On fluid limit for the semiconductors Boltzmann equation

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Abstract

This paper is devoted to the derivation of (non linear) drift-diffusion equations from the semiconductor Boltzmann equation. Collisions are taken into account through the non linear Pauli operator, but we do not assume relation on the cross section such as the so-called detailed balance principle. In turn, equilibrium states are implicitely defined. This article follows and completes the contribution of A. Mellet (Monatsh. Math. 134 (2002), no. 4, 305–329) where the electric field is given and does not depend on time. Here, we treat the self-consistent problem, the electric potential satisfying the Poisson equation. By mean of a Hilbert expansion, we shall formally derive the asymptotic model in the general case. We shall then rigorously prove the convergence in the one-dimensional case by using a modified Hilbert expansion.

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1 Introduction

Our study starts from the following system of equations, in which the unknowns are the function $f^\varepsilon(x,k,t)$ and the potential $V^\varepsilon(x,t)$:

$$\begin{cases} \partial_t f^\varepsilon + \dfrac{1}{\varepsilon}\left(v(k)\cdot\nabla_x f^\varepsilon + \nabla_x V^\varepsilon \cdot \nabla_k f^\varepsilon\right) = \dfrac{1}{\varepsilon^2} Q(f^\varepsilon) & \text{for } x\in\Omega,\ k\in B,\ t\in\mathbb{R}^+, \\ \Delta V^\varepsilon = \displaystyle\int_B f^\varepsilon(x,k,t)\ \mathrm{d}k - D(x) & \text{for } x\in\Omega,\ t\in\mathbb{R}^+, \\ f^\varepsilon(x,k,0) = f_{in}(x,k) & \text{for } x\in\Omega,\ k\in B. \end{cases} \tag{1}$$

Such a system naturally arises when modeling the electrons transport in a semiconductor device. Precisely, $f^\varepsilon(x,k,t)$ is the probability to find an electron with wave vector $k\in B$ and position $x\in\Omega$, at time $t\geq 0$. Here, the space variable x lies in some bounded subset Ω of $\mathbb{R}^N$. In order to avoid difficulties related to boundary conditions, we shall deal with periodic boundary conditions: we set $\Omega = \mathbb{R}^N/L$, where L is a lattice of $\mathbb{R}^N$. The wave vector variable k lies in the first Brillouin zone, B, which can be essentially seen as a torus in $\mathbb{R}^N$. Then, the set B is considered as endowed with the normalized Lebesgue measure

$$\int_B \mathrm{d}k = 1.$$

The velocity of the particles is defined by a smooth function of the wave vector, which derives from an energy functional, $k\in B\longmapsto v(k) = \nabla_k \mathcal{E}(k)\in\mathbb{R}^N$. The right hand side in the first equation of (1) describes interactions phenomena of the particles within the device, in particular collisions with impurities, phonons or other particles. It takes the form of a nonlinear operator, non local with respect to the variable k. It will be described more precisely in a few lines. An important feature is the mass conservation property which means that

$$\int_B Q(f)\ \mathrm{d}k = 0 \tag{2}$$

holds (at least formally). Accordingly, the solution satisfies

$$\int_\Omega\int_B f^\varepsilon\ \mathrm{d}k\ \mathrm{d}x = \int_\Omega\int_B f_{in}\ \mathrm{d}k\ \mathrm{d}x.$$

On the other hand, the electrons are also submitted to an electric field $E^\varepsilon = \nabla_x V^\varepsilon$, coupled to the probability f^ε via the Poisson equation. The function D is a given (nonnegative) doping profile. Throughout the paper, we assume that D is a regular function, say $C^\infty(\Omega)$. We therefore assume in what follows that

$$\int_\Omega D\ \mathrm{d}x = \int_\Omega\int_B f_{in}\ \mathrm{d}k\ \mathrm{d}x,$$

so that the Poisson equation with periodic boundary conditions makes sense (note however that the case of null boundary conditions can be treated similarly).

The right-hand-side in (1) is given by the following Boltzmann-Pauli operator

$$Q(f)(x,k) = \int_{k' \in B} \Big(\sigma(x,k,k')(1-f(k))f(k') - \sigma(x,k',k)(1-f(k'))f(k) \Big) \, \mathrm{d}k', \tag{3}$$

where we omitted the x-dependence of f for the sake of clarity and σ is a given non negative function. This operator models the collisions against other particles that electrons may suffer when crossing the device whereas the $(1-f)$ terms in Q take into account the Pauli exclusion principle. This leads us to deal with distribution functions satisfying naturally $0 \leq f \leq 1$. For such bounded functions, (3) clearly makes sense when σ is bounded, since B is finitely measured. Note also that the mass conservation relation (2) is satisfied. The cross-section $\sigma(x,k,k')$ represents the probability that the scattering event produces a transition of the state of the electron from the state k to the state k'. All the information on the scattering processes is embodied in this function. Our aim in the present paper is to deal with very general cross section $\sigma(x,k,k')$. We quote the papers of F. Poupaud [13], A. Mellet [12] and the references therein for details on the physical background concerning the equation; we also refer to the classical treatise of P. Markowich-C. Ringhofer-C. Schmeiser [11], and the recent lecture notes of P. Degond [3]. More details on solid state physics can be found for instance in N. Ashcroft-N. Mermin [1].

The parameter ε involved in (1) is a scaled version of the mean free path between two scattering events. In physical situation, this parameter is small and we are interested in the behaviour of the solutions of (1) as ε goes to 0. Precisely, we are interested in a physical situation where the observation length scale is large compared to the mean free path while the observation time scale is large compared to the characteristic time of evolution of the particles. In view of equation (1), one expects that f^ε converges to an equilibrium state, i.e. a function F satisfying $Q(F) = 0$. The characterisation of such functions is therefore an important issue in the asymptotic study of equation (1).

Usually, when modeling collision effects, the so-called detailed balance principle (or microreversibility condition) is assumed; namely, it is supposed that the collision kernel σ fulfills

$$\sigma(x,k,k') \exp(-\mathcal{E}(k)/(k_B T)) = \sigma(x,k',k) \exp(-\mathcal{E}(k')/(k_B T)), \tag{4}$$

where $\mathcal{E}$ is the energy functional, k_B is the Boltzmann constant and T, possibly depending on x, stands for the lattice temperature. In such a situation, the equilibrium states, solutions of $Q(F) = 0$, are the following Fermi-Dirac distributions

$$F(\mu, k) = \frac{1}{1 + \exp((\mathcal{E}(k) - \mu)/(k_B T))}$$

where μ is related to the density by $\rho = \int_B F(\mu, k)\, \mathrm{d}k$. Note that in this situation, $Q(F)$ vanishes since the integrand $\sigma(k, k')(1 - F(k))F(k') - \sigma(k', k)(1 - F(k'))F(k)$ vanishes. The corresponding asymptotic problem has been extensively investigated by F. Golse-F. Poupaud [7], assuming a constant temperature. Our main contribution in this paper is to remove the detailed balance assumption (4). We consider the Boltzmann equation written under general consideration as in (3) and we do not postulate that the system is driven by Fermi-Dirac statistics. Some results in this direction have been obtained in a linear situation by P. Degond-T. Goudon-F. Poupaud [4]. Reference [12] deals with the non-linear Boltzmann-Pauli equation (1-3), with a given time-independent electric potential V. In this work, we shall investigate the self-consistent case.

We now precise our assumptions on the cross-section $\sigma(x, k, k')$:

Hypothesis 1 *(i) There exist two positive constants* $\underline{\sigma}, \overline{\sigma} > 0$ *such that*

$$\underline{\sigma} \leq \sigma(x, k, k') \leq \overline{\sigma}$$

holds for almost all $x \in \Omega$, $k, k' \in B$.

(ii) The cross-section $\sigma(x, k, k')$ *is smooth with respect to* x, k, k'.

Note that Hypothesis 1-i) is fulfilled for instance when collisions are isotropic (in such a case we have $\sigma(x, k, k') = \sigma_0$). When Hypothesis 1 is satisfied, the existence and uniqueness of a solution f^ε of (1) are given by the following Proposition, for which we refer to F. Poupaud [13].

Proposition 2 *[13] For all* f_{in} *in* $W^{1,1}(\Omega \times B)$ *such that* $0 \leq f_{in} \leq 1$, *there exists a unique* $f^\varepsilon \in W^{1,1} \cap W^{1,\infty}(\Omega \times B \times (0, T))$ *satisfying* $0 \leq f^\varepsilon \leq 1$ *and* V^ε *with* $\nabla_x V^\varepsilon \in W^{1,\infty}(\Omega \times (0, T))$ *which solve (1) in the distributional sense.*

As mentioned above, when the detailed balance principle holds, the equilibrium states are the Fermi-Dirac distributions. In the general case, we do not have any explicit expression for equilibria and their existence is a non trivial point. However, the following statement says that equilibrium functions exist for the general Pauli operator (3), and they can be parametrized by their density.

Proposition 3 *For all* $\rho(x)$ *in* $L^\infty(\Omega)$ *satisfying* $0 \leq \rho(x) \leq 1$, *there exists a* *unique $F(\rho(x), x, k)$ in $L^\infty(\Omega, L^\infty(B))$ which verifies*

$$\begin{cases} 0 \le F(\rho(x), x, k) \le 1\,, \quad \displaystyle\int_B F(\rho(x), x, k)\ \mathrm{d}k = \rho(x) \qquad \text{a.e. } x \in \Omega, \\ Q(F(\rho)) = 0. \end{cases} \tag{5}$$

Proposition 3 has been announced in [8]; a detailed proof is given in the Appendix. Let us only say for the time being that the proof of Proposition 3 relies on the properties of the first derivative of Q with respect to f, which is the linear integral operator L_f on $L^\infty(B)$ defined by

$$\left(\frac{\partial Q}{\partial f}\right)_f (g) = L_f(g) = \int_B \Big(s_f(k, k')g(k') - s_f(k', k)g(k) \Big)\ \mathrm{d}k', \tag{6}$$

where we set

$$s_f(k, k') = \sigma(k, k')(1 - f(k)) + \sigma(k', k)f(k)\,, \tag{7}$$

for any function f satisfying $0 \le f(k) \le 1$ for almost all k in B. The properties of such a linear operator have been investigated in details in [4]. Here, we will essentially use the following Fredholm alternative (see Lemma 17 in the Appendix).

Lemma 4 *Let f satisfy $0 \le f \le 1$. Then, for any $h \in L^\infty(B)$ such that $\int_B h\ \mathrm{d}k = 0$ there exists a unique solution $g \in L^\infty(B)$ of $L_f(g) = h$ with $\int_B g\ \mathrm{d}k = 0$.*

Moreover, differentiating (5) with respect to ρ (in the distributional sense), we can check that

Lemma 5 *The function $F(\rho, x, k)$ is smooth with respect to ρ. More precisely, all the derivatives $\frac{\partial^n F}{\partial \rho^n}$ belong to $L^\infty((0, 1) \times \Omega \times B)$.*

In the sequel, we shall denote by $F'(\rho)$ the first derivative of F with respect to ρ. Important properties of $F'(\rho)$ are detailed in Remark 19. Finally, in the discussion of the asymptotics, one needs the following null flux hypothesis on the equilibrium state $F(\rho)$.

Hypothesis 6 *For all $\rho \in [0, 1]$, we assume that*

$$\int_B v(k)F(\rho)(x, k)\ \mathrm{d}k = 0$$

for almost all $x \in \Omega$.

In view of Lemma 4, this hypothesis will appear later on as a solvability condition for certain auxiliary equations (see (10) below). A formal analysis, performed in Section 2, shows that the system (1) tends to the following limit system, the macroscopic unknowns being the density $\rho(x,t)$ and the potential $V(x,t)$,

$$\begin{cases} \partial_t\rho - \mathrm{div}_x\left[\Pi(\rho,x)\nabla_x\rho + \Theta(\rho,x)\nabla_x V + \chi(\rho,x)\right] = 0 & \text{for } (x,t)\in Q_T, \\ \Delta V(x,t) = \rho(x,t) - D(x) & \text{for } (x,t)\in Q_T, \\ \rho(x,0) = \rho_{in}(x) & \text{for } x\in\Omega. \end{cases} \tag{8}$$

Here and below, we use the notation

$$Q_T = \Omega \times (0,T)$$

where $0 < T < \infty$. For $\rho \in [0,1]$, the matrices $\Pi(\rho,x)$, $\Theta(\rho,x)$, and the vector $\chi(\rho,x)$ are defined by

$$\begin{aligned} \Pi(\rho,x) &= -\int_B v(k)\otimes\lambda(\rho,x,k)\,\mathrm{d}k\,, \\ \Theta(\rho,x) &= -\int_B v(k)\otimes\nu(\rho,x,k)\,\mathrm{d}k\,, \\ \chi(\rho,x) &= -\int_B \mu(\rho,x,k)v(k)\,\mathrm{d}k, \end{aligned} \tag{9}$$

where $\lambda(\rho,x,k)$, $\nu(\rho,x,k)$, $\mu(\rho,x,k)$ solve the auxiliary equations

$$\begin{aligned} &L_{F(\rho)}(\lambda(\rho)) = v(k)F'(\rho)\,, && \int_B \lambda(\rho)\,\mathrm{d}k = 0, \\ &L_{F(\rho)}(\nu(\rho)) = \nabla_k F(\rho)\,, && \int_B \nu(\rho)\,\mathrm{d}k = 0, \\ &L_{F(\rho)}(\mu(\rho)) = v(k)\cdot(\nabla_x F)(\rho)\,, && \int_B \mu(\rho)\,\mathrm{d}k = 0. \end{aligned} \tag{10}$$

In (9), we have denoted, for $a, b \in \mathbb{R}^N$, $a\otimes b$ for the $N\times N$ matrix with coefficients $a_i b_j$. Recall that $F'(\rho)$ stands for the derivative of F with respect to ρ and, since Ω is a torus, equation (8) is understood with periodic boundary conditions. We shall prove the convergence of $f^\varepsilon, V^\varepsilon$ towards the solutions of (8) as ε goes to 0.

The rigorous proof of the convergence, however, involves some L^1 estimates for the Poisson equation that are available only in a one-dimensional setting. We shall therefore perform the proof only within this framework. More precisely, if Ω is a torus in $\mathbb{R}$, and Π_{11}, Θ_{11}, χ_1 denote the first coefficient of Π, Θ, χ, our main result states as follows.

Theorem 7 *Let f_{in} satisfy $0 \le f_{in} \le 1$ and $Q(f_{in}) = 0$ and $\rho_{in} = \int_B f_{in}\,\mathrm{d}k$ is a smooth function. Suppose that Hypotheses 1, 6 hold. Then, the solution $f^\varepsilon(x,k,t)$ of the 1-D Boltzmann equation (1) converges as $\varepsilon \to 0$ towards an equilibrium state $F(\rho(x,t),x,k)$, where the density $\rho(x,t)$ solves the following* *asymptotic equation*

$$\begin{cases} \partial_t \rho - \partial_x \left[\Pi_{11}(\rho,x)\partial_x \rho + \Theta_{11}(\rho,x)\partial_x V + \chi_1(\rho,x)\right] = 0 & \textit{for } (x,t) \in Q_T, \\ \partial_x^2 V(x,t) = \rho(x,t) - D(x) & \textit{for } (x,t) \in Q_T, \\ \rho(x,0) = \rho_{in}(x) & \textit{for } x \in \Omega. \end{cases} \tag{11}$$

Moreover for all $T > 0$ there exists a constant C_T such that

$$|f^\varepsilon(t) - F(\rho(t))|_{L^1(\Omega\times B)} \leq C_T\ \varepsilon \quad \textit{for all } t \in [0,T].$$

As in [12], the convergence proof relies on an asymptotic expansion of f^ε, and the Chapman-Enskog method (see [2]). This method has also been used to deal with a linearized version of (1) by F. Poupaud [14]. The main drawback of the Chapman-Enskog method is that it requires some regularity for the solution of the asymptotic equation, which is here a non-linear coupled system. It is therefore crucial to establish the existence and regularity of the solution of the asymptotic problem. This will be done in Section 3. In Section 2, we shall formally derive the asymptotic model (8) by means of a Hilbert expansion, and anticipating on the results of Section 3, we shall deduce Theorem 7.

2 Derivation of the asymptotic model

The first part of this section will be devoted to the formal derivation of (8), relying on the usual Hilbert expansion. Unfortunatly, the expansion of the quadratic term $\nabla_x V^\varepsilon \cdot \nabla_k f^\varepsilon$ gives rise to some singular term that prevents us from leading the usual estimate of the remainder. The proof of the Theorem 7 therefore relies on a new formal development of f^ε, first introduced by M. L. Tayeb in [16]. Note however that [16] uses crucially entropy estimates which are not available here; hence our proof needs another trick.

2.1 The formal derivation

For further purpose, let us introduce the following Taylor expansion of the collision operator Q

$$Q(f+g) = Q(f) + L_f(g) + R(g,g),$$

where L_f is the differential of Q with respect to f, introduced in (6)-(7). The remainder R is defined by the following bilinear operator

$$R(g,h) = \int_B \frac{1}{2}(\sigma(k',k) - \sigma(k,k'))\,[g(k)h(k') + g(k')h(k)]\ \mathrm{d}k'.$$

The usual way to derive the asymptotic model is to expand f^ε and V^ε as follows

$$\begin{aligned} f^\varepsilon &= f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + ... \\ V^\varepsilon &= V^0 + \varepsilon V^1 + \varepsilon^2 V^2 + ... \end{aligned}$$

We insert these expansions in (1). Identifying terms having the same order with respect to ε, we are led to

$$Q(f^0) = 0, \tag{12}$$

$$L_{f^0}(f^1) = v(k)\cdot\nabla_x f^0 + \nabla_x V^0 \cdot \nabla_k f^0, \tag{13}$$

$$L_{f^0}(f^2) = \partial_t f^0 + v(k)\cdot\nabla_x f^1 + \nabla_x V^0\cdot\nabla_k f^1 + \nabla_x V^1 \cdot \nabla_k f^0 - R(f^1,f^1), \tag{14}$$

while the potentials satisfy

$$\begin{aligned} \Delta V^0 &= \rho^0(x,t) - D(x) = \int_B f^0(x,k,t)\ \mathrm{d}k - D(x), \\ \Delta V^i &= \rho^i(x,t) = \int_B f^i(x,k,t)\ \mathrm{d}k. \end{aligned}$$

The first equality (12) reads

$$f^0(x,k,t) = F(\rho(x,t),x,k)\,,$$

where $\rho(x,t) = \int_B f^0(x,k,t)\ \mathrm{d}k$. Thus the right-hand side of (13) can be rewritten as follows

$$\begin{aligned} &\nabla_x V^0(x)\cdot(\nabla_k F)(\rho(x,t),x,k) \\ &+F'(\rho(x,t))(x,k)v(k)\cdot\nabla_x\rho + v(k)\cdot(\nabla_x F)(\rho(x,t),x,k). \end{aligned}$$

We recall that $F'(\rho)$ designates the derivative of F with respect to ρ. It can be shown that $F'(\rho)$ belongs to the kernel of the linear operator $L_{F(\rho)}$. Precisely, we have

$$\mathrm{Ker}(L_{F(\rho)}) = \mathrm{Span}(F'(\rho)), \qquad F'(\rho) > 0,$$

see Remark 19. We now define $\lambda(\rho,x,k)\,,\nu(\rho,x,k) \in (L^\infty(\Omega,L^\infty(B)))^N$ and $\mu(\rho,x,k) \in L^\infty(\Omega,L^\infty(B))$ by equations (10). In view of Lemma 4, it is worth pointing out that Hypothesis 6 is crucial to guarantee that these auxiliary equations admit solutions. We then set

$$f^1 = \lambda(\rho) \cdot \nabla_x \rho + \nu(\rho) \cdot \nabla_x V^0 + \mu(\rho). \tag{15}$$

The solvability condition for f^2 is obtained by integrating (15) with respect to k. Since $\int_B R(g,g)(k)\, \mathrm{d}k = 0$ for all g, we obtain

$$\partial_t \int_B f^0\, \mathrm{d}k + \mathrm{div}_x \left(\int_B v(k) f^1\, \mathrm{d}k \right) + \nabla_x V^0 \cdot \int_B \nabla_k f^1\, \mathrm{d}k + \nabla_x V^1 \cdot \int_B \nabla_k f^0\, \mathrm{d}k = 0. \tag{16}$$

The last two terms in the above expression vanish thanks to the periodicity with respect to k, while (15) gives

$$\int_B v(k) f^1\, \mathrm{d}k = -\left(\Pi(\rho)\nabla_x \rho + \Theta(\rho)\nabla_x V + \chi(\rho) \right), \tag{17}$$

with $\Pi(\rho)$, $\Theta(\rho)$ and $\chi(\rho)$ defined by (9), and $V(x,t) = V^0(x,t)$. Note that

$$\int_B f^0(x,k,t)\, \mathrm{d}k = \int_B F(\rho(x,t),x,k)\, \mathrm{d}k = \rho(x,t).$$

Then, inserting (17) in (16), the solvability condition becomes

$$\begin{cases} \partial_t \rho - \mathrm{div}_x \left(\Pi(\rho)\nabla_x \rho + \Theta(\rho)\nabla_x V + \chi(\rho) \right) = 0\,, \\ \Delta V = \rho - D(x)\,, \end{cases}$$

which is the asymptotic equation (8).

This equation is a non-linear diffusion equation, for which we are able to prove the following existence and uniqueness result, which applies in any dimension (in the statement Ω is a torus in $\mathbb{R}^N$).

Proposition 8 *For all $T > 0$, there exists a unique pair of functions $\rho(x,t) \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$ and $V(x,t) \in L^2(0,T;H^1(\Omega))$ solutions of the asymptotic equation:*

$$\begin{cases} \partial_t \rho - \mathrm{div}_x \left[\Pi(\rho,x)\nabla_x \rho + \Theta(\rho,x)\nabla_x V + \chi(\rho,x) \right] = 0 & \text{for } (x,t) \in Q_T, \\ \Delta V(x,t) = \rho(x,t) - D(x) & \text{for } (x,t) \in Q_T \\ \rho(x,0) = \rho_{in}(x) & \text{for } x \in \Omega. \end{cases} \tag{18}$$

Furthermore, this solution satisfies $0 \le \rho(x,t) \le 1$ for almost all $(x,t) \in Q_T$.

Moreover, we shall see that this solution is C^∞ smooth, as stated in the following lemma.

Lemma 9 *The solutions $(\rho(x,t), V(x,t))$ given by Proposition 8 are C^∞ with respect to (x,t).*

These statements are the cornerstone of the rigorous derivation of the asymptotic regime. The proofs are postponed to Section 3, we first derive the limit system.

2.2 Rigorous derivation: Proof of Theorem 7

A first attempt towards the rigorous proof.

As in [12], a first idea would be to estimate the remainder

$$r^\varepsilon = f^\varepsilon - F(\rho) - \varepsilon f^1 - \varepsilon^2 f^2 \,, \tag{19}$$

where $(\rho(x,t), V(x,t))$ are the solutions of the limit problem given by Proposition 8. Since the solution of (13) is defined up to an element of $\mathrm{Ker}(L_{F(\rho)})$, we define the first order corrector by:

$$f^1 = \lambda(\rho)\cdot\nabla_x\rho + \nu(\rho)\cdot\nabla_x V + \mu(\rho) + qF'(\rho)\,, \tag{20}$$

where $q(x,t)$ is an arbitrary (smooth) function that will be suitably chosen later on. Thanks to the null-flux assumption, this last term in (20) does not give any contribution in the expression of the flux $\int_B v(k) f^1 \,\mathrm{d}k$. Then we define the second order corrector by

$$L_{F(\rho)}(f^2) = \partial_t F(\rho) + v(k)\cdot\nabla_x f^1 + \nabla_x V\cdot\nabla_k f^1 - R(f^1,f^1)\,. \tag{21}$$

We also introduce a new splitting of the non-linear term $Q(f^\varepsilon)$:

$$\begin{aligned} Q(f+g) = Q(f) &+ \int_B \Big(\sigma(k,k')(1-(f+g))g' - \sigma(k',k)(1-(f'+g'))g\Big)\,\mathrm{d}k' \\ &+ \int_B \Big(\sigma(k',k)fg' - \sigma(k,k')f'g\Big)\,\mathrm{d}k' \end{aligned}$$

where f stands for $f(k)$ and f' for $f(k')$. We define the operators

$$\begin{aligned} N_f(g) &= \int_B \Big(\sigma(k,k')fg' - \sigma(k',k)f'g\Big)\,\mathrm{d}k'\,, \\ N_f'(g) &= \int_B \Big(\sigma(k',k)fg' - \sigma(k,k')f'g\Big)\,\mathrm{d}k' = -N_g(f). \end{aligned}$$

Then, writing $f^\varepsilon = (F(\rho) + \varepsilon f^1 + \varepsilon^2 f^2) + r^\varepsilon$, we get

$$Q(f^\varepsilon) = Q(F(\rho) + \varepsilon f^1 + \varepsilon^2 f^2) + N_{1-f^\varepsilon}(r^\varepsilon) + N'_{(F(\rho)+\varepsilon f^1+\varepsilon^2 f^2)}(r^\varepsilon).$$

Inserting the expansion (19) of f^ε in (1), simplifications arise from (13) and (21), and we see that $r^\varepsilon(x,k,t)$ solves

$$\begin{cases} \partial_t r^\varepsilon + \dfrac{1}{\varepsilon}\left(v(k)\cdot\nabla_x r^\varepsilon + \nabla_x V^\varepsilon \cdot \nabla_k r^\varepsilon\right) = \dfrac{1}{\varepsilon^2} N_{1-f^\varepsilon}(r^\varepsilon) + \dfrac{1}{\varepsilon^2} N'_{(F(\rho)+\varepsilon f^1+\varepsilon^2 f^2)}(r^\varepsilon) \\ \qquad\qquad\qquad\qquad + \varepsilon S^\varepsilon - \dfrac{1}{\varepsilon}\nabla_x(V^\varepsilon - V)\cdot\nabla_k(F(\rho)+\varepsilon f^1), \\ r^\varepsilon(t=0) = r^\varepsilon_{in}, \end{cases} \tag{22}$$

where the source term $S^\varepsilon(x,k,t)$ is given by

$$\begin{aligned} S^\varepsilon = &R(f^1,f^2) + R(f^2,f^1) + \varepsilon R(f^2,f^2) - \partial_t f^1 - \varepsilon \partial_t f^2 \\ &- v(k)\cdot\nabla_x f^2 - \nabla_x V^\varepsilon \cdot \nabla_k f^2. \end{aligned}$$

In [12], the potential V is given and does not depend on time. In this case, we are led to the same formulae with $V^\varepsilon = V$. Hence, the convergence to zero of the L^1-norm of r^ε can be established from the corresponding equation (22). But here, the method breaks down, due to the singular term

$$\frac{1}{\varepsilon}\nabla_x(V^\varepsilon - V)\cdot\nabla_k(F(\rho)+\varepsilon f^1).$$

In order to get rid of this term, we shall modify the expansion of f^ε.

A modified Hilbert expansion.

In the spirit of Tayeb [16], we use an hybrid Hilbert expansion for f^ε. We define new correctors $\widetilde{f}^{1,\varepsilon}$ and $\widetilde{f}^2$, solutions of

$$\begin{aligned} L_{F(\rho)}(\widetilde{f}^{1,\varepsilon}) &= v(k)\cdot\nabla_x F(\rho) + \nabla_x V^\varepsilon \cdot \nabla_k F(\rho), \\ L_{F(\rho)}(\widetilde{f}^{2}) &= \partial_t F(\rho) + v(k)\cdot\nabla_x f^1 + \nabla_x V\cdot\nabla_k f^1 - R(f^1,f^1)\,, \end{aligned}$$

leading to a new expansion of f^ε

$$f^\varepsilon = F(\rho) + \varepsilon \widetilde{f}^{1,\varepsilon} + \varepsilon^2 \widetilde{f}^2 + \widetilde{r}^\varepsilon. \tag{23}$$

With the notations of the previous sections, we have

$$\widetilde{f}^{1,\varepsilon} = \lambda(\rho)\cdot\nabla_x\rho + \nu(\rho)\cdot\nabla_x V^\varepsilon + \mu(\rho) + qF'(\rho), \tag{24}$$

which has to be compared with (20). The $\varepsilon-$dependence of the first corrector will help us to get rid of the singular term of the previous expansion. On the other hand, we point out that $\widetilde{f}^2$ does not depend on ε, and its definition involves the previous first order corrector f^1 defined in (20).

Then, a careful computation yields the following equation for the remainder

$$\begin{cases} \partial_t \widetilde{r}^\varepsilon + \dfrac{1}{\varepsilon}\left(v(k)\cdot\nabla_x \widetilde{r}^\varepsilon + \nabla_x V^\varepsilon \cdot \nabla_k \widetilde{r}^\varepsilon\right) \\ \qquad\qquad = \dfrac{1}{\varepsilon^2} N_{1-f^\varepsilon}(\widetilde{r}^\varepsilon) + \dfrac{1}{\varepsilon^2} N'_{(F(\rho)+\varepsilon \widetilde{f}^{1,\varepsilon}+\varepsilon^2 \widetilde{f}^2)}(\widetilde{r}^\varepsilon) + \varepsilon U^\varepsilon + W^\varepsilon, \\ \widetilde{r}^\varepsilon(t=0) = \widetilde{r}^\varepsilon_{in}. \end{cases} \quad (25)$$

The source term $U^\varepsilon(x,k,t)$ is now given by

$$U^\varepsilon = R(\widetilde{f}^{1,\varepsilon},\widetilde{f}^2) + R(\widetilde{f}^2,\widetilde{f}^{1,\varepsilon}) + \varepsilon R(\widetilde{f}^2,\widetilde{f}^2) - \varepsilon\partial_t \widetilde{f}^2 - v(k)\cdot\nabla_x \widetilde{f}^2 - \nabla_x V^\varepsilon\cdot\nabla_k \widetilde{f}^2,$$

while W^ε reads

$$\begin{aligned} W^\varepsilon = {} & v\cdot\nabla_x(f^1 - \widetilde{f}^{1,\varepsilon}) + \nabla_x(V - V^\varepsilon)\cdot\nabla_k f^1 + \nabla_x V^\varepsilon\cdot\nabla_k(f^1 - \widetilde{f}^{1,\varepsilon}) \\ & + R(\widetilde{f}^{1,\varepsilon},\widetilde{f}^{1,\varepsilon}) - R(f^1,f^1) - \varepsilon\partial_t \widetilde{f}^{1,\varepsilon}. \end{aligned} \quad (26)$$

As a first remark, we stress the fact that the source term $\varepsilon U^\varepsilon + W^\varepsilon$ in (25) does not contain (formally) singular terms. Next, in U^ε the only differentiated terms involve $\widetilde{f}^2$. Then, we can combine the regularity of the solutions of the auxiliary equations with respect to ρ, x, k as studied in [12], see Proposition 11, to the smoothness of $\rho(x,t)$, see Lemma 9. It leads to L^∞ estimates on $\partial^\alpha f^1$ and $\partial^\alpha \widetilde{f}^2$. Combining these informations with the natural estimates on V^ε and $\widetilde{f}^{1,\varepsilon}$, we can check that

$$\sup_{0\le t\le T} |U^\varepsilon(t)|_{L^1(\Omega\times B)} \le C.$$

The final remark is devoted to W^ε. The treatment of W^ε is not so easy, since the smoothness of $\widetilde{f}^{1,\varepsilon}$ is far from obvious and it is not clear at all that W^ε goes to 0 as $\varepsilon \to 0$. However, since R is bilinear and symmetric, we have

$$R(\widetilde{f}^{1,\varepsilon},\widetilde{f}^{1,\varepsilon}) - R(f^1,f^1) = R(\widetilde{f}^{1,\varepsilon} + f^1, \widetilde{f}^{1,\varepsilon} - f^1).$$

Hence, we realize that W^ε involves differences between $\widetilde{f}^{1,\varepsilon}$ and f^1, and between V and V^ε and ε times the time derivative of $\widetilde{f}^{1,\varepsilon}$. Restricting ourselves to the 1-Dimensional framework, we will establish that

$$\sup_{0\le t\le T} |W^\varepsilon(t)|_{L^1(\Omega\times B)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big)$$

holds. Here and below, C stands for a quantity which may vary from a line to another but remains independent on ε. We will see that this estimate suffices to conclude by using the Gronwall lemma.

From now on, we assume that the problem reduces to the one-dimensional Boltzmann equation (this occurs, for instance, when the cross-section and the initial data are invariant in the two other directions). Thus, Ω is a torus in $\mathbb{R}$ ($\Omega = \mathbb{R}/\mathbb{Z}$) and k_x, $v_x(k)$ denotes the first coordinates of the wave vector k and velocity $v(k)$. Equation (1) reads in this context

$$\begin{cases} \partial_t f^\varepsilon + \dfrac{1}{\varepsilon}\left(v_x(k)\partial_x f^\varepsilon + \partial_x V^\varepsilon \partial_{k_x} f^\varepsilon\right) = \dfrac{1}{\varepsilon^2} Q(f^\varepsilon) & \text{for } x \in \Omega,\ k \in B,\ t \in \mathbb{R}^+, \\ \partial_x^2 V^\varepsilon = \displaystyle\int_B f^\varepsilon(x,k,t)\ \mathrm{d}k - D(x) & \text{for } x \in \Omega,\ t \in \mathbb{R}^+, \\ f^\varepsilon(x,k,0) = f_{in}(x,k) & \text{for } x \in \Omega,\ k \in B. \end{cases}$$

Then, with the notation of the previous part, we have the following lemma.

Lemma 10 *There exists a constant $C > 0$ such that*

$$\begin{cases} \displaystyle\sup_{0\le t\le T} |U^\varepsilon(t)|_{L^1(\Omega\times B)} \le C \\ \displaystyle\sup_{0\le t\le T} |W^\varepsilon(t)|_{L^1(\Omega\times B)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big). \end{cases}$$

Note that we do not get the convergence to zero of the source term. However, Lemma 10 will be enough to conclude the proof by evoking the Gronwall Lemma.

PROOF. The key points in the proof rely on uniform estimates satisfied by V^ε and $V - V^\varepsilon$. First, the Poisson equation directly tells us that $\partial_x^2 V^\varepsilon = \rho^\varepsilon - D$ lies in a bounded set in $L^\infty(Q_T)$ and we have

$$|\partial_x V^\varepsilon|_{L^\infty(Q_T)} \le C.$$

(We note that the same estimate holds in any dimension by using Sobolev embedding). As a consequence, one obtains

$$|\widetilde{f}^{1,\varepsilon}|_{L^\infty(Q_T\times B)} \le C.$$

Moreover, f^1, $\widetilde{f}^2$ and their derivatives belong to L^∞. The estimate on U^ε follows easily.

Next, we use the following consequence of the Poisson equation

$$\begin{aligned}\partial_x^2(V-V^\varepsilon) &= \rho - \int_B f^\varepsilon \, \mathrm{d}k = \int_B (F(\rho)-f^\varepsilon)\, \mathrm{d}k \\ &= -\int_B \widetilde{r}^\varepsilon \, \mathrm{d}k - \varepsilon \int_B (\widetilde{f}^{1,\varepsilon} + \varepsilon \widetilde{f}^2)\, \mathrm{d}k.\end{aligned} \tag{27}$$

It yields, on the one hand

$$|\partial_x^2(V-V^\varepsilon)|_{L^1(\Omega)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big),$$

and, on the other hand,

$$|\partial_x(V-V^\varepsilon)|_{L^1(\Omega)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big).$$

By (20) and (24), we have

$$f^1 - \widetilde{f}^{1,\varepsilon} = \nu_1(\rho)\partial_x(V-V^\varepsilon),$$

so that, we get

$$|f^1 - \widetilde{f}^{1,\varepsilon}|_{L^1(\Omega\times B)} \le C|\partial_x(V-V^\varepsilon)|_{L^1(\Omega)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big).$$

Note that, by using the symmetry of the bilinear map R,

$$|R(\widetilde{f}^{1,\varepsilon},\widetilde{f}^{1,\varepsilon}) - R(f^1,f^1)|_{L^1(\Omega\times B)} \le C|\widetilde{f}^{1,\varepsilon} - f^1|_{L^1(\Omega\times B)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big).$$

Furthermore, we can also estimate

$$\begin{aligned}|\partial_{k_x}(f^1 - \widetilde{f}^{1,\varepsilon})|_{L^1(\Omega\times B)} &\le |\partial_x(V-V^\varepsilon)\ \partial_{k_x}\nu_1(\rho)|_{L^1(\Omega\times B)} \\ &\le C|\partial_x(V-V^\varepsilon)|_{L^1(\Omega)} \le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big),\end{aligned}$$

and

$$\begin{aligned}|\partial_x(f^1 - \widetilde{f}^{1,\varepsilon})|_{L^1(\Omega\times B)} &\le |\partial_x(V-V^\varepsilon)\ \partial_x\nu_1(\rho)|_{L^1(\Omega\times B)} \\ &\qquad + |\partial_x^2(V-V^\varepsilon)\ \nu_1(\rho)|_{L^1(\Omega\times B)} \\ &\le C\Big(|\partial_x(V-V^\varepsilon)|_{L^1(\Omega)} + |\partial_x^2(V-V^\varepsilon)|_{L^1(\Omega)}\Big) \\ &\le C\Big(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon\Big).\end{aligned}$$

It follows that the first four terms of (26) are bounded by $C(|\widetilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} + \varepsilon)$.

It remains to deal with $\partial_t \widetilde{f}^{1,\varepsilon}$. By differentiating (24), we obtain

$$\begin{aligned}|\partial_t \widetilde{f}^{1,\varepsilon}|_{L^1(\Omega\times B)} &\le |\partial_t(\lambda_1\partial_x\rho + \mu + qF'_\rho)|_{L^1(\Omega\times B)} + |\partial_t(\nu_1\partial_x V^\varepsilon)|_{L^1(\Omega\times B)} \\ &\le C\Big(1 + |\partial_t\partial_x V^\varepsilon|_{L^1(\Omega)}\Big).\end{aligned}$$

We can estimate the last term by using the Poisson equation again. Integration of (1) with respect to $k \in B$ gives the conservation relation

$$\partial_t \rho^\varepsilon + \partial_x j^\varepsilon = 0,$$

where the current j^ε is given by

$$j^\varepsilon = \frac{1}{\varepsilon} \int_B v(k) f^\varepsilon \, \mathrm{d}k = \int_B v(k) \, (\widetilde{f}^{1,\varepsilon} + \varepsilon \widetilde{f}^2) \, \mathrm{d}k + \frac{1}{\varepsilon} \int_B v(k) \, \widetilde{r}^\varepsilon \, \mathrm{d}k,$$

by using the null-flux assumption. Therefore the Poisson equation yields

$$\partial_t \partial_x^2 V^\varepsilon = \partial_x (\partial_t \partial_x V^\varepsilon) = \partial_t \rho^\varepsilon = -\partial_x j^\varepsilon$$

It follows that

$$|\partial_t \partial_x V^\varepsilon|_{L^1(\Omega)} \leq |j^\varepsilon|_{L^1(\Omega)} \leq C\Big(1 + \frac{1}{\varepsilon} |\widetilde{r}^\varepsilon|_{L^1(\Omega \times B)}\Big).$$

Hence, we conclude that

$$|\varepsilon \partial_t \widetilde{f}^{1,\varepsilon}|_{L^1(\Omega \times B)} \leq C\Big(\varepsilon + |\widetilde{r}^\varepsilon|_{L^1(\Omega \times B)}\Big).$$

This ends the proof of Lemma 10. □

The remainder of the proof follows now straightforwardly from [12] and the Gronwall lemma. We recall that $\partial_x V^\varepsilon$ and $\widetilde{f}^{1,\varepsilon}$ belong to bounded sets in $L^\infty(Q_T)$ and $L^\infty(\Omega \times B \times [0,T])$ respectively. According to [12], we have $F'(\rho) \geq \frac{\underline{\sigma}}{\overline{\sigma}}$ and we can choose the function $q(x,t)$ in (20) so that f^1 and $\widetilde{f}^{1,\varepsilon}$ remain non-negative. Repeating the reasoning for $\widetilde{f}^2$, we can choose $\widetilde{f}^2 \geq 0$ on $[0,T]$, by the addition of a constant times $F'(\rho)$. Hence, $F(\rho)+\varepsilon \widetilde{f}^{1,\varepsilon}+\varepsilon^2 \widetilde{f}^2 \geq 0$. Moreover, as stated in Proposition 2, $0 \leq f^\varepsilon \leq 1$. In turn, the kernels $\sigma(k,k')(1 - f^\varepsilon(k))$ and $\sigma(k',k)(F(\rho)(k) + \varepsilon \widetilde{f}^{1,\varepsilon}(k) + \varepsilon^2 \widetilde{f}^2(k))$ of the operators N_{1-f^ε} and $N'_{(F(\rho)+\varepsilon \widetilde{f}^{1,\varepsilon}+\varepsilon^2 \widetilde{f}^2)}$, respectively, are non-negative. Therefore, the classical L^1 estimate for the transport equation (25), together with Lemma 10 gives

$$\begin{aligned} |\widetilde{r}^\varepsilon(t)|_{L^1(\Omega \times B)} &\leq |\widetilde{r}^\varepsilon_{in}|_{L^1(\Omega \times B)} \\ &\qquad + \int_0^t \Big(\varepsilon |U^\varepsilon(s)|_{L^1(\Omega \times B)} + |W^\varepsilon(s)|_{L^1(\Omega \times B)}\Big)\, ds \\ &\leq |\widetilde{r}^\varepsilon_{in}|_{L^1(\Omega \times B)} + C\varepsilon + C \int_0^t |\widetilde{r}^\varepsilon(s)|_{L^1(\Omega \times B)}\, ds. \end{aligned} \tag{28}$$

The assumption on the preparation of the data means that the initial value

$$\widetilde{r}^\varepsilon_{in} = f_{in} - F(\rho_{in}) - \varepsilon(\widetilde{f}^{1,\varepsilon}|_{t=0} + \varepsilon \widetilde{f}^2|_{t=0}) \; = -\varepsilon(\widetilde{f}^{1,\varepsilon}|_{t=0} + \varepsilon \widetilde{f}^2|_{t=0})$$

has L^1 norm of order ε. By using the Gronwall lemma we conclude that

$$\sup_{0\le t\le T} |\tilde{r}^\varepsilon(t)|_{L^1(\Omega\times B)} \le C\,\varepsilon.$$

This completes the proof of Theorem 7. □

3 Existence and regularity of the solution of (8)

This section is devoted to the proofs of Proposition 8 and Lemma 9, namely, we justify existence and regularity of the solution of equation (8). Let $T>0$ be fixed; we recall the notation

$$Q_T = \Omega \times (0,T).$$

We firstly note that the coefficients $\Pi_{i,j}(\rho,x)$, $\Theta_{i,j}(\rho,x)$ and $\chi_j(\rho,x)$, defined by (9) are only defined on $[0,1]\times\Omega$, and satisfy

$$\Theta(\rho=0,x) = \Theta(\rho=1,x) \equiv 0\,, \qquad \text{and } \chi(\rho=0,x) = \chi(\rho=1,x) \equiv 0\,. \quad (29)$$

In order to deal within the general framework of the functional space $L^2(Q_T)$, we set $\Pi(\rho)=\Pi(0)$ if $\rho\le 0$, $\Pi(\rho)=\Pi(1)$ if $\rho\ge 1$, and we extend Θ and χ in the same way. We point out that this implies $\Theta(\rho)=0$ and $\chi(\rho)=0$, for all $\rho\notin[0,1]$. The following Proposition (the proof can be found in [12]) summarizes the properties of the coefficients.

Proposition 11 *(i)* $\Pi(\rho,x)$, $\Theta(\rho,x)$ *and* $\chi(\rho,x)$ *are Lipschitz continuous functions with respect to* $\rho\in\mathbb{R}$ *and measurable with respect to* $x\in\Omega$*. Moreover,* $\sigma(x,k,k')$ *being smooth with respect to* x*, we have the same regularity for the coefficients. Finally, these coefficients are smooth with respect to* ρ *in* $(0,1)$*.*
(ii) The coefficients $\Pi_{i,j}(\rho,x)$, $\Theta_{i,j}(\rho,x)$*, and* $\chi_j(\rho,x)$ *belong to* $L^\infty([0,1]\times\Omega)$*.*
(iii) $\Pi(\rho,x)$ *is a positive matrix, and there exists a positive constant* β *such that for all* $\xi\in\mathbb{R}^N$ *we have*

$$\Pi(\rho,x)\xi\cdot\xi \ge \beta|\xi|^2\,, \qquad \forall x\in\Omega\,,\ \forall\rho\in[0,1]\,. \quad (30)$$

Proof of Proposition 8. The existence part will be obtained by a two-step fixed-point procedure:

1) First, we solve a non-linear parabolic equation where both the potential V and the matrix Π are given; hence the nonlinearities appear only through Θ and χ. This step follows from a simple application of the Banach fixed point theorem, by using the fact that Θ and χ are Lipschitz functions with respect to ρ.

2) Second, we define, thanks to the first step, a mapping $\rho_1 \longmapsto \Pi(\rho_1), V(\rho_1) \longmapsto \rho$ and we show the existence of a fixed point by a Schauder argument. The regularity of this solution is the object of Lemma 9.

The following claim corresponds to the first step.

Lemma 12 *Let ρ_1 be a function in $L^2(Q_T)$ satisfying $0 \leq \rho_1(x,t) \leq 1$ and $\int_\Omega \rho_1(x,t)\, \mathrm{d}x = \int_\Omega \rho_{in}(x)\, \mathrm{d}x = \int_\Omega D(x)\, \mathrm{d}x$. Let $V \in L^\infty(0,T; H^2(\Omega))$ solves*

$$\Delta_x V(x,t) = \rho_1(x,t) - D(x). \tag{31}$$

Then there exists a unique solution $\rho(x,t) \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, of

$$\begin{cases} \partial_t \rho - div_x \left[\Pi(\rho_1)\nabla_x \rho + \Theta(\rho)\nabla_x V + \chi(\rho)\right] = 0 & \text{for } (x,t) \in Q_T \\ \rho(x,0) = \rho_{in}(x) & \text{for } x \in \Omega. \end{cases} \tag{32}$$

We point out that in equation (32), the matrix $\Pi(\rho_1)$ is fixed, so that the terms of order 2 are linear.

PROOF. The proof of Lemma 12 relies on the Banach fixed point theorem. Let $\widetilde{\rho}(x,t)$ in $L^2(Q_T)$, then there exists a unique $\rho(x,t) \in L^\infty(0,T;H^1(\Omega))$, solving the following parabolic linear equation

$$\begin{cases} \partial_t \rho - \mathrm{div}_x \left(\Pi(\rho_1)\nabla_x \rho\right) = \mathrm{div}_x \left[\Theta(\widetilde{\rho})\nabla_x V + \chi(\widetilde{\rho})\right] & \text{for } (x,t) \in Q_T, \\ \rho(x,0) = \rho_{in}(x) & \text{for } x \in \Omega. \end{cases}$$

We define the application $\Lambda : L^2(Q_T) \longmapsto L^2(Q_T)$ by $\Lambda(\widetilde{\rho}) = \rho$, and we assert that Λ is a contraction on $L^\infty(0,T;L^2(\Omega))$ endowed with a suitable norm.

First of all, we remark that since ρ_1 lies in $L^\infty(Q_T)$, it also belongs to $L^\infty(0,T;L^p(\Omega))$ for all $p \in [1,\infty]$. Therefore, $V(x,t)$ belongs to $L^\infty(0,T;W^{2,p}(\Omega))$ for all $p \in [1,\infty[$ (since V solves (31)). In particular, $\nabla_x V \in L^\infty(0,T;W^{1,p}(\Omega))$ for some $p > N$, and Sobolev's imbedding leads to $\nabla_x V \in L^\infty(Q_T)$.
Let now $\widetilde{\rho}$ and $\widetilde{\rho}'$ be two functions in $L^2(Q_T)$, we define $\rho = \Lambda(\widetilde{\rho})$ and $\rho' = \Lambda(\widetilde{\rho}')$. Then $\rho - \rho'$ solves

$$\begin{cases} \partial_t(\rho - \rho') - \mathrm{div}_x \left(\Pi(\rho_1)\nabla_x(\rho - \rho')\right) = \mathrm{div}_x \left[(\Theta(\widetilde{\rho}) - \Theta(\widetilde{\rho}'))\nabla_x V + \chi(\widetilde{\rho}) - \chi(\widetilde{\rho}')\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } (x,t) \in Q_T, \\ \rho(t=0) - \rho'(t=0) = 0 \qquad\qquad\qquad\qquad\qquad \text{for } x \in \Omega. \end{cases}$$

Multiplying by $\rho - \rho'$ and integrating with respect to $x \in \Omega$ we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t) - \rho'(t)|^2_{L^2(\Omega)} &+ \beta|\nabla_x(\rho - \rho')|^2_{L^2(\Omega)} \\
&\leq |(\Theta(\widetilde{\rho}) - \Theta(\widetilde{\rho}'))\nabla_x V + \chi(\widetilde{\rho}) - \chi(\widetilde{\rho}')|_{L^2(\Omega)}|\nabla_x(\rho - \rho')|_{L^2(\Omega)} \\
&\leq C_\beta|(\Theta(\widetilde{\rho}) - \Theta(\widetilde{\rho}'))\nabla_x V + \chi(\widetilde{\rho}) - \chi(\widetilde{\rho}')|^2_{L^2(\Omega)} + \frac{\beta}{2}|\nabla_x(\rho - \rho')|^2_{L^2(\Omega)}
\end{aligned}$$

by using (30), Cauchy-Schwarz and Young inequalities. Since Θ and χ are Lipshitz functions with respect to ρ, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t) - \rho'(t)|^2_{L^2(\Omega)} \leq C|\widetilde{\rho}(t) - \widetilde{\rho}'(t)|^2_{L^2(\Omega)}$$

holds where C is a constant depending on β, $|\nabla_x V|^2_{L^\infty(Q_T)}$ and the Lipschitz constants L_Θ, L_χ of the functions $\Theta(\rho)$ and $\chi(\rho)$, respectively. Integrating with respect to t, we get

$$\frac{1}{2}|\rho(t) - \rho'(t)|^2_{L^2(\Omega)} \leq C\int_0^t |\widetilde{\rho}(s) - \widetilde{\rho}'(s)|^2_{L^2(\Omega)}\, ds.$$

Let us introduce the following norm on $L^\infty(0, T; L^2(\Omega))$

$$|f|^2_b = \sup_{t\in(0,T)} \Big\{e^{-bt}|f(t)|^2_{L^2(\Omega)}\Big\}.$$

The above computation yields

$$\begin{aligned}
|\rho(t) - \rho'(t)|^2_{L^2(\Omega)} &\leq 2C\int_0^t e^{bs}e^{-bs}|\widetilde{\rho}(s) - \widetilde{\rho}'(s)|^2_{L^2(\Omega)}ds \\
&\leq 2C\int_0^t e^{bs}ds|\widetilde{\rho} - \widetilde{\rho}'|^2_b \leq C\frac{e^{bt} - 1}{b}|\widetilde{\rho} - \widetilde{\rho}'|^2_b
\end{aligned}$$

and therefore, we get

$$|\Lambda(\widetilde{\rho}) - \Lambda(\widetilde{\rho}')|^2_b \leq 2C\sup_{t\in(0,T)}\left(\frac{1 - e^{-bt}}{b}\right)|\widetilde{\rho} - \widetilde{\rho}'|^2_b \leq \frac{2C}{b}|\widetilde{\rho} - \widetilde{\rho}'|^2_b.$$

We now readily check that as soon as we have $b > 2C$, the application Λ is a contraction on $L^\infty(0, T; L^2(\Omega))$ for the norm $|\cdot|_b$. Therefore there exists a unique fixed point in $L^\infty(0, T; L^2(\Omega))$, which obviously is the unique solution $\rho(x, t)$ of equation (32). One also verifies easily that ρ belongs to $L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$. $\square$

Remark 13 *In view of (29), we check that* $\rho \equiv 0$ *and* $\rho \equiv 1$ *are respectively sub- and upper-solutions of equation (32). Therefore, the solution* ρ *given by Lemma (12) satisfies* $0 \leq \rho(x, t) \leq 1$ *for almost all* $(x, t) \in Q_T$*. We also have* $\int_\Omega \rho(x, t)\, \mathrm{d}x = \int_\Omega \rho_{in}(x)\, \mathrm{d}x$*.*

We can now deduce the existence part of Proposition 8. Let us introduce the following closed bounded convex subset of $L^2(Q_T)$

$$F = \left\{ g \in L^2(Q_T)\,;\, 0 \le g(x,t) \le 1 \text{ a.e. } x \in \Omega,\, t \in [0,T],\, \int_\Omega g(x,t)\,\mathrm{d}x = \int_\Omega D(x)\,\mathrm{d}x \right\}.$$

We define an application $\mathcal{F} : F \to F$ by $\mathcal{F}(\rho_1) = \rho$, where, for ρ_1 in F and $V(x,t)$ solving (31), ρ is the solution of (32) given by Lemma 12 (this solution still belongs to F, see Remark 13). Then we have the following lemma, which, together with the Schauder fixed point theorem easily yields the existence part of Proposition 8.

Lemma 14 *The application $\mathcal{F}$ is continuous and compact on F.*

PROOF. Let ρ_1 be in F, and $\rho = \mathcal{F}(\rho_1)$. We can estimate $\rho(x,t)$ by multiplying equation (32) by ρ, and integrating over Q_T. Precisely, one has

$$\begin{aligned} \tfrac{1}{2}|\rho(t)|^2_{L^2(\Omega)} + \tfrac{\beta}{2}|\nabla_x\rho|^2_{L^2(Q_T)} &\le C_\beta|\Theta(\rho)\nabla_x V + \chi(\rho)|^2_{L^2(Q_T)} + \tfrac{1}{2}|\rho_{in}|^2_{L^2(\Omega)} \\ &\le C_\beta|\Theta|^2_{L^\infty}|\nabla_x V|^2_{L^2(Q_T)} + \text{meas}(Q_T)^2|\chi|^2_{L^\infty} + \tfrac{1}{2}|\rho_{in}|^2_{L^2(\Omega)} \\ &\le C_\beta|\Theta|^2_{L^\infty}(|\rho_1|^2_{L^2(Q_T)} + |D|^2_{L^2(Q_T)}) \\ &\qquad + \text{meas}(Q_T)^2|\chi|^2_{L^\infty} + \tfrac{1}{2}|\rho_{in}|^2_{L^2(\Omega)}. \end{aligned} \tag{33}$$

Therefore, as ρ_1 lies in a bounded subspace of $L^2(Q_T)$, ρ belongs to a bounded subspace in $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, and Aubin's lemma provides the compactness of the application $\mathcal{F}$.

Let us now check that $\mathcal{F}$ is continuous for the $L^2(Q_T)$ norm. Let $\rho_{1,n}$ be a (strongly) convergent sequence in $L^2(Q_T)$ and set $\rho_n = \mathcal{F}(\rho_{1,n})$. Estimate (33) yields the boundedness of ρ_n in $L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$. Therefore, by using Aubin's lemma again, this sequence lies in a compact set of $L^2(Q_T)$ and, possibly at the cost of extracting subsequence, we can assume that

$$\begin{aligned} &\rho_{1,n} \xrightarrow[n\to\infty]{} \rho_1 \quad L^2(Q_T) \text{ strongly and almost everywhere,} \\ &\rho_n \xrightarrow[n\to\infty]{} \rho \quad\;\; L^2(Q_T) \text{ strongly and almost everywhere.} \end{aligned}$$

The continuity of Π and the almost everywhere convergence of $\rho_{1,n}$ gives

$$\Pi(\rho_{1,n}(x,t),x) \xrightarrow[n\to\infty]{} \Pi(\rho_1(x,t),x) \text{ almost everywhere.}$$

Since Π is bounded in L^∞, by applying the Lebesgue theorem we deduce that

$$\Pi(\rho_{1,n})\varphi \xrightarrow[n\to\infty]{} \Pi(\rho_1)\varphi \qquad L^2(Q_T) \text{ strong}$$

holds for all $\varphi(x,t) \in L^2(Q_T)$. Furthermore, since ρ_n is bounded in $L^2(0,T;H^1(\Omega))$, we also have

$$\nabla_x \rho_n \xrightarrow[n\to\infty]{} \nabla_x \rho \qquad L^2(Q_T) \text{ weak},$$

which yields

$$\Pi(\rho_{1,n})\nabla_x \rho_n \xrightarrow[n\to\infty]{} \Pi(\rho_1)\nabla_x \rho \qquad L^2(Q_T) \text{ weak}.$$

The same argument gives $\Theta(\rho_n)\nabla_x V_n \rightharpoonup \Theta(\rho)\nabla_x V$ and $\chi(\rho_n) \rightharpoonup \chi(\rho)$ in $L^2(Q_T)$. We deduce that ρ is the (unique) solution of (32), and therefore that $\rho = \mathcal{F}(\rho_1)$. By unicity of the solution of (32), a standard argument proves that the convergence applies to the whole sequence ρ_n, thus $\mathcal{F}$ is continuous. □

Before we prove the uniqueness of the so-obtained solution of equation (18), we have to investigate its regularity, as stated in Lemma 9. The proof of this Lemma essentially relies on results proved in the classical book of O.A. Ladyzenskaya-V.A Solonnikov-N.N. Ural'ceva [9]. Before beginning the proof, let us recall the definition of some Hölder spaces used in [9]. Let $\mathcal{H}^{l,\frac{l}{2}}$ be the space of functions $f(x,t)$ such that for all $r \in \mathbb{N}$ and $s \in \mathbb{N}^N$ satisfying $2r + |s| \le l$, we have $D_t^r D_x^s f \in L^\infty(Q_T)$, and for all $r \in \mathbb{N}$ and $s \in \mathbb{N}^N$ satisfying $2r+|s| = l$, the function $D_t^r D_x^s f$ is $(l-[l])$-Hölder with respect to x, and $\frac{l-[l]}{2}$-Hölder function with respect to t. We are now ready to prove Lemma 9.

Proof of Lemma 9. Throughout the proof, ρ and V will denote the solutions of equation (8), obtained in Proposition 8. First of all, ρ can be viewed as the solution of the following linear equation

$$\partial_t \rho - \operatorname{div}_x \left[\widetilde{\Pi}(x,t)\nabla_x \rho\right] = \operatorname{div}_x \left[\widetilde{\Theta}(x,t)\nabla_x V + \widetilde{\chi}(x,t)\right], \tag{34}$$

where the coefficients are actually defined by $\widetilde{\Pi}(x,t) = \Pi(\rho(x,t),x)$, $\widetilde{\Theta}(x,t) = \Theta(\rho(x,t),x)$ and $\widetilde{\chi}(x,t) = \chi(\rho(x,t),x)$. Note that the regularity of the coefficients appearing in equation (34) depends on that of $\rho(x,t)$ and $V(x,t)$.

Certainly, the coefficients belong to $L^\infty(Q_T)$. Then, Theorem 4.2, Chapter 3 in [9] yields that the generalized solution $\rho(x,t)$ given by Proposition 8 belongs to $\mathcal{C}^0(0,T;L^2(\Omega))$. We can therefore apply Theorem 12.1, Chapter 3 in [9] which says that $\rho(x,t)$, together with $\nabla_x \rho$ belong to $\mathcal{H}^{l,\frac{l}{2}}$ for some $l > 0$. Finally, Theorem 5.2, Chapter 4 in [9] states that, as soon as the coefficients, together with their derivatives, belong to $\mathcal{H}^{l,\frac{l}{2}}$ with $l \ge 0$, then the solution $\rho(x,t)$ belongs to $\mathcal{H}^{l+2,\frac{l}{2}+1}$.

In turn, we readily check (see [10]) that, if $\rho \in \mathcal{H}^{l,\frac{l}{2}}$, then the solution $V(x,t)$, of (31) satisfies $D_x^s V \in \mathcal{H}^{l,\frac{l}{2}}$ for all $s \in \mathbb{N}^N$ such that $|s| \le 2$. Therefore, as long as the derivative of Π, Θ and χ with respect to x are bounded in $[0,1]\times\Omega$, and that of ρ_{in}, are bounded in Ω, we can increase the regularity of the solution $\rho(x,t)$ of equation (8). $\square$

We can now obtain the proof of Proposition 8, by proving the uniqueness of the solution. In that purpose we remark that, as a consequence of the above Lemma 9, for all solution ρ of (18), $\nabla_x\rho$ and $\nabla_x V$ belong to $L^\infty(Q_T)$.

Let ρ and ρ' be two solutions of equation (18). Then, we have

$$\begin{cases} \partial_t(\rho-\rho') - \mathrm{div}_x\left(\Pi(\rho')\nabla_x(\rho-\rho')\right) = \mathrm{div}_x\left[(\Pi(\rho)-\Pi(\rho'))\nabla_x\rho + \Theta(\rho')\nabla_x(V-V')\right. \\ \qquad\qquad\qquad\qquad \left. +(\Theta(\rho)-\Theta(\rho'))\nabla_x V + \chi(\rho)-\chi(\rho')\right], \\ \qquad\qquad\qquad\qquad \text{for } x\in\Omega,\ t\in[0,+\infty[\\ \rho(t=0)-\rho'(t=0)=0 \qquad \text{for } x\in\Omega. \end{cases}$$

By standard manipulation, one obtains the following estimate

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)-\rho'(t)|^2_{L^2(\Omega)} + \beta|\nabla_x(\rho-\rho')|^2_{L^2(\Omega)} \\ &\qquad \le C_\beta\Big[|(\Pi(\rho)-\Pi(\rho'))|^2_{L^2(\Omega)}|\nabla_x\rho|^2_{L^\infty(Q_T)} + |(\Theta(\rho)-\Theta(\rho'))\nabla_x V|^2_{L^2(\Omega)} \\ &\qquad +|\Theta(\rho')\nabla_x(V-V')|^2_{L^2(\Omega)} + |\chi(\rho)-\chi(\rho')|^2_{L^2(\Omega)}\Big] + \frac{\beta}{2}|\nabla_x(\rho-\rho')|^2_{L^2(\Omega)}. \end{aligned}$$

Since Π, Θ and χ are Lipshitz functions, one deduces that

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho(t)-\rho'(t)|^2_{L^2(\Omega)} \le C|\rho(t)-\rho'(t)|^2_{L^2(\Omega)}\Big[|\nabla_x V|^2_{L^\infty(Q_T)} + |\nabla_x\rho|^2_{L^\infty(Q_T)} + C\Big]$$

holds. However, the initial condition $\rho(t=0)-\rho'(t=0)=0$ vanishes so that Gronwall's lemma leads to $|\rho(t)-\rho'(t)|^2_{L^2(\Omega)}=0$ for all $t\in[0,T]$. This completes the proof of Proposition 8, and ends this section. $\square$

A Appendix - Existence of equilibrium states.

This section is devoted to the proof of Proposition 3. For the sake of simplicity, throughout this section we shall drop the space dependence. We aim at justifying the existence of a family of equilibrium states parametrized by the mass, namely $\{F_\rho : B \to \mathbb{R},\ \rho \in [0,1]\}$ such that

$$Q(F_\rho) = 0, \qquad \int_B F_\rho \, \mathrm{d}k = \rho.$$

In this Section, the kernel σ is supposed to satisfy the following properties:

$(A1)$ σ is a measurable function defined on $B \times B$ with value in $\mathbb{R}$ such that

$$0 < \sigma(k,k') \qquad \text{for almost all } k, k',$$

$(A2)$ There exists a constant $\overline{\sigma} > 0$ such that

$$\sup_{k\in B} \int_B \sigma(k,k') \, \mathrm{d}k' \le \overline{\sigma}, \qquad \sup_{k\in B} \int_B \sigma(k',k) \, \mathrm{d}k' \le \overline{\sigma},$$

$(A3)$ For any measurable set $\mathcal{A} \subset B$ such that $\mathrm{meas}(\mathcal{A}) > 0$, one defines

$$\begin{cases} \underline{\Sigma}(\mathcal{A}) = \inf_{k\in B} \int_{\mathcal{A}} \sigma(k,k') \, \mathrm{d}k', & \overline{\Sigma}(\mathcal{A}) = \sup_{k\in B} \int_{\mathcal{A}} \sigma(k,k') \, \mathrm{d}k', \\ \underline{\Sigma}_*(\mathcal{A}) = \inf_{k\in B} \int_{\mathcal{A}} \sigma(k',k) \, \mathrm{d}k', & \overline{\Sigma}_*(\mathcal{A}) = \sup_{k\in B} \int_{\mathcal{A}} \sigma(k',k) \, \mathrm{d}k'. \end{cases}$$

Then, there exists a constant $\underline{\sigma} > 0$ satisfying for any such set $\mathcal{A}$,

$$\underline{\Sigma}_*(\mathcal{A})/\overline{\Sigma}(\mathcal{A}) \ge \underline{\sigma}, \qquad \underline{\Sigma}(\mathcal{A})/\overline{\Sigma}_*(\mathcal{A}) \ge \underline{\sigma},$$

$(A4)$ There exists a measurable negligeable set $N \subset B$ such that the families

$\{\sigma(k,\cdot),\ k \in B\backslash N\}$ and $\{\sigma(\cdot,k),\ k \in B\backslash N\}$ are relatively compact in $L^1(B)$.

In this general framework, we are able to show the existence of equilibrium states parametrized by the mass.

Theorem 15 *Suppose (A1-A4). Then, for any $\rho \in [0,1]$, there exists a unique $0 \le F_\rho \le 1$ verifying $\int_B F_\rho \, \mathrm{d}k = \rho$ and $Q(F_\rho) = 0$.*

Remark 16 *The assumptions introduced above need some comments:*

- *Since* $\mathrm{meas}(B) < \infty$, *the embeddings $L^p(B) \subset L^1(B)$ holds for any $1 \le p \le \infty$.*
- *Assumption (A2) implies that, for any $f \in L^\infty(B)$, $Q(f)$ is well defined in $L^\infty(B)$ (with $\|Q(f)\|_\infty \le 2\overline{\sigma}\|f\|_\infty(1 + \|f\|_\infty)$).*
- *Assumption (A3) is a non-degeneracy condition of the collision kernels. In particular, it is fulfilled when $0 < \sigma_1 \le \sigma(k,k') \le \sigma_2$ (with $\underline{\sigma} = \sigma_1/\sigma_2$), as*

assumed in Hypothesis 1.

- *According to the classical Weil criterion (see [5], Th. 4.20.1), assumption (A4) reads*

$$\begin{cases} \text{For any } \varepsilon > 0, \text{ there exists } \eta_\varepsilon > 0 \text{ such that, for any } h \in \mathbb{R}^N \text{ and a.a. } k \in B, \\ \text{if } |h| \le \eta_\varepsilon, \text{ then} \\ \displaystyle\int_{\mathbb{R}^N} |\sigma(k, k'+h) - \sigma(k,k')| \, \mathrm{d}k' < \varepsilon, \quad \int_{\mathbb{R}^N} |\sigma(k'+h, k) - \sigma(k', k)| \, \mathrm{d}k' < \varepsilon \end{cases} \tag{A.1}$$

(where σ has been extended by 0 out of $B \times B$ and $\mathrm{d}k$ stands for the Lebesgue measure divided by $\mathrm{meas}(B)$*).*

- *Assumption (A4) is fulfilled if one assumes*

$$\sigma(k,k') \in C^0(B_k; L^1(B_{k'})) \cap C^0(B_{k'}; L^1(B_k)).$$

In particular all assumptions (A1-A4) are satisfied when $\sigma \in C^0(B \times B)$. In this case, the equilibrium states F_ρ are continuous on B.

Let us introduce the following mapping

$$\Phi : \mathbb{R} \times L^\infty(B) \longrightarrow \mathbb{R} \times L^\infty(B)$$
$$\begin{pmatrix} \rho \\ f \end{pmatrix} \longmapsto \begin{pmatrix} \displaystyle\int_B f \, \mathrm{d}k - \rho \\ Q(f) \end{pmatrix}$$

defined on $[0,1] \times \{f \in L^\infty(B),\ 0 \le f \le 1\}$ with values in $[-1,1] \times L_0^\infty$. Here, L_0^∞ stands for the subspace

$$L_0^\infty = \Big\{ f \in L^\infty(B),\ \int_B f \, \mathrm{d}k = 0 \Big\}.$$

Determination of equilibrium state F having mass ρ reduces to searching for the zeros of Φ. First, note that $\Phi(0,0) = (0,0) = \Phi(1,1)$. We shall use the implicit function theorem to construct a family $\{(\rho, F_\rho),\ \rho \in [0,1]\}$ such that

$$\begin{cases} F_0 = 0, \qquad F_1 = 1, \\ \Phi(\rho, F_\rho) = (0,0). \end{cases}$$

furthermore, we will obtain that $0 \le F_\rho \le 1$ (precisely $0 < F_\rho < 1$ for $\rho \in (0,1)$).

The derivative of Φ with respect to f reads

$$\frac{\partial \Phi}{\partial f}(\rho, f)(h) = \begin{pmatrix} \displaystyle\int_B h \, \mathrm{d}k \\ L_f(h) \end{pmatrix}$$

where L_f is the operator from $L^\infty(B)$ to L_0^∞ defined by

$$\begin{cases} L_f(h) = K_f(h) - \nu_f(k)h, \\ K_f(h) = \int_B s_f(k,k')h(k')\ \mathrm{d}k', \\ \nu_f(k) = \int_B s_f(k',k)\ \mathrm{d}k'. \end{cases} \tag{A.2}$$

The kernel s_f depends on the function f as follows

$$s_f(k,k') = \sigma(k,k')(1-f(k)) + \sigma(k',k)f(k).$$

For $0 \le f \le 1$, notice that s_f is nothing but the barycenter of $\sigma(k,k')$ and $\sigma(k',k)$. We aim at proving that $\frac{\partial\Phi}{\partial f}$ is inversible from $L^\infty(B)$ to $\mathbb{R} \times L_0^\infty$. Let $(a,q) \in \mathbb{R} \times L_0^\infty(B)$. We should prove existence-uniqueness of $h \in L^\infty(B)$ verifying

$$\begin{pmatrix} \int_B h\ \mathrm{d}k \\ L_f(h) \end{pmatrix} = \begin{pmatrix} a \\ q \end{pmatrix}.$$

By linearity, and using $\int_B \mathrm{d}k = 1$, the problem reduces to the invertibility of L_f on L_0^∞.

Lemma 17 *For any f verifying $0 \le f \le 1$, the operator $L_f : L_0^\infty \longrightarrow L_0^\infty$ is invertible.*

The proof uses the following claim.

Lemma 18 *Let f verifying $0 \le f \le 1$. Then*

i) There exist constants $\overline{\nu} \ge \underline{\nu} > 0$ such that for a.a. $k \in B$,

$$0 < \underline{\nu} \le \nu_f(k) \le \overline{\nu} < \infty$$

(and $\overline{\nu}$ does not depend on f).

ii) The integral operator

$$T_f g(k) = \int_B t_f(k,k')g(k')\ \mathrm{d}k', \qquad t_f(k,k') = s_f(k,k')/\nu_f(k')$$

is compact on $L^\infty(B)$.

Proof of Lemma 17. Let $g(k) = \nu_f(k)h(k)$. The equation $L_f(h) = K_f(h) - \nu_f h = q$ recasts $(T_f - I)g = q$ where T_f is the integral operator involved in Lemma 18. One deduces from Lemma 18-ii) that $\mathrm{Ran}(T_f - I) = [\mathrm{Ker}(T_f^* - I)]^\perp$ where

$$T_f^*\phi(k) = \int_B t_f(k',k)\phi(k')\ \mathrm{d}k' = \int_B s_f(k',k)\ \left(\nu_f(k)\right)^{-1}\phi(k')\ \mathrm{d}k'.$$

Moreover, by (A1) and Lemma 18-ii), the Krein-Rutman Theorem (see [15], Th. 6.6) applies to T_f^*. The spectral radius r is a positive eigenvalue, associated to a eigenfunction > 0. The associated eigenspace has dimension 1 and the other eigenvalue have a modulus $< r$. Next, (A.2) gives $T_f^*(1) = 1$. Therefore, one obtains

$$\mathrm{Ker}(T_f^* - I) = \mathbb{R}, \qquad \mathrm{Ran}(T_f - I) = L_0^\infty.$$

This proves, for $q \in L_0^\infty$, the existence of a solution $g \in L^\infty(B)$ of $(T_f - I)g = q$ (respecively, with Lemma 2-i, $h \in L^\infty(B)$ solution of $L_f h = q$).

The Krein-Rutman Theorem also garantees that $T_f - I$ has a mono-dimensional kernel, spanned by a normalized eigenfunction $G_f > 0$. Hence, solutions $g \in L^\infty(B)$ of $(T_f - I)g = q$ can be written $g = g_0 + \alpha G_f$, $\alpha \in \mathbb{R}$, and the condition $h = g/\nu_f \in L_0^\infty$ yields uniqueness since $\int_B G_f/\nu_f \, \mathrm{d}k \neq 0$. Therefore,the operator L_f is invertible on L_0^∞. $\square$

Proof of Lemma 18. For $f = 0$, by using (A1-A3), one gets

$$0 < \underline{\sigma}\ \overline{\Sigma}(B) \leq \underline{\Sigma}_*(B) \leq \nu_0(k) \leq \overline{\sigma}.$$

A similar reasoning applies to $f = 1$.

Let $0 \leq f \leq 1$, with $f \neq 0$, $f \neq 1$. Then, there exist $\delta > 0$ and a measurable set $\mathcal{A}_\delta \subset B$ such that $f(k') \geq \delta$ on $\mathcal{A}_\delta$ and $\mathrm{meas}(\mathcal{A}_\delta) > 0$. It follows that

$$2\overline{\sigma} \geq \nu_f(k) \geq \delta\ \underline{\Sigma}(\mathcal{A}_\delta) > 0,$$

holds, still by using (A1-A3). This justifies Part i) of the statement.

Let $\varphi \in L^\infty(B)$ (which is destined to represent f or $1-f$). We shall show that the integral operator with kernel $\sigma(k,k')\varphi(k)\Big(\nu_f(k')\Big)^{-1}$ is compact on $L^\infty(B)$. According to the characterization given by Eveson [6], we have to establish that the set

$$\Big\{\frac{\sigma(k,k')\varphi(k)}{\nu_f(k')},\ k \in B\backslash N\Big\},$$

where $N \subset B$ is a negligeable set, is relatively compact in $L^1(B)$. Of course, the same reasoning applies to the operator with kernel $\sigma(k',k)\varphi(k)\Big(\nu_f(k')\Big)^{-1}$. Let $h \in \mathbb{R}^N$. Extending functions by 0 outside of B, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left| \frac{\sigma(k,k'+h)\varphi(k)}{\nu_f(k'+h)} - \frac{\sigma(k,k')\varphi(k)}{\nu_f(k')} \right| \, \mathrm{d}k' \\
&\qquad \leq \|\varphi\|_\infty \int_{\mathbb{R}^N} \left| \frac{\sigma(k,k'+h)\nu_f(k') - \sigma(k,k')\nu_f(k'+h)}{\nu_f(k'+h)\ \nu_f(k')} \right| \, \mathrm{d}k' \\
&\qquad \leq \frac{\|\varphi\|_\infty}{\underline{\nu}^2}\Big(I(k,h) + J(k,h)\Big)
\end{aligned}$$

where $\underline{\nu}$ is the bound from below on $\nu_f(k)$ obtained in Part i) and we have set

$$I(k,h) = \int_{\mathbb{R}^N} |\sigma(k,k'+h) - \sigma(k,k')| \ \nu_f(k') \ \mathrm{d}k',$$
$$J(k,h) = \int_{\mathbb{R}^N} \sigma(k,k') \ |\nu_f(k'+h) - \nu_f(k')| \ \mathrm{d}k'.$$

Let $\varepsilon > 0$. We shall exhibit $\eta_\varepsilon > 0$ such that, if $|h| \leq \eta_\varepsilon$, then for a.a. $k \in B$, $I(k,h) \leq \varepsilon$ and $J(k,h) \leq \varepsilon$. Estimation of I is a direct consequence of (A.1) since

$$\sup_{k\in B} I(k,h) \leq \overline{\nu} \ \sup_{k\in B} \int_{\mathbb{R}^N} |\sigma(k,k'+h) - \sigma(k,k')| \ \mathrm{d}k' \xrightarrow[|h|\to 0]{} 0.$$

To treat J, let us introduce the set $E_M(k) = \{k' \in \mathbb{R}^N, \ \sigma(k,k') \geq M\}$ where $M \geq 0$. We denote $F_M(k) = \complement(E_M(k))$ and we split

$$J(k,h) = \int_{E_M(k)} ... \ \mathrm{d}k' + \int_{F_M(k)} ... \ \mathrm{d}k' = K(k,h,M) + L(k,h,M).$$

The integral on $E_M(k)$ is estimated by

$$K(k,h,M) \leq 2\overline{\nu} \int_{E_M(k)} \sigma(k,k') \ \mathrm{d}k'.$$

Then, the compactness assumption (A4) guarantees in particular that the equi-integrability criterion

$$\sup_{k\in B} \int_{E_M(k)} \sigma(k,k') \ \mathrm{d}k' \xrightarrow[M\to\infty]{} 0$$

is fulfilled. Thus, we can choose $M = M_\varepsilon$ large enough to obtain $\sup_{k,h} K(k,h,M) < \varepsilon/2$. It remains to deal with the integral on $F_M(k)$. By definition of ν_f, with $0 \leq f \leq 1$, we get

$$\begin{aligned}
&|\nu_f(k'+h) - \nu_f(k')| \\
&= \left| \int_B \left\{ \Big(\sigma(k'',k'+h) - \sigma(k'',k')\Big) \ \Big(1 - f(k'')\Big) + \Big(\sigma(k'+h,k'') - \sigma(k',k'')\Big) \ f(k'') \right\} \ \mathrm{d}k'' \right| \\
&\leq \int_B |\sigma(k'',k'+h) - \sigma(k'',k')| \ \mathrm{d}k'' + \int_B |\sigma(k'+h,k'') - \sigma(k',k'')| \ \mathrm{d}k''.
\end{aligned}$$

Hence, we are led to

$$\begin{aligned}
L(k,h,M) &\leq \int_{F_M(k)} \int_B \sigma(k,k') \ |\sigma(k'',k'+h) - \sigma(k'',k')| \ \mathrm{d}k'' \ \mathrm{d}k' \\
&\qquad + \int_{F_M(k)} \int_B \sigma(k,k') \ |\sigma(k'+h,k'') - \sigma(k',k'')| \ \mathrm{d}k'' \ \mathrm{d}k' \\
&\leq M \ \int_B \mathrm{d}k'' \\
&\quad \times \left(\sup_{k''\in B} \int_{\mathbb{R}^N} |\sigma(k'',k'+h) - \sigma(k'',k')| \ \mathrm{d}k' + \sup_{k''\in B} \int_{\mathbb{R}^N} |\sigma(k'+h,k'') - \sigma(k',k'')| \ \mathrm{d}k' \right).
\end{aligned}$$

By (A.1), we can find $\eta = \eta_\varepsilon$ such that, for $|h| \le \eta_\varepsilon$, this quantity is $< \varepsilon/2$, uniformly with respect to $k \in B$. This ends the proof of ii). $\square$

In particular, Lemma 17, applies to $f(k) = 0$ (and of course also to $f(k) = 1$). Then, the implicit function theorem jusitifies the existence of an application $\rho \longmapsto F_\rho$, belonging to class C^1 on a certain interval $]-\rho_0, \rho_0[$, with values in $L^\infty(B)$, and such that F_ρ satisfies

$$\int_B F_\rho \, \mathrm{d}k = \rho, \qquad Q(F_\rho) = 0. \tag{A.3}$$

It remains to show that the function F_ρ verifies $0 \le F_\rho \le 1$, at least for "small enough" $\rho \ge 0$, and then to extend the function on the whole interval $\rho \in [0, 1]$. To this purpose, we will use the following remarks.

Remark 19 *By derivating (A.3) with respect to ρ, one is led to the following remarkable identities*

$$\int_B F'_\rho \, \mathrm{d}k = 1 \qquad et \qquad D_{F_\rho}(F'_\rho) = 0,$$

where $F'_\rho = \frac{\partial F_\rho}{\partial \rho}$. Thus, $F'_\rho(k)$ is a normalized eigenfunction of the operator D_{F_ρ}. Arguments used in the proof of Lemma 17, allows us to deduce that $F'_\rho(k) > 0$ for a.a. $k \in B$, as soon as $0 \le F_\rho \le 1$.

Remark 20 *The chemical potential associated to a function F is given by*

$$\Pi(F) = \frac{1 - F}{F}.$$

Then, the relation $Q(F) = 0$ leads to

$$\Pi(F)(k) = \frac{\displaystyle\int_B \sigma(k', k)(1 - F(k')) \, \mathrm{d}k'}{\displaystyle\int_B \sigma(k, k') F(k') \, \mathrm{d}k'}. \tag{A.4}$$

Now, we can complete the proof of Theorem 15. Since $\rho \longmapsto F_\rho$ is continuous, with values in $L^\infty(B)$, and $F_0 = 0$, we have $|F_\rho(k)| < 1$ for ρ small enough. Moreover, $\rho \longmapsto F_\rho$ being C^1, we have $F_\rho(k) = \rho(F'_0(k) + \varepsilon(\rho, k))$, with $\lim_{\rho \to 0} \|\varepsilon(\rho, \cdot)\|_\infty = 0$. If we can find some $\delta_0 > 0$ such that $F'_0(k) \ge \delta_0$ almost everywhere on B, then, we can deduce that $F_\rho(k) \ge \rho\delta_0/2 > 0$ for $\rho > 0$ small enough.

However, by definition, $F'_0 > 0$ satisfies (see the proof of Lemma 17)

$$K_0(F'_0)(k) = \nu_0(k) F'_0(k) = \int_B \sigma(k, k') F'_0(k') \, \mathrm{d}k' = \int_B \sigma(k', k) \, \mathrm{d}k' \, F'_0(k).$$

Suppose that for any $\delta > 0$, there exists a measurable set $\mathcal{A}_\delta \subset B$ such that $\mathrm{meas}(\mathcal{A}_\delta) > 0$ and $0 < F_0'(k) \leq \delta$ on $\mathcal{A}_\delta$. Then, for $k \in \mathcal{A}_\delta$, one has

$$\int_B \sigma(k,k')F_0'(k')\ \mathrm{d}k' \leq \delta \int_B \sigma(k',k)\ \mathrm{d}k'.$$

Integrate this relation over $\mathcal{A}_\delta$. Since $\int_B F_0'\ \mathrm{d}k = 1$, one obtains

$$\underline{\Sigma}_*(\mathcal{A}_\delta) \leq \int_B \left(\int_{\mathcal{A}_\delta} \sigma(k,k')\ \mathrm{d}k\right) F_0'(k')\ \mathrm{d}k' \leq \delta \int_B \left(\int_{\mathcal{A}_\delta} \sigma(k',k)\ \mathrm{d}k\right)\ \mathrm{d}k' \leq \delta\overline{\Sigma}(\mathcal{A}_\delta).$$

This contradicts (A3).

In this way, we construct for $\rho \geq 0$ in a neighborhood of 0, $F_\rho \in L^\infty(B)$ solving (A.3) and verifying $0 \leq F_\rho \leq 1$. Note also that $\rho \longmapsto F_\rho(k)$ is non decreasing (and bounded from below by a positive constant when $\rho > 0$). Furthermore, as soon as $0 \leq F_\rho \leq 1$, we can apply the implicit function theorem and extend the application $\rho \longmapsto F_\rho$. Let $\rho_0 \in (0,1]$ such that $[0,\rho_0]$ is the maximal interval on which F_ρ remains between 0 and 1. To complete the proof, we shall use the following argument.

Lemma 21 *If $0 \leq F_\rho \leq 1$ and $\rho \neq 1$, then, there exists $\delta > 0$ such that $F_\rho \leq 1-\delta$ almost everywhere on B.*

Before the proof of this claim, let us end the proof of Theorem 15. Since $0 \leq F_{\rho_0} \leq 1$, the operator $D_{F_{\rho_0}}$ is invertible. Hence, we can extend the application $\rho \longmapsto F_\rho$ on the interval $[0,\rho_1)$ for a certain $\rho_1 > \rho_0$. Suppose $\rho_0 \neq 1$. The function $\rho \longmapsto \Pi(F_\rho)$ lies in $C^0((0,\rho_1];L^\infty(B))$ (and is non increasing). By using Lemma 21, we check that $\Pi(F_{\rho_0}) \geq \frac{\delta}{1-\delta} > 0$. Then, by continuity with respect to ρ, the chemical potential $\Pi(F_\rho)$ remains non negative on an interval $\rho \in [\rho_0,\rho_1']$, $\rho_0 < \rho_1' \leq \rho_1$. Since $F_\rho \geq 0$, we have $F_\rho \leq 1$ for $\rho \in [0,\rho_1')$, where $\rho_1' > \rho_0$, which contradicts the definition of ρ_0. We conclude that $\rho_0 = 1$ and the proof of Theorem 15 is now finished. $\square$

Proof of Lemma 21. Suppose that for any $\delta > 0$, there exists a mesurable set $\mathcal{A}(\delta) \subset B$ such that $\mathrm{meas}(\mathcal{A}(\delta)) > 0$ and $F_\rho(k) \geq 1-\delta$ for all $k \in \mathcal{A}(\delta)$. Then, the left hand side in (A.4) leads to $0 \leq \Pi(F_\rho)(k) \leq \frac{\delta}{1-\delta}$ for $k \in \mathcal{A}(\delta)$. On the other hand, by using the right hand side in (A.4), one obtains

$$0 \leq \int_B \sigma(k',k)(1-F_\rho(k'))\ \mathrm{d}k' \leq \frac{\delta}{1-\delta}\int_B \sigma(k,k')F_\rho(k')\ \mathrm{d}k',$$

still for $k \in \mathcal{A}(\delta)$. Integration over $\mathcal{A}(\delta)$ yields

$$\begin{aligned}0 \leq \underline{\Sigma}_*(\mathcal{A}(\delta))(1-\rho) &\leq \int_B \left(\int_{\mathcal{A}(\delta)} \sigma(k', k)\ \mathrm{d}k \right) (1 - F_\rho)(k'))\ \mathrm{d}k' \\ &\leq \frac{\delta}{1-\delta} \int_B \left(\int_{\mathcal{A}(\delta)} \sigma(k, k')\ \mathrm{d}k \right) F_\rho(k')\ \mathrm{d}k' \\ &\leq \frac{\delta}{1-\delta}\ \overline{\Sigma}(\mathcal{A}(\delta))\ \rho.\end{aligned}$$

However, we have supposed meas$(\mathcal{A}(\delta)) > 0$; thus (A2) leads to

$$0 \leq \underline{\sigma}(1-\rho) \leq \frac{\delta\rho}{1-\delta}.$$

Letting δ go to 0, one is led to $\rho = 1$ which concludes the proof. □

Regularity of F with respect to ρ, see Lemma 5 is obtained reasoning by induction on the formulae satisfied by $\partial_\rho^n F(\rho)$. We have $\int_B \partial_\rho^n F(\rho)\ \mathrm{d}k = 0$ while $L_{F(\rho)}(\partial_\rho^n F(\rho))$ equals a term depending on the derivatives of order $< n$. □

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