0$. Assume that*

$$(4.8) \quad \frac{3\varepsilon^2 \rho_\star \bar{\nu}}{2(1-\tau)\underline{\kappa}} \leq h_t \leq \frac{3\rho_\star}{4(1-\tau)\bar{\kappa}} (h_x^2 + 2\varepsilon^2 \underline{\nu}).$$

Initially the data is required to satisfy $0 < \Theta_j^0 \leq \bar{\Theta}$. Then, the solution of the scheme (4.6) satisfies the same inequality.

Proof. Assuming that $\rho_j = \rho_\star$, we rewrite the scheme (4.6) as follows

$$(4.9) \quad (1 + a_j) \tilde{\Theta}_j^{n,r+1} = a_j^+ \tilde{\Theta}_{j+1}^{n,r+1} + a_j^- \tilde{\Theta}_{j-1}^{n,r+1} + (1 + \gamma_j) \Theta_j^n + b_j^+ \Theta_{j+1}^n + b_j^- \Theta_{j-1}^n$$

where we use the shorthand notation

$$\begin{aligned} a_j &= \frac{2\tau h_t}{3\rho_\star h_x^2} (\tilde{\kappa}_{j+1/2}^{n,r} + \tilde{\kappa}_{j-1/2}^{n,r}) + \frac{\varepsilon^2}{h_x^2} (\tilde{\nu}_{j+1/2}^{n,r} + \tilde{\nu}_{j-1/2}^{n,r}), \\ a_j^\pm &= \frac{2\tau h_t}{3\rho_\star h_x^2} \tilde{\kappa}_{j\pm 1/2}^{n,r} + \frac{\varepsilon^2}{h_x^2} \tilde{\nu}_{j\pm 1/2}^{n,r}, \\ \gamma_j &= -\frac{2(1-\tau)h_t}{3\rho_\star h_x^2} (\tilde{\kappa}_{j+1/2}^{n,r} + \tilde{\kappa}_{j-1/2}^{n,r}) + \frac{\varepsilon^2}{h_x^2} (\tilde{\nu}_{j+1/2}^{n,r} + \tilde{\nu}_{j-1/2}^{n,r}), \\ b_j^\pm &= \frac{2(1-\tau)h_t}{3\rho_\star h_x^2} \tilde{\kappa}_{j\pm 1/2}^{n,r} - \frac{\varepsilon^2}{h_x^2} \tilde{\nu}_{j\pm 1/2}^{n,r}. \end{aligned}$$

The condition (4.8), implies that the coefficients satisfy

$$a_j \geq 0, \quad a_j^\pm \geq 0, \quad b_j^\pm \geq 0, \quad 1 + \gamma_j \geq 0$$

with furthermore

$$1 + \gamma_j + b_j^+ + b_j^- = 1, \quad a_j = a_j^+ + a_j^-.$$

Let us denote $\mu^n = \sup_{j \in \mathbb{Z}} \Theta_j^n$ and $\mu^{n,r+1} = \sup_{j \in \mathbb{Z}} \tilde{\Theta}_j^{n,r+1}$. Since the coefficients in the right hand side of (4.9) are non negative, we have

$$\tilde{\Theta}_j^{n,r+1} \leq \frac{\mu^{n,r+1}(a_j^+ + a_j^-) + \mu^n}{1 + a_j}.$$

We argue by contradiction: let us assume that there exists $0 < \eta < 1$ such that $\mu^n = (1 - \eta)\mu^{n,r+1} < \mu^{n,r+1}$. Since the a_j 's are bounded (uniformly with respect to j),

$$0 \leq a_j \leq \frac{4\tau h_t}{3\rho_\star h_x^2} \bar{\kappa} + \frac{2\varepsilon^2}{h_x^2} \bar{\nu} = M,$$

the previous inequality becomes

$$\tilde{\Theta}_j^{n,r+1} \leq \mu^{n,r+1} \left(1 - \frac{\eta}{1+M}\right),$$

which would contradict the definition of $\mu^{n,r+1}$. \blacksquare

REMARK 4.2. *Note that the condition (4.8) is consistent when $\varepsilon = 0$ with the usual condition, which makes a proportionality constraint between h_t and h_x^2 appear, for proving the maximum principle (see [26, Theorem 2.2]).*

PROPOSITION 4.3 (Energy conservation). *The solution of the scheme satisfies the discrete analog of property (3.6):*

$$\sum_{j \in \mathbb{Z}} \rho_j \Theta_j^n = \sum_{i \in \mathbb{Z}} \rho_i \Theta_i^0.$$

Proof. Assume this equality is true at iteration n . Using (4.5), we have

$$\sum_{j \in \mathbb{Z}} \rho_j \tilde{\Theta}_j^{n,r} = \sum_{j \in \mathbb{Z}} \rho_j \Theta_j^n - \frac{2h_t}{h_x} \left(\sum_{j \in \mathbb{Z}} \tilde{Q}_{j+1/2}^{n,r} - \sum_{j \in \mathbb{Z}} \tilde{Q}_{j-1/2}^{n,r} \right) = \sum_{j \in \mathbb{Z}} \rho_j \Theta_j^n.$$

\blacksquare

PROPOSITION 4.4 (Entropy dissipation). *Let $\tau = 1/2$, and assume that $\rho = \rho_\star > 0$ is constant. The solution of the scheme satisfies the discrete analog of property (3.7):*

$$\sum_{j \in \mathbb{Z}} \left(\tilde{\Theta}_j^{n,r+1} \right)^2 + \varepsilon^2 \sum_{j \in \mathbb{Z}} \left(\tilde{\nu}_{j+1/2}^{n,r} |\nabla^D \tilde{\Theta}^{n,r+1}|_{j+1/2}^2 \right) \leq \sum_{j \in \mathbb{Z}} \left(\Theta_j^n \right)^2 + \varepsilon^2 \sum_{j \in \mathbb{Z}} \left(\tilde{\nu}_{j+1/2}^{n,r} |\nabla^D \tilde{\Theta}^n|_{j+1/2}^2 \right).$$

Proof. First, we make some general observation. Let $(A_j)_{j \in \mathbb{Z}}, (B_j)_{j \in \mathbb{Z}}$ and $(\alpha_{j+1/2})_{j \in \mathbb{Z}}$ be given sequences. Remark that we have

$$\begin{aligned} (4.10) \quad \sum_{j \in \mathbb{Z}} A_j \nabla^D \cdot (\alpha \nabla^D B)_j &= \sum_{j \in \mathbb{Z}} \frac{\alpha_{j+1/2}}{h_x^2} (A_{j+1} B_j + A_j B_{j+1} - A_j B_j - A_{j+1} B_{j+1}) \\ &= - \sum_{j \in \mathbb{Z}} \alpha_{j+1/2} (\nabla^D A)_{j+1/2} (\nabla^D B)_{j+1/2} \end{aligned}$$

Therefore, we multiply (4.6) by $(\tilde{\Theta}_j^{n,r+1} + \Theta_j^n)$ and we write the result as follows

$$\begin{aligned} & \left(\tilde{\Theta}_j^{n,r+1} + \Theta_j^n \right) \left(\tilde{\Theta}_j^{n,r+1} - \Theta_j^n \right) - \varepsilon^2 \left(\tilde{\Theta}_j^{n,r+1} + \Theta_j^n \right) \nabla^D \cdot \left(\tilde{\nu}^{n,r} \nabla^D (\tilde{\Theta}^{n,r+1} - \Theta^n) \right)_j \\ &= \frac{2h_t}{3\rho_\star} \left(\tilde{\Theta}_j^{n,r+1} + \Theta_j^n \right) \nabla^D \cdot \left(\tilde{\kappa}^{n,r} \nabla^D \left(\tau \tilde{\Theta}^{n,r+1} + (1-\tau) \Theta^n \right) \right)_j. \end{aligned}$$

Then using (4.10), we have

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \left(\left(\tilde{\Theta}_j^{n,r+1} \right)^2 - \left(\Theta_j^n \right)^2 \right) + \varepsilon^2 \sum_{j \in \mathbb{Z}} \left(\tilde{\nu}_{j+1/2}^{n,r} \left(\left| (\nabla^D \tilde{\Theta}^{n,r+1})_{j+1/2} \right|^2 - \left| (\nabla^D \Theta^n)_{j+1/2} \right|^2 \right) \right) \\ &= -\frac{2h_t}{3\rho_\star} \sum_{j \in \mathbb{Z}} \tilde{\kappa}_{j+1/2}^{n,r} \left(\tau \left| (\nabla^D \Theta^{n,r+1})_{j+1/2} \right|^2 + (1-\tau) \left| (\nabla^D \Theta^n)_{j+1/2} \right|^2 \right. \\ & \quad \left. + (\nabla^D \Theta^{n,r})_{j+1/2} (\nabla^D \Theta^n)_{j+1/2} \right). \end{aligned}$$

For $\tau = 1/2$, we conclude by factorization that

$$\begin{aligned} & \sum_{j \in \mathbb{Z}} \left(\left(\tilde{\Theta}_j^{n,r+1} \right)^2 - \left(\Theta_j^n \right)^2 \right) + \varepsilon^2 \sum_{j \in \mathbb{Z}} \left(\tilde{\nu}_{j+1/2}^{n,r} \left(\left| (\nabla^D \tilde{\Theta}^{n,r+1})_{j+1/2} \right|^2 - \left| (\nabla^D \Theta^n)_{j+1/2} \right|^2 \right) \right) \\ &= -\frac{4h_t}{3\rho_\star} \sum_{j \in \mathbb{Z}} \tilde{\kappa}_{j+1/2}^{n,r} \left| \left(\nabla^D \frac{\tilde{\Theta}^{n,r} + \Theta^n}{2} \right)_{j+1/2} \right|^2 \leq 0 \end{aligned}$$

holds. ■

REMARK 4.3. *The previous discussion does not account for boundary conditions. In practice of course the index j lies in a bounded domain, say $j \in \{1, \dots, J\}$. The scheme involves “ghost” points with indices out of this range, the values of which need to be prescribed by boundary conditions. It turns out that the formulation (4.5) is well adapted to Neumann-like conditions where the flux Q is prescribed at the boundary: we impose the values of $Q_{1/2}$ and $Q_{J+1/2}$ by using the prescribed flux. The formulation (4.6) is well adapted to Dirichlet conditions where the unknown Θ is imposed at the boundary: the value of Θ_0 and Θ_{J+1} is given by the Dirichlet conditions. The adaptation of Theorems 4.1 and 4.2, and Propositions 4.3 and 4.4 to these contexts is straightforward, with energy and entropy inequalities involving the prescribed boundary terms.*

4.2. Numerical results. A key features of ICF simulations is the possible occurrence of “antidiffusive effects”, see e. g. [14], such that the heat flux follows the direction of the temperature gradient. We illustrate this fact in Figure 2. This figure has been obtained by using the kinetic code `fpelec` developed at the Atomic Energy Commission [8]. In fact the code works on a simplified version of the Landau-Fokker-Planck equation considered here, based on a truncated expansion on spherical harmonics, together with the constraint of vanishing current. The boundary condition guarantees the conservation of energy. This model is more easily amenable to a numerical treatment but it preserves the main features of the original equation. The simulation is performed with a atomic number $Z_i = 4$, and a density constant equal to $2.5 \times 10^{21} \text{ g.cm}^{-3}$. The size of the domain is 10^{-2} cm . We plot the temperature $\Theta(t, x)$ for several times, as well as the product of the heat flux Q by the temperature gradient $\partial_x \Theta$. The antidiffusion is characterized by regions where the product $Q \partial_x \Theta$ takes positive values (see Figure 2(b)). The Spitzer-Härm model completely misses such a phenomena since, by definition, we have $Q_{SH} \cdot \partial_x \Theta = -\frac{\bar{q}}{2} \Theta^{5/2} |\partial_x \Theta|^2 < 0$. We illustrate the ability of the asymptotic model (4.1)-(4.2) in capturing the antidiffusive effect, see Figure 3. The comparison with the kinetic simulation cannot be fair since we do not have access to all physical data, nevertheless we can bring out

the main features of the nonlocal model. We consider the normalized space domain $(0, 1)$ and the initial data reads

$$\Theta_{init}(x) = (1 - (0.3 - 1) \times 2x) \mathbb{1}_{[0,0.5]}(x) + (1 - (0.3 - 1) \times 2(x - 0.5)) \mathbb{1}_{]0.5,1]}(x),$$

see Figure 3(a). We use the scheme (4.5). Of course, with $\varepsilon = 0$ it also provides a scheme for the Spitzer-Härm equation. We perform simulations for several values of the scaling parameter ε . The simulations are performed with $\tau = 1/2$, $h_x = 1/100$, and $j \in \llbracket 1, 99 \rrbracket$. In order to conserve the total energy, we impose that the boundary fluxes vanish ($Q_{1/2} = Q_{99+1/2} = 0$). The (common) time step is chosen so that the stability condition (4.8) for the maximum principle is satisfied. We remind that ν is a phenomenological nonlinearity introduced in many hydrodynamic codes to fit with kinetic simulations. Here we restrict to the simple case $\nu = 1$. As expected the smaller ε , the smoother the temperature profile, see Figures 3(c) and 3(e). In Figures 3(b), 3(d) and 3(f), we can observe the antidiffusive effects for positive values of ε , with quite sharp profiles for the largest ε 's. This effect reduces as time grows, as observed with the kinetic results. We point out that running the hydrodynamic code is very fast. This aspect is particularly important since for ICF simulation the computation of the electronic temperature is a small piece of a large hydrodynamic code involving coupled systems of PDEs.

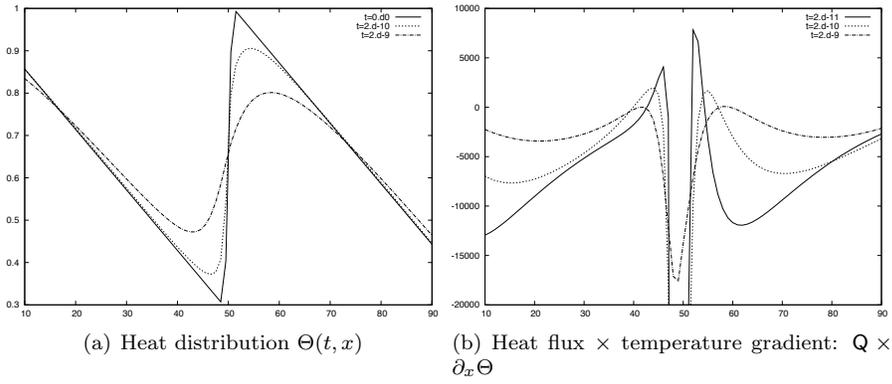


FIG. 2. Kinetic results: Simulation of a kinetic model (`fpelec` code from CEA)

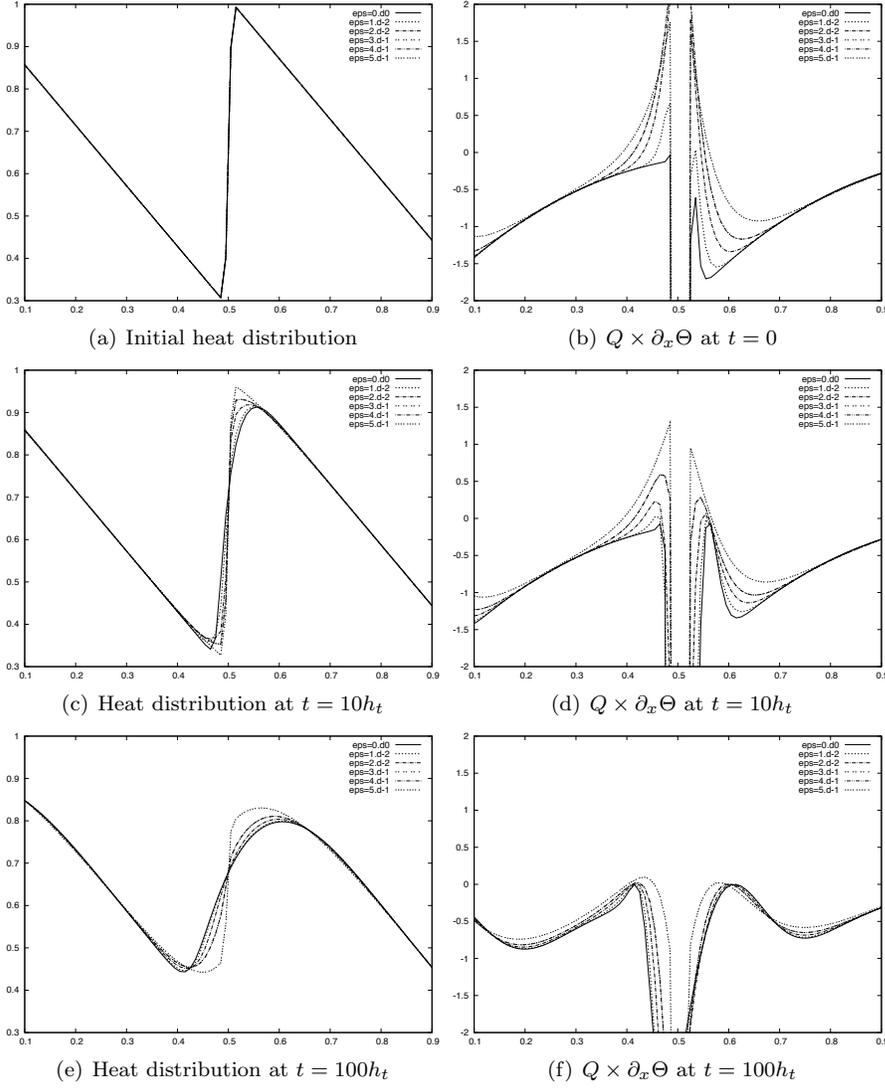


FIG. 3. Simulation of the hydrodynamic model (4.1)–(4.2): $h_t = 10^{-4}$, $h_x = 10^{-2}$, $\rho = 1$, $\kappa(\Theta) = \Theta^{5/2}$, $\nu = 1$

Conclusion. We have derived precisely the non linear diffusion equation for the electron temperature which arises in the Spritzer-Härm regime. The derivation starts from collisional models accounting for both electron-electron and electron-ion collisions. Due to the latter the collision operator preserves charge and energy only. The kinetic equation for the electron distribution is coupled to the Maxwell system. We discuss in details the scaling issues. The quasi-neutral regime implies that the asymptotic solution has a vanishing current, which can be interpreted as a constraint on the electric field. We identify, depending on the collision operator, the diffusion coefficient of the limit equation. Next, we revisit the Schurtz-Nicolai model, which is a non local macroscopic approximation of the kinetic model. In particular we justify the well-posedness of the non local model and prove the maximum principle. We design a specific numerical scheme for the Schurtz-Nicolai system and perform the stability analysis. Finally, we check numerically the ability of the model to capture anti-diffusive phenomena.

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Appendix A. Some useful properties of the linearized Landau-Fokker-Planck operator.

A.1. On spectral properties of the linearized Landau-Fokker-Planck operator. As said in section 2.3, the spectral properties of the linearized operator are crucial. In particular we use the Fredholm-like property (B2). However, spectral gap estimate, see (2.13), are usually not available for realistic collision kernels. Nevertheless, a fine estimate can be derived working with suitable weighted space; we refer to [1, 17, 25] for proofs, comments and applications of the following statement (see also [15] for the Boltzmann operator with soft potentials).

LEMMA A.1. *Let us consider the linearized Landau-Fokker-Planck operator. For any $\alpha \in [-4, 0]$, (B2) is satisfied in $\mathcal{D} = \{h : \mathbb{R}^3 \rightarrow \mathbb{R}, \text{ such that } \|\frac{h}{M}\|_{\mathcal{D}} < \infty\}$ where*

$$\|h\|_{\mathcal{D}}^2 = |h|_{L^2(\langle v \rangle^{-\alpha-1} M)} + |\Pi_v \nabla_v h|_{L^2(\langle v \rangle^{-\alpha-1} M)} + |(Id - \Pi_v) \nabla_v h|_{L^2(\langle v \rangle^{-\alpha-3} M)} + |\Pi_v \nabla_v h|_{L^2(|v|^{-\alpha-1} M)},$$

with the shorthand notation $\langle v \rangle = \sqrt{1 + |v|^2}$.

Proof. Here and below, the norm $|\cdot|_{L^2(w)}$ is associated to the weighted inner product

$$\langle \phi, \psi \rangle_{L^2(w)} = \int \phi(v) \psi(v) w(v) dv.$$

We write the problem as to find $h \in \mathcal{D}$ such that for any $g \in \mathcal{D}$ we have $a(h, g) = \int Fg dv$ with

$$a(h, g) = -\langle (L_{\rho, \Theta} + Z_i \tilde{C}_{ei})h, g \rangle_{L^2(M^{-1})},$$

and $F = |v|^\beta v M_{\rho, \Theta}(v)$ for some exponent $\beta \in \mathbb{R}$.

Firstly, we look at a coercivity estimate of the bilinear form a . Let us write

$$\begin{aligned} a(h, h) &= \frac{\Theta^{\alpha/2}}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left| \Pi_{v-v_*} \left(\nabla_v \frac{h}{M} - \nabla_{v_*} \frac{h_*}{M_*} \right) \right|^2 \frac{MM_* dv dv_*}{|v-v_*|^{\alpha+1}} \\ &\quad + \bar{Z} Z_i \rho_i \int_{\mathbb{R}^3} \left| \Pi_v \nabla_v \frac{h}{M} \right|^2 \frac{M dv}{|v|^{\alpha+1}}. \end{aligned}$$

According to [17] and to [25, Theorem 1.3], there exists $C_1 > 0$ with

$$\begin{aligned} &\frac{\Theta^{\alpha/2}}{2} \iint \left| \Pi_{v-v_*} \left(\nabla_v \frac{h}{M} - \nabla_{v_*} \frac{h_*}{M_*} \right) \right|^2 \frac{MM_* dv dv_*}{|v-v_*|^{\alpha+1}} \\ &\geq C_1 \left(\left\| \frac{h}{M} \right\|_{L^2(\langle v \rangle^{-\alpha-1} M)} + \left\| \Pi_v \nabla_v \frac{h}{M} \right\|_{L^2(\langle v \rangle^{-\alpha-1} M)} + \left\| (Id - \Pi_v) \nabla_v \frac{h}{M} \right\|_{L^2(\langle v \rangle^{-\alpha-3} M)} \right). \end{aligned}$$

Thus, there exists $C_2 > 0$ such that

$$a(h, h) \geq C_2 \left\| \frac{h}{M} \right\|_{\mathcal{D}}^2.$$

Then, we establish the continuity of a . Remark that

$$\begin{aligned} \langle L_{\rho_i} \Theta h, g \rangle_{L^2(M^{-1})} &= \frac{\Theta^{\alpha/2}}{2} \iint \Pi_{v-v_*} \left(\nabla_v \frac{h}{M} - \nabla_{v_*} \frac{h_*}{M_*} \right) \\ &\quad \cdot \Pi_{v-v_*} \left(\nabla_v \frac{g}{M} - \nabla_{v_*} \frac{g_*}{M_*} \right) \frac{MM_* dv dv_*}{|v-v_*|^{\alpha+1}}. \end{aligned}$$

Using the Cauchy-Schwartz inequality, we write

$$\begin{aligned} |a(h, g)| &\leq \frac{\Theta^{\alpha/2}}{2} \left(\iint \left| \Pi_{v-v_*} \left(\nabla_v \frac{h}{M} - \nabla_{v_*} \frac{h_*}{M_*} \right) \right|^2 \frac{MM_* dv dv_*}{|v-v_*|^{\alpha+1}} \right)^{1/2} \\ &\quad \times \left(\iint \left| \Pi_{v-v_*} \left(\nabla_v \frac{g}{M} - \nabla_{v_*} \frac{g_*}{M_*} \right) \right|^2 \frac{MM_* dv dv_*}{|v-v_*|^{\alpha+1}} \right)^{1/2} \\ &\quad + \bar{n} \rho_i Z_i \left\| \Pi_v \nabla_v \frac{h}{M} \right\|_{L^2(|v|^{-\alpha-1} M)} \left\| \Pi_v \nabla_v \frac{g}{M} \right\|_{L^2(|v|^{-\alpha-1} M)}. \end{aligned}$$

We set $w = \frac{v-v_*}{\sqrt{2}}$ and $w_* = \frac{v+v_*}{\sqrt{2}}$, so that

$$\begin{aligned} \iint \left| \Pi_{v-v_*} \left(\nabla_v \frac{h}{M} - \nabla_{v_*} \frac{h_*}{M_*} \right) \right|^2 \frac{MM_* dv dv_*}{|v-v_*|^{\alpha+1}} &= 2^{\frac{\alpha}{2}} \iint \left| \Pi_w \nabla_w \frac{h}{M} \right|^2 \frac{e^{-\frac{w^2}{2}} e^{-\frac{w_*^2}{2}} dv dv_*}{|w|^{\alpha+1}} \\ &= 2^{\frac{\alpha}{2}} (2\pi)^{\frac{3}{2}} \left\| \Pi_w \nabla_w \frac{h}{M} \right\|_{L^2(|v|^{-\alpha-1} M)}^2. \end{aligned}$$

Thus, we get

$$|a(h, g)| \leq \left(2^{\frac{\alpha}{2}} (2\pi)^{\frac{3}{2}} + \bar{n} \rho_i Z_i \right) \left\| \Pi_v \nabla_v \frac{h}{M} \right\|_{L^2(|v|^{-\alpha-1} M)} \left\| \Pi_v \nabla_v \frac{g}{M} \right\|_{L^2(|v|^{-\alpha-1} M)}.$$

Eventually, we apply the Lax-Milgram theorem to conclude that the problem is well-posed. In particular the conclusion applies for $\beta = 0$ and $\beta = 2$. \blacksquare

A.2. Proof of Lemma 2.4. We remark that

$$\begin{aligned}
L_{\rho,\Theta}G_{\rho,\Theta} &= \frac{\rho}{(2\pi\Theta)^{3/2}}v e^{-v^2/2\Theta} \\
&= \frac{\rho}{(2\pi\Theta)^{3/2}}\nabla_v \cdot \int \Theta^{\alpha/2} \frac{\Pi_{v-v_\star}}{|v-v_\star|^{\alpha+1}} \\
&\quad (\nabla_v - \nabla_{v_\star})(G_{\rho,\Theta}(v)e^{-v_\star^2/2\Theta} + G_{\rho,\Theta}(v_\star)e^{-v^2/2\Theta}) dv_\star \\
&\quad + Z_i \bar{n} \rho_i \nabla_v \cdot \left(\Theta^{\alpha/2} \frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v G_{\rho,\Theta} \right).
\end{aligned}$$

Since by (2.11), $\bar{Z}\rho_i = \rho$, dividing by ρ yields

$$\begin{aligned}
\frac{1}{(2\pi\Theta)^{3/2}}v e^{-v^2/2\Theta} &= Z_i \Theta^{\alpha/2} \nabla_v \cdot \left(\frac{\Pi_v}{|v|^{\alpha+1}} \nabla_v G_{\rho,\Theta} \right) \\
+ \frac{\Theta^{\alpha/2}}{(2\pi)^{3/2}} \nabla_v \cdot \int &\frac{\Pi_{v-v_\star}}{|v-v_\star|^{\alpha+1}} \left(\frac{v_\star}{\Theta} e^{-v_\star^2/2\Theta} G_{\rho,\Theta}(v) + e^{-v_\star^2/2\Theta} \nabla_v G_{\rho,\Theta}(v) \right. \\
&\quad \left. - \frac{v}{\Theta} e^{-v^2/2\Theta} G_{\rho,\Theta}(v_\star) - e^{-v^2/2\Theta} \nabla_{v_\star} G_{\rho,\Theta}(v_\star) \right) \frac{dv_\star}{\Theta^{3/2}}
\end{aligned}$$

which already tells us that $G_{\rho,\Theta}$ actually does not depend on ρ . We use the change of variable $v_\star/\sqrt{\Theta} \rightarrow v_\star$, and the relation $\sqrt{\Theta}(\partial_{v_k}\varphi)(v\sqrt{\Theta}) = \partial_{v_k}[\varphi(v\sqrt{\Theta})]$ so that evaluating the previous relation at $v\sqrt{\Theta}$, we are led to

$$\begin{aligned}
\frac{\Theta^{(\alpha-1)/2}}{(2\pi)^{3/2}} \nabla_v \cdot \int &\frac{\Pi_{\sqrt{\Theta}(v-v_\star)}}{|\sqrt{\Theta}(v-v_\star)|^{\alpha+1}} \left(-v_\star e^{-v_\star^2/2} \frac{G_{\rho,\Theta}(v\sqrt{\Theta})}{\sqrt{\Theta}} + e^{-v_\star^2/2} \nabla_v \left(\frac{G_{\rho,\Theta}(v\sqrt{\Theta})}{\sqrt{\Theta}} \right) \right. \\
&\quad \left. - v e^{-v^2/2} \frac{G_{\rho,\Theta}(v_\star\sqrt{\Theta})}{\sqrt{\Theta}} + e^{-v^2/2} \nabla_{v_\star} \left(\frac{G_{\rho,\Theta}(v_\star\sqrt{\Theta})}{\sqrt{\Theta}} \right) \right) dv_\star \\
&\quad + Z_i \Theta^{(\alpha-1)/2} \nabla_v \cdot \left(\frac{\Pi_{\sqrt{\Theta}v}}{|\sqrt{\Theta}v|^{\alpha+1}} \nabla_v \left(\frac{G_{\rho,\Theta}(v\sqrt{\Theta})}{\sqrt{\Theta}} \right) \right) \\
&= \frac{1}{(2\pi\Theta)^{3/2}} v \sqrt{\Theta} e^{-v^2/2} = \frac{1}{(2\pi)^{3/2}} \frac{1}{\Theta} v e^{-v^2/2}
\end{aligned}$$

with the property $\Pi_{\sqrt{\Theta}z} = \Pi_z$. We conclude that $\tilde{G}(v) = \frac{G_{\rho,\Theta}}{\sqrt{\Theta}}(v\sqrt{\Theta})$ is solution of

$$L_{1,1}\tilde{G} = \frac{1}{(2\pi)^{3/2}} v e^{-v^2/2}.$$

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