Mathematical Models and Methods in Applied Sciences

HYDRODYNAMIC LIMIT FOR THE VLASOV-POISSON-FOKKER-PLANCK SYSTEM: ANALYSIS OF THE TWO-DIMENSIONAL CASE

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We consider the hydrodynamic limit for the VPFP sytem in dimension two, dealing with general initial data having finite mass, energy and entropy. The limit equation consists in a drift-diffusion equation, where the drift velocity is defined by means of the Poisson relation. Our result is two-fold. In the case of repulsive (electrostatic) forces, we prove the convergence globally in time in a weak L^1 setting. Considering attractive (gravitational) forces, the same result applies provided a certain scaling parameter is large enough. This is precisely the assumption which prevents from the formation of Dirac masses in finite time in the limit equations, as recently shown by Dolbeault-Perthame.

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1. Introduction

We are interested in the behavior as the parameter $\epsilon > 0$ tends to 0 of the solution $(f_\epsilon, \Phi_\epsilon)$ of the following Vlasov-Poisson-Fokker-Planck system (VPFP for short)

$$\begin{cases} \partial_t f_\epsilon + \dfrac{1}{\epsilon}\, v \cdot \nabla_x f_\epsilon - \dfrac{1}{\chi\epsilon}\, \nabla_x \Phi_\epsilon \cdot \nabla_v f_\epsilon = \dfrac{1}{\chi\epsilon^2}\, L f_\epsilon \quad \text{for } t \geq 0,\, x \in \mathbb{R}^N,\, v \in \mathbb{R}^N, \\ Lf = \nabla_v \cdot (vf + \nabla_v f), \\ -\Delta \Phi_\epsilon = \gamma \rho_\epsilon, \qquad \rho_\epsilon(t,x) = \displaystyle\int_{\mathbb{R}^N} f_\epsilon(t,x,v)\,\mathrm{d}v. \end{cases} \tag{1.1}$$

The problem is supplemented with an initial data

$$f_{\epsilon,|t=0} = f_\epsilon^0 \geq 0. \tag{1.2}$$

This system models the evolution of Brownian particles, submitted to a self consistent force field: $f_\epsilon(t,x,v)$ stands for the density of particles in phase space (position x, velocity v) while the force derives from the potential $\Phi_\epsilon(t,x)$, defined self-consistently through the Poisson equation. Here, γ is either $+1$ or -1 depending on the physical context: $\gamma=+1$ corresponds to repulsive (electrostatic) forces while $\gamma=-1$ corresponds to attractive (gravitational) forces. The former arises in plasma physics, the latter arises in stellar dynamics. Of course, the equation is written in dimensionless form: $\epsilon>0$ is a "small" parameter destined to tend to 0 while $\chi>0$ is a fixed parameter. This comes from a discussion on the physical quantities arising in the system and a suitable choice of the units of observation of the phenomena described by (1.1). The Poisson equation which determines the potential by means of the macroscopic density ρ_ϵ should be understood in the sense that

$$\Phi_\epsilon(t,x)=\gamma\ \big(E*_x\rho_\epsilon(t,\cdot)\big)(x), \tag{1.3}$$

E being the elementary solution of the Laplace operator in $\mathbb{R}^N$. In this work, we shall concentrate on the two-dimensional case where

$$E(x)=-\frac{1}{2\pi}\ \ln(|x|). \tag{1.4}$$

The problem has been treated by Poupaud-Soler[28] in dimension 2 and 3. Remarking that the Fokker-Planck operator can be rewritten as follows

$$Lf=\nabla_v\cdot\big(e^{-v^2/2}\nabla_v(fe^{+v^2/2})\big),$$

we can guess on formal grounds that the penalization $\epsilon\to 0$ leads to

$$f_\epsilon(t,x,v)\simeq\rho(t,x)\ \frac{e^{-v^2/2}}{(2\pi)^{N/2}}. \tag{1.5}$$

Hence, the limit equation satisfied by the macroscopic density $\rho(t,x)$ can be obtained from the following system

$$\begin{cases}\partial_t\rho_\epsilon+\nabla_x\cdot J_\epsilon=0,\\ \epsilon^2\partial_t J_\epsilon+\mathrm{Div}_x\mathbb{P}_\epsilon=\dfrac{1}{\chi}(-\rho_\epsilon\nabla_x\Phi_\epsilon-J_\epsilon)\end{cases} \tag{1.6}$$

which is satisfied by the moments of the unknown f_ϵ

$$J_\epsilon(t,x)=\int_{\mathbb{R}^N}v\ f_\epsilon\,\mathrm{d}v,\qquad \mathbb{P}_\epsilon(t,x)=\int_{\mathbb{R}^N}v\otimes v\ f_\epsilon\,\mathrm{d}v.$$

Taking into account the formal asymptotics (1.5), we get $\mathbb{P}_\epsilon(t,x)\simeq\rho(t,x)\ \mathbb{I}$, so that, assuming that nonlinearities pass to the limit we are led to

$$\begin{cases}\partial_t\rho+\nabla_x\cdot J=0,\\ J=-\rho\nabla_x\Phi-\chi\nabla_x\rho,\end{cases} \tag{1.7}$$

coupled to the Poisson relation

$$-\Delta\Phi=\gamma\rho. \tag{1.8}$$

In the attractive case ($\gamma=-1$), this system is intended to describe the dynamics of collisionless stellar systems: it is referred to as the Smoluchowski equations.

We refer to Chandrasekhar[8] and to the more recent paper of Chavanis-Sommeria-Robert[9] and the references therein for a discussion on physical grounds of (1.7), (1.8) and its derivation from (1.1) presented there as a high friction asymptotics. These limit equations have also been proposed as a model describing the evolution of certain biological systems by Keller-Segel[20]. On a mathematical viewpoint, this system is very interesting since the solution can exhibit concentration as a Dirac mass in finite time. In particular, the value of the coefficient χ is crucial: a large enough χ prevents from these concentration effects. We refer on these questions to Jäger-Luckhaus, [19] Gajewski-Zacharias, [15] Herrero-Velazquez, [18] Rascle-Ziti, [31] Senba-Suzuki, [33] and to the very recent work of Dolbeault-Perthame [13]. A lot of informations both on the modeling in biology and on the mathematical results can be found in the survey of Perthame[26]. Concerning applications to biology note however that the Fokker-Planck operator should certainly be replaced by a more involved scattering operator, as in [1], [7]. Analysis of the diffusion limit in a linear situation goes back to [25]. Poupaud-Soler[28] established the convergence of $(f_\epsilon, \Phi_\epsilon)$ to solutions of (1.7-1.8) on a small enough interval of time $(0, T_*)$, under a suitable assumption on the initial data. Precisely, if, for some $p > N$, $e^{(p-1)v^2/2} f_\epsilon^0$ is bounded in $\mathrm{L}^p(\mathbb{R}^N \times \mathbb{R}^N)$, then, there exists $T_* > 0$ such that a subsequence satisfies

$$\begin{cases} \rho_\epsilon \to \rho \quad \text{in } \mathrm{L}^q(0, T_*; \mathrm{L}^r(\mathbb{R}^N)), \quad 1 \le r < p,\ 1 \le q < \infty, \\ f_\epsilon(t,x,v) \to \rho(t,x)\ (2\pi)^{-N/2} e^{-v^2/2} \quad \text{in } L^q(0, T_*; \mathrm{L}^r(\mathbb{R}^N \times \mathbb{R}^N)) \\ \hfill 2 \le r < p,\ 2 \le q < \infty. \end{cases}$$

Here, the question of the asymptotic behavior of (1.1) as ϵ goes to 0 is addressed again. The aim of this work is two-fold. First, we wish to prove a global convergence result, without any restriction on the time interval. Second, we weaken the assumption on the data, considering only bound on entropy and energy. This can be done when restricting to the two dimensional case, and, if $\gamma = -1$, for any data which do not develop singularities for the limit system (1.7), (1.8).

In dimension $N = 2$, a specific difficulty consists in obtaining useful estimates from the quantity

$$\int_{\mathbb{R}^2} \rho \Phi \, \mathrm{d}x$$

which appears naturally within the problem: it is physically interpreted as the potential energy of the system. It would be tempting to perform some integration by parts and to write it as the integral of $|\nabla_x \Phi|^2$. However, such a computation is misleading. Indeed, let $\rho \in \mathrm{L}^1(\mathbb{R}^2)$, with $\rho \ge 0$ and let Φ be a solution of $-\Delta\Phi = \gamma\rho$ in $\mathbb{R}^2$. Then $\nabla\Phi$ belongs to $\mathrm{L}^2(\mathbb{R}^2)$ iff $\rho = 0$. The main result of the paper states as follows.

Theorem 1.1. *Set $N = 2$. Let $f^0_\epsilon \geq 0$ satisfy*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^0_\epsilon \,\mathrm{d}v\,\mathrm{d}x = 1, \\ \displaystyle\sup_{\epsilon>0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^0_\epsilon(1 + v^2 + |x| + |\ln(f^0_\epsilon)|)\,\mathrm{d}v\,\mathrm{d}x = M_0 < \infty. \end{cases} \tag{1.9}$$

In the attractive case ($\gamma = -1$), we suppose furthermore that $\chi > 1/(8\pi)$. Let $0 < T < \infty$. Then, up to a subsequence, ρ_ϵ converges in $\mathrm{C}^0([0, T]; \mathrm{L}^1(\mathbb{R}^2) - weak)$ to ρ, solution of the limit system (1.7-1.8).

Remark 1.1. As it will be clear in the sequel, assumption (1.9) implies that

$$\sup_{\epsilon>0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho^0_\epsilon(x)\rho^0_\epsilon(y)\ \big|\ln(|x-y|)\big|\,\mathrm{d}y\,\mathrm{d}x$$

is finite. This quantity will be denoted by $M'_0 < \infty$.

The meaning of the convergence as well as the sense in which the limit equation should be understood will be precised later on (see Lemma 3.1 and eq. (3.1)). Focusing on the attractive case, the remarkable fact is that we obtain the convergence in a weak L^1 sense exactly in the same situation in which concentrations are avoided in the system (1.7-1.8). This is the meaning of the assumption $\chi > 1/(8\pi)$, see [13]. In the repulsive case, apparition of such singularities does not occur, and the convergence can be obtained without restriction on χ. Note also that the convergence result holds globally in time. Eventually, the paper is organized as follows. In Section 2, we derive the a priori estimates on the solutions from the evolution of physical quantities associated to the system. As mentioned above, the main difficulties rely on the treatment of the potential energy. Therefore, the main ingredient of the proof is the use of a trick due to Dolbeault[12] when dealing with the repulsive case, and the use of an entropy inequality due to Beckner[2] and Carlen-Loss[5] when dealing with the attractive case. Note that this argument appears also in the analysis of the limit system by Dolbeault-Perthame[13]. Then, we detail the passage to the limit in Section 3. The difficulty relies on the non linear term $\rho_\epsilon\nabla_x\Phi_\epsilon$. Then, we use a suitable weak formulation for this term, following the idea of Poupaud-Soler[28] which has been used in various contexts[30], [27], [17]...

2. A priori Estimates

Let us start with a few words about the existence theory for the system (1.1), with a given $\epsilon > 0$. The subject has been widely investigated. Global existence results have been obtained in the two dimensional framework by Neunzert-Pulvirenti-Triolo[22] who used a probabilistic approach. Considering a model where the friction force is neglected, Degond[11] proved with deterministic arguments global existence and uniqueness of smooth solutions, still in the 2D situation for the repulsive case. The system (1.1) is analyzed by Victory-O'Dwyer[24] who established existence-uniqueness results, globally in time in dimension two, locally in time in dimension three, for both cases $\gamma = \pm 1$. Global results are then extended to dimension three by Bouchut[3], and smoothing effects of the system are bring out in [4]. Concerning weak solutions, we refer to Carrillo-Soler[6] and Victory[34]. In what follows, we can use classical solutions as obtained by Victory-O'Dwyer[24] (see sp. Theorem III.2 and III.3, p. 149) which requires an initial data verifying $f_\epsilon^0 \in \mathrm{C}^1 \cap \mathrm{L}^1(\mathbb{R}^2 \times \mathbb{R}^2)$ and

$$(1+v^2)^{\gamma/2}\ (f_\epsilon^0 + |\nabla_x f_\epsilon^0| + |\nabla_v f_\epsilon^0|) \in \mathrm{L}^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$$

for some $\gamma > 2$.

The aim of the Section is the derivation of a priori estimates, uniform with respect to ϵ, on the solutions f_ϵ and the associated macroscopic density ρ_ϵ. Presicely, we will justify the following claim.

Proposition 2.1. *Suppose (1.9). In the attractive case, we suppose moreover that* $\chi > 1/(8\pi)$. *Let* $0 < T < \infty$. *Then,*
i) $\rho_\epsilon(1 + |x| + \ln(\rho_\epsilon))$ *is bounded in* $\mathrm{L}^\infty(0, T; \mathrm{L}^1(\mathbb{R}^2))$*;*
ii) $\left|\nabla_v \sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}$ *is bounded in* $\mathrm{L}^1((0,T) \times \mathbb{R}^2 \times \mathbb{R}^2)$.

These crucial estimates are deduced from the evolution of physical quantities associated to the VPFP system.

- Mass conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon \ \mathrm{d}v\ \mathrm{d}x = 0.$$

Hence, from now on, we assume that

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon \ \mathrm{d}v\ \mathrm{d}x = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon^0 \ \mathrm{d}v\ \mathrm{d}x = 1.$$

- Kinetic Energy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{v^2}{2} f_\epsilon \ \mathrm{d}v\ \mathrm{d}x = &-\frac{1}{\chi\epsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (vf_\epsilon + \nabla_v f_\epsilon)\cdot v \ \mathrm{d}v\ \mathrm{d}x \\ &-\frac{1}{\chi\epsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v\cdot\nabla_x\Phi_\epsilon\ f_\epsilon \ \mathrm{d}v\ \mathrm{d}x.\end{aligned}$$

- Potential Energy:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon\Phi_\epsilon \ \mathrm{d}v\ \mathrm{d}x &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} \rho_\epsilon\Phi_\epsilon \ \mathrm{d}x \\ &= \gamma \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} E(x-y)\Big(\rho(y)\partial_t\rho(x) + \rho(x)\partial_t\rho(y)\Big)\ \mathrm{d}y\ \mathrm{d}x \\ &= 2\gamma\int_{\mathbb{R}^2} \Phi_\epsilon\partial_t\rho_\epsilon \ \mathrm{d}x \\ &= \frac{2}{\epsilon}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v\cdot\nabla_x\Phi_\epsilon\ f_\epsilon \ \mathrm{d}v\ \mathrm{d}x.\end{aligned}$$

- Entropy:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon \ln(f_\epsilon) \ \mathrm{d}v\ \mathrm{d}x = -\frac{1}{\chi\epsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (vf_\epsilon + \nabla_v f_\epsilon)\cdot\frac{\nabla_v f_\epsilon}{f_\epsilon} \ \mathrm{d}v\ \mathrm{d}x.$$

Therefore, summing up these relations yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} f_\epsilon \ln(f_\epsilon)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} \frac{v^2}{2} f_\epsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\chi}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} f_\epsilon \Phi_\epsilon\,\mathrm{d}v\,\mathrm{d}x\right\}\\
&\quad = -\frac{1}{\chi\epsilon^2}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} (v f_\epsilon + \nabla_v f_\epsilon)^2\,\frac{1}{f_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = -\frac{1}{\chi\epsilon^2}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} (v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon})^2\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = -\frac{4}{\chi\epsilon^2}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} \left|\nabla_v\sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{2.1}$$

Note that this relation is consistent with the formal asymptotics (1.5).

In order to exploit this relation, we will also need some control on the behavior of f_ϵ at infinity (with respect to the space variable). To this end, we remark that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} |x| f_\epsilon\,\mathrm{d}v\,\mathrm{d}x = \frac{1}{\epsilon}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} v\cdot\frac{x}{|x|}\,f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = \frac{1}{\epsilon}\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} (v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon})\cdot\frac{x}{|x|}\,\sqrt{f_\epsilon}\,\mathrm{d}v\,\mathrm{d}x\\
&\quad \le \left(\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2} f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\right)^{1/2}\left(\int_{\mathbb{R}^2}\!\int_{\mathbb{R}^2}\left|\frac{v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon}}{\epsilon}\right|^2\mathrm{d}v\,\mathrm{d}x\right)^{1/2}.
\end{aligned} \tag{2.2}$$

Then, we will search for an estimate on the macroscopic density ρ_ϵ, while we remark that $\nabla_v\sqrt{f_\epsilon e^{v^2/2}}$ yet appears as a dissipation term in (2.1). Keeping this objective in mind, we evaluate the macroscopic entropy by means of the microscopic entropy.

Lemma 2.1. *Let $f : \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}$, with $f \ge 0$. Set $\rho(x) = \int_{\mathbb{R}^N} f(x,v)\,\mathrm{d}v$. Then, we have*

$$\begin{aligned}
\int_{\mathbb{R}^N} \rho\ln(\rho)\,\mathrm{d}x \le &\int_{\mathbb{R}^N}\!\int_{\mathbb{R}^N} f\ln(f)\,\mathrm{d}v\,\mathrm{d}x\\
&+\frac{N}{2}\ln(2\pi)\int_{\mathbb{R}^N}\!\int_{\mathbb{R}^N} f\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}\!\int_{\mathbb{R}^N}\frac{v^2}{2} f\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Proof. The statement is a direct consequence of the Jensen inequality applied to the convex function $\Psi(s) = s\ln(s)$ and the probability measure on $\mathbb{R}^N$ $(2\pi)^{-N/2}e^{-v^2/2}\,\mathrm{d}v = M(v)\,\mathrm{d}v$. We get

$$\begin{aligned}
\rho\ln(\rho) = \Psi(\rho) &= \Psi\Big(\int_{\mathbb{R}^N}\frac{f}{M}\,M\,\mathrm{d}v\Big)\\
&\le \int_{\mathbb{R}^N}\Psi\Big(\frac{f}{M}\Big)\,M\,\mathrm{d}v = \int_{\mathbb{R}^N} f\Big(\frac{v^2}{2} + \ln(f)\Big)\,\mathrm{d}v + \frac{N}{2}\ln(2\pi)\int_{\mathbb{R}^N} f\,\mathrm{d}v.
\end{aligned}$$

We conclude by integrating with respect to x. □

Integrating (2.1) with respect to time yields

$$\begin{aligned}&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon \ln(f_\epsilon)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{v^2}{2} f_\epsilon\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\chi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon \Phi_\epsilon\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad + \frac{4}{\chi\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v\sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon^0 \ln(f_\epsilon^0)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{v^2}{2} f_\epsilon^0\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2\chi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon^0 \Phi_\epsilon(0,x)\,\mathrm{d}v\,\mathrm{d}x\\ &\le M_0 + M_0'.\end{aligned}$$

Thus, using Lemma 2.1 leads to

$$\begin{aligned}&\int_{\mathbb{R}^2} \rho_\epsilon \ln(\rho_\epsilon)\,\mathrm{d}x + \frac{1}{2\chi}\int_{\mathbb{R}^2} \rho_\epsilon\Phi_\epsilon\,\mathrm{d}x\\ &\quad + \frac{4}{\chi\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v\sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \le M_0 + M_0' + \ln(2\pi).\end{aligned} \tag{2.3}$$

Next, we are left with the task of discussing a bound from below on the potential energy

$$\int_{\mathbb{R}^2} \rho\Phi\,\mathrm{d}x = -\frac{\gamma}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(x)\rho(y)\ln(|x-y|)\,\mathrm{d}y\,\mathrm{d}x.$$

Let us distinguish the repulsive and the attractive cases.

2.1. *Attractive case:* $\gamma = -1$

Lemma 2.2. *Let* $\rho : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\rho \ge 0$ *and* $\int_{\mathbb{R}^2} \rho\,\mathrm{d}x = 1$. *Then, there exists a constant* $C_* > 0$ *such that*

$$-4\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(x)\rho(y)\ln(|x-y|)\mathrm{d}y\,\mathrm{d}x \le C_* + 2\int_{\mathbb{R}^2} \rho\ln(\rho)\,\mathrm{d}x.$$

The proof follows from a direct application of Theorem 2 of [2] (see also [5]), the constant C_* being $C_* = 2\ln(\pi) + 2(\psi(2) - \psi(1)) - 2\ln(\Gamma(2)/\Gamma(1))$, $\psi(z) = \Gamma'(z)/\Gamma(z)$. (Hence, here $C_* = 2\ln(\pi) + 1 \simeq 3.289459772$). Accordingly, we get in the attractive case

$$\int_{\mathbb{R}^2} \rho\Phi\,\mathrm{d}x \ge -\frac{C_*}{8\pi} - \frac{1}{4\pi}\int_{\mathbb{R}^2} \rho\ln(\rho)\,\mathrm{d}x.$$

Then, using this inequality in (2.3), we get

$$\begin{aligned}&\left(1 - \frac{1}{8\pi\chi}\right)\int_{\mathbb{R}^2} \rho_\epsilon\ln(\rho_\epsilon)\,\mathrm{d}x + \frac{4}{\chi\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v\sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &\qquad \le M_0 + M_0' + \ln(2\pi) + \frac{C_*}{16\pi\chi}\end{aligned}$$

where the role of the constraint $\chi > 1/(8\pi)$ becomes clear. From now on, we assume that this property is fulfiled.

Besides, integrating (2.2) yields

$$\int_{\mathbb{R}^2} |x|\rho_\epsilon \,\mathrm{d}x \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |x| f_\epsilon^0 \,\mathrm{d}v\,\mathrm{d}x + \frac{t}{2} + \frac{1}{2}\frac{4}{\epsilon^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.$$

Let $\nu > 0$ to be determined later on. We have

$$\begin{aligned}&\Big(1-\frac{1}{8\pi\chi}\Big)\int_{\mathbb{R}^2} \rho_\epsilon \ln(\rho_\epsilon)\,\mathrm{d}x + \nu\int_{\mathbb{R}^2}|x|\rho_\epsilon\,\mathrm{d}x \\ &\qquad + \Big(\frac{1}{\chi}-\frac{\nu}{2}\Big)\frac{4}{\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\nabla_v \sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\ &\le (1+\nu)M_0 + M_0' + \ln(2\pi) + \frac{C_*}{16\pi\chi} + \frac{\nu t}{2}.\end{aligned}$$

To conclude, we use the classical trick of Carleman: for $\kappa > 0$, we write

$$\begin{aligned}\rho|\ln(\rho)| &= \rho\ln(\rho) - 2\rho\ln(\rho)\chi_{e^{-\kappa|x|}\le\rho\le 1} - 2\rho\ln(\rho)\chi_{0\le\rho\le e^{-\kappa|x|}} \\ &\le \rho\ln(\rho) + 2\kappa|x|\rho + K\sqrt{\rho}\chi_{0\le\rho\le e^{-\kappa|x|}} \\ &\le \rho\ln(\rho) + 2\kappa|x|\rho + Ke^{-\kappa|x|/2},\end{aligned}$$

for some $K > 0$. Hence, we are led to

$$\begin{aligned}&\Big(1-\frac{1}{8\pi\chi}\Big)\int_{\mathbb{R}^2} \rho_\epsilon|\ln(\rho_\epsilon)|\,\mathrm{d}x + \big(\nu - 2\kappa(1-1/(8\pi\chi))\big)\int_{\mathbb{R}^2}|x|\rho_\epsilon\,\mathrm{d}x \\ &\qquad + \Big(\frac{1}{\chi}-\frac{\nu}{2}\Big)\frac{4}{\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\nabla_v \sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\ &\le (1+\nu)M_0 + M_0' + \ln(2\pi) + \frac{C_*}{16\pi\chi} + \frac{\nu t}{2} + \Big(1-\frac{1}{8\pi\chi}\Big)K\int_{\mathbb{R}^2} e^{-\kappa|x|/2}\,\mathrm{d}x.\end{aligned}$$

Now, we choose the parameters as follows: first, pick $0 < \nu < 2/\chi$, then, fix $\kappa > 0$ such that $\nu - 2\kappa(1 - 1/(8\pi\chi)) > 0$. This concludes the proof of Proposition 2.1 in the attractive case.

2.2. *Repulsive case:* $\gamma = +1$

Lemma 2.3. *Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that $\rho \ge 0$. Then, for any $k > e$, we have*

$$\int_{\mathbb{R}^2} \rho\Phi\,\mathrm{d}x \ge -\frac{\ln(k)}{\pi}\left[\frac{1}{2}\left(\int_{\mathbb{R}^2}\rho\,\mathrm{d}x\right)^2 + \frac{1}{k}\int_{\mathbb{R}^2}\rho\,\mathrm{d}x\int_{\mathbb{R}^2}|x|\rho\,\mathrm{d}x\right].$$

Proof. We follow the reasoning of Dolbeault[12] by introducing the parameter $k > e$. Since the function $k \longmapsto \frac{\ln(k)}{k}$ is non increasing on $(e, +\infty)$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2}\rho\Phi\,\mathrm{d}x &= -\frac{1}{2\pi}\iint_{|x-y|\le k} ...\,\mathrm{d}y\,\mathrm{d}x - \frac{1}{2\pi}\iint_{|x-y|\ge k} ...\,\mathrm{d}y\,\mathrm{d}x \\ &\ge -\frac{\ln(k)}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x - \frac{\ln(k)}{2\pi k}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|\rho(x)\rho(y)\,\mathrm{d}y\,\mathrm{d}x \\ &\ge -\frac{\ln(k)}{2\pi}\left(\int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x\right)^2 - \frac{\ln(k)}{\pi k}\int_{\mathbb{R}^2}\rho(x)\,\mathrm{d}x\int_{\mathbb{R}^2}|x|\rho(x)\,\mathrm{d}x.\end{aligned}$$

□

Inserting this estimate in (2.3) leads to

$$\begin{aligned}&\int_{\mathbb{R}^2} \rho_\epsilon \ln(\rho_\epsilon)\,\mathrm{d}x + \frac{4}{\chi\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &\le M_0 + M_0' + \ln(2\pi) + \frac{\ln(k)}{4\pi\chi} + \frac{\ln(k)}{2\pi k\chi}\int_{\mathbb{R}^2} |x|\rho_\epsilon(x)\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Using (2.2), we obtain, for some $\nu > 0$,

$$\begin{aligned}&\int_{\mathbb{R}^2} \rho_\epsilon \ln(\rho_\epsilon)\,\mathrm{d}x + \left(\nu - \frac{\ln(k)}{2k\pi\chi}\right)\int_{\mathbb{R}^2} |x|\rho\,\mathrm{d}x\\ &\qquad +(1/\chi - \nu/2)\frac{4}{\epsilon^2}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left|\nabla_v \sqrt{f_\epsilon e^{v^2/2}}\right|^2 e^{-v^2/2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &\le (1+\nu)M_0 + M_0' + \frac{\ln(k)}{4\pi\chi} + \frac{\nu t}{2}.\end{aligned}$$

First, choose $0 < \nu < 2/\chi$, and then k large enough to obtain $\nu > \ln(k)/(2k\pi\chi)$. We end the proof of Proposition 2.1 in the repulsive case by reproducing the arguments of the previous subsection.

2.3. *Estimate on the Kinetic Energy*

As a consequence of Proposition 2.1, we deduce a bound on the kinetic energy.

Corollary 2.1. *Let the assumptions of Proposition 2.1 be fulfilled. Then* $v^2 f_\epsilon$ *is bounded in* $\mathrm{L}^1((0,T)\times\mathbb{R}^2\times\mathbb{R}^2)$.

Proof. We note that

$$\begin{aligned}0 &\le \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v^2 f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left(|v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon}|^2 - 4|\nabla_v\sqrt{f_\epsilon}|^2 - 4v\sqrt{f_\epsilon}\cdot\nabla_v\sqrt{f_\epsilon}\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &\le \int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon}|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + 0 + 4\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Hence, by using Proposition 2.1, this is dominated by $C_T\epsilon^2 + 4T$, with C_T depending only on T and (1.9). We point out that this is an estimate of the kinetic energy in L^1 norm with respect to time, and not in $\mathrm{L}^\infty(0,T)$. □

3. Passage to the Limit

Now, we aim at passing to the limit $\epsilon \to 0$ in the moments system (1.6). Of course, as a consequence of Corollary 2.1, J_ϵ and $\mathbb{P}_\epsilon$ are bounded in $\mathrm{L}^1((0,T)\times\mathbb{R}^2)$. More precisely, we can rewrite the kinetic pressure as follows

$$\begin{aligned}\mathbb{P}_\epsilon &= \int_{\mathbb{R}^2} v\sqrt{f_\epsilon}\otimes(v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon})\,\mathrm{d}v - 2\int_{\mathbb{R}^2} v\sqrt{f_\epsilon}\otimes\nabla_v\sqrt{f_\epsilon}\,\mathrm{d}v\\ &= \int_{\mathbb{R}^2} v\sqrt{f_\epsilon}\otimes(v\sqrt{f_\epsilon} + 2\nabla_v\sqrt{f_\epsilon})\,\mathrm{d}v + \rho_\epsilon\,\mathbb{I}.\end{aligned}$$

Then, we remark that the L^1 norm (with respect to time and space variables) of the first integral in the right hand side is dominated by

$$\left(\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} v^2 f_\epsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\left(\frac{|v\sqrt{f_\epsilon}+2\nabla_v\sqrt{f_\epsilon}|^2}{\epsilon^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2}\times\epsilon.$$

Therefore, according to Proposition 2.1 and Corollary 2.1, we have

$$\mathbb{P}_\epsilon = \rho_\epsilon\,\mathbb{I} + \mathcal{O}_{\mathrm{L}^1((0,T)\times\mathbb{R}^2)}(\epsilon).$$

The uniform bounds established on the macroscopic quantities allow us to consider converging subsequences

$$\rho_\epsilon \rightharpoonup \rho, \qquad J_\epsilon \rightharpoonup J,$$

the convergence being understood in the vague topology for bounded measures on $(0,T)\times\mathbb{R}^2$. (We will see that the convergence of the macroscopic density can be improved, and in particular that traces on the initial time make sense.) Letting $\epsilon\to 0$ in (1.6) yields

$$\begin{cases}\partial_t\rho + \nabla_x\cdot J = 0,\\ J + \chi\nabla_x\rho = -\lim\limits_{\epsilon\to 0}\rho_\epsilon\nabla_x\Phi_\epsilon\end{cases}$$

in the $\mathcal{D}'((0,T)\times\mathbb{R}^2)$ sense. It remains to deal with the nonlinear term $\rho_\epsilon\nabla_x\Phi_\epsilon$ and the Poisson relation.

To this end, we first improve the convergence property satisfied by the sequence of macroscopic densities.

Lemma 3.1. *Possibly at the price of extracting subsequences, ρ_ϵ converges to ρ in $\mathrm{C}^0([0,T];\mathrm{L}^1(\mathbb{R}^2)-weak)$, which means that for any test function $\varphi\in\mathrm{L}^\infty(\mathbb{R}^2)$, we have*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^2}\rho_\epsilon(t,x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^2}\rho(t,x)\varphi(x)\,\mathrm{d}x$$

uniformly on $[0,T]$.

Note in particular that for any time t, the limit $\rho(t)$ belongs to $\mathrm{L}^1(\mathbb{R}^2)$, and thus concentration effects do not appear in the limit system. This fits completely with the results in [13] concerning the attractive case: concentrations do not occur when $\chi > 1/(8\pi)$. We also point out that with this statement the initial condition for the limit problem makes sense: we have

$$\rho_{|t=0} = \lim_{\epsilon\to 0}\int_{\mathbb{R}^2} f_\epsilon^0\mathrm{d}v \qquad \text{weakly in } \mathrm{L}^1(\mathbb{R}^2).$$

We postpone the details of the proof to the Appendix.

In order to treat the nonlinear term, we need now to precise the meaning of the distribution $\rho_\epsilon\nabla_x\Phi_\epsilon$. According to an idea of Poupaud-Soler, [28] we exploit the symmetry properties of the Poisson kernel E and we write, for any test function $\varphi \in \left(\mathrm{C}_c^\infty(\mathbb{R}^2)\right)^2$,

$$\begin{aligned}\left\langle \rho_\epsilon \nabla_x \Phi_\epsilon, \varphi \right\rangle &= \frac{-\gamma}{2\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_\epsilon(t,x)\rho_\epsilon(t,y)\ \frac{x-y}{|x-y|^2} \cdot \varphi(x)\ \mathrm{d}y\ \mathrm{d}x \\ &= \frac{-\gamma}{4\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho_\epsilon(t,x)\rho_\epsilon(t,y)\ \frac{x-y}{|x-y|^2} \cdot \big(\varphi(x)-\varphi(y)\big)\ \mathrm{d}y\ \mathrm{d}x.\end{aligned}$$

This idea is reminiscient to the study of weak solutions of the two-dimensional Euler equations by Schochet[32]. It also appears as the basis of the notion of solutions introduced recently by Poupaud[27]. It has been used successfully by Nieto-Poupaud-Soler[30] and Goudon-Nieto-Poupaud-Soler[17] when performing the high-field limit from the VPFP. (This kind of idea also has also been used to define notion of weak solutions for the Boltzmann equation without cut-off, see e.g. [16].) Then, we realize that the function

$$\begin{aligned}\mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (x,y) \quad &\longmapsto \frac{x-y}{|x-y|^2} \cdot \big(\varphi(x)-\varphi(y)\big)\end{aligned}$$

belongs to $\mathrm{L}^\infty(\mathbb{R}^2\times\mathbb{R}^2)$ (it is bounded by $\|\nabla\varphi\|_{\mathrm{L}^\infty}$ and not well-defined on the diagonal $\{(x,x),\ x\in\mathbb{R}^2\}$ which is a negligeable set of $\mathbb{R}^2\times\mathbb{R}^2$). Therefore, by using the convergence stated in Lemma 3.1, we readily obtain

$$\lim_{\epsilon\to 0} \left\langle \rho_\epsilon \nabla_x \Phi_\epsilon, \varphi \right\rangle = \frac{-\gamma}{4\pi} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(t,x)\rho(t,y)\ \frac{x-y}{|x-y|^2} \cdot \big(\varphi(x)-\varphi(y)\big)\ \mathrm{d}y\ \mathrm{d}x,$$

for any time $t\in[0,T]$. We conclude that ρ is a solution of (1.7-1.8) in the sense that

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{\mathbb{R}^2} \rho(t,x)\varphi(x)\ \mathrm{d}x = \chi \int_{\mathbb{R}^2} \rho(t,x)\Delta\varphi(x)\ \mathrm{d}x \\ \qquad\qquad + \dfrac{\gamma}{4\pi} \displaystyle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \rho(t,x)\rho(t,y)\ \frac{x-y}{|x-y|^2} \cdot \big(\nabla\varphi(x)-\nabla\varphi(y)\big)\ \mathrm{d}y\ \mathrm{d}x, \\ \displaystyle\int_{\mathbb{R}^2} \rho(t,x)\varphi(x)\ \mathrm{d}x\Big|_{t=0} = \lim_{\epsilon\to 0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f_\epsilon^0(x)\varphi(x)\ \mathrm{d}v\ \mathrm{d}x. \end{cases} \tag{3.1}$$

holds in $\mathcal{D}'([0,+\infty))$ for any test function $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^2)$ (φ being of class C^2 would be enough...).

Let us end with the following consequence of our analysis, which provides some information on the behavior of the microscopic density.

Corollary 3.1. *Under the assumptions of Theorem 1.1, the corresponding subsequence f_ϵ converges to $\rho(t,x)\ \frac{1}{2\pi}e^{-v^2/2}$ in the following sense: for any test function $\varphi\in\mathrm{L}^\infty(\mathbb{R}^N)$, we have*

$$\lim_{\epsilon\to 0} \int_0^T \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f_\epsilon(t,x,v)\varphi(x)\ \mathrm{d}x - \int_{\mathbb{R}^2} \rho(t,x)\frac{e^{-v^2/2}}{2\pi}\varphi(x)\ \mathrm{d}x \right| \mathrm{d}v\ \mathrm{d}t = 0.$$

Note that this statement does not give a convergence pointwise with respect to time, but only in average.

Proof. Let us denote $M(v) = \frac{1}{2\pi} e^{-v^2/2}$ At first, we expand

$$f_\epsilon(t,x,v) - \rho(t,x)M(v) = \big(f_\epsilon(t,x,v) - \rho_\epsilon(t,x)M(v)\big) + \big(\rho_\epsilon - \rho\big)(t,x)M(v),$$

where we already know that $\rho_\epsilon - \rho$ tends to 0, in $\mathrm{C}^0([0,T];\mathrm{L}^1(\mathbb{R}^2) - weak)$. Hence, it remains to study the difference $f_\epsilon(t,x,v) - \rho_\epsilon(t,x)\ M(v)$. Second, we use the logarithmic Sobolev inequality, see e.g. [21] (Th. 8.14, p. 223), which yields

$$0 \leq \int_{\mathbb{R}^2} \left\{ \frac{f_\epsilon}{\rho_\epsilon M} \ln\left(\frac{f_\epsilon}{\rho_\epsilon M}\right) - \frac{f_\epsilon}{\rho_\epsilon M} + 1 \right\} \rho_\epsilon\ M\ \mathrm{d}v = \int_{\mathbb{R}^2} f_\epsilon \ln\left(\frac{f_\epsilon}{\rho_\epsilon M}\right) \mathrm{d}v$$
$$\leq 2 \int_{\mathbb{R}^2} \left| \nabla_v \sqrt{f_\epsilon e^{v^2/2}} \right|^2 e^{-v^2/2}\ \mathrm{d}v.$$

By Proposition 2.1-ii), after integration with respect to time and space, this quantity is dominated by $C_T\ \epsilon^2$. Eventually, we conclude by using the Csiszar-Kullback-Pinsker inequality [10], [29], which implies that

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |f_\epsilon - \rho_\epsilon M|\ \mathrm{d}v\ \mathrm{d}x \right)^2 \leq 4 \int_{\mathbb{R}^2} f_\epsilon \ln\left(\frac{f_\epsilon}{\rho_\epsilon M}\right) \mathrm{d}v\ \mathrm{d}x. \qquad \square$$

Remark 3.1. It remains an interesting and open issue to investigate the asymptotic problem when concentrations occur. A nice framework to define solutions in such a case, with stability properties, has been introduced by Poupaud[27] (see also Senba-Suzuki[33]) and one may wonder if the sequence ρ_ϵ converges to a solution defined in such a sense.

Appendix A. Proof of Lemma 3.1

Lemma 3.1 follows from the following general statement.

Lemma Appendix A.1. *Let* $n_\epsilon : (0,T) \times \mathbb{R}^N \to \mathbb{R}$ *such that* $\sup_{\epsilon>0,\ 0\leq t\leq T} \|n_\epsilon(t)\|_{\mathrm{L}^1(\mathbb{R}^N)} \leq M < \infty$. *We suppose also that there exists a weakly compact set* $\mathcal{K}$ *in* $\mathrm{L}^1(\mathbb{R}^N)$ *such that for a.a.* $t \in (0,T)$, $n_\epsilon(t)$ *belongs to* $\mathcal{K}$. *Furthermore, suppose that*

$$\partial_t n_\epsilon = \sum_{|\alpha|\leq k} \partial_x^\alpha g_\epsilon^{(\alpha)},$$

where, for any compact set $K \subset \mathbb{R}^N$,

$$\sup_{\epsilon>0} \left\{ \int_E\int_K |g_\epsilon^{(\alpha)}|\ \mathrm{d}x\ \mathrm{d}t \right\} \longrightarrow 0 \qquad \text{as } |E| \to 0.$$

Then, the sequence n_ϵ *is compact in* $\mathrm{C}^0([0,T];\mathrm{L}^1(\mathbb{R}^N) - weak)$.

In view of (1.6), we justify Lemma 3.1 by applying this result with $n_\epsilon = \rho_\epsilon$, $k = 1$ and g_ϵ given by the components of J_ϵ. Indeed, on the one hand, Proposition 2.1-i) garantees that $\{n_\epsilon(t),\ 0 \leq t \leq T,\ \epsilon > 0\}$ belongs to a weakly compact set of $\mathrm{L}^1(\mathbb{R}^N)$, as a standard consequence of the Dunford-Pettis theorem, see e.g. [14]

(Th. 4.21.2, p. 274). On the other hand, we are able to prove the following property, which strengthens the estimate on J_ϵ.

Lemma Appendix A.2. *Let the assumptions of Proposition 2.1 be fulfilled. Then, we have*

$$\sup_{\epsilon>0}\left\{\int_E\int_{\mathbb{R}^2}|J_\epsilon|\,\mathrm{d}x\,\mathrm{d}t\right\}\longrightarrow 0 \qquad \textit{as } |E|\to 0.$$

Proof. This is an immediate consequence of Proposition 2.1 and Corollary 2.1 since the Cauchy-Schwarz inequality yields, for any measurable set $E\subset(0,T)$,

$$\begin{aligned}\int_E\int_{\mathbb{R}^2}|J_\epsilon|\,\mathrm{d}x\,\mathrm{d}t &= \int_E\int_{\mathbb{R}^2}\left|\frac{1}{\epsilon}\int_{\mathbb{R}^2}(v\sqrt{f_\epsilon}+2\nabla_v\sqrt{f_\epsilon})\sqrt{f_\epsilon}\mathrm{d}v\right|\,\mathrm{d}x\,\mathrm{d}t\\ &\le\left(\int_E\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{|v\sqrt{f_\epsilon}+2\nabla_v\sqrt{f_\epsilon}|^2}{\epsilon^2}f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_E\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}f_\epsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\\ &\le C_T\,|E|^{1/2},\end{aligned}$$

where the constant C_T depends only on T and (1.9). □

Proof of Lemma Appendix A.1. Let $\varphi\in\mathrm{C}_c^\infty(\mathbb{R}^N)$. Clearly, we have

$$\sup_{\epsilon>0,\ 0\le t\le T}\left|\int_{\mathbb{R}^N}n_\epsilon(t,x)\varphi(x)\,\mathrm{d}x\right|\le M\|\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^N)}<\infty. \tag{A.1}$$

Furthermore, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^N}n_\epsilon(t+h,x)\varphi(x)\,\mathrm{d}x-\int_{\mathbb{R}^N}n_\epsilon(t,x)\varphi(x)\,\mathrm{d}x\right|\\ &\qquad\le\sum_{|\alpha|\le k}\|\partial^\alpha\varphi\|_{\mathrm{L}^\infty(\mathbb{R}^N)}\left|\int_t^{t+h}\int_{\mathrm{supp}(\varphi)}|g_\epsilon^{(\alpha)}|\,\mathrm{d}x\,\mathrm{d}s\right|,\end{aligned}$$

which proves that the family

$$\left\{\int_{\mathbb{R}^N}n_\epsilon(t,x)\varphi(x)\,\mathrm{d}x,\ \epsilon>0\right\}$$

is equicontinuous on $[0,T]$. Therefore, for a given test function φ, the family is compact in $\mathrm{C}^0([0,T])$, as a direct consequence of the Arzela-Ascoli theorem.

By density and using (A.1), the compactness property extends to any test function φ in $\mathrm{C}_c^0(\mathbb{R}^N)$. Since $\mathrm{C}_c^0(\mathbb{R}^N)$ is separable, by using a diagonal argument, we can extract a subsequence such that

$$\int_{\mathbb{R}^N}\varphi(x)n_{\epsilon_\ell}(t,x)\,\mathrm{d}x\longrightarrow\int_{\mathbb{R}^N}\varphi(x)n(t,x)\,\mathrm{d}x \qquad \text{as } \ell\to\infty \tag{A.2}$$

in $\mathrm{C}^0([0,T])$, for any test function φ in D, a demombrable dense subset of $\mathrm{C}_c^0(\mathbb{R}^N)$. (Note that for the time being, the limit n can only be considered as a family of measures on $\mathbb{R}^N$, parametrized by $t\in[0,T]$; we do not know if it is absolutely continuous with respect to the Lebesgue measure.) Coming back to (A.1) and by density, we realize that the convergence (A.2) applies to any test function $\varphi \in \mathrm{C}^0_c(\mathbb{R}^N)$.

Now we use the fact that n_ϵ enjoys better integrability property than the sole $\mathrm{L}^1(\mathbb{R}^N)$ bound. Indeed, $\{n_\epsilon(t,\cdot),\ \epsilon > 0,\ t \in [0,T]\} \subset \mathcal{K}$, a weakly compact set of $\mathrm{L}^1(\mathbb{R}^N)$. Consequently, the Dunford-Pettis theorem implies that

$$\sup_{\epsilon>0,\ 0\le t\le T} \left\{ \int_{|x|\le R,\ x\in E} |n(t,x)|\,\mathrm{d}x \right\} \longrightarrow 0 \qquad \text{as } |E| \to 0,$$

holds for any given $0 < R < \infty$, and

$$\sup_{\epsilon>0,\ 0\le t\le T} \left\{ \int_{|x|\ge R} |n(t,x)|\,\mathrm{d}x \right\} \longrightarrow 0 \qquad \text{as } R \to \infty.$$

Then, let $\phi \in \mathrm{L}^\infty(\mathbb{R}^N)$. It can be approached pointwise by a sequence of test functions $\varphi_m \in \mathrm{C}^0_c(\mathbb{R}^N)$ (with $\|\varphi_m\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \le \|\phi\|_{\mathrm{L}^\infty(\mathbb{R}^N)}$). By applying the Egoroff theorem, see e.g. [21] (Th. 1.16, p. 31), for any $0 < R < \infty$, we can find a measurable set $E \subset B(0,R)$ with arbitrarily small measure such that φ_m converges to φ uniformly on $\complement_{B(0,R)}(E)$. Hence, we split

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^N} n_\epsilon(t,x)\ \big(\phi(x) - \varphi_m(x)\big)\,\mathrm{d}x \Big| \\
&\qquad \le 2\|\phi\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \int_{|x|\ge R} |n_\epsilon(t,x)|\,\mathrm{d}x \\
&\qquad\qquad + 2\|\phi\|_{\mathrm{L}^\infty(\mathbb{R}^N)} \int_{|x|\le R,\ x\in E} |n_\epsilon(t,x)|\,\mathrm{d}x \\
&\qquad\qquad + \|\phi - \varphi_m\|_{\mathrm{L}^\infty(\complement_{B(0,R)}(E))} \int_{\mathbb{R}^N} |n_\epsilon(t,x)|\,\mathrm{d}x.
\end{aligned}$$

It follows that

$$\sup_{\epsilon>0,\ 0\le t\le T} \left\{ \int_{\mathbb{R}^N} n_\epsilon(t,x)\ \big(\phi(x) - \varphi_m(x)\big)\,\mathrm{d}x \right\} \longrightarrow 0$$

as $m \to \infty$. We deduce that $n(t) \in \mathrm{L}^1(\mathbb{R}^N)$ and that (A.2) can be extended to any test function $\phi \in \mathrm{L}^\infty(\mathbb{R}^N)$.

Appendix B. Dimension Analysis

A few words deserve to be said about the scaling of the equations and the physical meaning of the parameters ϵ, χ. The scaling discussion is made on the most physical case of the three space dimension; restrictions to the 2D framework can be made by standard arguments.

B.1. *Electrostatic case*

Let us write the equation in physical variables. The problem involve the following physical (positive) quantities

- ε_0, the vacuum permittivity,
- q, the elementary charge of the electrons,
- m_e the mass of the electrons,
- τ_e, the relaxation time characteristic of the interactions of the particles with the thermal bath,
- k_B, the Boltzmann constant,
- T_{th}, the temperature of the thermal bath.

The unknown $f(t,x,v)$ is the electron density in a plasma, which means that $\int_\Omega \int_{\mathcal{V}} f \mathrm{d}v\mathrm{d}v$ gives the number of electrons occupying at time t the domain $\Omega \times \mathcal{V}$ of the phase space $\mathbb{R}^3 \times \mathbb{R}^3$. The electrons are submitted to the force $-\frac{q}{m}\nabla_x\Phi$. Therefore, we get

$$\partial_t f + v \cdot \nabla_x f - \frac{q}{m}\nabla_x\Phi \cdot \nabla_v f = \frac{1}{\tau}\nabla_v \cdot \Big(vf + \frac{k_B T_{th}}{m_e}\nabla_v f\Big),$$

while the Poisson relation reads

$$-\Delta\Phi = \frac{q}{\varepsilon_0}\int f \mathrm{d}v.$$

Let us introduce time, length and velocity units denoted by T, L, V, respectively. Then, we define dimensionless variables and unknowns by the following relations

$$\begin{cases} t = \mathrm{T}\ t', \qquad x = \mathrm{L}\ x', \qquad v = \mathrm{V}\ v', \\ f(t,x,v) = \dfrac{\mathcal{N}}{\mathrm{L}^3\mathrm{V}^3}\ f'(t/\mathrm{T}, x/\mathrm{L}, v/\mathrm{V}), \qquad \Phi(t,x,v) = \mathcal{U}\ \Phi'(t/\mathrm{T}, x/\mathrm{L}) \end{cases}$$

where $\mathcal{U}$ stands for a typical value of the potential, and $\mathcal{N}$ stands for a typical value for the number of electrons within the plasma. (Hence, the primmed quantities are dimensionless and order one.) Writting the equations in dimensionless form, we obtain, dropping the primes,

$$\begin{cases} \partial_t f + \dfrac{\mathrm{VT}}{\mathrm{L}}\ v \cdot \nabla_x f - \dfrac{q\mathcal{U}\mathrm{T}}{m\mathrm{LV}}\ \nabla_x\Phi \cdot \nabla_v f = \dfrac{\mathrm{T}}{\tau}\ \nabla_v \cdot \Big(vf + \dfrac{k_B T_{th}}{m\mathrm{V}^2}\ \nabla_v f\Big), \\ -\dfrac{\varepsilon_0 \mathcal{U}}{q\mathcal{N}\mathrm{L}^{-3}\mathrm{L}^2}\ \Delta\Phi = \displaystyle\int f \mathrm{d}v. \end{cases}$$

Five dimensionless parameters appear in the system. We recover (1.1) when setting

$$\frac{\varepsilon_0\mathcal{U}\mathrm{L}}{q\mathcal{N}} = 1, \qquad \mathrm{V} = \sqrt{\frac{k_B T_{th}}{m}}, \qquad \frac{\mathrm{T}}{\tau} = \frac{1}{\chi\epsilon^2} \gg 1,$$
$$\frac{\mathrm{VT}}{\mathrm{L}} = \frac{1}{\epsilon}, \qquad \frac{q\mathcal{U}\mathrm{T}}{m\mathrm{LV}} = \frac{1}{\chi\epsilon}.$$

Note that the velocity unit coincides with the thermal velocity $\sqrt{\frac{k_B T_{th}}{m_e}}$.

The plasma is characterized by the mean free path $\ell = \sqrt{\frac{k_B T_{th}}{m_e}}\ \tau$ and the Debye length $\Lambda = \sqrt{\frac{\epsilon_0 k_B T_{th} \mathrm{L}^3}{q^2\mathcal{N}}}$. Hence, it turns out that in our analysis the mean free path is small compared to both the length of observation and the Debye length. The meaning of χ also becomes clear. Indeed, we get

$$\frac{\ell}{\mathrm{L}} = \frac{\tau}{\epsilon \mathrm{T}} = \chi\epsilon \ll 1, \qquad \frac{\Lambda}{\mathrm{L}} = \sqrt{\frac{k_B T_{th}}{q\mathcal{U}}} = \sqrt{\chi}, \qquad \frac{\ell}{\Lambda} = \sqrt{\chi}\ \epsilon \ll 1.$$

Note that these relations imply that $\mathcal{N}\mathrm{L}^{-3}\tau \ll \varepsilon_0 m_e/q^2$ where the value of this universal constant is 3.1 10^{-4} s^2 m^{-3}.

Eventually, we restrict to the 2D framework by considering solutions of the form $f(t,x,v) = \widetilde{f}(t,x_1,x_2,v_1,v_2)\ \frac{e^{-v_3^2/2}}{\sqrt{2\pi}}$.

B.2. *The gravitational case*

The problem involve the following physical (positive) quantities
- $\mathcal{G}$, the gravitational constant,
- m the mass of a particle,
- τ, the relaxation time due to collisions of the particles with the thermal bath,
- k_B, the Boltzmann constant,
- T_{th}, the temperature of the thermal bath.

We have

$$\partial_t f + v\cdot\nabla_x f - \frac{1}{m}\nabla_x\Phi\cdot\nabla_v f = \frac{1}{\tau}\nabla_v\cdot\Big(vf + \frac{k_B T_{th}}{m}\nabla_v f\Big),$$

and the Poisson relation reads

$$\Delta\Phi = m^2\mathcal{G}\int f\,\mathrm{d}v.$$

Therefore, we are led to

$$\frac{\mathcal{U}}{m^2\mathcal{G}} = \frac{\mathcal{N}}{\mathrm{L}}, \qquad V = \sqrt{\frac{k_B T_{th}}{m}}, \qquad \frac{T}{\tau} = \frac{1}{\chi\epsilon^2}, \qquad \frac{\mathrm{TV}}{\mathrm{L}} = \frac{1}{\epsilon}, \qquad \frac{\mathcal{U}\mathrm{T}}{m\mathrm{VL}} = \frac{1}{\chi\epsilon}.$$

We recover that χ is the ratio $k_B T_{th}/\mathcal{U}$ and we are dealing with a small mean free path asymptotics.

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