LIFSHITZ-SLYOZOV EQUATIONS: THE MODEL WITH ENCOUNTERS

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Abstract. We study the Lifshitz-Slyozov model of phase transitions, taking into account the effect of encounters between large clusters. For a wide class of initial data and kinetic coefficients, we prove the existence of solution for this system.
Key Words. Phase transitions, Lifshitz-Slyozov system, Integro-differential equations

1. INTRODUCTION

This paper is devoted to the existence of solutions for the following system

$$\begin{cases} \partial_t f(t,x) + \partial_x((k(x)c(t) - q(x))f(t,x)) = Q(f)(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times \mathbb{R}^+ \\ c(t) + \displaystyle\int_0^\infty x f(t,x)\,dx = \rho, \qquad\qquad \text{for } t\in(0,T). \end{cases} \tag{1.1}$$

This equation was proposed by Lifshitz-Slyozov [7] as a model for phase transitions phenomenom. Equation (1.1) describes the evolution of the density $f(t,x)$ of clusters having, at the time t, the size x, and which interact with a population of monomers, $c(t)$ being the monomers concentration at time t. The dynamics is governed by the assumption that the elementary process which modify the size of a cluster is the attachment or the evaporation of monomers. Thus, the kinetic coefficients $k(x)$ and $q(x)$ represent respectively the rates at which monomers are added to or removed from clusters of size x. The macroscopic quantities

$$M_0(t) = \int_0^\infty f(t,x)\,dx\ ;\ M_1(t) = \int_0^\infty x f(t,x)\,dx. \tag{1.2}$$

are interpreted as the total number of clusters and the total mass of the clusters, respectively. The first equation in (1.1) is the kinetic equation for the cluster density, while the second expresses the conservation of the total mass ρ of the population: the material exists either with the form of a monomer or with the form of a cluster. Actually, in [7] the discussion is essentially devoted to the case "without encounters" where $Q(f) = 0$. This model is studied in [5] (see also [9]). However, it is also noticed in [7] that the derivation of (1.1) neglects the local fluctuations of the monomer density in the neighborhood of the grains. Then, it is proposed to introduce the correction term $Q(f)$ which reads as follows

$$Q(f)(t,x) = \frac{1}{2}\int_0^x \omega(x-y,y)f(t,x-y)f(t,y)\,dy - \int_0^\infty \omega(x,y)f(t,x)f(t,y)\,dy. \tag{1.3}$$

Let us split this so called Smoluchowsky encounters integral into a gain term

$$Q^+(f) = \frac{1}{2}\int_0^x \omega(x-y,y)f(t,x-y)f(t,y)\,dy$$

and a loss term

$$Q^-(f) = f(t,x)\int_0^\infty \omega(x,y)f(t,y)\,dy = f(t,x)Lf(t,x),$$

respectively. This additional term describes some collision phenomenom: when the distance between two clusters is of the order of the size of the clusters, then the greater absorbs the lesser. On the one hand, the gain term $Q^+(f)$ describes the situation where a cluster of size $x-y$ meets a cluster having size y and they form a cluster with size x. The encounter y with $x-y$ produces the same result, which explains the $\frac{1}{2}$ factor. On the other hand, the concentration of clusters with size x decreases as such a cluster coalesces with a cluster of size y, which clearly leads to a cluster of size $x+y$; this situation is described by the loss term $Q^-(f)$. The quantity $\omega(x,y)f(t,x)f(t,y)dy$ is the number of encounters between grains with size x and grains with a size enclosed between y and

$y+dy$ by time and volume unit. We can write $\omega(x,y) = \frac{1}{\tau}\sigma(x,y)$, where σ has the physical meaning of a volume centred on a grain x over which interaction with a grain y may be held to occur and τ is a relaxation time. It seems rather natural to assume that the following symmetry property

$$\omega(x,y) = \omega(y,x) \tag{1.4}$$

is fulfilled. Hence, we notice that (at least formally)

$$\int_0^\infty Q(f)(t,x)\,dx \leq 0 \tag{1.5}$$

which means that the number of clusters is decreased by the collisions while

$$\int_0^\infty xQ(f)(t,x)\,dx = 0 \tag{1.6}$$

which implies that the mass of the encountering clusters is preserved. For more details on the model, we refer to the original paper [7], or for a recent review on the subject to [11]. An interesting variant to (1.1) involving a Fokker-Planck like second order term is introduced in [6]. Let us also point out that equations which look like those discussed here also appear in the description of film growth kinetics, see [2], [4]. We also quote [8] and [1] where the situation without interaction with monomers ($k = 0 = q$) is investigated.

The paper is organized as follows. Section 2 contains some preliminaries: we recall some results from [5] on the characteristics curves associated to $kc - q$ and we collect some estimates on the operator Q. Next, in Section 3, we introduce a suitable mild formulation of the problem, involving these characteristics, and we discuss some properties of these solutions, in particular we detail the sense of the trace at $\{x = 0\}$. The fourth section is devoted to the proof of the existence of solutions for system (1.1). Actually, our strategy follows closely those of [5] and uses the characteristics associated to c. However, due to the additional non linear collision term, the proof requires a double fixed point step. First, we assume that c is a given function of time and we solve the first equation (1.1), with the quadratic right hand side (1.3) by the classical Banach fixed point theorem. Next, we get the mass conservation relation, obtaining c as a fixed point of a suitable mapping which satisfies the requirements of Schauder's theorem. We conclude this Introduction with the statement of our main result.

Theorem 1 *Assume that the kinetic coefficients satisfy (2.1)-(2.3) and belong to* $W^{2,\infty}(\mathbb{R})$. *Assume that* ω *satisfies* $0 \le \omega(x,y) = \omega(y,x) \le \Omega$. *Let the initial data* f_0 *be nonnegative and satisfy*

$$\int_0^\infty f_0(x)\,dx < \infty, \qquad \int_0^\infty x f_0(x)\,dx \le \rho.$$

Then there exists a solution $f \in C^0([0,T]; L^1(\mathbb{R}^+))$ *to the system (1.1).*

2. PRELIMINARIES

First of all, one needs to recall some facts on the characteristic curves naturally arising in (1.1); we also establish some estimates on the operator Q.

2.1. *Characteristic curves*

Let us recall the assumptions made in [5] on the kinetic coefficients. In the sequel, we denote $a(t,x) = k(x)c(t) - q(x)$ and q and k are required to satisfy

$$\text{for a.e. } x \ge 0 : \ |k'(x)|, |q'(x)| \le K, \tag{2.1}$$

$$\text{for a.e.} \quad x \ge 0, \quad k(x) \ge 0, \quad q(x) > 0, \tag{2.2}$$

$$q(0), k(0) \le K, \quad k(0)\rho - q(0) \le 0. \tag{2.3}$$

In fact, it will be necessary to require a little bit more regularity on the coefficients. However, at the moment, the regularity assumption (2.1) suffices to define the characteristic curves

$$\begin{cases} \frac{d}{ds}X(s;t,x) = a(s, X(s;t,x)), \\ X(t;t,x) = x \end{cases} \tag{2.4}$$

associated to a given function of time c which belongs to the set

$$\mathcal{B} = \{c \in \mathcal{C}^0([0,T]),\ 0 \leq c(t) \leq \rho\}. \tag{2.5}$$

It is worth pointing out that assumption (2.3) implies that the characteristic curves which intersect the axis $x = 0$ never come back to the domain $x > 0$. This fact is related to the tendancy of small grains to dissociate, see [7]. Then, the following useful estimates satisfied by the characteristic curves are established in [5].

Lemma 1 *Let c be in the set $\mathcal{B}$. Assume that (2.1)-(2.3) hold true. Then for $x \geq 0, t > 0$, as long as it does not exit the domain $X > 0$, the characteristic curve $X(s;t,x)$ defined by (2.4) satisfies*

$$\text{for} \quad s_1 \leq s_2 \leq t: \quad 0 \leq X(s_2;t,x) \leq X(s_1;t,x)e^{A(s_2-s_1)}, \tag{2.6}$$

$$\text{if} \quad x_n \to \infty, \quad \text{then for all}\, t > 0: \quad X(0;t,x_n) \to \infty, \tag{2.7}$$

$$\text{for} \quad s \geq t: \quad X(s;t,x) \leq xe^{A(s-t)}, \tag{2.8}$$

$$\text{for all} \quad t \geq 0: \quad X(t;0,y) \leq ye^{At}, \tag{2.9}$$

$$\text{for all} \quad s,t: \quad |a(s, X(t;0,y)| \leq A(1 + ye^{At}), \tag{2.10}$$

$$\text{for} \quad s_2 \leq s_1 \leq t: \quad X(s_2;t,x) \leq (X(s_1;t,x) + 1)e^{A(s_1-s_2)} - 1. \tag{2.11}$$

Remark 1 *In the statement of all the estimates given above, it is always implicitly assumed that s, t, x are such that $X(s;t,x) > 0$, that is, we consider characteristic curves before their exit time. Indeed, all the bounds used further below for deriving existence of solutions never involve characteristics beyond their exit time.*

We also recall the following continuity result, still referring to [5] for the proof.

Lemma 2 *Consider the characteristic curves X_1, X_2 associated to given functions c_1, c_2 and let $z(s;t,x) = X_2(s;t,x) - X_1(s;t,x)$. Then the following inequalities*

$$\text{for} \quad t \le t_1 \le t_2 : \qquad |z(t_2;t,x)| \le |z(t_1;t,x)| e^{A_2(t_2-t_1)} + K e^{A_2 t_2} \int_{t_1}^{t_2} (1 + x e^{A_1(s-t)}) |c_2(s) - c_1(s)| e^{-A_2 s}\, ds, \tag{2.12}$$

$$\text{for} \quad t_2 \le t_1 \le t : \qquad |z(t_2;t,x)| \le |z(t_1;t,x)| e^{A_2(t_1-t_2)} + K(x+1) e^{(A_1 t - A_2 t_2)} \int_{t_2}^{t_1} |c_2(s) - c_1(s)| e^{(A_2-A_1)s}\, ds \tag{2.13}$$

hold true. Here $A_{1,2}$ denotes any constant greater than $K(1+ \sup_{t\in(0,T)} |c_{1,2}|)$.

Finally, we will use the following classical formulae

$$X(t;s,X(s;t,x)) = x, \tag{2.14}$$

$$J(s;t,x) := \frac{\partial X}{\partial x}(s;t,x) = \exp\left(\int_s^t b(\sigma, X(\sigma;t,x))\, d\sigma\right), \tag{2.15}$$

$$\frac{\partial X}{\partial t}(s;t,x) = -a(t,x) \exp\left(\int_s^t b(\sigma, X(\sigma;t,x))\, d\sigma\right) \tag{2.16}$$

where $b(t,x) = -\partial_x a(t,x) = q'(x) - c(t)k'(x)$ belongs to $L^\infty((0,T) \times \mathbb{R}^+)$ by (2.1).

2.2. *Properties of the collision operator*

In the sequel, we assume that collision kernel satisfies

$$0 \le \omega(x,y) \le \Omega. \tag{2.17}$$

Combined with (1.4), this yields to the following properties on the collision operator.

Lemma 3 *Let f, g be in $L^1(\mathbb{R}^+)$. Then, one has*

$$\begin{cases} 0 \leq Lf \leq \Omega \|f\|_{L^1(\mathbb{R}^+)}, \\ \|Q(f)\|_{L^1(\mathbb{R}^+)} \leq 2\Omega \|f\|^2_{L^1(\mathbb{R}^+)}, \\ \|Q(f) - Q(g)\|_{L^1(\mathbb{R}^+)} \leq 2\Omega(\|f\|_{L^1(\mathbb{R}^+)} + \|g\|_{L^1(\mathbb{R}^+)})\|f - g\|_{L^1(\mathbb{R}^+)}. \end{cases} \tag{2.18}$$

Moreover, as f lies in $L^1((1+x)dx, \mathbb{R}^+)$, we set

$$[f] = \int_{\mathbb{R}^+} (1+x)|f|dx,$$

and we get

$$\int_0^\infty x|Q(f)|dx \leq 2\Omega[f]^2, \tag{2.19}$$

and

$$\int_0^\infty x|Q(f) - Q(g)|dx \leq 4\Omega([f] + [g])[f - g]. \tag{2.20}$$

The proof is more or less obvious and is left to the reader.

3. MILD FORMULATION

We will used the mild formulation which naturally arises by integrating (1.1) along the characteristic curves associated to c. This formulation allows us to define the outgoing trace of the solution on $\{x = 0\}$.

3.1. *Definition*

It will be convenient to write (1.1) in the following mild formulation

$$f(t,x) = f_0(X(0;t,x))J(0;t,x) + \int_0^t Q(f)(s, X(s;t,x))J(s;t,x)\,ds \tag{3.21}$$

which is obtained by considering the differential equation satisfied by $f(s, X(s;t,x))$. The connection with the distribution notion of solution is ensured by the following

Proposition 1 *Let $c \in \mathcal{B}$ and assume (2.1-2.3). Let $Q \in L^1((0,T) \times \mathbb{R}^+)$ and $f_0 \in L^1(\mathbb{R}^+)$. The following assertions are equivalent:*

i) $f \in C^0([0,T]; L^1(\mathbb{R}^+))$ satisfies

$$\partial_t f + \partial_x(af) = Q \tag{3.22}$$

in $\mathcal{D}'((0,T) \times \mathbb{R}^+)$ and $f_{|t=0} = f_0$.

ii) f is a mild solution of (3.22) in the sense that f is given by

$$f(t,x) = f_0(X(0;t,x))J(0;t,x) + \int_0^t Q(s, X(s;t,x))J(s;t,x)\,ds. \tag{3.23}$$

Remark 2 *The assumption of outgoing characteristics (2.3) means that if $X(s;t,x) = 0$ then for all time $\sigma > s$, one has $X(\sigma;t,x) \leq 0$. In other words, a positive position may be reach only by following a characteristic curve emanating from a positive point. In particular, for $x \geq 0$, one has $X(s;t,x) \geq 0$ for all $0 \leq s \leq t$, so that (3.23) makes sense for functions f_0 and Q which are only defined on $\mathbb{R}^+$.*

The proof of Proposition 1 is postponed in the Appendix, we only emphasize in this Section that f defined by (3.23) satisfies the following estimate

$$\|f\|_{L^\infty(0,T;L^1(\mathbb{R}^+))} \leq \|f_0\|_{L^1(\mathbb{R}^+)} + \|Q\|_{L^1((0,T)\times\mathbb{R}^+)}, \tag{3.24}$$

by using (2.15) and we have $f \geq 0$ as the data f_0 and Q are non negative. Note also that we can actually multiply a mild solution f by a test function ψ say in $C_0^\infty([0,T] \times \mathbb{R}^+)$ (which is possibly non vanishing on $0, T$, but with compact support in $\mathbb{R}^+$). In this case, computations made when proving Proposition 1 lead to the following relation

$$\begin{aligned} &\int_0^\infty f(T,x)\psi(T,x)dx - \int_0^\infty f_0(x)\psi(0,x)dx \\ &- \int_0^T \int_0^\infty Q(t,x)\psi(t,x)\,dxdt \\ &= \int_0^T \int_0^\infty f(t,x)(\partial_t + a(t,x)\partial_x)\psi(t,x)\,dxdt \end{aligned} \tag{3.25}$$

which will be used to establish some moment properties of the solutions of (1.1).

3.2. *Trace properties*

Since the characteristics go outside the domain $\{x \geq 0\}$, it seems rather natural to discuss the properties of the "outgoing trace" at $\{x = 0\}$ of a mild solution. We get

Proposition 2 *Assume that f is given by (3.23). Then, the outgoing trace $-a(t,0)f(t,0)$ is defined as a $L^1(0,T)$ function, nonnegative as f_0 and Q are nonnegative and*

$$\lim_{h\to 0} \int_0^T \frac{1}{h} \int_0^h (-a(t,x)f(t,x))\,dxdt = \int_0^T (-a(t,0)f(t,0))\,dt. \tag{3.26}$$

Proof. We assume that f_0 and Q are regular functions, and then, we will conclude by a simple density argument. Formula (3.23) gives for

$$\begin{aligned}
\int_0^T -a(t,0)|f(t,0)|\,dt &\leq \int_0^T |f_0(X(0;t,0))|(-a(t,0)J(0;t,0))\,dt \\
&\quad + \int_0^T \int_0^t |Q(s,X(s;t,0))|(-a(t,0)J(s;t,0))\,dsdt \\
&\leq \int_0^T |f_0(X(0;t,0))|(-a(t,0)J(0;t,0))\,dt \\
&\quad + \int_0^T \int_\sigma^T |Q(\sigma,X(\sigma;\tau,0))|(-a(\tau,0)J(\sigma;\tau,0))\,d\tau d\sigma \\
&\leq \int_0^{X(0;T,0)} |f_0(y)|\,dy + \int_0^T \int_0^{X(\sigma;T,0)} |Q(\sigma,y)|\,dy,
\end{aligned}$$

where, first, we exchanged the order of integration in the integral of Q, and, next, we used the change of variable $y = X(0;t,0)$ or $y = X(\sigma;\tau,0)$, the jacobian being defined by (2.16). We conclude by extending the domain of integration over $\mathbb{R}^+$

$$\int_0^T -a(t,0)|f(t,x)|\,dt \leq \int_0^\infty |f_0(y)|\,dy + \int_0^T \int_0^\infty |Q(\sigma,y)|\,dy.$$

Next, we write

$$\int_0^T \frac{1}{h} \int_0^h (-a(t,x)f(t,x))\,dxdt = I_{0,h} + I_{Q,h},$$

with the following definition

$$I_{0,h} = \frac{1}{h}\int_0^h \int_0^T f_0(X(0;t,x))((-a(t,x)J(0;t,x))\,dtdx$$

and

$$\begin{aligned} I_{Q,h} &= \frac{1}{h}\int_0^h \int_0^T \int_0^t Q(s,(X(s;t,x))((-a(t,x)J(s;t,x))\,dsdtdx \\ &= \frac{1}{h}\int_0^h \int_0^T \int_\sigma^T Q(\sigma,X(\sigma;\tau,x))(-a(\tau,x)J(\sigma;\tau,x))\,d\tau d\sigma dx. \end{aligned}$$

By using again the change of variable $y = X(0;t,x)$ or $y = X(\sigma;\tau,x)$ and (2.16), we transform these quantities as

$$\begin{aligned} I_{0,h} &= \frac{1}{h}\int_0^h \int_x^{X(0;T,x)} f_0(y)\,dydx, \\ I_{Q,h} &= \int_0^T \frac{1}{h}\int_0^h \int_x^{X(\sigma;T,x)} Q(\sigma,y)\,dydxd\sigma, \end{aligned}$$

respectively. Then, we write $I_{0,h} = \frac{1}{h}\int_0^h \Phi(x)\,dx$ where

$$\Phi(x) = \int_0^{X(0;T,x)} f_0(y)\,dy - \int_0^x f_0(y)\,dy$$

is a continuous function of x on $\mathbb{R}^+$. Then, it is a standard fact, see [10], that as h goes to 0, we obtain

$$\lim_{h\to 0} I_{0,h} = \Phi(0) = \int_0^{X(0;T,0)} f_0(y)\,dy.$$

Similarly, we have $I_{Q,h} = \int_0^T \frac{1}{h}\int_0^h \Psi(\sigma,x)\,dxd\sigma$, where

$$\Psi(\sigma,x) = \int_0^{X(\sigma;T,x)} Q(\sigma,y)\,dy - \int_0^x Q(\sigma,y)\,dy.$$

Reasoning as before, for a.e. $\sigma \in (0,T)$,

$$\frac{1}{h}\int_0^h \Psi(\sigma,x)\,dx \to \int_0^{X(\sigma;T,0)} Q(\sigma,y)\,dy$$

as $h \to 0$ and is dominated as follows

$$\frac{1}{h}\left|\int_0^h \Psi(\sigma,x)\,dx\right| \le \frac{1}{h}\int_0^h \int_0^\infty |Q(\sigma,y)|\,dydx = \int_0^\infty |Q(\sigma,y)|\,dy$$

the right hand side belonging to $L^1((0,T),d\sigma)$. Therefore, we conclude that

$$\lim_{h\to 0}(I_{0,h}+I_{Q,h}) \;=\; \int_0^{X(0;T,0)} f_0(y)\,dy + \int_0^T \int_0^{X(\sigma;T,0)} Q(\sigma,y)\,dyd\sigma,$$

where we recognize $\displaystyle\int_0^T (-a(t,0)f(t,0))\,dt$. □

4. EXISTENCE

As mentioned above, the existence result is obtained by solving two successive fixed point problems. First, we assume that c is a given function belonging to the set $\mathcal{B}$ and we search for the solution f of the problem

$$\begin{cases} \partial_t f(t,x) + \partial_x((k(x)c(t)-q(x))f(t,x)) = Q(f), & \text{in} \quad (0,T)\times I\!R^+ \\ f_{t=0} = f_0, & \text{in} \quad I\!R^+. \end{cases} \tag{4.1}$$

Note that, in view of the physical meaning of f, we should prove the non negativeness of the solution. According to the strategy adopted in [5], this will allow us to deal with the map

$$\mathcal{S} : c \in \mathcal{C}^0([0,T]) \longmapsto \tilde{c}(t) = \left[\rho - \int_0^\infty xf(t,x)dx\right]_+, \tag{4.2}$$

where f denotes the solution of (4.1) just obtained and $[\cdot]_+$ stands for the positive part ($[s]_+ = \max(s,0)$). The proof falls into four steps as follows :

1) Local in time existence for the problem with c given in $\mathcal{B}$,
2) Extension on any time interval,
3) Moment properties of the obtained solution,
4) Fixed point on c.

4.1. *Problem with given c, local existence*

At the moment, we recall that c is a given function which lies in the set $\mathcal{B}$. Hence, we are interested in the problem (4.1). In order to preserve non negativeness properties, we introduce a parameter $K > 0$, to be determined. We consider the map $\mathcal{T}(f_0, \cdot)$ defined by

$$f(t,x) = \mathcal{T}(f_0, g)(t,x) = f_0(X(0;t,x))J(0;t,x)e^{-Kt}$$
$$+ \int_0^t (Q(g)(s, X(s;t,x)) + Kg(s, X(s;t,x)))J(s;t,x)e^{-K(t-s)}\, ds \tag{4.3}$$

This formula coresponds to the mild formulation of

$$\begin{cases} \partial_t f + \partial_x(af) + Kf = Q(g) + Kg, \\ f_{t=0} = f_0. \end{cases} \tag{4.4}$$

The constant K is chosen in such a way that the right hand side in (4.4) is a non negative operator on the set

$$\mathcal{X}_M = \{f \in L^\infty(0,T; L^1(\mathbb{R}^+)),\ f \geq 0,\ \|f\|_{L^1(\mathbb{R}^+)} \leq M\}. \tag{4.5}$$

Indeed, as $K \geq \Omega M$, one has

$$Q(g) + Kg = Q^+(g) + (K - Lg)g \geq 0$$

for all $g \in \mathcal{X}_M$, by (2.18). Furthermore, still by using (2.18), we get on the time interval $(0, T)$

$$\begin{aligned} \|\mathcal{T}(f_0, g)(t, \cdot)\|_{L^1(\mathbb{R}^+)} &\leq \|f_0\|_{L^1(\mathbb{R}^+)} \\ &\quad + \int_0^t (\|Q(g)(s,\cdot)\|_{L^1(\mathbb{R}^+)} + K\|g(s,\cdot)\|_{L^1(\mathbb{R}^+)})\, ds \\ &\leq \|f_0\|_{L^1(\mathbb{R}^+)} + T\|g\|_{L^\infty(0,T;L^1(\mathbb{R}^+))}(K + 2\Omega\|g\|_{L^\infty(0,T;L^1(\mathbb{R}^+))}). \end{aligned} \tag{4.6}$$

Hence, we define M depending on the initial data so that

$$\|f_0\|_{L^1(\mathbb{R}^+)} \leq \frac{M}{2},$$

and, then, we fix T such that

$$T(K + 2\Omega M) \leq \frac{1}{2} \tag{4.7}$$

holds. Thus, (4.6) ensures that $\mathcal{T}(f_0, g) \in \mathcal{X}_M$ provided $g \in \mathcal{X}_M$. Notice also that, in view of Proposition 2, $\mathcal{T}(f_0, g)$ has a nonnegative outgoing trace on $\{x = 0\}$.

Now, it remains to show that $\mathcal{T}(f_0, \cdot)$ is a contraction on the set $\mathcal{X}_M$. Indeed, let us denote $f_i = \mathcal{T}(f_0, g_i)$, for $i \in \{1, 2\}$. We get

$$\begin{aligned}\|(f_1 - f_2)(t)\|_{L^1(\mathbb{R}^+)} \leq \int_0^t & \|(Q(g_1) - Q(g_2))(s)\|_{L^1(\mathbb{R}^+)} \\ & + K\|(g_1 - g_2)(s)\|_{L^1(\mathbb{R}^+)}\, ds.\end{aligned}$$

Thus, (2.18) leads to

$$\begin{aligned}&\|(f_1 - f_2)(t)\|_{L^1(\mathbb{R}^+)} \\ &\quad \leq T(K + 2\Omega(\|g_1\|_{L^\infty(0,T;L^1(\mathbb{R}^+))} + \|g_2\|_{L^\infty(0,T;L^1(\mathbb{R}^+))})) \\ &\qquad\qquad \|g_1 - g_2\|_{L^\infty(0,T;L^1(\mathbb{R}^+))},\end{aligned}$$

which implies that

$$\|f_1 - f_2\|_{L^\infty(0,T;L^1(\mathbb{R}^+))} \leq T(K + 4\Omega M)\|g_1 - g_2\|_{L^\infty(0,T;L^1(\mathbb{R}^+))}$$

holds for all g_1, g_2 in $\mathcal{X}_M$. Modifying if necessary the value of T, we can assume that $T(K + 4\Omega M) < 1$ which implies that $\mathcal{T}(f_0, \cdot)$ is a contraction on $\mathcal{X}_M$. By construction, the unique fixed point of this mapping is a non negative integrable solution of (4.1), the collision term $Q(f)$ lies in $L^\infty(0, T; L^1(\mathbb{R}^+))$ and the outgoing trace $-a(t, 0)f(t, 0)$ belongs to $L^1(0, T)$ and is non negative.

4.2. *Extension on any time interval*

Property (1.5) of the collision integral combined to the non negativeness of the outgoing trace lead formally to

$$0 \leq \int_0^\infty f(t, x)\, dx \leq \int_0^\infty f_0(x)\, dx, \tag{4.8}$$

by integrating (3.23). If we are able to establish rigorously this inequality, then we can use $f(T)$ as initial data and reproduce the previous reasoning to extend the solution on the interval $(T, 2T)$ and repeat the procedure... We prove in this way the following

Proposition 3 *Let c be given in $\mathcal{B}$ and assume (2.1-2.3). Let f_0 be an non negative integrable data. Then, for any $T > 0$, there exists a unique non negative solution $f \in C^0([0,T]; L^1(\mathbb{R}^+))$ of (4.1). This solution satisfies (4.8).*

Proof. It remains to justify (4.8). To this end, we use the formulation (3.25), choosing as test function $\psi(t,x) = 1_t\, \eta_R(x)$ where

$$\begin{cases} 0 \leq \eta_R(x) \leq 1, \ supp(\eta_R) \subset (0, 2R), \\ \eta_R(x) = 1 \ for\ 1/R \leq x \leq R. \end{cases} \tag{4.9}$$

This implies that $\eta_R'(x) = R$ for $x \in (0, 1/R)$ and $\eta_R'(x) = -1/R$ for $x \in (R, 2R)$, 0 otherwise. We have

$$\begin{aligned} \int_0^\infty f(t,x)\eta_R(x)dx = \int_0^\infty f_0(x)\eta_R(x)dx + \int_0^t \int_0^\infty Q(f)(s,x)\eta_R(x)\,dxds \\ + \int_0^t \int_0^\infty f(s,x)a(s,x)\eta_R'(x)\,dxds \end{aligned} \tag{4.10}$$

Since $f(t,x)$, $f_0(x)$ and $Q(f)(s,x)$ are integrable and $\eta_R(x) \to 1$ a.e., we can pass to the limit in the first terms by the Lebesgue theorem. Next, we split

$$\begin{aligned} \int_0^t \int_0^\infty f(s,x)a(s,x)\eta_R'(x)\,dxds = \int_0^t \int_0^{1/R} f(s,x)a(s,x)\eta_R'(x)\,dxds \\ + \int_0^t \int_R^{2R} f(s,x)a(s,x)\eta_R'(x)\,dxds. \end{aligned}$$

The behaviour of the first integral is given by Proposition 2; we get

$$\begin{aligned} \int_0^t \int_0^{1/R} f(s,x)a(s,x)\eta_R'(x)\,dxds = \int_0^t R \int_0^{1/R} f(s,x)a(s,x)\,dxds \\ \to \int_0^t f(s,0)a(s,0)\,ds, \end{aligned}$$

as $R \to \infty$. The second integral is evaluated as follows

$$\left| \int_0^t \int_R^{2R} f(s,x)a(s,x)\eta_R'(x)\,dxds \right| \leq \int_0^t \int_R^{2R} A(1+x)\,1/R\, f(s,x)\,dxds$$
$$\leq \frac{A(1+2R)}{R} \int_0^t \int_R^\infty f(s,x)\,dxds \leq 3A \int_0^t \int_R^\infty f(s,x)\,dxds$$

which goes to 0 as $R \to \infty$ since f is integrable. Therefore, passing to the limit in (4.10), we finally obtain

$$\int_0^\infty f(t,x)\,dx = \int_0^\infty f_0(x)\,dx$$
$$+\int_0^t \int_0^\infty Q(f)(s,x)\,dxds + \int_0^t f(s,0)a(s,0)\,ds.$$

By (1.5), the integral of $Q(f)$ is non positive while $a(t,0)f(t,0) \le 0$; this concludes the proof. $\square$

4.3. *Moment properties*

Next, in preparation to the study of the second equation in (1.1), we are interested in the first moment of the obtained solution.

Lemma 4 *If, moreover* $xf_0 \in L^1(I\!R^+)$*, then* $xf(t,x)$ *belongs to* $L^\infty(0,T;$ $L^1(I\!R^+))$ *and we have*

$$M_1(t) = \int_0^\infty xf(t,x)\,dx \le e^{AT}(M_1(0) + AM_0(0)). \tag{4.11}$$

Furthermore, the time derivative of M_1 *is given by*

$$\frac{dM_1(t)}{dt} = \frac{d}{dt}\left(\int_0^\infty xf(t,x)\,dx\right) = \int_0^\infty a(t,x)f(t,x)\,dx. \tag{4.11}$$

Proof. Since $xQ(f)$ is not a priori integrable, one needs to introduce a truncated kernel $\omega_m(x,y) = \omega(x,y)$ if $0 \le x \le m, 0 \le y \le m$, and $\omega_m(x,y) = 0$ if $x > m$ or $y > m$. Notice in particular that $\omega_m(x-y,y) \neq 0$ if $0 \le x-y \le m, 0 \le y \le m$ which leads to $0 \le x \le 2m$. Thus, for f in $L^1(I\!R^+)$, $Q_m(f) \in L^1(I\!R^+)$ has its support in $(0,2m)$ and $xQ_m(f)$ is also integrable. The previous proof of existence-uniqueness, as c is given in $\mathcal{B}$, works with Q_m replacing Q and provides a solution f_m. We claim that f_m approaches f. Indeed, the mild formulation gives

$$f(t,x) - f_m(t,x)$$
$$= \int_0^t (Q(f)(s,X(s;t,x)) - Q_m(f_m)(s,X(s;t,x)))\,J(s;t,x)\,ds$$

and it follows that

$$\int_0^\infty |f(t,x)-f_m(t,x)|\,dx \le \int_0^t \int_{X(s;t,0)}^\infty |Q(f)(s,y)-Q_m(f_m)(s,y)|\,dyds$$
$$\le \int_0^t \int_0^\infty |Q(f)(s,y)-Q_m(f)(s,y)|\,dyds$$
$$+\int_0^t \int_0^\infty |Q_m(f)(s,y)-Q_m(f_m)(s,y)|\,dyds.$$

By combining (2.18) which is satisfied by Q_m with the same constant 2Ω independent on m with (4.8), we get

$$\int_0^\infty |Q_m(f)(s,y)-Q_m(f_m)(s,y)|\,dy \le 4\Omega\|f_0\|_{L^1(\mathbb{R}^+)}\|(f-f_m)(s)\|_{L^1(\mathbb{R}^+)},$$

and then it follows that

$$\|(f-f_m)(t)\|_{L^1(\mathbb{R}^+)} \le \int_0^t \|(Q(f)-Q_m(f))(s)\|_{L^1(\mathbb{R}^+)}\,ds$$
$$+\int_0^t C_0\|(f-f_m)(s)\|_{L^1(\mathbb{R}^+)}\,ds,$$

with $C_0 = 4\Omega\|f_0\|_{L^1(\mathbb{R}^+)}$. Gronwall's lemma leads to

$$\|(f-f_m)(t)\|_{L^1(\mathbb{R}^+)} \le \int_0^t e^{C_0(t-s)}\|(Q(f)-Q_m(f))(s)\|_{L^1(\mathbb{R}^+)}\,ds$$

and it suffices now to apply the dominated convergence theorem to conclude that $f_m \to f$ in $L^\infty(0,T;L^1(\mathbb{R}^+))$.

Since xQ_m is integrable, we can use the mild formulation for f_m which justifies that f_m has a finite first moment, the bound depending on m. Indeed, we have

$$\int_0^\infty xf_m(t,x)\,dx = \int_0^\infty xf_0(X(0;t,x))J(0;t,x)\,dx$$
$$+\int_0^t \int_0^\infty xQ_m(f_m)(s,X(s;t,x))J(s;t,x)\,dxds$$
$$=\int_{X(0;t,0)}^\infty X(t;0,y)f_0(y)\,dy$$
$$+\int_0^t \int_{X(s;t,0)}^\infty X(t;s,y)Q_m(f_m)(s,y)\,dyds$$
$$\le e^{AT}\left(\int_0^\infty yf_0(y)\,dy + \int_0^t \int_0^\infty yQ_m(f_m)(s,y)\,dyds\right)$$
$$\le e^{AT}\left(M_1(f_0) + 2mT\|Q_m(f_m)\|_{L^\infty(0,T;L^1(\mathbb{R}^+))}\right)$$
$$\le e^{AT}\left(M_1(f_0) + 4mT\|f_0\|^2_{L^1(\mathbb{R}^+)}\right).$$

Next, we turn to formula (3.25), choosing now as test function $\psi(t,x) = 1_t\,\zeta_R(x)$ with $\zeta_R(x) = x$ when $0 \le x \le R$, $\zeta_R(x) = 0$ when $x \ge 2R$ and $0 \le \zeta_R(x) \le x$, $|\zeta_R'(x)| \le 1$. We get

$$\int_0^\infty f_m(t,x)\zeta_R(x)dx = \int_0^\infty f_0(x)\zeta_R(x)dx \\ + \int_0^t \int_0^\infty Q_m(f_m)(s,x)\zeta_R(x)\,dxds + \int_0^t \int_0^\infty f_m(s,x)a(s,x)\zeta_R'(x)dxds.$$

Since $|a(s,x)| \le A(1+x)$ and $(1+x)f_m$, $xQ_m(f_m)$ are integrable, by using Lebesgue's theorem, we can pass to the limit $R \to \infty$ in each term. Recalling that the integral of $xQ_m(f_m)$ vanishes, see (1.6), we obtain

$$\int_0^\infty xf_m(t,x)\,dx = \int_0^\infty xf_0(x)\,dx + \int_0^t \int_0^\infty f_m(s,x)a(s,x)dxds \\ \le \int_0^\infty xf_0(x)\,dx + \int_0^t \int_0^\infty A(1+x)f_m(s,x)dxds.$$

By applying Gronwall's lemma, we conclude that (4.11) is satisfied by f_m. Finally, we have

$$\int_0^\infty \zeta_R(x)f_m(t,x)\,dx \le \int_0^\infty xf_m(t,x)\,dx \le e^{AT}(M_1(0) + AM_0(0)).$$

We first let m go to ∞ by using the strong convergence $f_m \to f$ in L^1; this leads to

$$\int_0^\infty \zeta_R(x)f_m(t,x)\,dx \le e^{AT}(M_1(0) + AM_0(0)).$$

Then, Fatou's lemma allows us to pass to the limit $R \to \infty$ and gives (4.11).

Consequently, (2.19) ensures that $xQ^+(f)$ and $xfL(f)$ are integrable on $\mathbb{R}^+$. Hence, we use (3.25) with $\psi(t,x) = \phi(t)\zeta_R(x)$ where $\phi \in C_0^\infty(]0,T[)$. We obtain

$$\langle \tfrac{d}{dt}\int_0^\infty f(t,x)\zeta_R(x)\,dx, \phi\rangle = -\int_0^T \phi(t)\int_0^\infty f(t,x)\zeta_R(x)\,dx\,dt \\ = \int_0^T \int_0^\infty a(t,x)f(t,x)\zeta_R'(x)\phi(t)\,dxdt \\ + \int_0^T \int_0^\infty Q(f)(t,x)\zeta_R(x)\phi(t)\,dxdt.$$

Since $xQ(f)$ and af are integrable, Lebesgue's theorem allows us to pass to the limit as $R \longrightarrow \infty$ in this expression. By (1.6), this leads to the required formula

$$\begin{aligned}\langle \tfrac{d}{dt}\int_0^\infty xf(t,x)\,dx, \phi\rangle &= \int_0^T \phi(t)\int_0^\infty a(t,x)f(t,x)\,dxdt \\ &= \langle \int_0^\infty a(t,x)f(t,x)\,dx, \phi\rangle.\end{aligned}$$

4.4. *Fixed point on c*

According to [5], existence of a solution of (1.1) will result from the following Lemma.

Lemma 5 *Assume that (2.1)-(2.3) hold true. Let $f_0 \geq 0$ belong to $L^1(\mathbb{R}^+, xdx) \cap L^1(\mathbb{R}^+, dx)$. Then, the map $\mathcal{S}$ defined in (4.2) maps the set $\mathcal{B}$ into itself, is continuous, and the image of $\mathcal{S}$ is a precompact subset of $\mathcal{C}^0([0,T])$.*

Remark 3 *Here, continuity and compactness are understood for the uniform norm on $[0,T]$.*

Proof. Proposition 3 allows us to consider the map $\mathcal{S}$ defined in (4.2). By construction, it is obvious that $\mathcal{S}(\mathcal{B}) \in \mathcal{B}$. Let $\tilde{c}(t) = \mathcal{S}(c)(t)$. Stampacchia's theorem yields

$$\frac{d}{dt}\tilde{c}(t) = \begin{cases} 0 & \text{if } \displaystyle\int_0^\infty xf(t,x)\,dx > \rho, \\ -\dfrac{d}{dt}\left(\displaystyle\int_0^\infty xf(t,x)dx\right) & \text{otherwise.}\end{cases}$$

Thus Lemma 4 implies that $\tilde{c}$ belongs to $W^{1,\infty}(0,T)$ and Ascoli's theorem leads to the compactness statement. It remains to show the continuity of $\mathcal{S}$. Let c_n be a sequence of elements in $\mathcal{B}$ converging to $c \in \mathcal{B}$. We need to show that

$$\lim_{n\to\infty} \|\mathcal{S}(c_n) - \mathcal{S}(c)\|_{L^\infty(0,T)}. = 0. \tag{4.12}$$

Clearly, this follows from the convergence

$$\lim_{n\to\infty} \sup_{0\le t\le T} \left| \int_0^\infty x f_n(t,x)dx - \int_0^\infty x f(t,x)dx \right| = 0. \tag{4.13}$$

Nevertheless, contrary to the collisionless case where it is sufficient to prove this convergence of the moment of f_n to the moment of f, see [5]; here, due to the additional non linear term, it seems necessary to show also the convergence of the following norm

$$\begin{aligned} N(f_n - f) &= \lim_{n\to\infty} \sup_{0\le t\le T} [f_n - f](t) \\ &= \lim_{n\to\infty} \sup_{0\le t\le T} \int_0^\infty (1+x)|f_n(t,x) - f(t,x)|\, dx = 0. \end{aligned} \tag{4.14}$$

This stronger convergence requires more regularity on the kinetic coefficents (even if one wants to prove it for the collisionless case) and we assume from now on

$$k \text{ and } q \text{ belong to } W^{2,\infty}(I\!\!R^+). \tag{4.15}$$

The expression $[f_n - f](t)$ naturally splits into a term related to the initial data

$$I_n^0(t) = \int_0^\infty (1+x)|f_0(X_n(0;t,x))J_n(0;t,x) - f_0(X(0;t,x))J(0;t,x)|\, dx \tag{4.16}$$

and a term related to the collision integral

$$\begin{aligned} I_n^Q(t) = \int_0^\infty \int_0^t & (1+x)|Q(f_n)(s,X_n(s;t,x))J_n(s;t,x) \\ & -Q(f)(s,X(s;t,x))J(s;t,x)|\, ds dx. \end{aligned} \tag{4.17}$$

We will approximate f_0 and $Q(f)$ by regular functions f_0^δ and Q^δ respectively and our aim is to obtain estimates like

$$\begin{aligned} & I_n^0(t) \le \Delta(T,\delta) + C(T,\delta,n), \\ & I_n^Q(t) \le \Delta(T,\delta) + C(T,\delta,n) + e^{AT}\int_0^t [f_n - f](s)\, ds \end{aligned} \tag{4.18}$$

with the meaning that

$$\lim_{\delta\to 0}\Delta(T,\delta)=0,\ \lim_{\delta\to 0}C(T,\delta,n)=+\infty,\ \lim_{n\to\infty}C(T,\delta,n)=0, \tag{4.19}$$

(but these quantities blow up as T goes to ∞...). From now on we will use these notations to designate various quantities, which can be different from a line to another, emphasizing in this way their behaviour with respect to the arguments. Let us assume temporarily (4.18). We get

$$[f_n-f](t)\le\Delta(T,\delta)+C(T,\delta,n)+e^{AT}\int_0^t[f_n-f](s)\,ds$$

which permits us to apply Gronwall's lemma, so that

$$N(f_n-f)\le(\Delta(T,\delta)+C(T,\delta,n))\Phi(T).$$

Fix $0<T<\infty$ and let $\varepsilon>0$ be given. We can choose $\delta>0$ such that $\Delta(T,\delta)\Phi(T)\le\varepsilon$, and, then, for n large enough, depending on T and ε, we finally get $N(f_n-f)\le 2\varepsilon$ which is the required estimate, proving the continuity of $\mathcal{S}$.

Before we detail the proof of (4.18), we claim the existence of solutions for bounded times.

Lemma 6 *Under the assumptions of Theorem 1, for any $T>0$, the problem (1.1) has at least one solution on the time interval* $[0,T]$.

Proof. By Schauder's theorem, we obtain a fixed point $c\in\mathcal{B}$ of the map $\mathcal{S}$ as a consequence of Lemma 5. To ensure that the fixed point actually is a solution of the Lifshitz-Slyozov system (1.1), we should remove the truncation $[\cdot]_+$ in (4.2). This is obtained exactly as in [5], using the expression (4.11) for the time derivative of $M_1(t)$. Suppose there exists a time $t_*\in[0,T]$ such that

$$\begin{cases} M_1(t_*)=\displaystyle\int_0^\infty xf(t_*,x)dx=\rho, \\ M_1(t)=\displaystyle\int_0^\infty xf(t,x)dx<\rho \ for\ t<t_*, \end{cases} \tag{4.20}$$

f being the solution of (4.1) associated to c. Note that (4.20) makes sense in view of the regularity result in Lemma 4. Thus, one has $\mathcal{S}(c)(t_*) = c(t_*) = 0$ and it follows that

$$\frac{d}{dt}M_1(t_*) = \int_0^\infty a(t_*, x) f(t_*, x)dx = -\int_0^\infty q(x)f(t_*, x)dx. \qquad (4.21)$$

Since $q(x)f(t_*, x) \geq 0$ a.e. in $I\!R^+$ the quantity (4.21) is non positive and vanishes only if $q(x)f(t_*, x) = 0$ a.e. in $I\!R^+$, which would contradict (2.2) and (4.20). Thus, we actually have $\frac{d}{dt}M_1(t_*) < 0$. If $t_* = 0$, one deduces that $M_1(t) < \rho$ in a neighborhood of the initial time. If $t_* > 0$, then $t \longmapsto M_1(t)$ is non increasing for $t < t_*$ in a neighborhood of t_* which contradicts (4.20). Finally, one obtains $M_1(t) < \rho$ for all $t > 0$, and the second equation in (1.1) holds. □

We are left with the task of proving (4.18). First, we deal with the "initial data" term. We introduce a regular approximation f_0^δ of f_0, namely $f_0^\delta \in C_0^\infty(I\!R^+)$ (say) with $supp f_0^\delta \subset (0, R^\delta)$ and

$$\lim_{\delta \to 0}[f_0 - f_0^\delta] = \lim_{\delta \to 0} \int_0^\infty (1+x)|f_0 - f_0^\delta|\, dx = 0. \qquad (4.22)$$

Hence, we get the following estimate

$$I_n^0 \leq J_n^{1,\delta} + J_n^{2,\delta} + J^{3,\delta} \qquad (4.23)$$

where

$$\begin{aligned}
J_n^{1,\delta} &= \int_0^\infty (1+x)|f_0(X_n(0;t,x)) - f_0^\delta(X_n(0;t,x))| J_n(0;t,x)\, dx \\
J_n^{2,\delta} &= \int_0^\infty (1+x)|f_0^\delta(X_n(0;t,x)) J_n(0;t,x) - f_0^\delta(X(0;t,x)) J(0;t,x)|\, dx \\
J^{3,\delta} &= \int_0^\infty (1+x)|f_0(X(0;t,x)) - f_0^\delta(X(0;t,x))| J(0;t,x)\, dx.
\end{aligned} \qquad (4.24)$$

By using the change of variable $y = X_n(0;t,x)$ or $y = X(0;t,x)$ and (2.9), we easily obtain

$$J_n^{1,\delta} + J^{3,\delta} \leq 2\int_0^\infty (1 + e^{AT}y)|f_0(y) - f_0^\delta(y)|\, dy = \Delta(T, \delta).$$

Now, we write

$$\begin{aligned} J_n^{2,\delta} &\leq \int_0^\infty (1+x)|f_0^\delta(X_n(0;t,x))||J_n(0;t,x)-J(0;t,x)|\,dx \\ &+ \int_0^\infty (1+x)|f_0^\delta(X(0;t,x))-f_0^\delta(X_n(0;t,x))|J(0;t,x)\,dx. \end{aligned}$$

We designate by $K_n^{1,\delta}$ and $K_n^{2,\delta}$ the integral involved in the r.h.s., respectively. Since f_0^δ has its support in $(0,R^\delta)$, by (2.6) with $s_2=t, s_1=0$, the integrand in $K_n^{2,\delta}$ vanishes for $x>\exp(AT)R^\delta$. Thus, we can estimate

$$K_n^{2,\delta} \leq \int_0^{\exp(AT)R^\delta} (1+x)|\int_{X_n(0;t,x)}^{X(0;t,x)} \frac{d}{dz}f_0^\delta(z)\,dz|J(0;t,x)\,dx.$$

Since it is assumed that k' and q' are bounded, then, we bound $J(0;t,x)\leq e^{AT}$ and, by (2.13), it follows that

$$\begin{aligned} K_n^{2,\delta} &\leq e^{AT}\|\tfrac{d}{dz}f_0^\delta\|_{L^\infty(\mathbb{R}^+)}\int_0^{\exp(AT)R^\delta}(1+x)|X_n(0;t,x)-X(0;t,x)|\,dx \\ &\leq e^{AT}\|\tfrac{d}{dz}f_0^\delta\|_{L^\infty(\mathbb{R}^+)}\int_0^{\exp(AT)R^\delta}(1+x)^2KTe^{AT}\|c_n-c\|_{L^\infty(0,T)}\,dx \end{aligned}$$

and we can denote the r.h.s. by $C(T,\delta,n)$, with the behaviour mentioned above. Finally, setting $y=X_n(0;t,x)$ gives

$$\begin{aligned} K_n^{1,\delta} &\leq \int_{X_n(0;t,0)}^\infty (1+X_n(t;0,y))|f_0^\delta(y)||1-\frac{J(0;t,X_n(t;0,y))}{J_n(0;t,X_n(t;0,y))}|\,dy \\ &\leq \int_0^\infty (1+e^{AT}y)|f_0^\delta(y)||1-\exp(\phi_n(t,y))|\,dy \end{aligned}$$

with

$$\phi_n(t,y)=\int_0^t (b(\sigma,X(\sigma;t,X_n(t;0,y)))-b_n(\sigma,X_n(\sigma;t,X_n(t;0,y))))\,d\sigma.$$

Obviously, we can write $\phi_n(t,y)=\alpha_n(t,y)+\beta_n(t,y)+\gamma_n(t,y)$ where

$$\left\{\begin{aligned} \alpha_n(t,y) &= \int_0^t (k'(X(\sigma;t,X_n(t;0,y)))-k'(X_n(\sigma;t,X_n(t;0,y))))c(\sigma)\,d\sigma, \\ \beta_n(t,y) &= \int_0^t k'(X_n(\sigma;t,X_n(t;0,y)))(c(\sigma)-c_n(\sigma))\,d\sigma, \\ \gamma_n(t,y) &= \int_0^t (q'(X(\sigma;t,X_n(t;0,y)))-q'(X_n(\sigma;t,X_n(t;0,y))))\,d\sigma, \end{aligned}\right.$$

and it suffices to show that

$$\sup_{0\le t\le T, 0\le y\le R^\delta} |\alpha_n(t,y)| \longrightarrow 0$$

as n goes to ∞ and similarly for β_n and γ_n. For β_n the result is clear from the estimate

$$|\beta_n(t,y)| \le \|k'\|_{L^\infty(\mathbb{R}^+)} T \|c_n - c\|_{L^\infty(0,T)}.$$

For α_n and γ_n, this follows by using the fact that k' and q' are lipschitzian functions. For instance, we get

$$\begin{aligned}
&|\alpha_n(t,y)| \le \rho L_{k'} \int_0^t |X(\sigma;t,X_n(t;0,y))) - X_n(\sigma;t,X_n(t;0,y))|\,d\sigma \\
&\le \rho L_{k'} K(1+X_n(t;0,y)) \int_0^t e^{A(t-\sigma)} \int_\sigma^t |c_n(\tau)-c(\tau)| e^{A\tau}\,d\tau\,d\sigma \\
&\le \rho L_{k'} K(1+e^{AT}y) T e^{2AT} \|c_n - c\|_{L^\infty(0,T)} = C(T,\delta)\|c_n - c\|_{L^\infty(0,T)}
\end{aligned}$$

$L_{k'}$ being the Lipschitz constant of k'.

It remains to show the collision part of (4.18). We write

$$I_n^Q(t) \le L_n^1(t) + L_n^2(t)$$

where

$$\begin{aligned}
L_n^1(t) &= \int_0^\infty \int_0^t (1+x)|Q(f_n)(X_n(s;t,x)) \\
&\qquad -Q(f)(X_n(s;t,x))|J_n(s;t,x)\,dsdx, \\
L_n^2(t) &= \int_0^\infty \int_0^t (1+x)|Q(f)(X_n(s;t,x))J_n(s;t,x) \\
&\qquad -Q(f)(X(s;t,x)J(s;t,x))|\,dsdx.
\end{aligned}$$

The first term is easily estimated by using the Lipschitz properties (2.18) and (2.20) of the collision operator. We get

$$\begin{aligned}
L_n^1(t) &\le \int_0^t \int_{X_n(s;t,0)}^\infty (1+X_n(t;s,y))|Q(f_n)(y) - Q(f)(y)|\,dyds \\
&\le \int_0^t \int_0^\infty (1+e^{AT}y)|Q(f_n)(y) - Q(f)(y)|\,dyds \\
&\le 4\Omega e^{AT} \int_0^t ([f_n(s)] + [f(s)])[(f_n - f)(s)]\,ds \\
&\le C(T,[f_0]) \int_0^t [(f_n - f)(s)]\,ds.
\end{aligned}$$

Finally, we treat the second term $L_n^2(t)$ by using an approximation Q^δ of $Q(f)$ (in the space $L^1((0,T)\times I\!R^+)$) and reproducing the argument used for the initial data term. This achieves the proof of Theorem 1. $\square$

APPENDIX : PROOF OF PROPOSITION 1

The statement of Proposition 1 is more or less standard, see for instance [3]; however for the sake of completness, let us detail a little bit the proof in this appendix. Let f be a mild solution. We can show from the definition that f belongs to $C^0([0,T];L^1(I\!R^+))$ (the proof follows arguments similar to those in the last Section) and satisfies $f_{|t=0}=f_0$. Let $\psi\in C_0^\infty((0,T)\times I\!R^+)$. One has

$$\langle \partial_t f+\partial_x(af),\psi\rangle = -\int_0^T\int_0^\infty f(t,x)(\partial_t + a(t,x)\partial_x)\psi(t,x)\,dxdt. \tag{4.24}$$

We naturally split this integral into a term I_0 involving the initial data f_0 and a term I_Q involving the right hand side Q. First, we get

$$\begin{aligned} I_0 &= -\int_0^T\int_0^\infty f_0(X(0;t,x))J(0;t,x)(\partial_t + a(t,x)\partial_x)\psi(t,x)\,dxdt \\ &= -\int_0^T\int_{X(0;t,0)}^\infty f_0(y)(\partial_t + a(t,X(t;0,y))\partial_x)\psi(t,X(t;0,y))\,dydt, \end{aligned}$$

with the change of variable $y=X(0;t,x)$ and using (2.15). The outgoing characteristic property implies that $X(0;t,0)\geq 0$ and, moreover for $y\in[0,X(0;t,0)]$, one has $X(t;0,y)\leq 0$. Since $\psi(t,\cdot)$ has its support in $I\!R^+$, we can expand the previous $y-$ integral on $I\!R^+$ and, then, we exchange the order of integration with respect to t and y. By using (2.4), this yields

$$\begin{aligned} I_0 &= -\int_0^T\int_0^\infty f_0(y)\frac{d}{dt}[\psi(t,X(t;0,y))]\,dydt \\ &= -\int_0^\infty f_0(y)(\psi(T,X(T;0,y)) - \psi(0,X(0;0,y)))\,dydt = 0 \end{aligned}$$

since ψ has a compact support in $(0,T)\times I\!R^+$. It thus remains to show that

$$I_Q = \int_0^T \int_0^\infty Q(t,y)\psi(t,y)\,dydt = \langle Q, \psi \rangle.$$

Indeed, one has

$$\begin{aligned} I_Q = &- \int_0^T \int_0^\infty \Big(\int_0^t Q(s, X(s;t,x)) J(s;t,x) ds \Big) \\ &\qquad\qquad (\partial_t + a(t,x)\partial_x)\psi(t,x)\,dxdt \\ = &- \int_0^T \int_0^t \int_{X(s;t,0)}^\infty Q(s,y) \\ &\quad (\partial_t + a(t, X(t;s,y))\partial_x)\psi(t, X(t;s,y))\,dydsdt. \end{aligned}$$

By reasoning as above, one may integrate over $I\!R^+$ and write

$$\begin{aligned} I_Q &= -\int_0^T \int_0^t \int_0^\infty Q(s,y) \frac{d}{dt}[\psi(t, X(t;s,y))]\,dydsdt \\ &= -\int_0^T \int_\sigma^T \int_0^\infty Q(\sigma,y) \frac{d}{d\tau}[\psi(\tau, X(\tau;\sigma,y))]\,dyd\tau d\sigma \end{aligned}$$

by inverting the integration with respect to s and t. It follows that

$$\begin{aligned} I_Q &= -\int_0^T \int_0^\infty Q(\sigma,y)(\psi(T, X(T;\sigma,y)) - \psi(\sigma, X(\sigma;\sigma,y)))\,dyd\sigma \\ &= \int_0^T \int_0^\infty Q(\sigma,y)\psi(\sigma,y)\,dyd\sigma \end{aligned}$$

holds, by using the support property of ψ. Therefore, we proved that a mild solution is also a distribution solution. Let us prove now that the converse also holds.

Let f satisfy i). We expand f and Q by 0 for $x < 0$, denoting these functions, defined on all $I\!R$ by $\tilde{f}$, $\tilde{Q}$ respectively. We set

$$g(t,x) = \tilde{f}(t, X(t;0,x))J(t;0,x) - \int_0^t \tilde{Q}(s, X(s;0,x))J(s;0,x)\,ds. \tag{4.24}$$

Obviously, g is defined on $(0,T) \times I\!R^+$ and lies in $C^0([0,T]; L^1(I\!R^+))$. Let us assume temporarily that

$$\partial_t g = 0 \tag{4.24}$$

holds in $\mathcal{D}'((0,T)\times \mathbb{R}^+)$. This leads to

$$g(t,x) = g_0(x) = f(0, X(0;0,x))J(0;0,x) - 0 = f_0(x).$$

Thus, let $y \geq 0$ and write (4.24) with $x = X(0;t,y) \geq 0$ (i.e. $y = X(t;0,x)$). This gives

$$\begin{aligned}
f(t,y) &= f_0(X(0;t,y))\exp\Big(-\int_t^0 b(\sigma, X(\sigma;0,X(0;t,y))\,d\sigma\Big)\\
&\quad + \int_0^t Q(s, X(s;0,X(0;t,y)))\\
&\qquad\qquad \exp\Big(-\int_s^0 b(\sigma, X(\sigma;0,X(0;t,y)))\,d\sigma\Big)\\
&\qquad\qquad \exp\Big(-\int_0^t b(\sigma, X(\sigma;0,X(0;t,y)))\,d\sigma\Big)\,ds\\
&= f_0(X(0;t,y))J(0;t,y) + \int_0^t Q(s,X(s;t,y))J(s;t,y)\,ds
\end{aligned}$$

which means that f satisfies the mild formulation. It remains to show (4.24). Let $\psi \in C_0^\infty((0,T)\times \mathbb{R}^+)$. One has

$$\begin{aligned}
\langle \partial_t g, \psi\rangle &= -\int_0^T\int_0^\infty \tilde f(t, X(t;0,x))J(t;0,x)(\partial_t\psi)(t,x)\,dxdt\\
&\quad + \int_0^T\int_0^\infty\int_0^t \tilde Q(s, X(s;0,x))J(s;0,x)ds(\partial_t\psi)(t,x)\,dxdt.
\end{aligned}$$

Integration by parts in time in the second integral allows us to write it as

$$\int_0^T\int_0^\infty \tilde Q(t, X(t;0,x))J(t;0,x)\psi(t,x)\,dxdt.$$

Then, we get

$$\begin{aligned}
\langle \partial_t g, \psi\rangle = &- \int_0^T\int_{X(t;0,0)}^\infty \tilde f(t,y)(\partial_t\psi)(t, X(0;t,y))dydt\\
&- \int_0^T\int_{X(t;0,0)}^\infty \tilde Q(t,y)\psi(t, X(0;t,y))dydt.
\end{aligned}$$

The outgoing characteristic property implies that $X(t;0,0) \leq 0$ for $t \in (0,T)$. Since $\tilde f$ and $\tilde Q$ vanish for $y \leq 0$, we can only integrate on the non negative domain and delete the $\tilde{}$ marks, so that we are led to

$$\begin{aligned}
\langle \partial_t g, \psi\rangle = &- \int_0^T\int_0^\infty f(t,y)(\partial_t\psi)(t, X(0;t,y))dydt\\
&- \int_0^T\int_0^\infty Q(t,y)\psi(t, X(0;t,y))dydt.
\end{aligned}$$

We set $\tilde\psi(t,y) = \psi(t, X(0;t,y))$. Then, by (3.22), the second integral in the r.h.s. becomes

$$-\langle Q, \tilde\psi\rangle = -\langle \partial_t f + \partial_y(af), \tilde\psi\rangle = \langle f, (\partial_t + a(t,y)\partial_y)\tilde\psi\rangle$$

where on the one hand

$$\begin{array}{l}\partial_t\tilde\psi(t,y) = \partial_t[\psi(t,X(0;t,y))] \\ = (\partial_t\psi)(t,X(0;t,y)) - a(t,y)J(0;t,y)(\partial_x\psi)(t,X(0;t,y))\end{array}$$

by (2.16), and, on the other hand

$$\begin{array}{rcl}\partial_y\tilde\psi(t,y) = \partial_y[\psi(t,X(0;t,y))] & = & \dfrac{\partial X(0;t,y)}{\partial y}(\partial_X\psi)(t,X(0;t,y)) \\ & = & J(0;t,y)(\partial_X\psi)(t,X(0;t,y)),\end{array}$$

by (2.15). It follows that

$$-\langle Q, \tilde\psi\rangle = \int_0^T\int_0^\infty f(t,y)(\partial_t\psi)(t,X(0;t,y))dydt$$

which allows to conclude that (4.24) is satisfied. □

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