Hydrodynamic Limit for the Vlasov-Navier-Stokes Equations. Part II: Fine Particles Regime

Thierry GOUDON[1], Pierre-Emmanuel JABIN[2] and Alexis VASSEUR[3]

[1] Labo. Paul Painlevé, UMR 8524
CNRS-Université des Sciences et Technologies de Lille
Cité Scientifique
F-59655 Villeneuve d'Ascq cedex , France
thierry.goudon@univ-lille1.fr

[2] Département de Mathématiques et Applications, ENS
45, rue d'Ulm,
F-75232 Paris
jabin@dma.ens.fr

[3] Labo. J.A. Dieudonné, UMR 6621
Université Nice-Sophia Antipolis, Parc Valrose
F-06108 Nice cedex 02
vasseur@math.unice.fr

Abstract

The paper is devoted to the analysis of a hydrodynamic limit for the Vlasov-Navier-Stokes equations. This system is intended to model the evolution of particles interacting with a fluid. The coupling arises from the force terms. The limit problem is the Navier-Stokes system with non constant density. The density which is involved in this system is the sum of the (constant) density of the fluid and of the macroscopic density of the particles. The proof relies on a relative entropy method.

Key words. Fluid-particles interaction. Vlasov-Navier-Stokes equation. Hydrodynamic limits. Relative entropy method.

AMS Subject classification. 35Q99 35B25

1 Introduction

In this paper, we investigate models of particles dispersed in an incompressible viscous fluid. The particles are described through a density function $f(t,x,v) \geq 0$. The evolution of f is governed by the following Vlasov-type equation

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v(Ff) = r\Delta_v f. \tag{1.1}$$

The particles evolve in a fluid which is described by its velocity field $u(t,x) \in \mathbb{R}^N$. The cloud of particules is assumed highly dilute so that we can suppose that the density of the gas remains constant. Accordingly, u verifies the following incompressible Navier-Stokes equation

$$\begin{cases} \partial_t u + \mathrm{Div}_x(u \otimes u) + \nabla_x p - \Delta_x u = \mathfrak{F}, \\ \mathrm{div}_x(u) = 0. \end{cases} \tag{1.2}$$

Here and below, for $u = (u_1, ...u_N) \in \mathbb{R}^N$, we use the notation $u \otimes u$ to denote the matrix with components $u_i u_j$ whereas, A being a matrix valued function, $\mathrm{Div}_x(A) = \sum_{j=1}^N \partial_{x_j} A_{ij} \in \mathbb{R}^N$. In view of the incompressibility condition, we have of course (at least if u is regular) $\mathrm{Div}_x(u \otimes u) = (u \cdot \nabla_x)u$.

Here, equation (1.1) and (1.2) are written in dimensionless form. We refer to the companion paper [15] for a dimension analysis. In (1.1), $F(t,x,v)$ is associated to the forces acting on the particle while the right hand side models Brownian motion. In (1.2) the function $\mathfrak{F}(t,x)$ is associated to the density of forces exerted on the fluid. Equations (1.1) and (1.2) are coupled through these force terms. The forces acting on the particles are supposed to reduce to the friction force exerted by the fluid, assumed to be proportional to the relative velocity

$$F = F_0(u - v), \qquad F_0 > 0.$$

The right hand side for the fluid equation is therefore given by the sum

$$\mathfrak{F} = -\int_{\mathbb{R}^N} Ff \, \mathrm{d}v = F_0 \int_{\mathbb{R}^N} f(v-u) \, \mathrm{d}v. \tag{1.3}$$

This paper is devoted to the study of the asymptotic behavior of this coupled system when both the force terms and the brownian effects are very strong, namely:

$$r = F_0 = 1/\varepsilon \gg 1.$$

Coming back to the identification of dimensionless physical parameters in [15] (see also [4]), the scaling corresponds to suppose that:
- the size of the particles is small compared to the observation length scale ($\mathcal{L} \gg a$ with the notation of [15]),
- the densities of the fluid phase and of the particles have the same order ($\rho_p \approx \rho_g$),
- a certain relaxation time, which depends on the physical characteristics of the fluid and the particles, is small compared to the observation time scale ($\mathcal{T} \gg \tau$).
We refer for dimensionless form of the equations to [15] where another ordering is dealt with. More details on the model can be found in Caflisch-Papanicolaou [4], and Williams [28] for the physical framework of combustion theory. Slightly different models describing fluid-particles interactions are presented in Jabin-Perthame [22], Herrero-Lucquin-Perthame [18], Russo-Smerecka [26], Clouet-Domelevo [5], Gavrilyuck-Teshukhov [10]. Readers interested in mathematical studies of the system (1.1, 1.2) should consult Hamdache [16], who also introduced singular perturbation problems in [17]. Asymptotic results concerning some simplified situations can be found in Berthonnaud [1], Domelevo-Roquejoffre [8], Domelevo-Vignal [9], Goudon [14], Jabin [20, 21]...

Hence, we aim at describing the behavior of $(f^\varepsilon, u^\varepsilon)$ solution of the following system

$$\begin{cases} \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon = -\dfrac{1}{\varepsilon}\ \mathrm{div}_v\Big((u^\varepsilon - v) f^\varepsilon - \nabla_v f^\varepsilon\Big), \\ \partial_t u^\varepsilon + \mathrm{Div}_x(u^\varepsilon \otimes u^\varepsilon) + \nabla_x p^\varepsilon - \Delta_x u^\varepsilon = \dfrac{1}{\varepsilon}\left(\displaystyle\int_{\mathbb{R}^N} v f^\varepsilon \,\mathrm{d}v - u^\varepsilon \int_{\mathbb{R}^N} f^\varepsilon \,\mathrm{d}v\right), \\ \mathrm{div}_x(u^\varepsilon) = 0, \\ f^\varepsilon_{|t=0} = f^\varepsilon_0, \qquad u^\varepsilon_{|t=0} = u^\varepsilon_0 \end{cases} \tag{1.4}$$

as the small parameter ε goes to 0. The paper is organized as follows. First, we present heuristically the limit problem which can be expected. It consists of the Navier Stokes system with non constant density. The density which is involved in this system is the sum of the (constant) density of the fluid and the macroscopic density of the particles. To justify the asymptotics we use the relative entropy method (see e.g. [29], [11]). Section 3 introduces a relative entropy which is intended to compare $(f^\varepsilon, u^\varepsilon)$ to the solution of the limit

problem. Then, we will state precisely the result of convergence, whose proof can be found in Section 4. Throughout the paper, we work on weak solutions $f^\varepsilon \in C^0([0,T];L^1(\mathbb{R}^N\times\mathbb{R}^N))$, $u^\varepsilon \in C^0([0,T];L^2(\mathbb{R}^N))\cap L^2(0,T;H^1(\mathbb{R}^2))$ of (1.4) verifying certain energy estimate (see section 4). We refer on existence of such solutions to [16].

2 Formal Derivation of the Limit Problem

It is worth rewriting the right hand side in the kinetic equation in (1.4) as

$$-\frac{1}{\varepsilon}\,\mathrm{div}_v\big((u^\varepsilon - v)f^\varepsilon - \nabla_v f^\varepsilon\big) = \frac{1}{\varepsilon}\,\mathrm{div}_v\Big(M^\varepsilon\,\nabla_v\Big(\frac{f^\varepsilon}{M^\varepsilon}\Big)\Big)$$

where M^ε is the (normalized) Maxwellian with velocity u^ε:

$$M^\varepsilon(t,x,v) = (2\pi)^{-N/2}\exp(-|v-u^\varepsilon|^2/2).$$

Let us introduce the quantity

$$d^\varepsilon = (v-u^\varepsilon)\sqrt{f^\varepsilon} + 2\nabla_v\sqrt{f^\varepsilon} = 2\sqrt{M^\varepsilon}\nabla_v\Big(\sqrt{\frac{f^\varepsilon}{M^\varepsilon}}\Big). \tag{2.1}$$

The cornerstone of the analysis relies on the fact that d^ε is $\mathcal{O}(\sqrt{\varepsilon})$ in L^2:

$$\int_0^T\!\!\int_{\mathbb{R}^N}\!\!\int_{\mathbb{R}^N} |d^\varepsilon|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \le C\varepsilon. \tag{2.2}$$

It will appear as the dissipation of some free energy associated to the system (1.4). Formally, this estimate illustrates the trend of the kinetic equation to relax to the Maxwellian with the velocity of the fluid. If we assume that $u^\varepsilon \to u$, then, we can expect that

$$f^\varepsilon \to M_{\rho,u}(t,x,v) = \frac{\rho(t,x)}{(2\pi)^{N/2}}\ \exp(-|v-u(t,x)|^2/2),$$

with ρ the limit (which is supposed to exist) of $\int_{\mathbb{R}^N} f^\varepsilon\,\mathrm{d}v$. Now, we aim at describing the limit equations satisfied by (ρ,u).

Let us introduce the macroscopic density, velocity and kinetic pressure associated to the particles density

$$\rho^\varepsilon(t,x) = \int_{\mathbb{R}^N} f^\varepsilon\,\mathrm{d}v, \quad J^\varepsilon(t,x) = \int_{\mathbb{R}^N} v f^\varepsilon\,\mathrm{d}v, \quad \mathbb{P}^\varepsilon(t,x) = \int_{\mathbb{R}^N} v\otimes v f^\varepsilon\,\mathrm{d}v.$$

Integration of the kinetic equation with respect to v yields the following moment equations

$$\left\{\begin{array}{l} \partial_t \rho^\varepsilon + \mathrm{div}_x(J^\varepsilon) = 0, \\ \partial_t J^\varepsilon + \mathrm{Div}_x(\mathbb{P}^\varepsilon) = \dfrac{1}{\varepsilon}\Big(\rho^\varepsilon u^\varepsilon - J^\varepsilon\Big). \end{array}\right. \tag{2.3}$$

By using the fluid equation, the current equation can be rewritten as

$$\partial_t(u^\varepsilon + J^\varepsilon) + \mathrm{Div}_x(u^\varepsilon \otimes u^\varepsilon + \mathbb{P}^\varepsilon) + \nabla_x p^\varepsilon - \Delta u^\varepsilon = 0.$$

Then, we remark that the kinetic pressure can be split as follows

$$\mathbb{P}^\varepsilon = \int_{\mathbb{R}^N} d^\varepsilon \otimes v\sqrt{f^\varepsilon}\,\mathrm{d}v + \int_{\mathbb{R}^N} u^\varepsilon \otimes v\ f^\varepsilon\,\mathrm{d}v - 2\int_{\mathbb{R}^N} \nabla_v\sqrt{f^\varepsilon} \otimes v\sqrt{f^\varepsilon}\,\mathrm{d}v. \tag{2.4}$$

After integration with respect to x, by combining the Cauchy-Schwarz inequality to (2.2), we see that the first term in (2.4) is $\mathcal{O}(\sqrt{\varepsilon})$ provided the kinetic energy $\iint v^2 f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x$ can be bounded uniformly with respect to ε. Besides, the last integral in (2.4) is nothing but $\rho^\varepsilon \mathbb{I}$ and the second one is $u^\varepsilon \otimes J^\varepsilon$. Hence, we have

$$\partial_t(u^\varepsilon + J^\varepsilon) + \mathrm{Div}_x(u^\varepsilon \otimes (u^\varepsilon + J^\varepsilon)) + \nabla_x(p^\varepsilon + \rho^\varepsilon) - \Delta u^\varepsilon = \mathcal{O}(\sqrt{\varepsilon}). \tag{2.5}$$

On the other hand, we remark that

$$J^\varepsilon - \rho^\varepsilon u^\varepsilon = \int_{\mathbb{R}^N} f^\varepsilon(v - u^\varepsilon)\,\mathrm{d}v = \int_{\mathbb{R}^N} \sqrt{f^\varepsilon}\ d^\varepsilon\,\mathrm{d}v.$$

Hence, (2.2) implies that $J^\varepsilon - \rho^\varepsilon u^\varepsilon$ tends to 0, as $\varepsilon \to 0$. Consequently, if we assume that ρ^ε and u^ε admit limits ρ, u, and the product also passes to the limit

$$\rho^\varepsilon u^\varepsilon \to \rho u$$

then, we deduce that $J^\varepsilon \to \rho u$ too. Passing also to the limit formally in the product $u^\varepsilon \otimes (u^\varepsilon + J^\varepsilon)$ we are finally led to the following incompressible Navier-Stokes system, with $\widetilde{\rho} = 1 + \rho$,

$$\left\{\begin{array}{l} \partial_t \widetilde{\rho} + \mathrm{div}_x(\widetilde{\rho} u) = 0, \\ \partial_t(\widetilde{\rho} u) + \mathrm{Div}_x(\widetilde{\rho} u \otimes u) - \Delta u + \nabla_x P = 0, \\ \mathrm{div}_x(u) = 0. \end{array}\right. \tag{2.6}$$

We shall prove, under an assumption of preparation of the data, that:

$$\|\rho^\varepsilon - (\tilde\rho - 1)\|_{L^\infty(0,T;L^1(\mathbb{R}^N))} \xrightarrow[\varepsilon\to 0]{} 0,$$
$$\|u^\varepsilon - u\|_{L^\infty(0,T;L^2(\mathbb{R}^N))} \xrightarrow[\varepsilon\to 0]{} 0,$$

where $(\tilde\rho, u)$ is solution to (2.6). The precise statement can be found in the following section.

3 Entropy Method

Here, it seems far from obvious to justify by a compactness argument the convergence of $(\rho^\varepsilon, u^\varepsilon)$ to $(\widetilde\rho - 1, u)$, with $(\widetilde\rho, u)$ solution of (2.6): the difficulty relies on the non linear passage to the limit in the products $\rho^\varepsilon u^\varepsilon$ and, much more difficult, $\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon$. The estimates we are able to derive for the system (1.4) are not sufficient to obtain the needed strong convergences. Instead we shall use a relative entropy method. Starting from a smooth initial data $(\widetilde\rho_0, u_0)$, the limit problem (2.6) admits a smooth solution $(\widetilde\rho, u)$ at least on a small interval of time $[0, T]$. We refer on this aspect to the paper of Itoh-Tani [19]; further details and references can be found in the up-to-date review by Danchin [7]. We set $\rho = \widetilde\rho - 1 > 0$ on this time interval. We aim at comparing in some sense the sequence $(\rho^\varepsilon, u^\varepsilon)$ to $(\widetilde\rho - 1, u)$. This method has been introduced by Yau [29]. It is reminiscent to weak-strong uniqueness principle (see Dafermos [2] and Lions [25]). It has been successfully used to derive the incompressible Euler equation from the Vlasov-Poisson system by Brenier [3], to investigate hydrodynamic limits of the Boltzmann equation by Golse-Levermore-Saint-Raymond [12], Saint-Raymond [27] or to study gyrokinetic limits by Brenier [3] and Golse-Saint-Raymond [13].

Let $h : \mathbb{R} \to \mathbb{R}$ be a strictly convex function. The quantity

$$H(x|y) = h(x) - h(y) - h'(y)(x - y)$$

can be used as a way to evaluate how far x is from y. Indeed, by convexity, we have

$$H(x|y) = \int_y^x \Big(h'(z) - h'(y)\Big)\,\mathrm{d}z = \int_y^x \int_y^z h''(r)\,\mathrm{d}r\,\mathrm{d}z \geq 0$$

and it vanishes iff $x = y$. Let $M_{\rho,u}$ stand for the Maxwellian with density $\rho = \widetilde{\rho} - 1$ and velocity u

$$M_{\rho,u}(t,x,v) = (2\pi)^{-N/2}\ \rho(t,x)\ \exp(-|v-u(t,x)|^2/2).$$

Let us introduce the relative entropy

$$\mathcal{H}(f^\varepsilon, u^\varepsilon) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(f^\varepsilon|M_{\rho,u})\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} |u^\varepsilon - u|^2\,\mathrm{d}x.$$

The former integral evaluates how far f^ε is from the Maxwellian $M_{\rho,u}$, while the latter is nothing but the (squared) L^2 norm between u^ε and u. We have in mind to obtain a relation looking like

$$\mathcal{H}(f^\varepsilon, u^\varepsilon)(t) \le \mathcal{H}(f^\varepsilon, u^\varepsilon)(0) + C\int_0^t \mathcal{H}(f^\varepsilon, u^\varepsilon)(s)\,\mathrm{d}s + r^\varepsilon(t), \tag{3.1}$$

with a constant C which does not depend on ε. Then, the conclusion follows by means of an application of the Gronwall lemma provided:
- the initial data is well prepared in the sense that $\mathcal{H}(f^\varepsilon, u^\varepsilon)(0) \to 0$ as ε goes to 0,
- the remainder r^ε tends to 0 as $\varepsilon \to 0$. This will appear as a consequence of the dissipation estimate (2.2).

Now, let us precise the definition of the relative entropy. Obviously, $h(s) = s^2$ can be used to define the relative entropy. Here, it is well adapted to use instead $s = s\ln(s)$. Accordingly, we have

$$H(f^\varepsilon|M_{\rho,u}) = f^\varepsilon \ln\big(\frac{f^\varepsilon}{M_{\rho,u}}\big) + M_{\rho,u} - f^\varepsilon = M_{\rho,u}\ \overline{h}\big(\frac{f^\varepsilon}{M_{\rho,u}}\big),$$

with $\overline{h}(s) = s\ln(s) - s + 1 \ge 0$. As a preliminary, let us discuss some properties of the relative entropy which will be useful for our purposes. First, we remark that the relative entropy between macroscopic quantities is dominated by the relative entropy of microscopic quantities. Second, we are interested in estimates of $|x-y|$ in terms of $H(x|y)$.

Lemma 1 *For $i \in \{1,2\}$, let $f_i : \mathbb{R}^N \to \mathbb{R}^+$. We set $\rho_i = \int_{\mathbb{R}^N} f_i\,\mathrm{d}v$. Then, we have*

$$H(\rho_1|\rho_2) \le \int_{\mathbb{R}^N} H(f_1|f_2)\,\mathrm{d}v.$$

Proof. We write

$$H(\rho_1|\rho_2) = \rho_2\ \overline{h}(\rho_1/\rho_2) = \rho_2\ \overline{h}\left(\int_{\mathbb{R}^N} \frac{f_1}{f_2}\ \frac{f_2}{\rho_2}\,\mathrm{d}v\right).$$

Since $\frac{f_2}{\rho_2}\,\mathrm{d}v$ is a probability measure, the Jensen inequality applies and we get

$$H(\rho_1|\rho_2) \le \rho_2\ \int_{\mathbb{R}^N} \overline{h}\Big(\frac{f_1}{f_2}\Big)\ \frac{f_2}{\rho_2}\,\mathrm{d}v = \int_{\mathbb{R}^N} H(f_1|f_2)\,\mathrm{d}v.$$

∎

Lemma 2 *Let $x, y \ge 0$. There exists a constant C such that*

$$\begin{cases} \text{If } |x-y| \le y, \text{ then } |x-y|^2 \le C\ y\ H(x|y), \\ \text{If } |x-y| \ge y, \text{ then } |x-y| \le C\ H(x|y). \end{cases}$$

Proof. We start by rewriting $H(x|y) = y\overline{h}(x/y)$ so that these estimates can be deduced from elementary properties of the function

$$\overline{h}(z) = \int_1^z \ln(s)\,\mathrm{d}s.$$

First, consider $z \ge 2$ so that $(z+1)/2 \ge 3/2 \ge 1$ and

$$\overline{h}(z) \ge \int_{(z+1)/2}^z \ln(s)\,\mathrm{d}s \ge \ln(3/2)\ \frac{z-1}{2}.$$

Next, for $0 \le z \le 2$, we get

$$\overline{h}(z) = \int_1^z \int_1^s \frac{1}{r}\,\mathrm{d}r\,\mathrm{d}s \ge \frac{1}{2}\int_1^z \int_1^s \mathrm{d}r\,\mathrm{d}s = \frac{|z-1|^2}{4}.$$

∎

We can now state precisely the main result of the paper.

Theorem 1 *Let $(f_0^\varepsilon, u_0^\varepsilon)$ be the initial data for (1.4) such that $f_0^\varepsilon \ge 0$ and*

$$\sup_{\varepsilon>0}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_0^\varepsilon(1+x^2+v^2+|\ln(f_0^\varepsilon)|)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N} |u_0^\varepsilon|^2\,\mathrm{d}x\right) \le C < \infty. \tag{3.2}$$

Let $(\widetilde{\rho}_0, u_0)$ be a smooth initial data for the limit problem (2.6) such that $\widetilde{\rho}_0 > 1$ and

$$\int_{\mathbb{R}^N} (\widetilde{\rho}_0 - 1)\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_0^\varepsilon \,\mathrm{d}v\,\mathrm{d}x = C_0.$$

Let $(\widetilde{\rho}, u)$ be the corresponding smooth solution on $[0,T]$. Finally, we suppose that

$$\mathcal{H}(f_0^\varepsilon, u_0^\varepsilon) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(f_0^\varepsilon | M_{\widetilde{\rho}_0 - 1, u_0})\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} |u_0^\varepsilon - u_0|^2\,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} 0.$$

Then, we have

$$\sup_{0\le t\le T} \mathcal{H}(f^\varepsilon, u^\varepsilon) \xrightarrow[\varepsilon\to 0]{} 0.$$

Remark 1 *In view of the Csiszar-Kullback-Pinsker inequality [6], [23], the integral $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(f^\varepsilon | M_{\rho,u})\,\mathrm{d}v\,\mathrm{d}x$ dominates the square of the L^1 norm of $f^\varepsilon - M_{\rho,u}$. Hence, we have the convergences $f^\varepsilon \to M_{\rho,u}$ and $u^\varepsilon \to u$ in $L^\infty(0,T;L^1(\mathbb{R}^N\times\mathbb{R}^N))$ and $L^\infty(0,T;L^2(\mathbb{R}^N))$ norms respectively. Combining Lemma 1 and the Csiszar-Kullback-Pinsker inequality with the convergence in Theorem 1, we have also $\rho^\varepsilon \to \rho = \widetilde{\rho} - 1$ strongly in $L^\infty(0,T;L^1(\mathbb{R}^N))$.*

Remark 2 *We shall see that the remainder r^ε in (3.1) is $\mathcal{O}(\sqrt{\varepsilon})$, which gives the rate of convergence, up to the initial data term.*

4 Proof of Theorem 1

We divide the proof into two parts. First, we discuss the a priori estimates satisfied by the solutions of (1.4). Second, we establish relation (3.1), the remainder r^ε being $\mathcal{O}(\sqrt{\varepsilon})$.

4.1 A priori Estimates

We start by establishing preliminaries estimates on the microscopic quantity f^ε and the velocity field u^ε. The crucial estimate (2.2) is also contained in this statement. Throughout the paper, we use the convention that C denotes a constant depending on (3.2), $\widetilde{\rho}_0$, u_0 and T but not on ε, even if the value of the constant may vary from a line to another.

Proposition 1 *Let $(f^\varepsilon, u^\varepsilon)$ be the solution of (1.4) associated to an initial data verifying (3.2). Let $0 < T < \infty$. Then, the following assertions hold*
i) $f^\varepsilon(1 + x^2 + v^2 + |\ln(f^\varepsilon)|)$ is bounded in $L^\infty(0, T; L^1(\mathbb{R}^N \times \mathbb{R}^N))$,
ii) u^ε is bounded in $L^\infty(0, T; L^2(\mathbb{R}^N))$ and in $L^2(0, T; H^1(\mathbb{R}^N))$,
iii) The quantity $\frac{1}{\sqrt{\varepsilon}}\big((v-u^\varepsilon)\sqrt{f^\varepsilon}+2\nabla_v\sqrt{f^\varepsilon}\big) = \frac{1}{\sqrt{\varepsilon}}d^\varepsilon$ is bounded in $L^2((0,T)\times \mathbb{R}^N \times \mathbb{R}^N))$.

Proof. Let us derive formally these estimates; the rigorous proof can be obtained with an appropriate approximate argument, or in the construction of the solution, see [16]. Of, course, the total mass is conserved

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_0^\varepsilon \,\mathrm{d}v\,\mathrm{d}x,$$

which gives immediately the L^1 bound on f^ε. (Note that it is assumed to be equal to $\int_{\mathbb{R}^N}(\widetilde{\rho}_0 - 1)\,\mathrm{d}x$.)
Next, we consider the evolution of the following free energy associated to the system (1.4):

$$\mathcal{E}(f^\varepsilon, u^\varepsilon) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon\Big(\frac{v^2}{2} + \ln(f^\varepsilon)\Big)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}\frac{|u^\varepsilon|^2}{2}\,\mathrm{d}x.$$

It is the sum of the entropy of the particles with the kinetic energy of the particles and the fluid. We show that this quantity is dissipated, due to nice combinations between the fluid and the kinetic equation in (1.4); the dissipation rate is precisely given by the L^2 norm of $d^\varepsilon/\sqrt{\varepsilon}$ plus those of $\nabla_x u^\varepsilon$. Indeed, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f^\varepsilon, u^\varepsilon) + \int_{\mathbb{R}^N}|\nabla_x u^\varepsilon|^2\,\mathrm{d}x \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big((u^\varepsilon - v)f^\varepsilon - \nabla_v f^\varepsilon\Big)\cdot\Big(v + \frac{\nabla_v f^\varepsilon}{f^\varepsilon}\Big)\,\mathrm{d}v\,\mathrm{d}x \\
&\qquad + \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(v - u^\varepsilon)f^\varepsilon\cdot u^\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\
&= -\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big|d^\varepsilon\big|^2\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{4.1}$$

Besides, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x &= 2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x\cdot v\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

Therefore, Gronwall's lemma yields

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x \le e^T\Big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x^2\ f_0^\varepsilon\,\mathrm{d}v\,\mathrm{d}x + \int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\Big). \tag{4.2}$$

Then, we use classical tricks of kinetic theory (see e.g. [24]). We write $s|\ln(s)| = s\ln(s) - 2s\ln(s)\chi_{0\le s\le 1}$. Let $\omega \ge 0$. We split

$$\begin{aligned} -s\ln(s)\chi_{0\le s\le 1} &= -s\ln(s)\chi_{e^{-\omega}\le s\le 1} - s\ln(s)\chi_{e^{-\omega}\ge s} \\ &\le s\omega + C\sqrt{s}\chi_{e^{-\omega}\ge s} \le s\omega + Ce^{-\omega/2}. \end{aligned}$$

We use these relations with $s = f^\varepsilon$, $\omega = (x^2+v^2)/8$. We are led to

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon|\ln(f^\varepsilon)|\,\mathrm{d}v\,\mathrm{d}x \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon\ln(f^\varepsilon)\,\mathrm{d}v\,\mathrm{d}x \\ +\frac14\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(x^2+v^2)\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x + 2C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-(x^2+v^2)/16}\,\mathrm{d}v\,\mathrm{d}x. \end{aligned}$$

Then, combining this to (4.1) and (4.2) yields

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon(1+|\ln(f^\varepsilon)|)\,\mathrm{d}v\,\mathrm{d}x + \frac14\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(x^2+v^2)\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\ &\quad+\frac12\int_{\mathbb{R}^N}\left|u^\varepsilon\right|^2\mathrm{d}x + \frac1\varepsilon\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|d^\varepsilon\right|^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_{\mathbb{R}^N}\left|\nabla_x u^\varepsilon\right|^2\mathrm{d}x \\ &\le \mathcal{E}(f^\varepsilon,u^\varepsilon)(t) + \frac12\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} x^2 f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x \\ &\quad+C + \frac1\varepsilon\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|d^\varepsilon\right|^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + \int_0^T\int_{\mathbb{R}^N}\left|\nabla_x u^\varepsilon\right|^2\mathrm{d}x \\ &\le \mathcal{E}(f^\varepsilon,u^\varepsilon)(0) + C\Big(1+\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} v^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\Big), \end{aligned}$$

where C depends on (3.2) and T. We conclude by using the Gronwall lemma. ∎

Next, we wish to discuss some estimates on the macroscopic quantities associated to f^ε. To this end, it is convenient to establish the following claim.

Corollary 1 *Let the assumptions of Proposition 1 be fulfilled. The quantity $|u^\varepsilon - v|^2 f^\varepsilon$ is bounded in $L^1((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$.*

Proof. We rewrite

$$\begin{aligned}\int_{\mathbb{R}^N} |v-u^\varepsilon|^2 f^\varepsilon \,\mathrm{d}v &= \int_{\mathbb{R}^N} \Big(\big|d^\varepsilon\big|^2 - 4\big|\nabla_v \sqrt{f^\varepsilon}\big|^2 - 4\nabla_v \sqrt{f^\varepsilon}\cdot (v-u^\varepsilon)\sqrt{f^\varepsilon}\Big)\,\mathrm{d}v \\ &\le \int_{\mathbb{R}^N} \big|d^\varepsilon\big|^2 \,\mathrm{d}v + 2N\int_{\mathbb{R}^N} f^\varepsilon \,\mathrm{d}v,\end{aligned}$$

where we used an integration by parts for the last term. The result follows from Proposition 1. ∎

Corollary 2 *Let the assumptions of Proposition 1 be fulfilled. Then, the following assertions hold*
i) ρ^ε *is bounded in* $L^\infty(0,T;L^1(\mathbb{R}^N))$,
ii) $\rho^\varepsilon\big|u^\varepsilon\big|^2$ *(and* $\rho^\varepsilon u^\varepsilon$*) is bounded in* $L^1((0,T)\times\mathbb{R}^N)$,
iii) $J^\varepsilon - \rho^\varepsilon u^\varepsilon$, $\mathbb{P}^\varepsilon - \rho^\varepsilon(\mathbb{I}+u^\varepsilon\otimes u^\varepsilon)$ *and* $(J^\varepsilon-\rho^\varepsilon u^\varepsilon)\otimes u^\varepsilon$ *are* $\mathcal{O}(\sqrt{\varepsilon})$ *in* $L^1((0,T)\times\mathbb{R}^N)$ *norm.*

Proof. The bound on ρ^ε is an immediate consequence of Proposition 1-i). Actually, it can be shown, see e.g. [24], that $\rho^\varepsilon(1+x^2+|\ln(\rho^\varepsilon)|)$ is bounded in $L^\infty(0,T;L^1(\mathbb{R}^N))$. This is well known to imply weak compactness in L^1, but we shall not use such kind of information.

Next, we remark that

$$\rho^\varepsilon\big|u^\varepsilon\big|^2 = \int_{\mathbb{R}^N} f^\varepsilon\big|u^\varepsilon\big|^2\,\mathrm{d}v \le 2\int_{\mathbb{R}^N} f^\varepsilon\,\Big(|v-u^\varepsilon|^2+v^2\Big)\,\mathrm{d}v$$

and ii) follows from Proposition 1 and Corollary 1.

As mentioned in Section 2, we have

$$J^\varepsilon - \rho^\varepsilon u^\varepsilon = \int_{\mathbb{R}^N} \sqrt{f^\varepsilon}\; d^\varepsilon \,\mathrm{d}v,$$

so that we conclude by applying the Cauchy-Schwarz inequality. Similarly, we rewrite

$$\begin{aligned}&\mathbb{P}^\varepsilon - \rho^\varepsilon(\mathbb{I}+u^\varepsilon\otimes u^\varepsilon) = \int_{\mathbb{R}^N}(v\otimes v - \mathbb{I} - u^\varepsilon\otimes u^\varepsilon) f^\varepsilon\,\mathrm{d}v \\ &= \int_{\mathbb{R}^N}\Big(d^\varepsilon\otimes v\sqrt{f^\varepsilon} + u^\varepsilon\sqrt{f^\varepsilon}\otimes d^\varepsilon - \mathbb{I}f^\varepsilon \\ &\qquad\qquad -2\nabla_v\sqrt{f^\varepsilon}\otimes v\sqrt{f^\varepsilon} - 2u^\varepsilon\sqrt{f^\varepsilon}\otimes\nabla_v\sqrt{f^\varepsilon}\Big)\,\mathrm{d}v \\ &= \int_{\mathbb{R}^N}(d^\varepsilon\otimes v\sqrt{f^\varepsilon} + u^\varepsilon\sqrt{f^\varepsilon}\otimes d^\varepsilon)\,\mathrm{d}v.\end{aligned}$$

After integration with respect to t, x it can be estimated by

$$\left(\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|d^\varepsilon\right|^2\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(v^2+\left|u^\varepsilon\right|^2\right)f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.$$

We conclude by combining Proposition 1 and and ii).

Finally, we treat similarly the expression

$$(J^\varepsilon-\rho^\varepsilon u^\varepsilon)\otimes u^\varepsilon=\int_{\mathbb{R}^N}(v-u^\varepsilon)\otimes u^\varepsilon f^\varepsilon=\int_{\mathbb{R}^N}d^\varepsilon\otimes u^\varepsilon\sqrt{f^\varepsilon}\,\mathrm{d}v.$$

■

This statement makes rigorous the argument presented in Section 2. Coming back to the momentum equation (2.5), we are led to

$$\partial_t(u^\varepsilon+\rho^\varepsilon u^\varepsilon)+\mathrm{Div}_x\big(u^\varepsilon\otimes(u^\varepsilon+\rho^\varepsilon u^\varepsilon)\big)+\nabla_x(p^\varepsilon+\rho^\varepsilon)-\Delta u^\varepsilon\xrightarrow[\varepsilon\to0]{}0$$

at least in the distributions sense. Furthermore, we know that each term involved in this relation admits a limit (at least for a subsequence) but the obtained estimates are not enough to identify the limits by passing to the limit in the non linear terms.

4.2 Evolution of the Relative Entropy

We recall that $(\widetilde{\rho},u)$ is the smooth solution of (2.6), defined on the time interval $[0,T]$, which corresponds to the initial data $(\widetilde{\rho}_0,u_0)$, see e.g. [19]. Precisely, we shall use

$$\left\{\begin{array}{ll}\widetilde{\rho}\in L^\infty(0,T;W^{1,\infty}(\mathbb{R}^N)), & \widetilde{\rho}(t,x)>1,\\ u\in L^\infty(0,T;W^{2,\infty}(\mathbb{R}^N)), & \nabla P\in L^\infty((0,T)\times\mathbb{R}^N).\end{array}\right.$$

We set $\rho=\widetilde{\rho}-1$ and we aim at comparing u^ε to u and f^ε to the Maxwellian $M_{\rho,u}$ in the sense of the relative entropy

$$\mathcal{H}(f^\varepsilon,u^\varepsilon)=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(f^\varepsilon\ln\big(\frac{f_\varepsilon}{M_{\rho,u}}\big)-f^\varepsilon+M_{\rho,u}\right)\,\mathrm{d}v\,\mathrm{d}x+\frac12\int_{\mathbb{R}^N}\left|u^\varepsilon-u\right|^2\mathrm{d}x.$$

The objective in this section is to justify the following claim, which immediately leads to the conclusion of Theorem 1.

Proposition 2 *There exists a constant C, depending on (3.2), $\widetilde{\rho}_0$, u_0 and T such that*

$$\mathcal{H}(f^\varepsilon, u^\varepsilon)(t) \le C\Big(\mathcal{H}(f^\varepsilon, u^\varepsilon)(0) + \sqrt{\varepsilon}\Big).$$

Indeed, the time evolution of the relative entropy can be evaluated as follows.

Lemma 3 *The relative entropy satisfies*

$$\begin{aligned}\mathcal{H}(f^\varepsilon, u^\varepsilon)(t) + \frac{1}{\varepsilon}\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |d^\varepsilon|^2 \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{R}^N} |\nabla_x(u^\varepsilon - u)|^2\,\mathrm{d}x\,\mathrm{d}s \\ \le \mathcal{H}(f^\varepsilon, u^\varepsilon)(0) + \int_0^t \left|A^\varepsilon + B^\varepsilon + C^\varepsilon + D^\varepsilon\right|\mathrm{d}s\end{aligned} \tag{4.3}$$

with

$$\begin{cases} A^\varepsilon = -\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon (v-u)\otimes(v-u) : \nabla_x u \,\mathrm{d}v\,\mathrm{d}x \\ B^\varepsilon = -\displaystyle\int_{\mathbb{R}^N} (u^\varepsilon - u)\otimes(u^\varepsilon - u) : \nabla_x u\,\mathrm{d}x \\ C^\varepsilon = \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon (v-u^\varepsilon)\cdot(F - \nabla_x \ln(\rho))\,\mathrm{d}v\,\mathrm{d}x \\ D^\varepsilon = \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (\rho^\varepsilon - \rho)(u^\varepsilon - u)\cdot(F - \nabla_x \ln(\rho))\,\mathrm{d}x \end{cases}$$

where we used the notation $F(t,x) = (\nabla_x P - \Delta u)/(1+\rho) = (\nabla_x P - \Delta u)/\widetilde{\rho}$.

Therefore, we obtain Proposition 2 by a simple application of the Gronwall lemma once we are able to establish that

$$\int_0^t |A^\varepsilon|\,\mathrm{d}s, \quad \int_0^t |B^\varepsilon|\,\mathrm{d}s, \quad \int_0^t |C^\varepsilon|\,\mathrm{d}s, \quad \int_0^t |D^\varepsilon|\,\mathrm{d}s \le C\Big(\int_0^t \mathcal{H}(f^\varepsilon, u^\varepsilon)\,\mathrm{d}s + \sqrt{\varepsilon}\Big) \tag{4.4}$$

holds.

Remark 3 *Keeping in mind the term*

$$\int_0^t\int_{\mathbb{R}^N} |\nabla(u^\varepsilon - u)|^2\,\mathrm{d}x\,\mathrm{d}s$$

in (4.3), we also prove that

$$u^\varepsilon \xrightarrow[\varepsilon\to 0]{} u \quad \text{strongly in } L^2(0,T;H^1(\mathbb{R}^N)).$$

Proof of Lemma 3. We compute the time derivative of $\mathcal{H}(f^\varepsilon, u^\varepsilon)$. By using the mass conservation

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_0^\varepsilon \,\mathrm{d}v\,\mathrm{d}x = \int_{\mathbb{R}^N} \rho\,\mathrm{d}x = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} M_{\rho,u}\,\mathrm{d}v\,\mathrm{d}x$$

we remark that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(f^\varepsilon|M_{\rho,u})\,\mathrm{d}v\,\mathrm{d}x\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon\Big(\ln(f^\varepsilon) + \frac{|v-u|^2}{2}\Big)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^N}\rho^\varepsilon\ln(\rho)\,\mathrm{d}x\right).
\end{aligned}$$

Then, by using the equations satisfied by f^ε and u and integration by parts, we are led to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon\Big(\ln(f^\varepsilon) + \frac{|v-u|^2}{2}\Big)\,\mathrm{d}v\,\mathrm{d}x\right)\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big((u^\varepsilon - v)f^\varepsilon - \nabla_v f^\varepsilon\Big)\cdot\Big(\frac{\nabla_v f^\varepsilon}{f^\varepsilon} + (v-u)\Big)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon(v-u)\cdot F\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^N}(v-u)\otimes(v-u)f^\varepsilon : \nabla_x u\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{4.5}$$

The first term in the right hand side recasts as

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|d^\varepsilon\right|^2\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big((u^\varepsilon - v)f^\varepsilon - \nabla_v f^\varepsilon\Big)\cdot(u^\varepsilon - u)\,\mathrm{d}v\,\mathrm{d}x\\
&= -\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left|d^\varepsilon\right|^2\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u^\varepsilon - v)\cdot(u^\varepsilon - u)f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{4.6}$$

Next, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathbb{R}^N}\rho^\varepsilon\ln(\rho)\,\mathrm{d}x\right) = \int_{\mathbb{R}^N}(J^\varepsilon - \rho^\varepsilon u)\cdot\nabla_x\ln(\rho)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon(v-u^\varepsilon)\cdot\nabla_x\ln(\rho)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}\rho^\varepsilon(u^\varepsilon - u)\cdot\nabla_x\ln(\rho)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon(v-u^\varepsilon)\cdot\nabla_x\ln(\rho)\,\mathrm{d}v\,\mathrm{d}x + \int_{\mathbb{R}^N}(\rho^\varepsilon - \rho)(u^\varepsilon - u)\cdot\nabla_x\ln(\rho)\,\mathrm{d}x,
\end{aligned} \tag{4.7}$$

where the incompressibility condition $\mathrm{div}_x(u) = 0 = \mathrm{div}_x(u^\varepsilon)$ has been used to obtain the last equality.

Finally, for the fluid part, we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}^N}|u^\varepsilon - u|^2\,\mathrm{d}x\right)\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon(v-u^\varepsilon)\cdot(u^\varepsilon - u)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}(u^\varepsilon - u)\cdot F\,\mathrm{d}x + \int_{\mathbb{R}^N}(u^\varepsilon - u)\cdot(u\cdot\nabla_x u - u^\varepsilon\cdot\nabla_x u^\varepsilon)\,\mathrm{d}x\\
&\quad + \int_{\mathbb{R}^N}(u^\varepsilon - u)\cdot\Delta u^\varepsilon\,\mathrm{d}x.
\end{aligned} \tag{4.8}$$

The first term in (4.8) will compensate the last one in (4.6). In the last term of (4.8), we expand $\Delta u^\varepsilon = \Delta u + \Delta(u^\varepsilon - u)$. Then, by incompressibility we have

$$\int_{\mathbb{R}^N}(u^\varepsilon - u)\cdot\Delta u\,\mathrm{d}x = -\int_{\mathbb{R}^N}(u^\varepsilon - u)\cdot F(1+\rho)\,\mathrm{d}x.$$

Therefore, we can rewrite

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\int_{\mathbb{R}^N}|u^\varepsilon - u|^2\,\mathrm{d}x\right)\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f^\varepsilon(v-u^\varepsilon)\cdot(u^\varepsilon - u)\,\mathrm{d}v\,\mathrm{d}x - \int_{\mathbb{R}^N}|\nabla_x(u^\varepsilon - u)|^2\,\mathrm{d}x.\\
&\quad - \int_{\mathbb{R}^N}(u^\varepsilon - u)\cdot F\rho\,\mathrm{d}x - \int_{\mathbb{R}^N}(u^\varepsilon - u)\otimes(u^\varepsilon - u) : \nabla_x u\,\mathrm{d}x.
\end{aligned} \tag{4.9}$$

Putting all the pieces (4.5), (4.6), (4.7), (4.9) together yields the announced equality. ∎

We are thus left with the task of evaluating $A^\varepsilon, B^\varepsilon, C^\varepsilon, D^\varepsilon$. The easiest part is the following.

Lemma 4 *We have*

$$\begin{cases}
|B^\varepsilon| \le \|\nabla_x u\|_\infty \int_{\mathbb{R}^N}|u^\varepsilon - u|^2\,\mathrm{d}x \le C\mathcal{H}(f^\varepsilon, u^\varepsilon)\\
\displaystyle\int_0^t |C^\varepsilon|\,\mathrm{d}s \le C\,\sqrt{\varepsilon}.
\end{cases}$$

Proof. The estimate on B^ε is immediate. Let us set $G(t,x) = F - \nabla_x \ln(\rho)$. We evaluate C^ε by remarking that

$$C^\varepsilon = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} d^\varepsilon \cdot G\sqrt{f^\varepsilon}\,\mathrm{d}v\,\mathrm{d}x$$

and we conclude by using the Cauchy-Schawarz inequality and Proposition 1. ∎

Estimations of the other terms in Lemma 3 require intermediate manipulations.

Lemma 5 *We have*

$$\int_0^t |A^\varepsilon|\,\mathrm{d}s \le C\left(\sqrt{\varepsilon} + \int_0^t\int_{\mathbb{R}^N} \rho^\varepsilon |u-u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s\right).$$

Proof. We split A^ε into four pieces by expanding

$$\begin{aligned}(v-u)\otimes(v-u) &= (v-u^\varepsilon)\otimes(v-u^\varepsilon) + (v-u^\varepsilon)\otimes(u^\varepsilon-u)\\ &\quad +(u^\varepsilon-u)\otimes(v-u^\varepsilon) + (u^\varepsilon-u)\otimes(u^\varepsilon-u).\end{aligned}$$

Let us denote by $I_1^\varepsilon, ..., I_4^\varepsilon$ the corresponding integrals. We evaluate readily

$$|I_4^\varepsilon| \le \|\nabla_x u\|_\infty \int_0^t\int_{\mathbb{R}^N} \rho^\varepsilon |u-u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s.$$

Next, we show that the other terms are of order $\sqrt{\varepsilon}$. To this end, we use the entropy dissipation d^ε. Indeed, for the crossed terms, we have

$$I_2^\varepsilon = \int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} d^\varepsilon \otimes (u^\varepsilon - u)\sqrt{f^\varepsilon} : \nabla_x u\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.$$

Hence, we deduce that

$$|I_2^\varepsilon| \le \|\nabla_x u\|_\infty\ \sqrt{\varepsilon}\ \left(\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (|u|^2 + |u^\varepsilon|^2)\rho^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \le C\ \sqrt{\varepsilon}$$

where we used the estimates in Proposition 1 and Corollary 2-i), ii). Integrals I_3^ε, as well as I_1^ε (by using the incompressiblity of u), can be treated similarly. ∎

We shall combine Lemma 5 with the following claim.

Lemma 6 *The following estimate*

$$\int_0^t \int_{\mathbb{R}^N} \rho^\varepsilon |u^\varepsilon - u|^2 \,\mathrm{d}x\,\mathrm{d}s \leq C\sqrt{\varepsilon} + \int_0^t \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} H(f^\varepsilon | M_{\widetilde{\rho},u}) \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$

holds.

Proof. The proof relies on the following expansion

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \rho^\varepsilon |u^\varepsilon - u|^2 \,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(|u^\varepsilon - u|^2 - |v-u|^2 + |v-u^\varepsilon|^2\right) f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\ln(f^\varepsilon) + |v-u|^2\right) f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad - \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(\ln(f^\varepsilon) + |v-u^\varepsilon|^2\right) f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&= -2\int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} (u^\varepsilon - u)\cdot(v-u^\varepsilon) f^\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \left(H(f^\varepsilon|M_{\rho,u}) - H(f^\varepsilon|M_{\rho,u^\varepsilon})\right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

The first integral can be shown to be of order $\sqrt{\varepsilon}$ by using the entropy dissipation as in the proof of Lemma 5. Remarking that $H(f^\varepsilon|M_{\rho,u^\varepsilon}) \geq 0$ ends the proof. ∎

Eventually, we end the proof of (4.4) with the following statement.

Lemma 7 *We have*

$$\int_0^t |D^\varepsilon| \,\mathrm{d}s \leq C\left(\sqrt{\varepsilon} + \int_0^t \mathcal{H}(f^\varepsilon, u^\varepsilon)\,\mathrm{d}s\right).$$

Proof. The proof uses the fundamental properties of the relative entropy discussed in Lemma 1 and 2. Let us split the integral as follows

$$\begin{aligned}
\int_0^t |D^\varepsilon| \,\mathrm{d}s &\leq \|G\|_\infty \int_0^t \int_{\mathbb{R}^N} |\rho^\varepsilon - \rho| \;|u^\varepsilon - u| \,\mathrm{d}x\,\mathrm{d}s \\
&\leq \|G\|_\infty \left(\int_{|\rho-\rho^\varepsilon|\leq\rho} \ldots \,\mathrm{d}x\,\mathrm{d}s + \int_{|\rho-\rho^\varepsilon|\geq\rho} \ldots \,\mathrm{d}x\,\mathrm{d}s\right).
\end{aligned}$$

Cauchy-Schwarz and Young inequalities yield

$$\begin{aligned}\int_{|\rho-\rho^\varepsilon|\le\rho}\dots\,\mathrm{d}x\,\mathrm{d}s &\le \frac{1}{2}\left(\int_0^t\int_{|\rho-\rho^\varepsilon|\le\rho}|\rho^\varepsilon-\rho|^2\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^N}|u^\varepsilon-u|^2\,\mathrm{d}x\,\mathrm{d}s\right)\\ &\le C\int_0^t\int_{\mathbb{R}^N}\rho\;H(\rho^\varepsilon|\rho)\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^N}\frac{1}{2}\;|u^\varepsilon-u|^2\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

by using Lemma 2. On the other hand, we get

$$\begin{aligned}\int_{|\rho-\rho^\varepsilon|\ge\rho}\dots\,\mathrm{d}x\,\mathrm{d}s &\le \int_{|\rho-\rho^\varepsilon|\ge\rho,\ |u^\varepsilon-u|\le1}\dots\,\mathrm{d}x\,\mathrm{d}s+\int_{|u^\varepsilon-u|\ge1}\dots\,\mathrm{d}x\,\mathrm{d}s\\ &\le \int_{|\rho-\rho^\varepsilon|\ge\rho}|\rho^\varepsilon-\rho|\,\mathrm{d}x\,\mathrm{d}s+\int_0^t\int_{\mathbb{R}^N}(\rho^\varepsilon+\rho)|u^\varepsilon-u|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\le C\int_0^t\int_{\mathbb{R}^N}H(\rho^\varepsilon|\rho)\,\mathrm{d}x\,\mathrm{d}s+\|\rho\|_\infty\int_0^t\int_{\mathbb{R}^N}|u^\varepsilon-u|^2\,\mathrm{d}x\,\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}^N}\rho^\varepsilon|u^\varepsilon-u|^2\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

We note that the last integral in the right hand side can be evaluated by using Lemma 6. Then, an application of Lemma 1 ends the proof. ∎

Now, we are in position of concluding. Combining Lemma 4, 5, 6 and 7 proves (4.4). Coming back to Lemma 3, we are led to

$$\mathcal{H}(f^\varepsilon,u^\varepsilon)(t)\le\mathcal{H}(f^\varepsilon,u^\varepsilon)(0)+C\int_0^t\mathcal{H}(f^\varepsilon,u^\varepsilon)(s)\,\mathrm{d}s+C\;\sqrt{\varepsilon}.$$

Applying the Gronwall lemma finishes the proof of Proposition 2. ∎

References

[1] Berthonnaud P. *Limites fluides pour des modèles cinétiques de brouillards de gouttes monodispersés*, C. R. Acad. Sci., 331 (2000) 651–654.

[2] Boillat G., Dafermos C., Lax P., Liu T.P., **Recent mathematical methods in nonlinear wave propagation,** Lecture Notes in Math., 1640 (Springer, Berlin, 1996).

[3] Brenier, Y., *Convergence of the Vlasov-Poisson system to the incompressible Euler equations*, Comm. Partial Differential Equations, 25 (2000) 737–754.

[4] Caflisch R., Papanicolaou G., *Dynamic theory of suspensions with Brownian effects*, SIAM J. Appl. Math., 43 (1983) 885–906.

[5] Clouet J.F., Domelevo K., *Solutions of a kinetic stochastic equation modeling a spray in a turbulent gas flow*, Math. Models Methods Appl. Sci., 7 (1997) 239–263.

[6] Csiszar I., *Information-type measures of difference of probability distributions and indirect observations*, Studia Sci. Math. Hungar., 2 (1967) 299–318.

[7] Danchin R., *Fluides incompressibles à densité variable*, Séminaire Equations aux Derivées Partielles, 2002–2003, Exposé XI. (Ecole Polytechnique, Palaiseau, 2003).

[8] Domelevo K., Roquejoffre J. M., *Existence and stability of traveling wave solutions in a kinetic model of two-phase flows*, Comm. PDE, 24 (1999) 61–108.

[9] Domelevo K., Vignal M.-H., *Limites visqueuses pour des systèmes de type Fokker-Planck-Burgers unidimensionnels*, C. R. Acad. Sci., 332 (2001) 863–868.

[10] Gavrilyuck S., Teshukhov V., *Kinetic model for the motion of compressible bubbles in a perfect fluid*, Eur. J. Mech. B/Fluids, 21 (2002) 469–491.

[11] Golse F., *From kinetic to macroscopic models* in **Kinetic equations and asymptotic theory**, Perthame B., Desvillettes L. Eds., series in Appl. Math, vol. 4 (Gauthier-Villars, 2000) pp. 41–121.

[12] Golse F., Levermore C. D., Saint-Raymond L., *La méthode de l'entropie relative pour les limites hydrodynamiques de modèles cinétiques*, Séminaire Equations aux Derivées Partielles, 1999–2000, Exposé XIX (Ecole Polytechnique, Palaiseau, 2000).

[13] Golse F., Saint-Raymond L., *The Vlasov-Poisson system with strong magnetic field in quasineutral regime*, to appear in Math. Meth. Models in Appl. Sci.

[14] Goudon T., *Asymptotic problems for a kinetic model of two-phase flow*, Proc. Royal Soc. Edimburgh. 131, (2001) 1371–1384.

[15] Goudon T., Jabin P.-E., Vasseur A., *Hydrodymamic limit for the Vlasov-Navier-Stokes equations: Light particles regime*, Preprint.

[16] Hamdache K., *Global existence and large time behaviour of solutions for the Vlasov-Stokes equations*, Japan J. Industrial and Appl. Math., 15 (1998) 51–74.

[17] Hamdache K., Unpublished work, Personal Communication.

[18] Herrero H., Lucquin-Desreux B., Perthame B., *On the motion of dispersed balls in a potential flow: a kinetic description of the added mass effect*, SIAM J. Appl. Math., 60 (1999) 61–83.

[19] Itoh S., Tani A., *Solvability of nonstationary problems for nonhomogeneous incompressible fluids and the convergence with vanishing viscosity*, Tokyo J. Math., 22 (1999) 17–42.

[20] Jabin P.E., *Large time concentrations for solutions to kinetic equations with energy dissipation*, Comm. PDE., 25 (2000) 541–557.

[21] Jabin P.E., *Macroscopic limit of Vlasov type equations with friction*, Ann. IHP Anal. Non Linéaire, 17 (2000) 651–672.

[22] Jabin P.E., Perthame B., *Notes on mathematical problems on the dynamics of dispersed particles interacting through a fluid* in **Modeling in applied sciences, a kinetic theory approach**, N. Bellomo, M. Pulvirenti Eds. (Birkhäuser, 2000) pp. 111–147.

[23] Kullback S., *A lower bound for discrimination information in terms of variation*, IEEE Trans. Information Theory, 4 (1967) 126–127.

[24] Lions P.L., *Compactness in Boltzmann's equation via Fourier integral operators and applications. I, II, III*, J. Math. Kyoto Univ. 34 (1994) 391–461, 1994 and 34 (1994) 539–584.

[25] Lions P.L., **Mathematical topics in fluid mechanics** (Vol. 2. Compressible models) Oxford Lecture Series in Mathematics and its Applications, 10 (Oxford University Press, 1996).

[26] Russo G., Smereka P., *Kinetic theory for bubbly flows I, II*, SIAM J. Appl. Math., 56 (1996) 327–371.

[27] Saint-Raymond L., *The incompressible Euler limit of the Boltzmann equation: convergence proof*, Preprint 2002.

[28] Williams F. A., **Combustion theory** (Benjamin Cummings Publ., 2nd ed., 1985).

[29] Yau H.T., *Relative entropy and hydrodynamics of Ginzburg-Landau models,* Lett. Math. Phys., 22(1) (1991) 63–80.