Analysis of the $M1$ model: well–posedness and diffusion asymptotics

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Abstract

This paper is devoted to the analysis of the $M1$ model which arises in radiative transfer theory. The derivation of the model is based on the entropy minimization principle, which leads to a hyperbolic system of balance laws with relaxation. In the multi-dimensional case, we establish the existence-uniqueness of a globally defined smooth solution under a suitable smallness condition on the initial data. In the one-dimensional case we show that the smallness condition does not depend on the particles mean free path so that we can also rigorously justify the consistency of the model with the diffusion asymptotics. The result extends the analysis of *[J-F Coulombel, F. Golse and T. Goudon, Asymptotic Analysis, 45, 1–39, 2005]* to the case where the entropy functional accounts for relaxation towards the Planckian state, which is physically more relevant.

Key words. $M1$ model, radiative transfer, diffusion approximation, hyperbolic systems, relaxation, initial value problem, global existence of smooth solutions

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1 Introduction and main results

The modeling of radiative transfer phenomena leads to consider PDEs which look like

$$\varepsilon\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon = \frac{1}{\varepsilon}Q(f_\varepsilon). \tag{1}$$

The unknown $f_\varepsilon(t,x,v,\nu)$ is the specific energy of the radiation: it depends on the time variable $t\geq 0$, the space variable $x\in\mathbb{R}^n$, the direction variable $v\in V$ and the frequency variable $\nu\in\mathbb{R}^+$. We will be specifically interested in the physically relevant cases $n=3$ and $n=1$. In the 3-dimensional case ($n=3$), we take $V=\mathbb{S}^2$, endowed with the normalized Lebesgue measure, that we denote $\mathrm{d}v$. In the 1-dimensional case ($n=1$), we set $V=[-1,1]$, and $\mathrm{d}v$ stands for the normalized Lebesgue measure over $[-1,1]$. Up to a suitable change of variable, the unknown f_ε can be related to the distribution of photons in phase space, with the momentum variable proportional to $\nu\ v$. The right hand side $Q(f_\varepsilon)$ in (1) involves a complex operator describing the interactions the particles are subject to: scattering, absorption, emission... The equation has to be understood here in dimensionless form and the scaling parameter $0<\varepsilon\ll 1$ represents a dimensionless version of the mean free path of the particles, that is the average distance that a particle may fly between such scattering events. For many relevant models, equation (1) is coupled to equations describing the evolution of the material with which the radiation are exchanging energy and possibly momentum. The interaction operator Q induces relaxation properties, so that as ε goes to 0 the particle distribution function relaxes towards a distribution with a prescribed dependence with respect to the variables v and ν. In turn, in such a regime, the dynamics can be described by means of macroscopic quantities, depending on time and space variables only. Typically, it turns out that the macroscopic energy $R(t,x)$, corresponding to the limit of $\int_{\mathbb{R}^+}\int_V f_\varepsilon(t,x,v,\nu)\,\mathrm{d}\nu\,\mathrm{d}v$ as $\varepsilon\to 0$, satisfies a mere diffusion equation, the coefficient of which depends on the details of the operator Q. However, for many situations of practical interest, one is rather interested in regimes where ε is small... but not too much so that discrepancies with the diffusion behavior are significant. Motivated by simulations purposes, a large

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amount of literature is devoted to the design of relevant models of macroscopic nature, intended to preserve the main features of the microscopic problem (1), but which remain accessible to a numerical treatment for a reasonable cost. Indeed, simulating directly (1) at resolved scales, with mesh size and time steps small compared to ε and an unknown depending also on the momentum variable becomes rapidly prohibitive. In this paper, we are concerned with a specific family of such macroscopic models, the so–called $M1$ models, which are obtained by closing the moment system derived by integrating (1) with respect to the momentum variable. The closure is based on a entropy minimization principle. The idea consists in minimizing a certain convex functional, with the constraint of prescribed moments. The functional is precisely constructed so that the minimizers have the same shape as the equilibrium distribution functions which make the interaction operator Q vanish, and it is expected to mimic the relaxation properties of Q. The model obtained by this procedure has relevant structure properties, guaranteed by the construction: hyperbolicity, maximum principle and limited flux.

We refer for a detailed introduction to radiative transfer theory to the classical treatise [33]. A recent overview of related mathematical problems, including numerical topics, can be found in [21]. The analysis of the diffusion regime, and the justification of the so–called Rosseland approximation, dates back to [1, 2]. In particular the problem has motivated the introduction of compactness techniques specific to kinetic equations, the average lemma being at the basis of the results in [2]. Alternative approaches and further details can be found in [13, 20, 22] and the references therein. For entropy-based moment closures we refer to [21] again, and to [10, 7, 18, 24] for other approaches. The entropy minimization principle has been developed by D. Levermore [27, 28, 29] for general kinetic equations, in particular for application in gas dynamics, while for the specific derivation in radiative transfer theory we refer to [17] and [14, 15, 16]. The $M1$ model is widely used for industrial purposes; it has motivated several extensions and the design of specific numerical schemes [3, 6, 7, 8, 19, 36]. Here, we wish to investigate well-posedness and asymptotic issues. As we shall see below, the $M1$ model can be cast into the form of a hyperbolic problem with relaxation. Proving the existence of a smooth solution defined on a small enough time interval is thus the matter of classical techniques, in the spirit of T. Kato and A. Majda, see [31, 32, 26]. Furthermore, we can expect that the relaxation acts to prevent blow–up formation, so that, up to a smallness condition on the data, the solution is globally defined. A quite general framework to address such questions is due to B. Hanouzet and R. Natalini [23] (see also W.-A. Yong [37]): since the relaxation does not act on all the components of the system, we need a strong enough coupling condition, the so–called Kawashima-Shizuta condition [35]. However, following directly these arguments it is not clear whether or not the smallness condition depends on the scaling parameter ε: refined estimates are necessary to make it uniform with respect to ε. This is the program followed in [10] for a simplified version of the $M1$ model. These techniques have been adopted also to treat gas dynamics equations in the strong relaxation limit, for both the isothermal case [11], and the isentropic case [12]. Therefore, we wish to extend the analysis of [10] towards a physically more relevant model of radiative transfer: the entropy functional we consider encodes relaxation towards Planckian states, and, in turn, we are led to more intricate closure relations. In [10], the standard Godunov symmetrization is used to obtain a convenient symmetric form of the system. Uniform energy estimates can be established on the corresponding symmetric system. For the complex closure we are dealing with, we shall use Godunov's symmetrization in order to justify the global existence of smooth solutions in any space dimension. However, we are not able to establish energy estimates uniform with respect to the scaling parameter. For this reason, we introduce, for the one-dimensional case, a new symmetrizor with better structure properties, so that we can rigorously justify in this case the consistency of the model with the diffusion asymptotics. However, we point out that, by contrast to the analysis in [10], the coefficients of the symmetric system depend on the full set of the independent variables, not only on the velocity-type (relaxed) quantities, see Remark 4.1 below, which motivates a specific study.

The paper is organized as follows. We start by introducing precisely in Section 2 the entropy minimization principle and the system of PDEs we are interested in. Then, we state the main results of the paper, summarized in Theorem 2.4 (global existence of smooth solutions in the 3-D case), Theorems 2.6 and 2.7 (global existence and asymptotic behavior of the solutions in the 1-D case) below. Next, we give some hints in Section 3 for the proof of Theorem 2.4 in the 3-D case, which follows from adaptations of [10], and we detail in Section 4 the proof of the global existence in 1-D case, which relies on a uniform non linear estimate satisfied by a suitable energy functional. The key of the proof consists in writing the system in a symmetric form and identifying the relevant coupling structure. Finally, using the stream function as in [10], we discuss in Section 5 the asymptotic issues as the parameter ε goes to 0, justifying the strong relaxation limit of the solution and the consistency of the model with the diffusion regime.

2 The $M1$ model: definition and main results

The derivation of the $M1$ model relies on moment closure assumptions. Let us assume that the interaction term in (1) satisfies

$$\begin{aligned}&\int_V Q(f)\,\mathrm{d}v = 0,\\ &Q(f) = 0 \quad \text{iff } f \text{ does not depend on } v,\\ &\int_V vQ(f)\,\mathrm{d}v = -\int_V vf\,\mathrm{d}v.\end{aligned} \tag{2}$$

The first relation means that interactions do not affect the total energy balance: it certainly applies to scattering events. Dealing with absorption-emission phenomena leads to consider a coupling of (1) with equations prescribing the properties of the material. Having in mind numerical purposes and splitting techniques, it also makes sense to neglect such phenomena in non equilibrium regimes, see [4, 3, 6]. The second assumption prescribes the kernel of the interaction operator. The third assumption is usually related to the identification of the eigenspace associated to the first non-zero eigenvalue of Q, see [10]. Let us introduce the following macroscopic quantities

$$\begin{aligned}\rho_\varepsilon(t,x) &= \int_{\mathbb{R}^+}\int_V f_\varepsilon(t,x,v,\nu)\,\mathrm{d}v\,\mathrm{d}\nu\ ,\\ J_\varepsilon(t,x) &= \int_{\mathbb{R}^+}\int_V \frac{v}{\varepsilon} f_\varepsilon(t,x,v,\nu)\,\mathrm{d}v\,\mathrm{d}\nu\ ,\\ P_\varepsilon(t,x) &= \int_{\mathbb{R}^+}\int_V v\otimes v f_\varepsilon(t,x,v,\nu)\,\mathrm{d}v\,\mathrm{d}\nu.\end{aligned}$$

Therefore, integrating (1) with respect to the direction and frequency variables, we get

$$\left\{\begin{array}{l}\partial_t\rho_\varepsilon + \nabla_x\cdot J_\varepsilon = 0,\\ \varepsilon^2\partial_t J_\varepsilon + \mathrm{Div}_x P_\varepsilon = -J_\varepsilon.\end{array}\right. \tag{3}$$

The system (3) is not closed since it is made of $n+1$ equations and it involves $(n+1)(n+2)/2$ unknowns, since ρ_ε, J_ε and P_ε are scalar, n-components vector and symmetric $n\times n$ matrix, respectively. Nevertheless the asymptotic behavior can be understood looking at (3). Assuming that $\rho_\varepsilon, J_\varepsilon, P_\varepsilon$ admit limits, denoted $R, \bar J, \bar P$ respectively, we obtain as ε goes to 0

$$\partial_t R + \nabla_x\cdot \bar J = 0, \qquad \mathrm{Div}_x \bar P = -\bar J.$$

But, coming back to (2), we expect that f_ε tends to an element $\bar f$ of the kernel of Q, which does not depend on v. Accordingly we guess that $P_\varepsilon = \int_{\mathbb{R}^+}\int_V v\otimes v f_\varepsilon\,\mathrm{d}v\,\mathrm{d}\nu \to \bar P = \int_V v\otimes v\,\mathrm{d}v\int_{\mathbb{R}^+}\bar f\,\mathrm{d}\nu = \frac{R}{3}\mathbb{I}$. Consequently, we end up with

$$\partial_t R - \frac{1}{3}\triangle_x R = 0, \tag{4}$$

that is a mere diffusion equation for the radiation energy. The above formal analysis applies in 3-D and in 1-D, owing to the assumptions on V.

A closure method consists in defining a closed system by imposing in (3) a relation which defines P_ε as a functional of ρ_ε and J_ε. Therefore, we are led to consider the approximate system

$$\left\{\begin{array}{l}\partial_t\widehat\rho_\varepsilon + \nabla_x\cdot \widehat J_\varepsilon = 0,\\ \varepsilon^2\partial_t \widehat J_\varepsilon + \mathrm{Div}_x \widehat P_\varepsilon(\widehat\rho_\varepsilon, J_\varepsilon) = -\widehat J_\varepsilon.\end{array}\right. \tag{5}$$

The construction is expected to preserve some fundamental properties of the original model (1): as a matter of fact the energy density f_ε being non negative, we have

$$\rho_\varepsilon \geq 0, \qquad |J_\varepsilon| \leq \frac{1}{\varepsilon}\int_V |v| f_\varepsilon\,\mathrm{d}v\,\mathrm{d}\nu \leq \frac{\rho_\varepsilon}{\varepsilon}.$$

In particular, it is important for the applications to satisfy the flux limited condition (which is related to the finite speed of propagation). Furthermore, as ε goes to 0, the approximate model should reproduce the asymptotic behavior of the solutions of (1). The entropy minimization principle works as follows. Given $\widehat\rho_\varepsilon, \widehat J_\varepsilon$, and a convex function H, we define $\widehat{f_\varepsilon}$ which minimizes

$$f \longmapsto \int_{\mathbb{R}^+}\int_V H(f)\,\mathrm{d}v\,\mathrm{d}\nu$$

with the constraint of prescribed zeroth and first moments

$$\int_{\mathbb{R}^+}\int_V f\,\mathrm{d}v\,\mathrm{d}\nu = \widehat{\rho}_\varepsilon, \qquad \int_{\mathbb{R}^+}\int_V \frac{v}{\varepsilon} f\,\mathrm{d}v\,\mathrm{d}\nu = \widehat{J}_\varepsilon.$$

Then (5) is closed by setting

$$\widehat{P}_\varepsilon(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon) = \int_{\mathbb{R}^+}\int_V v\otimes v \widehat{f}_\varepsilon\,\mathrm{d}v\,\mathrm{d}\nu.$$

It is convenient to rewrite the kinetic pressure as follows

$$\widehat{P}_\varepsilon = \widehat{\rho}_\varepsilon D(\widehat{u}_\varepsilon), \qquad \widehat{u}_\varepsilon = \varepsilon\frac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}$$

where D is referred to as the Eddington tensor. Of course, changing the function H modifies the macroscopic model; several relevant entropy functionals are discussed in [10]. In radiative transfer theory it turns out that it makes sense to consider the following entropy

$$\frac{15}{4\pi^5}\int_{\mathbb{R}^+}\int_V \nu^2\big[n\ln n - (n+1)\ln(n+1)\big]\,\mathrm{d}v\,\mathrm{d}\nu, \qquad n = \frac{4\pi^5}{15\nu^3}f.$$

We refer to [17, 9, 14, 15, 16, 5] for a justification of this choice of entropy. The main motivation comes from the fact that this quantity is dissipated by the kinetic model (1). It prescribes the form of the equilibrium functions, which are minimizers of H. Assuming $\frac{15\nu^2}{4\pi^5}\frac{\mathrm{d}}{\mathrm{d}f}\big[n\ln n - (n+1)\ln(n+1)\big] = \frac{1}{\nu}\ln\big(\frac{n}{n+1}\big) = \lambda$, a constant, yields $f = \frac{15\nu^3}{4\pi^5}\big(e^{-\nu\lambda}-1\big)^{-1}$. Prescribing the value, say ρ, of the total energy associated to such an equilibrium uniquely defines λ by the relation

$$\int_{\mathbb{R}^+}\int_V f\,\mathrm{d}v\,\mathrm{d}\nu = \rho = \int_{\mathbb{R}^+}\frac{15\nu^3}{4\pi^5}\frac{\mathrm{d}\nu}{e^{-\nu\lambda}-1}.$$

In the literature on the $M1$ model, it is convenient to define the radiation temperature $T_r = \rho^{1/4}$ and the previous relation recasts as $\lambda = -1/T_r$. The corresponding equilibrium is the Planckian function with temperature T_r, see e. g. [14, Proposition 1]. It explains the definition of the entropy because Planckian functions are natural equilibrium states in radiative transfer, playing in some sense the same role as the Maxwellian in gas dynamics: interaction processes are intended to make the energy density relax to such a state, with a prescribed dependence with respect to both direction and frequency variables. Coming back to the moment closure problem, we can repeat similar manipulations, making the Lagrange multipliers associated to the moments $\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon$ appear. We obtain that way generalized Planckian functions, as explained in the following claim [14, Section 3], [15, 16]. It is worthwhile mentioning that many details are discussed with a different viewpoint, combining information theory and statistical physics, in [17]. These considerations make the $M1$ model a very popular macroscopic system to describe the dynamics of radiative fields, see [5, 21].

Lemma 2.1. *Let $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon) \in \mathbb{R}^+\times\mathbb{R}^3$ such that $|\varepsilon J_\varepsilon| < \rho_\varepsilon$. The minimizer function $\widehat{f}_\varepsilon$ is given by*

$$\widehat{f}_\varepsilon(v,\nu) = \frac{15\nu^3}{4\pi^5}\frac{1}{\exp((\alpha_0+\alpha_1\cdot v)\nu)-1}$$

where $\alpha_0 > 0$, and $\alpha_1 \in \mathbb{R}^3$, verifying $|\alpha_1| < \alpha_0$, are Lagrangian multiplier. Then, the second moment of the closure $\widehat{P}_\varepsilon = \int_{\mathbb{R}^+}\int_{\mathbb{S}^2} v\otimes v\widehat{f}_\varepsilon(v,\nu)\,\mathrm{d}v\,\mathrm{d}\nu$ can be written in the Eddington form:

$$\widehat{P}_\varepsilon = \widehat{\rho}_\varepsilon D(\widehat{u}_\varepsilon), \qquad \widehat{u}_\varepsilon = \varepsilon\frac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}$$

where the Eddington tensor is given by

$$D(u) = \frac{1-\chi(u)}{2}\mathbb{I}_3 + \frac{3\chi(u)-1}{2}\frac{u\otimes u}{|u|^2}, \tag{6}$$

with $\mathbb{I}_3$ standing for the 3×3 identity matrix and

$$\chi(u) = \frac{3+4|u|^2}{5+2\sqrt{4-3|u|^2}} = \frac{5-2\sqrt{4-3|u|^2}}{3}.$$

Remark 2.2. *Since $\widehat{\rho}_\varepsilon$ and $\widehat{J}_\varepsilon$ are the zeroth and first order moments of the minimizer function $\widehat{f}_\varepsilon$, they can be expressed as functions of the scalar α_0 and the vector α_1; namely, we have*

$$\widehat{\rho}_\varepsilon = \frac{3\alpha_0^2 + |\alpha_1|^2}{3(\alpha_0^2 - |\alpha_1|^2)^3}, \qquad \varepsilon\widehat{J}_\varepsilon = \frac{-4\alpha_0\alpha_1}{3(\alpha_0^2 - |\alpha_1|^2)^3}. \tag{7}$$

Conversely, knowing $\widehat{\rho}_\varepsilon$ and $\widehat{u}_\varepsilon$, with $\widehat{u}_\varepsilon = \varepsilon\widehat{J}_\varepsilon/\widehat{\rho}_\varepsilon$, the Lagrangian multiplier α_0 and α_1 are given by

$$\alpha_0 = \frac{1}{2\widehat{\rho}_\varepsilon^{1/4}} \frac{(\sqrt{4-3|\widehat{u}_\varepsilon|^2}+2)^{1/2}}{(\sqrt{4-3|\widehat{u}_\varepsilon|^2}-1)^{3/4}} \quad \alpha_1 = -\frac{3\widehat{u}_\varepsilon}{2\widehat{\rho}_\varepsilon^{1/4}} \frac{(\sqrt{4-3|\widehat{u}_\varepsilon|^2}+2)^{-1/2}}{(\sqrt{4-3|\widehat{u}_\varepsilon|^2}-1)^{3/4}}. \tag{8}$$

Furthermore we have the following relation

$$\frac{\alpha_1}{\alpha_0} = -\frac{3\widehat{u}_\varepsilon}{2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}}.$$

Remark 2.3. *Note that $D(u)$ defined by (6) satisfies $\mathrm{tr}(D(u)) = 1$ and the relations*

$$1-\chi(u) = \frac{2}{3}\big(\sqrt{4-3|u|^2}-1\big) \geq 0, \qquad \chi(u) - |u|^2 = \frac{1}{3}\big(1-\sqrt{4-3|u|^2}\big)^2 \geq 0$$

imply $D(u) - \frac{u\otimes u}{|u|^2} \geq 0$. Let us assume that $|\widehat{u}_\varepsilon| \leq \eta < 1$ holds. According to Remark 2.2, we deduce that $|\alpha_1|/\alpha_0 \leq 3\eta/(2+\sqrt{4-3\eta^2}) < 1$. Therefore, we have $\alpha_0 + \alpha_1 \cdot v > 0$. Observe that

$$\int_{\mathbb{R}^+} \widehat{f}_\varepsilon \,\mathrm{d}\nu = \frac{1}{(\alpha_0 + \alpha_1\cdot v)^4}.$$

Furthermore we can rewrite the Eddington tensor $D(u)$ as

$$D(u) = \frac{1}{3}\mathbb{I}_3 + \frac{-|u|^2\mathbb{I}_3 + 3u\otimes u}{2+\sqrt{4-3|u|^2}}.$$

Accordingly, for any $\xi \in \mathbb{R}^3$, we remark that $D(0) = \frac{1}{3}\mathbb{I}_3$ and

$$\frac{-4|u|^2|\xi|^2}{2+\sqrt{4-3|u|^2}} \leq \xi^T\big(D(u) - \frac{1}{3}\mathbb{I}_3\big)\xi \leq \frac{2|u|^2|\xi|^2}{2+\sqrt{4-3|u|^2}}.$$

Therefore the main results of the paper state as follows. The first statement is concerned with the global existence–uniqueness of smooth solutions in the multi-dimensional case, under a suitable smallness condition.

Theorem 2.4 (Global Existence in 3-D)**.** *Let $\bar{\rho} > 0$ be a constant and let $s \in \mathbb{N}$ with $s > \frac{5}{2}$. Let $0 < \varepsilon < 1$ be a fixed constant. There exist positive constants $0 < \delta < 1$ and $C > 0$ such that for any (ρ_0, J_0) verifying $\|\rho_0 - \bar{\rho}\|_{H^s(\mathbb{R}^3)} + \|J_0\|_{H^s(\mathbb{R}^3)} \leq \delta$, there exists a unique smooth solution $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon)$ to the system (5) with the initial data (ρ_0, J_0), that satisfies $(\widehat{\rho}_\varepsilon - \bar{\rho}, \widehat{J}_\varepsilon) \in C(\mathbb{R}^+, H^s(\mathbb{R}^3)) \bigcap C^1(\mathbb{R}^+, H^{s-1}(\mathbb{R}^3))$. Furthermore this global solution satisfies*

$$\begin{aligned} \sup_{t>0}\Big(\|\widehat{\rho}_\varepsilon(t) - \bar{\rho}\|^2_{H^s(\mathbb{R}^3)} + \|\widehat{J}_\varepsilon(t)\|^2_{H^s(\mathbb{R}^3)}\Big) + \int_0^\infty \|\widehat{J}_\varepsilon(\tau)\|^2_{H^s(\mathbb{R}^3)}\,\mathrm{d}\tau \\ \leq C\Big(\|\rho_0 - \bar{\rho}\|^2_{H^s(\mathbb{R}^3)} + \|J_0\|^2_{H^s(\mathbb{R}^3)}\Big). \end{aligned} \tag{9}$$

Remark 2.5. *By using standard Sobolev embedding and the energy inequality (9), the velocity $\widehat{u}_\varepsilon$ is uniformly bounded; here and below, we shall use the fact that we are working with solutions verifying $\|\widehat{u}_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^3)} \leq \delta < 1$. Accordingly, the pair $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon = \widehat{\rho}_\varepsilon\widehat{u}_\varepsilon/\varepsilon)$ is admissible in the sense that its components are respectively the zeroth and the first moments of some distribution function, see Lemma 2.1.*

The above theorem shows the global existence of smooth solutions to the relaxation system (5). The proof relies on a result of [37], by using Godunov's symmetrization, energy estimates and the so-called Shizuta-Kawashima condition. Some basic facts will be given in Section 3. It is not clear whether or not the constant δ and C can be made independent of ε. Indeed, compared to [10], we lose some structure properties in choosing the more physically relevant entropy functional. Consequently we lack the uniform estimates with respect to ε necessary to handle the strong relaxation limit of (5) as in the case of compressible Euler equations [11, 12].

However, the difficulty can be treated within the one dimensional framework. This is the object of the following stetaments, where we investigate the asymptotic behavior of the solutions to (5) as ε tends to zero in the one dimensional case. A new symmetrization will be used to get the uniform estimates, which allows to prove the convergence of the density $\widehat{\rho}_\varepsilon$ to the solution of the heat equation. We start with a refined version of Theorem 2.4 for the one dimensional case.

Theorem 2.6 (Global Existence in 1-D)**.** *Let $\bar{\rho} > 0$ be a constant and let $s \in \mathbb{N}$ with $s > \frac{3}{2}$. There exist positive constants $0 < \delta < 1$ and $C > 0$ such that for any $\varepsilon \in (0,1)$, and for any (ρ_0, J_0) verifying $\|\rho_0 - \bar{\rho}\|_{H^s(\mathbb{R})} + \|J_0\|_{H^s(\mathbb{R})} \leq \delta$, there exists a unique smooth solution $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon)$ to system (5) with the initial data (ρ_0, J_0), that satisfies $(\widehat{\rho}_\varepsilon - \bar{\rho}, \widehat{J}_\varepsilon) \in C(\mathbb{R}^+, H^s(\mathbb{R})) \bigcap C^1(\mathbb{R}^+, H^{s-1}(\mathbb{R}))$. Furthermore this global solution satisfies*

$$\sup_{t>0} \left(\|\widehat{\rho}_\varepsilon(t) - \bar{\rho}\|^2_{H^s(\mathbb{R})} + \varepsilon^2 \|\widehat{J}_\varepsilon(t)\|^2_{H^s(\mathbb{R})} \right) + \int_0^\infty \|\widehat{J}_\varepsilon(\tau)\|^2_{H^s(\mathbb{R})} \, \mathrm{d}\tau \leq C \left(\|\rho_0 - \bar{\rho}\|^2_{H^s(\mathbb{R})} + \|J_0\|^2_{H^s(\mathbb{R})} \right). \tag{10}$$

Observe that $D(0) = \chi(0) = \frac{1}{3}$ so that the $M1$ model is formally consistent with the diffusion asymptotic. A detailed formal analysis, by means of Chapman-Enskog expansion, can be found for instance in [5, Section 4, sp. Lemma 7]. Therefore, it can be expected that $\widehat{\rho}_\varepsilon$ obtained in Theorem 2.6 converges to the solution of the heat equation as ε goes to zero. This is indeed the case, as shown in the following statement.

Theorem 2.7 (Asymptotic Behavior)**.** *Consider an initial data $(\rho_0, u_0) \in (\bar{\rho} + H^s(\mathbb{R})) \times H^s(\mathbb{R})$ independent of ε. Let the assumptions of Theorem 2.4 be fulfilled and let $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon = \widehat{\rho}_\varepsilon \widehat{u}_\varepsilon / \varepsilon)$ be the unique solution of (5) obtained in Theorem 2.4. Let $(t,x) \mapsto R(t,x) \in C(\mathbb{R}^+, \bar{\rho} + H^s(\mathbb{R}))$ be the unique solution to the heat equation (4) with initial value $R|_{t=0} = \rho_0$. Then we have*

$$\|\widehat{\rho}_\varepsilon - R\|_{L^2(\mathbb{R}^+ \times \mathbb{R})} \leq C\varepsilon,$$

for some numerical constant C.

3 Global existence in 3-D case: proof of Theorem 2.4

In this section we prove the global existence of classical solution to (5) for a given $\varepsilon > 0$. For the sake of convenience, we drop the dependence of the unknowns $\widehat{\rho}$ and $\widehat{u}$ on the parameter ε.

Since $\widehat{f}$ is a function of $\alpha = (\alpha_0, \alpha_1)$, also of $(\widehat{\rho}, \widehat{J})$, the entropy $\widehat{\eta} = \int_{V \times \mathbb{R}^+} H(\widehat{f}) \, \mathrm{d}v \, \mathrm{d}\nu$ is also a function of $\alpha = (\alpha_0, \alpha_1)$. From Remark 2.2 the entropy $\widehat{\eta}$ is function of $(\widehat{\rho}, \widehat{J})$ too. Then the entropic variable is defined as $\nabla_{\widehat{\rho}, \widehat{J}} \widehat{\eta}(\widehat{\rho}, \widehat{J})$. By a tedious calculation we check that

$$\nabla_{\widehat{\rho}, \widehat{J}} \widehat{\eta}(\widehat{\rho}, \widehat{J}) = -(\alpha_0, \alpha_1)^t = -\alpha.$$

In this Section we choose α as defining the independent variables. Immediately from Lemma 2.1 we have

$$D_\alpha \widehat{f} = -\frac{15\nu^4}{4\pi^5} \widehat{n}(\widehat{n}+1)(1, v^t), \text{ with } \widehat{n} = \frac{1}{\exp((\alpha_0 + \alpha_1 \cdot v)\nu) - 1}.$$

As $\widehat{\rho}$, $\widehat{J}$, $\widehat{P}$ are the moments of $\widehat{f}$, the momentum system (5) can be written as

$$\varepsilon A_0(\alpha) \partial_t \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} + \sum_j A_j(\alpha) \partial_{x_j} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix} = -\frac{1}{\varepsilon} \begin{pmatrix} 0 \\ \Phi(\alpha)\alpha_1 \end{pmatrix}, \tag{11}$$

with

$$\begin{aligned}
A_0(\alpha) &= \frac{15}{4\pi^5} \left\langle \nu^4 \widehat{n}(\widehat{n}+1) \begin{pmatrix} 1 \\ v \end{pmatrix} (1, v^t) \right\rangle_{v,\nu} \\
A_j(\alpha) &= \frac{15}{4\pi^5} \left\langle v_j \nu^4 \widehat{n}(\widehat{n}+1) \begin{pmatrix} 1 \\ v \end{pmatrix} (1, v^t) \right\rangle_{v,\nu}, \ j = 1,2,3. \\
\Phi(\alpha) &= \frac{4\alpha_0}{3(\alpha_0^2 - |\alpha_1|^2)},
\end{aligned}$$

where $< \cdot >_{v,\nu}$ denotes the integration with respect to v, ν. It is easy to check that

$$\int_{\mathbb{R}^+} \nu^4 \widehat{n}(\widehat{n}+1) \, \mathrm{d}\nu = \frac{4}{(\alpha_0 + \alpha_1 \cdot v)^5} \frac{4\pi^5}{15}.$$

Hence the matrices A_0, A_j have the following simple form

$$\begin{aligned}
A_0(\alpha) &= \left\langle \frac{4}{(\alpha_0 + \alpha_1 \cdot v)^5} \begin{pmatrix} 1 \\ v \end{pmatrix} (1, v^t) \right\rangle_v, \\
A_j(\alpha) &= \left\langle \frac{4 v_j}{(\alpha_0 + \alpha_1 \cdot v)^5} \begin{pmatrix} 1 \\ v \end{pmatrix} (1, v^t) \right\rangle_v, \ j = 1,2,3,
\end{aligned}$$

where $< \cdot >_v$ denotes the integration with respect to v only. Note that the matrices A_0, A_j, fully depend on α. This is by contrast to [10] where the matrices obtained by Godunov's symmetrization only depend on α_1, the velocity type variable, which is the relaxed variable. Hence the strategy used in [10] can not be applied directly here, even for the one-dimensional model. Here we give the global existence of smooth solutions to (11) by using the known results for hyperbolic systems with relaxation. For any constant state $\overline{\alpha} = (\overline{\alpha}_0, 0)$ with $\overline{\alpha}_0 > 0$, we set

$$A_0(\overline{\alpha}) = \frac{4}{\overline{\alpha}_0^5}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3}\mathbb{I}_3 \end{pmatrix}, A_1(\overline{\alpha}) = \frac{4}{3\overline{\alpha}_0^5}\begin{pmatrix} 0 & e_j^t \\ e_j & 0_{3\times 3} \end{pmatrix}.$$

The system (11) with unknown α fulfills the following properties:

- symmetrizability in Friedrich's sense; namely, the matrix $A_0(\alpha)$ is symmetric and positive definite, the matrices $A_j(\alpha)$, $j = 1, 2, 3$, are symmetric, for any $\alpha = (\alpha_0, \alpha_1)$ verifying $\alpha_0 > 0$, $|\alpha_1| < \alpha_0$.
- Shizuta-Kawashima stability condition: for any $\mu \in \mathbb{R}$, and $\omega = (\omega_1, \omega_2, \omega_3) \in S^2$

$$\left(\mu A_0(\overline{\alpha}) + \sum_j \omega_j A_j(\overline{\alpha})\right)\begin{pmatrix} X \\ 0_{3\times 1} \end{pmatrix} = 0 \text{ iff } X = 0.$$

Now we can use general results on the global existence of smooth solutions to relaxation hyperbolic systems in multi-dimension [37, page 255, Theorem 3.1]. We omit the details here.

It remains to check the admissibility condition for the Lagrangian multiplier α :

$$\alpha_0 > 0, \quad |\alpha_1| < \alpha_0.$$

It can be verified by choosing $\bar{\alpha}_0 > 0$ large enough and the initial data sufficiently small. Once we get the solutions α of (11) verifying the admissibility condition, then $(\widehat{\rho}, \widehat{J})$, solutions to (5), is immediately obtained owing to Remark 2.2. Hence the proof of Theorem 2.4 is complete. For more details we refer to [37, 10].

4 Global existence in 1-D case: proof of Theorem 2.6

In this section, we restrict to the one-dimensional framework and we shall prove the global existence of smooth solutions to the $M1$ model (5). The argument relies on the possibility of identifying some key structure properties of the system (5). These properties, which is the missing ingredient in 3-D, allows to prove the global existence of smooth solutions with uniform a priori estimates with respect to ε. Having disposed of a few preliminary observations, we will next discuss the a priori estimates which lead to the global existence of solutions.

4.1 Preliminary transformations

For the sake of convenience, we drop the dependence of the unknowns $\widehat{\rho}$ and $\widehat{J}$ on the parameter ε, and we recall that $\widehat{u} = \varepsilon\widehat{J}/\widehat{\rho}$. We also introduce a new rescaled time variable: we set

$$\rho(t, x) = \widehat{\rho}(\varepsilon t, x), \quad u(t, x) = \widehat{u}(\varepsilon t, x).$$

and the system (5) becomes

$$\begin{cases} \partial_t \rho + \partial_x (\rho u) = 0 \ , \\ \partial_t(\rho u) + \partial_x [\rho D(u)] = -\dfrac{1}{\varepsilon}\rho u \ . \end{cases} \tag{12}$$

In the 1-D case, the Eddington tensor reads $D(u) = \chi(u)$, see (6). We rewrite (12) in the quasi-linear form: let $W = (\rho, u)^T$, which satisfies

$$\widetilde{A}_0(W)\partial_t W + \widetilde{A}_1(W)\partial_x W = -\frac{1}{\varepsilon}\begin{pmatrix} 0 \\ \rho u \end{pmatrix},$$

where

$$\widetilde{A}_0(W) = \begin{pmatrix} 1 & 0 \\ u & \rho \end{pmatrix},$$

and

$$\widetilde{A}_1(W) = \begin{pmatrix} u & \rho \\ D(u) & \rho D'(u) \end{pmatrix}.$$

The system (12) is hyperbolic, symetrizable, see [14, Proposition 3]. Actually, this is a direct consequence of the entropy minimization principle, see [27, 28]. Hence by the standard theory of hyperbolic system, there exists a unique smooth local in time solution, [32, 31, 26]. The next ingredient will be to check that the system satisfies a stability propriety, the so-called Shizuta-Kawashima condition, which will allow to show that the obtained solution is globally defined. Let us start by writing the system in symmetric form.

We set

$$S(W) = \begin{pmatrix} 1 & 0 \\ -3\rho u\phi(u) & 3\rho\phi(u) \end{pmatrix},$$

with $\phi(u) = \frac{1}{(\sqrt{4-3u^2}-1)^2}$. Multiply system (12) from the left by $S(W)$; we have

$$A_0(W)\partial_t W + A_1(W)\partial_{x_j} W = -\frac{3}{\varepsilon}\begin{pmatrix} 0 \\ \rho^2\phi(u)u \end{pmatrix}, \tag{13}$$

with

$$A_0(W) = \begin{pmatrix} 1 & 0 \\ 0 & 3\rho^2\phi(u) \end{pmatrix}, \quad A_1(W) = \begin{pmatrix} u & \rho \\ \rho & \dfrac{3\rho^2\phi(u)u(2-\sqrt{4-3u^2})}{\sqrt{4-3u^2}} \end{pmatrix}. \tag{14}$$

Let $\rho = \bar\rho > 0$ and $u = 0$: it defines a particular solution of (13) that we denote $\overline{W} = (\bar\rho, 0)^T$. It is easy to check that this 1-D system verifies the same properties as the 3-D model, namely, symmetrizability in Friedrichs sense and the so-called Shizuta-Kawashima condition. Furthermore, inspired form [10, 11, 12], this system verifies some more particular properties, which allows us to get the uniform estimates with respect to ε.

Remark 4.1. *In [10], the symmetric form of the moment system is obtained by using Godunov's symmetrization. As pointed in Section 3, Godunov's symmetrization only yields the global existence result for fixed $\varepsilon > 0$, but, as far as we have checked, it does not provide uniform estimates with respect to ε. To circumvent this difficulty, we use for the 1-D case a new symmetrization defined by $S(W)$. A remarkable feature of the problem investigated in [10] is the fact that the matrices A_0 and A_j do not fully depend on W: they only depend on the velocity-type variable, that is the relaxed variable. This observation greatly simplifies the analysis. For the $M1$ model under consideration, it is not clear that such a simplified framework can be exhibited. Nevertheless, we shall see below how the difficulty can be treated. Notice that a similar difficulty arises when dealing with the isentropic Euler system with strong relaxation [12]. In fact, in these cases we need a more accurate analysis in the energy estimates, see Remark 4.4.*

Local existence of a smooth solution to (13) can be obtained by using an iterative scheme and the Picard fixed point theorem. We skip the proof of local existence which follows the arguments detailed in the classical references [26, 32, 31]. We wish to show that the solution is actually globally defined. To this end, we need to establish uniform a priori estimates. As in [10] let us introduce the following energy functional

$$\begin{aligned} N_\varepsilon(T)^2 &= \sup_{0\le t\le T}\Big(\|\rho(t)-\bar\rho\|^2_{H^s(\mathbb{R})} + \|u(t)\|^2_{H^s(\mathbb{R})}\Big) \\ &\quad + \frac{1}{\varepsilon}\int_0^T \|u(t)\|^2_{H^s(\mathbb{R})}\,\mathrm{d}t + \varepsilon\int_0^T \|\partial_x\rho(t)\|^2_{H^{s-1}(\mathbb{R})}\,\mathrm{d}t. \end{aligned}$$

where $s > \frac{3}{2}$ is an integer. The following theorem is the key ingredient in the proof of the global existence.

Theorem 4.2. *Let $s > \frac{3}{2}$ be an integer and let $T > 0$. Let $0 < \varepsilon < 1$. Let $\rho \in C([0,T];\bar\rho + H^s(\mathbb{R}))$ and $u \in C([0,T];H^s(\mathbb{R}))$ be a local solution to (13) associated to the initial data $(\rho_0, u_0) \in (\bar\rho + H^s(\mathbb{R})) \times H^s(\mathbb{R})$. Then, there exists a non decreasing function $\mathcal{C} : \mathbb{R}^+ \to \mathbb{R}^+$, which is independent of ε, T, ρ, and u, such that the following inequality holds*

$$N_\varepsilon(T)^2 \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})(N_\varepsilon(0)^2 + N_\varepsilon(T)^3).$$

Once Theorem 4.2 is proven, the proof of Theorem 2.6 follows from quite standard arguments, which dates back to [34]. We refer for details in a similar context to [10, 11]. We split the proof into two steps: firstly, we establish the $L^\infty(H^s)$ estimate of (ρ, u) and the $L^2(H^s)$ estimate of u, and, secondly, we turn to the $L^2(H^{s-1})$ estimate of $\partial_x\rho$. Based on these estimates, we will get the proof of Theorem 4.2 directly.

4.2 Energy estimate I: $L^\infty(H^s)$ estimate

This Section is devoted to the proof of the following claim.

Proposition 4.3. *Under the hypotheses stated in Theorem 4.2, for any $t \in [0,T]$, there holds*

$$\|\rho(t)-\bar\rho\|^2_{H^s(\mathbb{R})} + \|u(t)\|^2_{H^s(\mathbb{R})} + \frac{1}{\varepsilon}\int_0^t \|u(\tau)\|^2_{H^s(\mathbb{R})}\,\mathrm{d}\tau \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})(N^2_\varepsilon(0) + N^3_\varepsilon(t)). \tag{15}$$

Proof. The proof splits into several steps. We start with the zeroth order estimate, and next we treat the higher order derivatives. Finally based on these estimates, we will finish the proof.

Step 1: Zeroth-order estimate. We remind that $\bar\rho$ is a fixed positive constant and we set $\overline{W} = (\bar\rho, 0)^T$. Take the inner product of the system (13) with the vector $W - \overline{W}$, then integrate over $[0,t] \times \mathbb{R}$. We obtain

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2} A_0(W)(W-\overline{W})\cdot(W-\overline{W})\,\mathrm{d}x\Big|_0^t + \frac{3}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}} \rho^2\phi(u)u^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad = \int_0^t\!\!\int_{\mathbb{R}} \frac{1}{2}\left\{\partial_t\{A_0(W)\} + \partial_x\{A_1(W)\}\right\}(W-\overline{W})\cdot(W-\overline{W})\,\mathrm{d}x\,\mathrm{d}\tau\ . \end{aligned} \tag{16}$$

Owing to the definition of $A_0(W)$ we get

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2} A_0(W(t))(W(t)-\overline{W})\cdot(W(t)-\overline{W})\,\mathrm{d}x + \frac{3}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}} \rho^2\phi(u)u^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad \geq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\left(\|W(t)-\overline{W}\|^2_{L^2(\mathbb{R})} + \frac{1}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}}\|u(\tau)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}\tau\right), \\ &\int_{\mathbb{R}} \frac{1}{2} A_0(W)(W-\overline{W})\cdot(W-\overline{W})\Big|_{t=0}\,\mathrm{d}x \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})N_\varepsilon^2(0). \end{aligned}$$

Using the specific structure of the matrices $A_0(W)$ and $A_1(W)$, the right-hand side of (16) can be estimated as follows

$$\begin{aligned} &\int_0^t\!\!\int_{\mathbb{R}} \frac{1}{2}\partial_t\{A_0(W)\}(W-\overline{W})\cdot(W-\overline{W})\,\mathrm{d}x \\ &\qquad \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}|\partial_t W||u|^2\,\mathrm{d}\tau\,\mathrm{d}x \\ &\qquad \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}\left(|\partial_x W| + \frac{1}{\varepsilon}|u|\right)|u|^2\,\mathrm{d}\tau\,\mathrm{d}x \\ &\qquad \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})N_\varepsilon^3(t), \\ &\int_0^t\!\!\int_{\mathbb{R}} \frac{1}{2}\partial_x\{A_1(W)\}(W-\overline{W})\cdot(W-\overline{W})\,\mathrm{d}x \\ &\qquad \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}(|\partial_x u||\rho-\bar\rho|^2 + |\partial_x\rho||\rho-\bar\rho||u| + |\partial_x W||u|^2)\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})N_\varepsilon^3(t). \end{aligned}$$

For obtaining the last inequality, we have used the following estimate

$$\begin{aligned} \int_0^t\!\!\int_{\mathbb{R}}|\partial_x\rho||W-\overline{W}||u| &\leq CN_\varepsilon(t)\int_0^t\left(\varepsilon\|\partial_x\rho(\tau)\|^2_{L^2(\mathbb{R})} + \frac{1}{\varepsilon}\|u(\tau)\|^2_{L^2(\mathbb{R})}\right)\mathrm{d}x\,\mathrm{d}\tau \\ &\leq CN_\varepsilon^3(t). \end{aligned}$$

Coming back to (16) yields

$$\|\rho(t)-\bar\rho\|^2_{L^2(\mathbb{R})} + \|u(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{\varepsilon}\int_0^t\|u(\tau)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}\tau \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\left(N_\varepsilon^2(0) + N_\varepsilon^3(t)\right), \tag{17}$$

which corresponds to (15) for $s=0$.

Step 2: Higher-order estimates. Let k be an integer with $1 \leq k \leq s$. Apply ∂_x^k to system (13), take the inner product with $\partial_x^k W$, then integrate the resulting equality over $[0,t]\times\mathbb{R}$. We have

$$\begin{aligned} &\int_{\mathbb{R}} \frac{1}{2} A_0(W)\partial_x^k W\cdot\partial_x^k W\,\mathrm{d}x\Big|_0^t + \frac{3}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}}\rho^2\phi(u)|\partial_x^k u|^2\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad = \int_0^t\!\!\int_{\mathbb{R}}\left[\frac{I_1+I_2}{2} - (I_3+I_4+I_5)\right]\mathrm{d}x\,\mathrm{d}\tau, \end{aligned} \tag{18}$$

with

$$\begin{aligned} &I_1 = \partial_t\{A_0(W)\}\partial_x^k W\cdot\partial_x^k W, \quad I_2 = \partial_x\{A_1(W)\}\partial_x^k W\cdot\partial_x^k W, \\ &I_3 = [\partial_x^k, A_0(W)]\partial_t W\cdot\partial_x^k W, \quad I_4 = [\partial_x^k, A_1(W)]\partial_x W\cdot\partial_x^k W, \\ &I_5 = \frac{3}{\varepsilon}\{[\partial_x^k, \rho^2\phi(u)]u\}\partial_x^k u, \end{aligned}$$

where $[a,b]$ denotes the commutator $ab-ba$. We wish to estimate the integrals of I_1 to I_5. As already remarked, $\partial_t A_0(W)$ has only one non-zero block. Then the integral of I_1 can be estimated as follows

$$\left|\int_0^t\!\!\int_{\mathbb{R}} I_1 \,\mathrm{d}x\,\mathrm{d}\tau\right| \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}} |\partial_t W||\partial_x^\alpha u|^2 \,\mathrm{d}x\,\mathrm{d}tau \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})N_\varepsilon^3(t),$$

where we have also used the same strategy as for the zeroth-order term. In order to estimate I_2 we exploit the expression of $A_1(W)$ and we get

$$\begin{aligned}&\left|\int_0^t\!\!\int_{\mathbb{R}} I_2 \,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\quad\leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}} \left(|\partial_x u||\partial_x^k\rho|^2 + |\partial_x\rho||\partial_x^k\rho||\partial_x^k u| + |\partial_x W||\partial_x^k u|^2\right) \mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

The conclusion follows by using the inequalities

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |\partial_x u||\partial_x^k\rho|^2 \,\mathrm{d}x\,\mathrm{d}\tau &\leq \int_0^t \|\partial_x u(\tau)\|_{L^\infty(\mathbb{R})}\|\partial_x^k\rho\|^2_{L^2(\mathbb{R})}\,\mathrm{d}\tau \\ &\leq CN_\varepsilon(t)\int_0^t \|\partial_x u(\tau)\|_{H^{s-1}(\mathbb{R})}\|\partial_x^k\rho\|_{L^2(\mathbb{R})}\,\mathrm{d}\tau \\ &\leq CN_\varepsilon(t)\int_0^t \left(\frac{1}{\varepsilon}\|\partial_x u(\tau)\|^2_{H^{s-1}(\mathbb{R})} + \varepsilon\|\partial_x^k\rho\|^2_{L^2(\mathbb{R})}\right)\mathrm{d}\tau \\ &\leq CN_\varepsilon^3(t),\end{aligned}$$

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |\partial_x\rho||\partial_x^k\rho||\partial_x^k u| \,\mathrm{d}x\,\mathrm{d}\tau &\leq \|\partial_x\rho\|_{L^\infty([0,t]\times\mathbb{R})}\int_0^t \left(\frac{1}{\varepsilon}\|\partial_x^k u(\tau)\|^2_{L^2(\mathbb{R})} + \varepsilon\|\partial_x^k\rho\|^2_{L^2(\mathbb{R})}\right)\mathrm{d}\tau \\ &\leq CN_\varepsilon^3(t),\end{aligned}$$

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |\partial_x W||\partial_x^k u|^2 \,\mathrm{d}x\,\mathrm{d}\tau &\leq \|\partial_x W\|_{L^\infty([0,t]\times\mathbb{R})}\frac{1}{\varepsilon}\int_0^t \|\partial_x^k u(\tau)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}\tau \\ &\leq CN_\varepsilon^3(t).\end{aligned}$$

Thus for I_2 we are led to

$$\left|\int_0^t\!\!\int_{\mathbb{R}} I_2 \,\mathrm{d}x\,\mathrm{d}\tau\right| \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})N_\varepsilon^3(t).$$

Using the expression of $A_0(W)$ and the Cauchy-Schwarz inequality, the integral of I_3 can be estimated as follows

$$\left|\int_0^t\!\!\int_{\mathbb{R}} I_3 \,\mathrm{d}x\,\mathrm{d}\tau\right| \leq \int_0^t \|[\partial_x^k, \rho^2\phi(u)]\partial_t u\|_{L^2(\mathbb{R})}\|\partial_x^k u\|_{L^2(\mathbb{R})}\,\mathrm{d}\tau. \tag{19}$$

Then using the classical estimate for commutators, see [31, Proposition 2.1], we obtain for any $\tau \in [0,t]$,

$$\begin{aligned}\|[\partial_x^k, \rho^2\phi(u)]\partial_t u\|_{L^2(\mathbb{R})} \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\Big(&\|\partial_t u(\tau)\|_{L^\infty(\mathbb{R})}\|\partial_x(\rho^2\phi(u))\|_{H^{s-1}(\mathbb{R})} \\ &+\|\partial_t u(\tau)\|_{H^{s-1}(\mathbb{R})}\|\partial_x(\rho^2\phi(u))\|_{L^\infty(\mathbb{R})}\Big).\end{aligned} \tag{20}$$

By using the second equation in (12), we can express $\partial_t u$ in terms of $\partial_x W$ and u/ε. Namely, we can get the following estimates, for any $\tau \in [0,t]$,

$$\|\partial_t u(\tau)\|_{L^\infty(\mathbb{R})} \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\left(\|\partial_x W(\tau)\|_{L^\infty(\mathbb{R})} + \frac{1}{\varepsilon}\|u(\tau)\|_{L^\infty(\mathbb{R})}\right),$$

and

$$\|\partial_t u(\tau)\|_{H^{s-1}(\mathbb{R})} \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\left(\|\partial_x W(\tau)\|_{H^{s-1}(\mathbb{R})} + \frac{1}{\varepsilon}\|u(\tau)\|_{H^{s-1}(\mathbb{R})}\right),$$

where $0 < \varepsilon < 1$ has been used. We insert these estimates in (20), we use the classical tame estimate for the composition of functions together with the Sobolev embedding theorem, and we arrive at

$$\begin{aligned}\|[\partial_x^k, \rho^2\phi(u)]\partial_t u\|_{L^2(\mathbb{R})} \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})&\|\partial_x W(\tau)\|_{H^{s-1}(\mathbb{R})} \\ &\left(\|\partial_x W(\tau)\|_{H^{s-1}(\mathbb{R})} + \frac{1}{\varepsilon}\|u(\tau)\|_{H^{s-1}(\mathbb{R})}\right).\end{aligned}$$

Eventually using this inequality in (19) we conclude that

$$\left| \int_0^t \int_{\mathbb{R}} I_3 \, \mathrm{d}x \, \mathrm{d}\tau \right| \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) N_\varepsilon^3(t)$$

holds.

Next we treat the integral of I_4. Note that the component located at the first row and the first column of the matrix $A_1(W)$ depends only on u. Hence I_4 involves sums of the following terms:

$$\{[\partial_x^k, u]\partial_x \rho\}\partial_x^k \rho, \quad \{[\partial_x^k, \rho]\partial_x \rho\}\partial_x^k u, \quad \{[\partial_x^k, \rho]\partial_x u\}\partial_x^k \rho, \quad \{[\partial_x^k, A_1^{22}(W)]\partial_x u\}\partial_x^k u,$$

where $A_1^{22}(W)$ denotes the element of the matrix A_1 located at the second line and the second column. By using the Cauchy-Schwarz' inequality and the classical estimate for commutators as we did when dealing with I_3, the L_x^1 norm of the above terms can be dominated by $\|\partial_x W(\tau)\|_{H^{s-1}(\mathbb{R})} \, \|\partial_x \rho(\tau)\|_{H^{s-1}(\mathbb{R})} \, \|\partial_x u(\tau)\|_{H^{s-1}(\mathbb{R})}$. Hence we have

$$\left| \int_0^t \int_{\mathbb{R}} I_4 \, \mathrm{d}x \, \mathrm{d}\tau \right| \quad \le \quad \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) N_\varepsilon^3(t).$$

We treat similarly the last term, obtaining

$$\left| \int_0^t \int_{\mathbb{R}} I_5 \, \mathrm{d}x \, \mathrm{d}\tau \right| \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) N_\varepsilon^3(t).$$

We skip the details.

Combining the estimates obtained on $I_1 - I_5$ we go back to (18), and we conclude that

$$\|\partial_x^k W(t)\|_{L^2(\mathbb{R})}^2 + \frac{1}{\varepsilon} \int_0^t \|\partial_x^k u(\tau)\|_{L^2(\mathbb{R})}^2 \, \mathrm{d}\tau \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) \left(N_\varepsilon^2(0) + N_\varepsilon^3(t) \right) \tag{21}$$

holds for any $1 \le k \le s$.

Step 3: Conclusion. Using the above estimates (17) and (21), the derivation of (15) is direct. Hence the proof of Proposition 4.3 is complete. □

Remark 4.4. *For the simple model dealt with in [10], Godunov's symmetrization leads to*

$$A_0(W_2)\partial_t W + A_1(W_2)\partial_x W = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ W_2 \end{pmatrix},$$

with $W = (W_1, W_2)^t$ and the matrices

$$A_0(W_2) = \begin{pmatrix} 1 & G(W_2) \\ G(W_2) & A_0^{22}(W_2) \end{pmatrix}, \quad A_1(W_2) = \begin{pmatrix} G(W_2) & A_0^{22}(W_2) \\ A_0^{22}(W_2) & A_1^{22}(W_2) \end{pmatrix},$$

only depend on W_2, which is precisely the relaxed variable, through the functions G, A_0^{22}, A_1^{22}. It greatly simplifies the analysis of the integrals of $I_1 - I_5$. For example, when dealing with I_2, we only have to estimate a product which looks like $|\partial_x W_2||\partial_x^\alpha W_2||\partial_x^\alpha W|$. Here, a refined analysis is necessary, term by term, in order to isolate the term involving the density-like variable, that is the non relaxed variable.

4.3 Energy estimates II: $L^2(H^{s-1})$ estimates of $\partial_x \rho$

The next step consists in deriving the $L^2(H^{s-1})$ estimates of the density variable $\partial_x \rho$. We adapt the method introduced in [35] and further developed [23, 37] (see also [10, 11, 30] for further applications). The proof uses a consequence of the Kawashima-Shizuta property: the existence of a compensating matrix having structure properties which allow to derive the desired estimates.Therefore the following statement has a central role within the proof.

Lemma 4.5. *Let $\bar\rho > 0$ and $\overline{W} = (\bar\rho, 0)^T$. Let us define the so–called compensating matrix*

$$K = \begin{pmatrix} 0 & \dfrac{1}{3\bar\rho^2} \\ -1 & 0 \end{pmatrix}.$$

Then $KA_0(\overline{W})$ is skew-symmetric and

$$KA_1(\overline{W}) = \begin{pmatrix} \dfrac{1}{3\bar{\rho}} & 0 \\ 0 & -\bar{\rho} \end{pmatrix}.$$

Proposition 4.6. *Let the assumptions of Theorem 4.2 be fulfilled. Then there exists a function $\mathcal{C} : \mathbb{R}^+ \to \mathbb{R}^+$ that is independent of ε, such that, for any $0 \le t \le T$, we have*

$$\varepsilon \int_0^t \|\partial_x \rho(\tau)\|^2_{H^{s-1}(\mathbb{R})} \, \mathrm{d}\tau \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) \, \left(N^2_\varepsilon(0) + N^3_\varepsilon(t)\right). \tag{22}$$

Proof. The proof begins by linearizing the system (13) around the constant state $\overline{W} = (\bar{\rho}, 0)^T$: we write

$$A_0(\overline{W})\partial_t(W - \overline{W}) + A_1(\overline{W})\partial_x(W - \overline{W}) = -\frac{3\bar{\rho}^2}{\varepsilon}\begin{pmatrix} 0 \\ u \end{pmatrix} + h, \tag{23}$$

with

$$\begin{aligned} h = &-A_0(\overline{W})[A_0^{-1}(W)A_1(W) - A_0^{-1}(\overline{W})A_1(\overline{W})]\partial_x W \\ &-\frac{1}{\varepsilon}\begin{pmatrix} 0 \\ A_0(\overline{W})[A_0^{-1}(W)\rho^2\phi(u) - A_0^{-1}(\overline{W})3\bar{\rho}^2)]u \end{pmatrix}. \end{aligned}$$

Since $s - 1 > \frac{1}{2}$, the Sobolev space $H^{s-1}(\mathbb{R})$ is an algebra. Thus we have

$$\|h\|_{H^{s-1}(\mathbb{R})} \le \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) \left(\|W - \overline{W}\|_{H^{s-1}(\mathbb{R})}\|\partial_x W\|_{H^{s-1}(\mathbb{R})} + \frac{1}{\varepsilon}\|W - \overline{W}\|_{H^{s-1}}\|u\|_{H^{s-1}(\mathbb{R})} \right). \tag{24}$$

Next we apply ∂_x^k, $0 \le k \le s-1$, to the linearized system (23). Then, the compensated matrix K being given in Lemma 4.5, we multiply the equation from the left by εK. Finally, we take the inner product of the resulting equality with $\partial_x^{k+1}W$ and integrate over $[0,t] \times \mathbb{R}$. We obtain

$$\begin{aligned} &\int_0^t\!\!\int_{\mathbb{R}} \varepsilon K A_0(\overline{W})\partial_t[\partial_x^k(W - \overline{W})] \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\!\!\int_{\mathbb{R}} \varepsilon K A_1(\overline{W})\partial_x^{k+1}(W - \overline{W}) \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad = -3\bar{\rho}^2 \int_0^t\!\!\int_{\mathbb{R}} K\begin{pmatrix} 0 \\ u \end{pmatrix} \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon \int_0^t\!\!\int_{\mathbb{R}} K\partial_x^k h \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{25}$$

Owing to the properties of the compensating matrix K and using the estimates obtained in Proposition 4.3, the integrals in the left side can be estimated as follows

$$\begin{aligned} &\int_0^t\!\!\int_{\mathbb{R}} \varepsilon K A_0(\overline{W})\partial_t[\partial_x^k(W - \overline{W})] \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad = -\frac{\varepsilon}{2}\int_{\mathbb{R}} \varepsilon K A_0(\overline{W})\partial_x^k(W - \overline{W}) \cdot \partial_x^{k+1}W \,\mathrm{d}x \Big|_0^t \\ &\qquad \ge -C\varepsilon(\|\partial_x^k(W(t) - \overline{W})\|^2_{H^1(\mathbb{R})} + \|\partial_x^k(W(0) - \overline{W})\|^2_{H^1(\mathbb{R})}) \\ &\qquad \ge -\mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) \left(N^2_\varepsilon(0) + N^3_\varepsilon(t)\right), \end{aligned}$$

$$\begin{aligned} &\int_0^t\!\!\int_{\mathbb{R}} \varepsilon K A_1(\overline{W})\partial_x^{k+1}(W - \overline{W}) \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad = \frac{\varepsilon}{3\bar{\rho}}\int_0^t \|\partial_x^{k+1}\rho(\tau)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}\tau - \varepsilon\bar{\rho}\int_0^t \|\partial_x^{k+1}u(\tau)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}\tau \\ &\qquad \ge \frac{\varepsilon}{3\bar{\rho}}\int_0^t \|\partial_x^{k+1}\rho(\tau)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}\tau - \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) \left(N^2_\varepsilon(0) + N^3_\varepsilon(t)\right), \end{aligned}$$

where the constant C depends only on $\bar{\rho}$. Similarly we can estimate the terms arising in the right-hand side of (25) as follows

$$\begin{aligned} \left|\int_0^t\!\!\int_{\mathbb{R}} K\begin{pmatrix} 0 \\ u \end{pmatrix} \cdot \partial_x^{k+1}W \,\mathrm{d}x\,\mathrm{d}\tau\right| &\le C\int_0^t\!\!\int_{\mathbb{R}} |\partial_x^k u| \cdot |\partial_x^{k+1}\rho| \,\mathrm{d}x\,\mathrm{d}\tau \\ &\le \frac{1}{2}\frac{1}{3\bar{\rho}}\int_0^t \|\partial_x^{k+1}\rho(\tau)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}\tau + \frac{C}{\varepsilon}\int_0^t \|\partial_x^k u(\tau)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}\tau \\ &\le \frac{1}{2}\frac{1}{3\bar{\rho}}\int_0^t \|\partial_x^{k+1}\rho(\tau)\|^2_{L^2(\mathbb{R})} \,\mathrm{d}\tau + \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})}) \left(N^2_\varepsilon(0) + N^3_\varepsilon(t)\right), \end{aligned}$$

and

$$\begin{aligned}\left|\varepsilon\int_0^t\int_{\mathbb{R}} K\partial_x^k h\cdot\partial_x^{k+1}W\,\mathrm{d}x\,\mathrm{d}\tau\right| &\leq C\varepsilon\int_0^t \|\partial_x^k h(\tau)\|_{L^2(\mathbb{R})}\|\partial_x^{k+1}W(\tau)\|_{L^2(\mathbb{R})}\,\mathrm{d}\tau\\ &\leq C\varepsilon\int_0^t \|h(\tau)\|_{H^{s-1}(\mathbb{R})}\|\partial_x^{k+1}W(\tau)\|_{L^2(\mathbb{R})}\,\mathrm{d}\tau\\ &\leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\left(N_\varepsilon^2(0)+N_\varepsilon^3(t)\right),\end{aligned}$$

where we use the estimate (24) for the last inequality.

Using the above estimates in (25), together with Proposition 4.3, we are led to

$$\varepsilon\int_0^t \|\partial_x^{k+1}\rho(\tau)\|_{L^2(\mathbb{R})}^2\,\mathrm{d}\tau \leq \mathcal{C}(\|W\|_{L^\infty([0,T]\times\mathbb{R})})\left(N_\varepsilon^2(0)+N_\varepsilon^3(t)\right). \tag{26}$$

Then Proposition 4.6 is proved by summing (26) over all $0\leq k\leq s-1$.

□

5 Asymptotic analysis and the proof of the Theorem 2.7

We now wish to discuss the behavior as ε goes to 0 of the solutions to (5) obtained in Theorem 2.6; we shall show that the $M1$ model is indeed consistent with the diffusion asymptotic: as $\varepsilon\to 0$, $\widehat{\rho}_\varepsilon$ converges to the solution of the heat equation while u_ε vanishes. This is the consequence of the estimate (10) which is uniform with respect to ε. Since we are concerned with the one-dimensional case only, following [10], we apply the stream function trick, directly inspired from [25].

Proof. We rewrite the momentum system (5) as

$$\left\{\begin{aligned}\partial_t\widehat{\rho}_\varepsilon+\partial_x\widehat{J}_\varepsilon &= 0,\\ \varepsilon^2\partial_t\widehat{J}_\varepsilon+\partial_x\left[\widehat{\rho}_\varepsilon D(\widehat{u}_\varepsilon)\right] &= -\widehat{J}_\varepsilon.\end{aligned}\right. \tag{27}$$

From the a priori estimate (10) in Theorem 2.6, the following estimates

$$\|\widehat{\rho}_\varepsilon-\bar{\rho}\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+\varepsilon\|\widehat{J}_\varepsilon\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}))}+\|\widehat{u}_\varepsilon\|_{L^\infty(\mathbb{R};L^2(\mathbb{R}))} \leq C, \tag{28}$$

$$\|\widehat{J}_\varepsilon\|_{L^2(\mathbb{R}^+\times\mathbb{R})}+\frac{1}{\varepsilon}\|\widehat{u}_\varepsilon\|_{L^2(\mathbb{R}^+\times\mathbb{R})} \leq C, \tag{29}$$

hold. Here and below, we denote with the same letter C positive constants independent of ε and t. Let $R(t,x)$ be the solution of the heat equation (4) (in the 1-D case) with the initial data ρ_0. We can write

$$\partial_t(\widehat{\rho}_\varepsilon-R)+\partial_x(\widehat{J}_\varepsilon+\frac{1}{3}\partial_x R)=0.$$

This equality can be seen as a divergence free condition satisfied by a stream function

$$\begin{aligned}\partial_x z_\varepsilon &= \widehat{\rho}_\varepsilon-R,\\ \partial_t z_\varepsilon &= -(\widehat{J}_\varepsilon+\frac{1}{3}\partial_x R).\end{aligned}$$

The definition of z_ε is completed by imposing

$$z_\varepsilon\big|_{t=0}=0.$$

Let us multiply the second equation in (27) by the stream function z_ε, and integrate the resulting equality over $[0,T]\times\mathbb{R}$. We get the following relation

$$\varepsilon^2\int_0^T\int_{\mathbb{R}} z_\varepsilon\partial_t\widehat{J}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{R}} z_\varepsilon\partial_x(\widehat{\rho}_\varepsilon D(\widehat{u}_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t=-\int_0^T\int_{\mathbb{R}} z_\varepsilon\widehat{J}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t. \tag{30}$$

Owing to the definition of z_ε and using integration by parts, we can rewrite the integrals as follows

$$\varepsilon^2\int_0^T\int_{\mathbb{R}} z_\varepsilon\partial_t\widehat{J}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=\varepsilon^2\int_{\mathbb{R}}(z_\varepsilon\widehat{J}_\varepsilon)\big|_{t=T}\,\mathrm{d}x+\varepsilon^2\int_0^T\int_{\mathbb{R}}\widehat{J}_\varepsilon\left(\widehat{J}_\varepsilon+\frac{1}{3}\partial_x R\right)\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T\!\!\int_{\mathbb{R}} z_\varepsilon \partial_x(\widehat{\rho}_\varepsilon D(\widehat{u}_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{3}\int_0^T\!\!\int_{\mathbb{R}} (\widehat{\rho}_\varepsilon - R)^2\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_{\mathbb{R}} \frac{1}{3}R(\widehat{\rho}_\varepsilon - R)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_{\mathbb{R}} \widehat{\rho}_\varepsilon \left(D(\widehat{u}_\varepsilon) - \frac{1}{3}\right)(\widehat{\rho}_\varepsilon - R)\,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T\!\!\int_{\mathbb{R}} z_\varepsilon \widehat{J}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\!\!\int_{\mathbb{R}} z_\varepsilon\left(\partial_t z_\varepsilon + \frac{1}{3}\partial_x R\right)\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2}\int_{\mathbb{R}} z_\varepsilon^2\big|_{t=T}\,\mathrm{d}x + \frac{1}{3}\int_0^T\!\!\int_{\mathbb{R}} R(\widehat{\rho}_\varepsilon - R)\,\mathrm{d}x\,\mathrm{d}t.$$

The Cauchy-Schwarz inequality combined to the estimates (28)-(29) yields

$$\begin{aligned} \varepsilon^2\left|\int_{\mathbb{R}} \widehat{J}_\varepsilon z_\varepsilon(T,x)\,\mathrm{d}x\right| &\leq \frac{1}{4}\|z_\varepsilon(T)\|^2_{L^2(\mathbb{R})} + \varepsilon^4\|\widehat{J}_\varepsilon(T)\|^2_{L^2(\mathbb{R})} \leq \frac{1}{4}\|z_\varepsilon(T)\|^2_{L^2(\mathbb{R})} + C\varepsilon^2, \\ \frac{\varepsilon^2}{3}\left|\int_0^T\!\!\int_{\mathbb{R}} \widehat{J}_\varepsilon \partial_x R\,\mathrm{d}x\,\mathrm{d}t\right| &\leq C\varepsilon^2\|\widehat{J}_\varepsilon\|_{L^2([0,T]\times\mathbb{R})}\|\partial_x R\|_{L^2([0,T]\times\mathbb{R})} \leq C\varepsilon^2. \end{aligned}$$

Furthermore, we remind that $D(0) = 1/3$, and $D'(0) = 0$, which leads to

$$\left|\int_0^T\!\!\int_{\mathbb{R}} \widehat{\rho}_\varepsilon\left(D(\widehat{u}_\varepsilon) - \frac{1}{3}\right)(\widehat{\rho}_\varepsilon - R)\,\mathrm{d}x\,\mathrm{d}t\right| \leq C\varepsilon^2.$$

Coming back to (30), these estimates allow us to deduce that

$$\int_0^T \|(\widehat{\rho}_\varepsilon - R)(t)\|^2_{L^2(\mathbb{R})}\,\mathrm{d}t \leq C\varepsilon^2$$

holds. We end the proof of Theorem 2.7 by letting $T \to +\infty$. □

In [10], the convergence statement is completed by an estimate of the difference between the approximate distribution $\widehat{f}_\varepsilon$ provided by the $M1$ model, and the solution of (1) in the specific case where Q is a mere relaxation operator describing isotropic scattering events. Of course, the generalization of such an estimate highly depends on the details of the collision processes. Let us give some hints in this direction.

As detailed above, the construction of the $M1$ model is based on an entropy functional which is dissipated by the microscopic model: any solution f_ε of (1) satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\int H(f)\,\mathrm{d}\nu\,\mathrm{d}v\,\mathrm{d}x \leq 0$. See for instance [5] for detailed computations on a complex model including relativistic effects. The approximate distribution $\widehat{f}_\varepsilon$ provided by the entropy closure is a generalized Planckian state. However, dealing with scattering-dominated dynamics there is no reason why the leading term in the expression of f_ε, solution of (1), has a Planckian shape; it is only known to be isotropic. Nevertheless the $M1$ model is advocated to be a relevant macroscopic model for describing such non equilibrium regimes [5, 21]. Let us discuss further this aspect. To this end, we consider the average over frequencies

$$\widehat{g}_\varepsilon(t,x,v) = \int_{\mathbb{R}^+} \widehat{f}_\varepsilon(t,x,v,\nu)\,\mathrm{d}\nu, \quad g_\varepsilon(t,x,v) = \int_{\mathbb{R}^+} f_\varepsilon(t,x,v,\nu)\,\mathrm{d}\nu.$$

The triangle inequality yields

$$|\widehat{g}_\varepsilon - g_\varepsilon| \leq |\widehat{g}_\varepsilon - \widehat{\rho}_\varepsilon| + |\widehat{\rho}_\varepsilon - R| + |R - g_\varepsilon|.$$

We already know that $\widehat{\rho}_\varepsilon - R$ tends to 0 as $\varepsilon \to 0$. We consider now $\widehat{g}_\varepsilon - \widehat{\rho}_\varepsilon$. Owing to the formulae in Remarks 2.2 and 2.3 we obtain

$$|\widehat{g}_\varepsilon - \widehat{\rho}_\varepsilon| = \frac{1}{\alpha_0^4}\,(2+\sqrt{4-3|\widehat{u}_\varepsilon|^2})^4\,\left|\frac{1}{(2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}-3v\cdot\widehat{u}_\varepsilon)^4} - \frac{2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}}{16(2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}-3|\widehat{u}_\varepsilon|^2)^3}\right|.$$

On the one hand, we have

$$\frac{1}{\alpha_0^4}\,(2+\sqrt{4-3|\widehat{u}_\varepsilon|^2})^4 \leq C(\|\widehat{\rho}_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}, \|\widehat{u}_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}).$$

On the other hand, the Taylor formula allows us to estimate

$$\left|\frac{1}{(2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}-3v\cdot\widehat{u}_\varepsilon)^4} - \frac{2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}}{16(2+\sqrt{4-3|\widehat{u}_\varepsilon|^2}-3|\widehat{u}_\varepsilon|^2)^3}\right| \leq C(\|\widehat{u}_\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})})\,|\widehat{u}_\varepsilon|.$$

Coming back to (29) we deduce that

$$\|\widehat{g}_\varepsilon - \widehat{\rho}_\varepsilon\|_{L^2(\mathbb{R}^+\times\mathbb{R}\times V)} \leq C\varepsilon.$$

Finally, we conclude

$$\lim_{\varepsilon\to 0}\|\widehat{g}_\varepsilon - g_\varepsilon\|_{L^2((0,T)\times\mathbb{R}\times V)} = 0$$

provided we are able to justify that $\|R - g_\varepsilon\|_{L^2((0,T)\times B_X\times V)}$ tends to 0. This is precisely the statement of the diffusion asymptotics, with arguments depending on the details of the collision operator.

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