Hydrodynamic Limit for the Vlasov-Navier-Stokes Equations. Part I: Light Particles Regime

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Abstract

The paper is devoted to the analysis of a hydrodynamic limit for the Vlasov-Navier-Stokes equations. This system is intended to model the evolution of particles interacting with a fluid. The coupling arises from the force terms. The limit problem consists of an advection-diffusion equation for the macroscopic density of the particles, the drift velocity being solution of the incompressible Navier-Stokes equation.

Key words. Fluid-particles interaction. Vlasov-Navier-Stokes equation. Hydrodynamic limits.

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1 Introduction

We consider a cloud of particles interacting with a fluid. The evolution of the particles is described through the density function $f(t, x, v) \geq 0$. Precisely, the integral

$$\int_\Omega \int_{\mathcal{V}} f(t,x,v)\,\mathrm{d}v\,\mathrm{d}x$$

is interpreted as the probable number of particles occupying, at time $t \geq 0$, a position in the set $\Omega \subset \mathbb{R}^N$, and having velocity in $\mathcal{V} \subset \mathbb{R}^N$. This quantity obeys the following Vlasov-type equation

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v(Ff) = r\Delta_v f. \tag{1.1}$$

Here, denoting by M the mass of a particle, MF is the force acting on the particle. The right hand side in (1.1), with a velocity diffusivity $r > 0$, describes the Brownian motion of the particles. In other words, (v, F) is the acceleration of the particles in phase space and particles follow the trajectories X, V solution of the ODEs system

$$\frac{\mathrm{d}}{\mathrm{d}t}X = V, \qquad \mathrm{d}V = F(X,V)\mathrm{d}t + \sqrt{r}\;\mathrm{d}B,$$

where $\mathrm{d}B$ is a white noise. Indeed, considering any family of solutions $(X_i,\ V_i)$ to the ODEs system, the associated distribution function $f = \sum_i \delta(x - X_i)\,\delta(v - V_i)$ satisfies equation (1.1).

On the other hand, the fluid is described by its velocity field $u(t,x) \in \mathbb{R}^N$. We assume that the cloud of particles is highly dilute so that we can suppose that the density of the gas remains constant $\rho_g > 0$. Accordingly, u verifies the following incompressible Navier-Stokes equation

$$\left\{ \begin{array}{l} \rho_g(\partial_t u + \mathrm{Div}_x(u \otimes u) + \nabla_x p) - \mu\Delta_x u = \mathfrak{F}, \\ \mathrm{div}_x(u) = 0. \end{array} \right. \tag{1.2}$$

Here, for $u = (u_1, ...u_N) \in \mathbb{R}^N$, we use the notation $u \otimes u$ to designate the matrix with components $u_i u_j$ whereas, A being a matrix valued function, $\mathrm{Div}_x(A) = \sum_{j=1}^N \partial_{x_j} A_{ij} \in \mathbb{R}^N$. In view of the incompressibility condition, we have of course $\mathrm{Div}_x(u \otimes u) = (u \cdot \nabla_x)u$. One denotes by $\mu > 0$ the dynamic viscosity of the fluid, and $\mathfrak{F}$ stands for the force density exerted on the fluid. Coupling between (1.1) and (1.2) is created through the force terms.

Of course the natural framework is $N = 3$. Let us describe further the model in this context. From now on, we suppose that the particles are spherically shaped with a constant radius $a > 0$. We denote by ρ_p the mass density of the particle, so that $M = \frac{4}{3}\pi a^3 \rho_p$. Neglecting gravity effects (particles are said neutrally buoyant), forces submitted by the particles reduce to the drag Stokes force, which is proportional to the relative velocity with the fluid: $MF = 6\pi\mu a\ (u - v)$. Therefore, we have

$$F(t, x, v) = \frac{9\mu}{2a^2\rho_p}\ \big(u(t, x) - v\big). \tag{1.3}$$

On the other hand, the diffusivity is given by the following Einstein formula

$$r = \frac{kT}{M}\ \frac{6\pi\mu a}{M} = \frac{kT}{M}\ \frac{9\mu}{2a^2\rho_p}, \tag{1.4}$$

which involves the Boltzmann constant k and the temperature $T > 0$ of the suspension, assumed constant. Finally, the effect of the particles motion on the fluid is obtained by summing the contributions of all the particles; we get

$$\mathfrak{F} = -\int_{\mathbb{R}^N} MFf\,\mathrm{d}v = 6\pi\mu a\ \int_{\mathbb{R}^3} f(v - u)\,\mathrm{d}v. \tag{1.5}$$

Therefore, we are concerned with the following system of partial differential equations

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v\Big(\dfrac{9\mu}{2a^2\rho_p}(u - v)f\Big) = \dfrac{kT}{M}\ \dfrac{9\mu}{2a^2\rho_p}\ \Delta_v f, \\ \\ \rho_g(\partial_t u + \mathrm{Div}_x(u \otimes u) + \nabla_x p) - \mu\Delta_x u = 6\pi\mu a\ \displaystyle\int_{\mathbb{R}^3} (v - u)\ f\,\mathrm{d}v, \\ \\ \mathrm{div}_x(u) = 0, \end{cases} \tag{1.6}$$

equipped with initial data f_0, u_0. We refer for details on the model to Caflisch-Papanicolaou [3], and to Williams [29] for application to combustion theory. Related models describing fluid/particles interactions can be found in the papers of Clouet-Domelevo [4], with a probablistic approach, Russo-Smerecka [26], Herrero-Lucquin-Perthame [17], Jabin-Perthame [20] with a special interest to potential flows. More recently, we also mention the work of Gavrilyuck-Teshukhov, [9]. Readers interested in mathematical studies of the system (1.6) should consult Hamdache [15] who investigated well-posedness and large time asymptotics. Here, we will deal with singular perturbation questions. Such kind of questions have been introduced by Hamdache [16], and results can be found in Berthonnaud [2], Domelevo-Vignal [7], Goudon [13], Jabin [18, 19] for some simplified situations. It is also worth mentioning that similar problems arise in plasma physics or in astrophysics, see Poupaud-Soler [25], Nieto-Poupaud-Soler [23].

The paper is organized as follows. In the next section, we write the equations in dimensionless form, identifying relevant dimensionless parameters. Then, we will give the precise statement of the main result of the paper: it is concerned with an asymptotic regime which leads to a convection-diffusion equation for the macroscopic density of the particles, coupled to the incompressible Navier-Stokes equation. In Section 3, we derive the fundamental a priori estimates satisfied by the sequence of solutions $(f^\varepsilon, u^\varepsilon)$, $\varepsilon > 0$ being the characteristic parameter of the regime. The crucial fact relies on the dissipation of a certain free energy associated to the whole system, as well as a control on its dissipation rate. Next, in Section 4, we present the details of the passage to the limit in the macroscopic equation.

2 Dimension Analysis and Main Result

First of all, we write the system (1.1) in a dimensionless form. Then, we discuss the ordering of the dimensionless parameters which appear in the equations. Eventually, we give the statement of the main result of the paper which establishes the convergence properties of the solutions.

2.1 Dimensionless Equations

Let us introduce time and length units, denoted by $\mathfrak{T}$ and $\mathcal{L}$ respectively. They naturally define the velocity unit $\mathcal{U} = \mathcal{L}/\mathfrak{T}$. The quantity

$$\tau = \frac{M}{6\pi\mu a} = \frac{2a^2\rho_p}{9\mu}$$

defines a relaxation time, the so-called Stokes settling time, which has to be compared to $\mathfrak{T}$. Next, we set

$$\sqrt{\theta} = \Big(\frac{kT}{M}\Big)^{1/2}$$

for a fluctuation of the thermal velocity which has to be compared to $\mathcal{U} = \mathcal{L}/\mathfrak{T}$. Then, we deal with dimensionless variables by writting

$$t = \mathfrak{T}\bar{t}, \qquad x = \mathcal{L}\,\bar{x}, \qquad v = \sqrt{\theta}\,\bar{v}, \qquad u(\mathfrak{T}\bar{t}, \mathcal{L}\bar{x}) = \mathcal{U}\,\bar{u}(\bar{t}, \bar{x}) = \frac{\mathcal{L}}{\mathfrak{T}}\,\bar{u}(\bar{t}, \bar{x})$$

and

$$\overline{f}(\bar{t}, \bar{x}, \bar{v}) = \frac{4}{3}\pi a^3\,\sqrt{\theta}^3\,f(\mathfrak{T}\bar{t}, \mathcal{L}\bar{x}, \sqrt{\theta}\bar{v}).$$

(Recall that $f\,\mathrm{d}x\,\mathrm{d}v$ has the meaning of a number of particles, so that $\overline{f}$ is dimensionless.) Note also that (1.5) rewrites as follows

$$\mathfrak{F} = \frac{\rho_p}{\tau}\int_{\mathbb{R}^3}\frac{4\pi a^3}{3}f(v-u)\,\mathrm{d}v.$$

Similarly, we set $\frac{\mathcal{L}^2}{\mathfrak{T}^2}\,\bar{p}(\bar{t}, \bar{x}) = p(\mathfrak{T}\bar{t}, \mathcal{L}\bar{x})$, which means that we use $\rho_g\mathcal{L}^2/\mathfrak{T}^2$ as a pressure unit.
Therefore, we get from (1.6)

$$\begin{cases} \dfrac{1}{\mathfrak{T}}\partial_{\bar{t}}\overline{f} + \dfrac{\sqrt{\theta}}{\mathcal{L}}\,\bar{v}\cdot\nabla_{\bar{x}}\overline{f} + \dfrac{1}{\sqrt{\theta}}\,\mathrm{div}_{\bar{v}}\Big(\dfrac{1}{\tau}\big(\dfrac{\mathcal{L}}{\mathfrak{T}}\bar{u} - \sqrt{\theta}\bar{v}\big)\overline{f}\Big) = \dfrac{\theta}{(\sqrt{\theta})^2\tau}\,\Delta_{\bar{v}}\overline{f}, \\ \dfrac{\mathcal{L}}{\mathfrak{T}^2}\,\partial_{\bar{t}}\bar{u} + \dfrac{\mathcal{L}}{\mathfrak{T}^2}\,\mathrm{Div}_{\bar{x}}(\bar{u}\otimes\bar{u}) + \dfrac{\mathcal{L}}{\mathfrak{T}^2}\,\nabla_{\bar{x}}\bar{p} - \dfrac{\mu}{\rho_g\mathcal{L}\mathfrak{T}}\,\Delta_{\bar{x}}\bar{u} \\ \qquad\qquad = \dfrac{1}{\tau}\,\dfrac{\rho_p}{\rho_g}\displaystyle\int_{\mathbb{R}^N}\big(\sqrt{\theta}\bar{v} - \dfrac{\mathcal{L}}{\mathfrak{T}}\bar{u}\big)\,\overline{f}\,\mathrm{d}\bar{v}, \\ \mathrm{div}_{\bar{x}}(\bar{u}) = 0. \end{cases} \tag{2.1}$$

Let us define the following dimensionless quantities (we keep the letter C for generic constants)

$$\begin{cases} A = \dfrac{\mathfrak{T}}{\mathcal{L}}\,\sqrt{\theta}, & B = \dfrac{\mathfrak{T}}{\tau}, \\ D = \dfrac{\mathfrak{T}}{\tau}\,\dfrac{\rho_p}{\rho_g} = B\,\dfrac{\rho_p}{\rho_g}, & E = \dfrac{2}{9}\Big(\dfrac{a}{\mathcal{L}}\Big)^2\dfrac{\mathfrak{T}}{\tau}\,\dfrac{\rho_p}{\rho_g} = \dfrac{2}{9}\Big(\dfrac{a}{\mathcal{L}}\Big)^2\,D. \end{cases} \tag{2.2}$$

Hence, dropping the overlines in (2.1), we are led to

$$\begin{cases} \partial_t f + Av\cdot\nabla_x f + B\,\mathrm{div}_v\Big(\big(\dfrac{1}{A}u - v\big)f - \nabla_v f\Big) = 0, \\ \partial_t u + \mathrm{Div}_x(u\otimes u) + \nabla_x p - E\Delta_x u = D(J - \rho u), \\ \mathrm{div}_x(u) = 0, \end{cases} \tag{2.3}$$

where we have used the notation

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)\,\mathrm{d}v,$$
$$J(t,x) = A \int_{\mathbb{R}^N} v\; f(t,x,v)\,\mathrm{d}v.$$

It remains to discuss the ordering of the quantities (2.2) with respect to some small parameter $\varepsilon > 0$, which leads to singular perturbation problems. In this paper, we are interested in the following ordering in (2.2)

$$\begin{cases} A = \varepsilon^{-1}, & B = \varepsilon^{-2}, \\ D = 1 = E. \end{cases} \tag{2.4}$$

Let us make a couple of comments about this scaling. The assumption on A means that $\sqrt{\theta} \gg \mathcal{U}$. Hence, this assumption means that the velocity of the gas flow is small compared to the molecular velocity $\sqrt{\theta}$. This is very close to the low Mach number regime in fluid hydrodynamics, see e.g. [12]. The assumption on B says that the relaxation time is far smaller than the typical time scale; i.e. the time scale of the interactions is very fast, which looks like the low Knudsen number regime. Finally, assumptions on D and E depends on the physical characteristics of the particles. Since D has order 1, we deduce that $\rho_p/\rho_g = \mathcal{O}(\varepsilon^2)$ which means that particles are very light. Furthermore, E being $\mathcal{O}(1)$, we deduce that $a \sim \mathcal{L}$. The companion paper [14] deals with the scaling $A = 1 = E$, $A = 1/\varepsilon = D$ where the size of the particles is very small.

2.2 Main Result

Therefore, here we are concerned with the behavior as ε goes to 0 of the solution $(f^\varepsilon, u^\varepsilon)$ of the following system

$$\begin{cases} \partial_t f^\varepsilon + \dfrac{1}{\varepsilon} v\cdot\nabla_x f^\varepsilon = \dfrac{1}{\varepsilon^2}\,\mathrm{div}_v\Big((v-\varepsilon u^\varepsilon)f^\varepsilon + \nabla_v f^\varepsilon\Big), \\ \partial_t u^\varepsilon + \mathrm{Div}_x(u^\varepsilon\otimes u^\varepsilon) + \nabla_x p^\varepsilon - \Delta_x u^\varepsilon = J^\varepsilon - \rho^\varepsilon u^\varepsilon, \\ \mathrm{div}_x(u^\varepsilon) = 0, \\ \rho^\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^N} f^\varepsilon(t,x,v)\,\mathrm{d}v,\;\; J^\varepsilon(t,x) = \int_{\mathbb{R}^N} \frac{v}{\varepsilon} f^\varepsilon(t,x,v)\,\mathrm{d}v, \end{cases} \tag{2.5}$$

endowed with initial conditions:

$$f^\varepsilon_{|t=0} = f^\varepsilon_0, \qquad u^\varepsilon_{|t=0} = u^\varepsilon_0.$$

Our analysis of the singularly perturbed problem is restricted to the two-dimensional case. We have presented the modeling in the three-dimensional case, which is certainly the physically most relevant one. Of course, similar reasoning applies to the two-dimensional situation, with slight modifications in the formula. For instance, the drag Stokes force would be like $C\mu(u-v)/(M|\ln a|)$ in 2D. Then, we are led to the same scaling discussion.

Next, let us make a few comments on the boundary conditions for this problem. Of course, for the velocity field u^ε the natural boundary condition would be the usual adhesion relation $u(t,x)=0$ on the boundary. The boundary condition for the density f^ε is more questionable. Indeed, it only prescribes the incoming trace $\gamma_- f^\varepsilon(t,x,v)$ for $v\cdot\nu(x)<0$, $\nu(x)$ being the outer normal at the point x. It can be simply a given non negative function, but it is certainly more realistic to suppose that it depends on the outgoing trace $\gamma_+ f^\varepsilon(t,x,v)$ for $v\cdot\nu(x)>0$ in some complicated way intended to describe how particles are reflected by the wall. As ε tends to 0, we are led to a macroscopic model, where the velocity variable v disappears (see (2.7) below). Hence, complicated phenomena can arise at the boundary which might require a tedious boundary layer analysis, as performed in e.g. [1], [24]. If the problem is completed with a boundary condition implying mass conservation, then it would be tempting to infer the Fourier-like boundary condition $\rho u - \nabla_x \rho = 0$ for the limit equation. But a rigorous proof would lead to technical difficulties due to a lack of estimates on traces. We mention with this respect the recent work by Masmoudi-Saint Raymond [22] about the Stokes limit from the Boltzmann equation.

Here, we avoid these boundary difficulties, and we consider the problem in the torus $\mathbb{T}^2$, with periodic boundary conditions. We work on weak solutions $f^\varepsilon \in C^0([0,T];L^1(\mathbb{T}^2\times\mathbb{R}^2))$, $u^\varepsilon \in C^0([0,T];L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2))$ of (2.5) verifying certain energy estimates (see section 3). We refer on existence of such solutions to [15]. Then, the main result of the paper states as follows.

Theorem 1 *Let the initial data* $f^\varepsilon_0 \geq 0$, $f^\varepsilon_0 \in L^1(\mathbb{T}^2\times\mathbb{R}^2)$ *and* $u^\varepsilon_0 \in L^2(\mathbb{T}^2)$

satisfy

$$\begin{cases} \displaystyle\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f_0^\varepsilon(1+v^2+|\ln(f_0^\varepsilon)|)\,\mathrm{d}v\,dx \le C_0, \\ \displaystyle\int_{\mathbb{T}^2} |u_0^\varepsilon|^2\,\mathrm{d}x \le C_0, \end{cases} \tag{2.6}$$

for some $C_0 > 0$, independent on ε. Let $0 < T < \infty$. Then, up to a subsequence, the macroscopic density ρ^ε converges weakly in $L^1((0,T)\times\mathbb{T}^2)$ to ρ, and u^ε converges weakly in $L^2(0,T;H^1(\mathbb{T}^2))$ and strongly in $L^2((0,T)\times\mathbb{T}^2)$ to u where (ρ,u) satisfies

$$\begin{cases} \partial_t\rho + \mathrm{div}_x(u\rho - \nabla\rho) = 0, \\ \partial_t u + \mathrm{Div}_x(u\otimes u) + \nabla_x p - \Delta_x u = 0, \\ \mathrm{div}_x(u) = 0. \end{cases} \tag{2.7}$$

If the initial data converge:

$$\rho_0^\varepsilon \rightharpoonup \rho_0 \quad \text{in } L^1(\mathbb{T}^2),$$

and

$$u_0^\varepsilon \rightharpoonup u_0 \quad \text{in } L^2(\mathbb{T}^2),$$

then the entire sequence $(\rho^\varepsilon, u^\varepsilon)$ converges to (ρ,u), which is the unique solution of (2.7) lying in $C^0([0,T];L^1(\mathbb{T})-weak) \times (L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2)))$ with $\nabla\rho \in L^2(0,T;\mathcal{M}^1(\mathbb{T}^2))$.

The limit problem consists of the incompressible Navier-Stokes equation for the velocity field u (with constant density), while the macroscopic density of the particles verifies an advection-diffusion equation, driven by the velocity u. This is referred to as the Kramer-Smoluchowski equation, see e.g. [3].

3 A priori Estimates

The proof starts with the following estimates on the microscopic quantity f^ε and the velocity field. Throughout the paper, we use the convention that C denotes a constant depending on (2.6) and T but not on ε, even if the value of the constant may vary from a line to another.

Proposition 1 *Let $(f^\varepsilon, u^\varepsilon)$ be the solution of (2.5) associated to initial data verifying (2.6). Then, the following assertions hold:*

i) $f^\varepsilon(1+v^2+|\ln(f^\varepsilon)|)$ *is bounded in* $L^\infty(0,\infty;L^1(\mathbb{T}^2\times\mathbb{R}^2))$*,*
ii) u^ε *is bounded in* $L^\infty(0,\infty;L^2(\mathbb{T}^2))$ *and in* $L^2(0,\infty;H^1(\mathbb{T}^2))$*,*
iii) The quantity $\frac{1}{\varepsilon}\big((v-\varepsilon u^\varepsilon)\sqrt{f^\varepsilon}+2\nabla_v\sqrt{f^\varepsilon}\big)=\frac{1}{\varepsilon}d^\varepsilon$ *is bounded in* $L^2((0,\infty)\times\mathbb{T}^2\times\mathbb{R}^2))$*.*

Remark 1 *We can rewrite the dissipation term* $\frac{1}{\varepsilon}\,d^\varepsilon$ *by means of the Maxwellian*

$$M^\varepsilon(t,x,v)=(2\pi)^{-1}\ \exp\Big(-\frac{|v-\varepsilon u^\varepsilon(t,x)|^2}{2}\Big).$$

We have

$$\frac{1}{\varepsilon}\ d^\varepsilon=\frac{2}{\varepsilon}\sqrt{M^\varepsilon}\ \nabla_v\sqrt{\frac{f^\varepsilon}{M^\varepsilon}}.$$

Formally, assuming that $\varepsilon u^\varepsilon\to 0$*,* $f^\varepsilon\to f$ *strongly enough, relation iii) would imply that*

$$(v-\varepsilon u^\varepsilon)\sqrt{f^\varepsilon}+2\nabla_v\sqrt{f^\varepsilon}\to 0=v\sqrt{f}+2\nabla_v\sqrt{f}$$

holds, which means that f *is nothing but a centred Maxwellian*

$$f(t,x,v)=\frac{\rho(t,x)}{2\pi}\ \mathrm{e}^{-v^2/2}.$$

Proof. We only give a formal derivation of these a priori estimates, which can be completely justified by suitable truncation, regularization argument, or within the construction of solutions, see e.g. [15]. First, we notice that the total mass is conserved

$$\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x=\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f_0^\varepsilon\,\mathrm{d}v\,\mathrm{d}x.$$

Thus, f^ε is bounded in $L^\infty(\mathbb{R}^+;L^1(\mathbb{T}^2\times\mathbb{R}^2))$. Next, we compute the time derivative of the free energy

$$\mathcal{E}(f^\varepsilon,u^\varepsilon)=\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon\big(\frac{v^2}{2}+\ln(f^\varepsilon)\big)\,\mathrm{d}v\,\mathrm{d}x+\int_{\mathbb{T}^2}\frac{(u^\varepsilon)^2}{2}\,\mathrm{d}x.$$

(It is the sum of the kinetic energy of both the particles and the fluid with the entropy of the particles.) We get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(f^\varepsilon,u^\varepsilon)+\int_{\mathbb{T}^2}|\nabla_x u^\varepsilon|^2\,\mathrm{d}x\\&=-\frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\Big((v-\varepsilon u^\varepsilon)f^\varepsilon+\nabla_v f^\varepsilon\Big)\cdot\Big(\frac{\nabla_v f^\varepsilon}{f^\varepsilon}+v\Big)\,\mathrm{d}v\,\mathrm{d}x\\&\quad+\frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\varepsilon u^\varepsilon\cdot(v-\varepsilon u^\varepsilon)f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

Remarking that $\int_{\mathbb{R}_2} \nabla_v f^\varepsilon \cdot u^\varepsilon \,\mathrm{d}v = 0$, we rewrite the right hand side as

$$-\frac{1}{\varepsilon^2}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \left|d^\varepsilon\right|^2 \mathrm{d}v\,\mathrm{d}x.$$

Therefore, integration with respect to time yields the following fundamental relation

$$\mathcal{E}(f^\varepsilon, u^\varepsilon)(t) + \int_0^t\int_{\mathbb{T}^2} |\nabla_x u^\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}s + \int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \frac{\left|d^\varepsilon\right|^2}{\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x \leq \mathcal{E}(f_0^\varepsilon, u_0^\varepsilon) \leq C_0. \tag{3.1}$$

We aim at deducing an estimate on the non negative quantity $f^\varepsilon|\ln(f^\varepsilon)|$. To this end, see e.g. [21], we write $s|\ln(s)| = s\ln(s) - 2s\ln(s)\chi_{0\leq s\leq 1}$, and, for $\omega \geq 0$, we split

$$\begin{aligned}-s\ln(s)\chi_{0\leq s\leq 1} &= -s\ln(s)\chi_{e^{-\omega}\leq s\leq 1} - s\ln(s)\chi_{e^{-\omega}\geq s}\\ &\leq s\omega + C\sqrt{s}\chi_{e^{-\omega}\geq s} \leq s\omega + Ce^{-\omega/2}.\end{aligned}$$

Let us use it with $s = f^\varepsilon$ and $\omega = v^2/8$. We are led to

$$\begin{aligned}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon|\ln(f^\varepsilon)|\,\mathrm{d}v\,\mathrm{d}x \leq& \int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon\ln(f^\varepsilon)\,\mathrm{d}v\,\mathrm{d}x\\ &+\frac{1}{4}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} v^2\; f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x + 2C\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} e^{-v^2/16}\,\mathrm{d}v\,\mathrm{d}x.\end{aligned}$$

We deduce that

$$\begin{aligned}&\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} f^\varepsilon(1+|\ln(f^\varepsilon)|)\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} v^2\; f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\\ &\qquad+\frac{1}{2}\int_{\mathbb{T}^2}\left|u^\varepsilon\right|^2\mathrm{d}x + \int_0^t\int_{\mathbb{T}^2}\left|\nabla_x u^\varepsilon\right|^2\mathrm{d}x\,\mathrm{d}s + \frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\left|d^\varepsilon\right|^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\\ &\leq \mathcal{E}(f^\varepsilon, u^\varepsilon)(t) + \int_0^t\int_{\mathbb{T}^2}\left|\nabla_x u^\varepsilon\right|^2\mathrm{d}x\,\mathrm{d}s + \frac{1}{\varepsilon^2}\int_0^t\int_{\mathbb{T}^2}\int_{\mathbb{R}^2}\left|d^\varepsilon\right|^2\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + C\end{aligned}$$

holds; and, by (3.1), it is bounded uniformly with respect to ε. This ends the proof. ∎

Next, for the macroscopic quantities, we have

Lemma 1 *Let the assumptions of Proposition 1 be fulfilled. Then,*
i) $\rho^\varepsilon(1+|\ln(\rho^\varepsilon)|)$ *is bounded in* $L^\infty(0,\infty; L^1(\mathbb{T}^2))$.
ii) $J^\varepsilon - \rho^\varepsilon u^\varepsilon$ *is bounded in* $L^2(0,\infty; L^1(\mathbb{T}^2))$.

Proof. By Proposition 1-i), ρ^ε is bounded in $L^\infty(0,\infty;L^1(\mathbb{T}^2))$. Next, since $h(s)=s\ln(s)$ is convex, the Jensen inequality yields

$$h(\rho^\varepsilon)=h\Big(\int_{\mathbb{R}^2}\frac{f^\varepsilon}{M}\;M\,\mathrm{d}v\Big)\le\int_{\mathbb{R}^2}h\Big(\frac{f^\varepsilon}{M}\Big)\;M\,\mathrm{d}v,$$

with $M(v)=\frac{1}{2\pi}\mathrm{e}^{-v^2/2}$. The right hand side is $\int_{\mathbb{R}^2}f^\varepsilon\big(\ln f^\varepsilon+v^2/2\big)\,\mathrm{d}v+\ln(2\pi)\rho^\varepsilon$ which is bounded in $L^\infty(0,\infty;L^1(\mathbb{T}^2))$. Hence, we have the bound from above

$$\int_{\mathbb{T}^2}\rho^\varepsilon\ln(\rho^\varepsilon)\,\mathrm{d}x\le C.$$

However, $-s\ln(s)\le 1/\mathrm{e}$ and we deduce that

$$\int_{\mathbb{T}^2}\rho^\varepsilon|\ln(\rho^\varepsilon)|\,\mathrm{d}x=\int_{\mathbb{T}^2}\rho^\varepsilon\ln(\rho^\varepsilon)\,\mathrm{d}x-2\int_{\mathbb{T}^2}\rho^\varepsilon\ln(\rho^\varepsilon)\chi_{0\le\rho^\varepsilon\le 1}\,\mathrm{d}x\le C+2|\mathbb{T}^2|/\mathrm{e}.$$

The proof of ii) starts with the remark

$$J^\varepsilon-\rho^\varepsilon u^\varepsilon=\frac{1}{\varepsilon}\int_{\mathbb{R}^2}(v-\varepsilon u^\varepsilon)f^\varepsilon\,\mathrm{d}v=\frac{1}{\varepsilon}\int_{\mathbb{R}^2}d^\varepsilon\;\sqrt{f^\varepsilon}\,\mathrm{d}v.$$

Then, the conclusion follows by applying the Cauchy-Schwarz inequality and using Proposition 1-i) and iii). ∎

To go further, we shall use the restriction to the two dimensional framework. Indeed, we can deduce additional bounds by using the following claim.

Lemma 2 *Let Ω be a bounded domain in $\mathbb{R}^2$. Let $\Phi\in L^2(0,T;H^1_0(\Omega))$. Let $\rho\ge 0$ verify*

$$\sup_{0\le t\le T}\int_\Omega\rho|\ln(\rho)|\,\mathrm{d}x<\infty.$$

Then there exists a constant C depending only on Ω such that

$$\int_0^T\int_\Omega\rho|\Phi|^2\,\mathrm{d}x\,\mathrm{d}t\le C\left(1+\sup_{0\le t\le T}\int_{\mathbb{T}^2}\rho|\ln(\rho)|\,\mathrm{d}x\right)\;\|\Phi\|^2_{L^2(0,T;H^1(\Omega))}.$$

Proof. The proof relies on the Trudinger inequality (see [10], Th. 7.15 p. 162): there exist two constants, $0<\sigma,K<\infty$, such that for every function $\phi\in H^1_0(\Omega)$ we have

$$\int_\Omega\exp\Big(\sigma\;\frac{\phi^2}{\|\nabla\phi\|^2_{L^2(\Omega)}}\Big)\,\mathrm{d}x\le K.\tag{3.2}$$

Let us set

$$\overline{\Phi}(t,x) = \frac{\Phi(t,x)}{\|\nabla\Phi(t,\cdot)\|_{L^2(\Omega)}}.$$

We split the integral to be evaluated as follows

$$\int_\Omega \rho|\Phi|^2\,\mathrm{d}x = \int_{\{\rho\le 1\}} \rho|\Phi|^2\,\mathrm{d}x + \|\nabla\Phi\|^2_{L^2(\Omega)} \int_{\{\rho\ge 1\}} \rho|\overline{\Phi}|^2\,\mathrm{d}x.$$

The first term is obviously bounded by $\|\Phi\|^2_{L^2(\Omega)}$. On the other hand, we can split the second integral as

$$\begin{aligned}\int_{\{\rho\ge 1\}} \rho|\overline{\Phi}|^2\,\mathrm{d}x &= \int_{\{1\le\rho\le\exp(\sigma|\overline{\Phi}|^2/2)\}} \ldots\,\mathrm{d}x + \int_{\{\rho\ge\exp(\sigma|\overline{\Phi}|^2/2)\}} \ldots\,\mathrm{d}x \\ &\le \int_\Omega \exp\left(\frac{\sigma}{2}|\overline{\Phi}|^2\right)|\overline{\Phi}|^2\,\mathrm{d}x + \frac{2}{\sigma}\int_\Omega \rho|\ln(\rho)|\,\mathrm{d}x\end{aligned}$$

Since $y \le e^y$ for every y, using (3.2), we find

$$\int_\Omega \exp\left(\frac{\sigma}{2}|\overline{\Phi}|^2\right)\frac{\sigma}{2}\,|\overline{\Phi}|^2\,\mathrm{d}x \le \int_{\mathbb{T}^2} \exp\left(\sigma|\overline{\Phi}|^2\right)\mathrm{d}x \le K.$$

Hence, it follows that

$$\begin{aligned}&\int_0^T\int_\Omega \rho|\Phi|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\le \|\Phi\|^2_{L^2(0,T;L^2(\Omega))} + \frac{2}{\sigma}\left(K + \sup_{0\le t\le T}\int_\Omega \rho|\ln\rho|\,\mathrm{d}x\right)\int_0^T\int_\Omega |\nabla_x\Phi|^2\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

holds. ∎

Corollary 1 *Let the assumptions of Proposition 1 be fulfilled. Then, the following assertions hold*
i) $\rho^\varepsilon|u^\varepsilon|^2$ *is bounded in* $L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{T}^2)$,
ii) $\rho^\varepsilon u^\varepsilon$ *and* J^ε *are bounded in* $L^2(0,T;L^1(B_R))$, *for any* $0 < T < \infty$, $B_R \subset \mathbb{T}^2$.

Proof. The first claim is an immediate consequence of Lemma 2 (applied with ρ^ε and $\Phi = u^\varepsilon\varphi$, where $\varphi \in C_c^\infty((0,\infty)\times\mathbb{T}^2)$), combined to Lemma 1-i) and Proposition 1-ii). Since $\rho^\varepsilon|u^\varepsilon| = \sqrt{\rho^\varepsilon}\,\sqrt{\rho^\varepsilon|u^\varepsilon|^2}$, we can apply the Cauchy-Schwarz inequality to obtain the estimate on $\rho^\varepsilon u^\varepsilon$ by using i) and Lemma 1-i). The estimate on $J^\varepsilon = (J^\varepsilon - \rho^\varepsilon u^\varepsilon) + \rho^\varepsilon u^\varepsilon$ then follows from Lemma 1-ii). ∎

4 Passing to the limit

From the estimates discussed above, we can suppose, possibly at the cost of extracting subsequences, that

$$\begin{cases} \rho^\varepsilon \rightharpoonup \rho \quad \text{weakly in } L^1((0,T)\times\mathbb{T}^2), \\ J^\varepsilon \rightharpoonup J \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^2), \\ u^\varepsilon \rightharpoonup u \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^2), \\ \nabla_x u^\varepsilon \rightharpoonup \nabla_x u \quad \text{weakly in } L^2((0,T)\times\mathbb{T}^2), \\ \rho^\varepsilon u^\varepsilon \rightharpoonup \nu \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{T}^2). \end{cases} \tag{4.1}$$

However, it seems difficult in view of the available bounds to discuss the behavior of the microscopic quantity f^ε. Instead, we look at the moment equations.

Step 1- Moment Equations

Multiplying the Vlasov equation by 1 and integrating with respect to v give the continuity equation

$$\partial_t \rho^\varepsilon + \mathrm{div}_x J^\varepsilon = 0. \tag{4.2}$$

As ε goes to 0, it becomes, at least in the $\mathcal{D}'((0,\infty)\times\mathbb{T}^2)$ sense,

$$\partial_t \rho + \mathrm{div}_x J = 0. \tag{4.3}$$

It is also worth noting that (4.2), combined to the bound on J^ε in L^1_{loc}, tells us that the sequence $\left(\int_{\mathbb{T}^2} \rho^\varepsilon\, \varphi \,\mathrm{d}x\right)_{\varepsilon>0}$ lies in a compact set of $C^0([0,T])$, for any $\varphi \in C_c^\infty(\mathbb{T}^2)$. By using an approximation argument, one deduces that ρ^ε converges to ρ in $C^0([0,T]; L^1(\mathbb{R}^2) - weak)$.

Next, multiplying the Vlasov equation by v and integrating, we are led to

$$\varepsilon^2 \partial_t J^\varepsilon + \mathrm{Div}_x \mathbb{P}^\varepsilon = \frac{1}{\varepsilon}\int_{\mathbb{R}^2} \big((\varepsilon u^\varepsilon - v) f^\varepsilon - \nabla_v f^\varepsilon\big)\,\mathrm{d}v = \rho^\varepsilon u^\varepsilon - J^\varepsilon, \tag{4.4}$$

where the pressure tensor reads

$$\mathbb{P}^\varepsilon = \int_{\mathbb{R}^2} v\otimes v\ f^\varepsilon \,\mathrm{d}v.$$

Step 2- Limit of the Kinetic Pressure
In (4.4), the pressure term recasts as

$$\mathbb{P}^\varepsilon = \int_{\mathbb{R}^2} d^\varepsilon \otimes v\sqrt{f^\varepsilon}\,\mathrm{d}v + \int_{\mathbb{R}^2} \varepsilon u^\varepsilon \otimes v f^\varepsilon\,\mathrm{d}v - 2\int_{\mathbb{R}^2} \nabla_v\sqrt{f^\varepsilon} \otimes v\sqrt{f^\varepsilon}\,\mathrm{d}v. \quad (4.5)$$

Actually, the last term is nothing but $\rho^\varepsilon \mathbb{I}$. By using the Cauchy-Schwarz inequality and Proposition 1-iii), we realize that the $L^1((0,T)\times\mathbb{T}^2)$ norm of the first term is dominated by

$$\left(\int_0^T\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} \left|d^\varepsilon\right|^2 \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \left(\int_0^T\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} v^2\ f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s\right)^{1/2} \le C\,\varepsilon.$$

Let us show that the second term in the right hand side of (4.5) tends to 0 in $\mathcal{D}'((0,\infty)\times\mathbb{T}^2)$. Let $\varphi \in C_c^\infty((0,\infty)\times\mathbb{T}^2)$. We notice that

$$\begin{aligned}
&\int_0^T\int_{\mathbb{T}^2}\left|\varphi\int_{\mathbb{R}^2} u^\varepsilon\otimes v f^\varepsilon\,\mathrm{d}v\right|\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \int_0^T\int_{\mathbb{T}^2} |\varphi u^\varepsilon| \left(\int_{\mathbb{R}^2} f^\varepsilon\,\mathrm{d}v\right)^{1/2}\left(\int_{\mathbb{R}^2} v^2 f^\varepsilon\,\mathrm{d}v\right)^{1/2}\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \int_0^T\int_{\mathbb{T}^2} \sqrt{\rho^\varepsilon}\ |\varphi u^\varepsilon| \left(\int_{\mathbb{R}^2} v^2 f^\varepsilon\,\mathrm{d}v\right)^{1/2}\mathrm{d}x\,\mathrm{d}t\\
&\quad\le \left(\int_0^T\int_{\mathbb{T}^2} \rho^\varepsilon|\varphi u^\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int_0^T\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} v^2 f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
\end{aligned}$$

As a consequence of Corollary 1 and Proposition 1-i), the right hand side is bounded uniformly with respect to ε. Therefore, we proved

$$\mathbb{P}^\varepsilon \rightharpoonup \rho\mathbb{I} \qquad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{T}^2).$$

Coming back to (4.4), we get, as $\varepsilon \to 0$

$$\nabla_x \rho = \nu - J \quad (4.6)$$

in $\mathcal{D}'((0,T)\times\mathbb{T}^2)$. We are thus left with the task of identifying the limit ν of $\rho^\varepsilon u^\varepsilon$.

Step 3- Fluid Equation
Let us go back to the fluid equation. In view of Lemma 1, the right hand side $J^\varepsilon - \rho^\varepsilon u^\varepsilon$ of the fluid equation in (2.5) is in a bounded set of $L^2(0,T;L^1(\mathbb{T}^2))$.

Actually, the previous step shows that it converges to $\nabla_x \rho$ (at least in $\mathcal{D}'((0,\infty)\times\mathbb{T}^2)$). Hence, provided we are able to pass to the limit in the non linear convective term, u satisfies the usual incompressible Navier-Stokes equation

$$\partial_t u + \mathrm{Div}_x(u\otimes u) - \Delta u + \nabla_x p = 0$$

with $\mathrm{div}_x(u) = 0$. This conclusion follows from the following compactness property.

Lemma 3 *The sequence u^ε is relatively compact in $L^2((0,T)\times\mathbb{T}^2)$ and in $C^0([0,T];L^2(\mathbb{T}^2)-weak)$.*

Proof. We know from Proposition 1 that u^ε belongs to a bounded set in $L^2(0,T;H^1(\mathbb{T}^2))$ and in $L^\infty(0,T;L^2(\mathbb{T}^2))$. Then, the conclusion follows from an application of Aubin's compactness lemma, see e. g. [27], which requires a bound on $\partial_t u^\varepsilon$. Precisely, we shall show that the pressure term $\nabla_x p^\varepsilon$ lies in a bounded set of $L^2(0,T;W^{-1,1}(\mathbb{T}^2))$. In turn, coming back to the Navier-Stokes equation in (2.5), we realize that $\partial_t u^\varepsilon$ is also bounded in $L^2(0,T;W^{-1,1}(\mathbb{T}^2))$. To obtain the announced bound, we apply the div_x operator to the Navier-Stokes equation in (2.5). The incompressibility condition allows us to get rid of the time derivative and we get

$$\Delta p^\varepsilon = \mathrm{div}_x(R^\varepsilon)$$

with $R^\varepsilon(t,x) = (J^\varepsilon - \rho^\varepsilon u^\varepsilon) - (u^\varepsilon\cdot\nabla_x)u^\varepsilon$. Thus, bounds in Proposition 1 garantee that R^ε lies in a bounded set in $L^2(0,T;L^1(\mathbb{T}^2))$. We deduce that p^ε is bounded in $L^2(0,T;L^1(\mathbb{T}^2))$. This remark ends the proof of Lemma 3.
∎

Step 4- Macroscopic Density Equation
We shall end the proof of Theorem 1 by using the following compactness argument.

Lemma 4 *The sequence $\rho^\varepsilon u^\varepsilon$ converges to $\nu = \rho u$ in $\mathcal{D}'((0,\infty)\times\mathbb{T}^2)$.*

Let us postpone temporarily the proof. Coming back to (4.6), we obtain $J = \rho u - \nabla_x \rho$. Inserting this relation in the mass conservation equation leads to

$$\partial_t \rho + \mathrm{div}_x(\rho u - \nabla_x \rho) = 0,$$

which ends the proof of the convergence announced in Theorem 1.

Proof of Lemma 4. First of all, we check that $\rho\ln(\rho)$, with ρ the weak limit of ρ^ε in $L^\infty(0,T;L^1(\mathbb{T}^2))$, belongs to $L^\infty(0,T;L^1(\mathbb{T}^2))$. (This is a standard consequence of the convexity of $s\ln(s)$, combined to Lemma 1-i).) Let $\varphi \in C_c^\infty((0,\infty)\times\mathbb{T}^2)$. For $M>0$, let us write

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{T}^2}(\rho^\varepsilon u^\varepsilon-\rho u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_0^\infty\int_{\mathbb{T}^2}(\rho^\varepsilon-\rho)\ u\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\int_{\mathbb{T}^2}\rho^\varepsilon(u^\varepsilon-u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{\{|u\varphi|\le M\}}(\rho^\varepsilon-\rho)\ u\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\{|u\varphi|>M\}}(\rho^\varepsilon-\rho)\ u\varphi\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_{\{\rho^\varepsilon\le M\}}\rho^\varepsilon(u^\varepsilon-u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\{\rho^\varepsilon>M\}}\rho^\varepsilon(u^\varepsilon-u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

For $M>0$, fixed, $u\varphi\chi_{|u\varphi|\le M}$ lies in $L^\infty((0,T)\times\mathbb{T}^2)$; hence the weak convergence of ρ^ε to ρ implies that

$$\lim_{\varepsilon\to 0}\int_{\{|u\varphi|\le M\}}(\rho^\varepsilon-\rho)\ u\varphi\,\mathrm{d}x\,\mathrm{d}t=0.$$

Similarly, by the Cauchy-Schwarz inequality, we get

$$\left|\int_{\{\rho^\varepsilon\le M\}}\rho^\varepsilon(u^\varepsilon-u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t\right|\le M\ \|\varphi\|_{L^2((0,T)\times\mathbb{T}^2)}\ \|u^\varepsilon-u\|_{L^2((0,T)\times\mathbb{T}^2)}\xrightarrow[\varepsilon\to 0]{}0$$

by using Lemma 3. Therefore, the conclusion

$$\lim_{\varepsilon\to 0}\int_0^\infty\int_{\mathbb{T}^2}(\rho^\varepsilon u^\varepsilon-\rho u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t=0$$

follows provided we are able to justify that

$$\begin{cases}\displaystyle\sup_{\varepsilon>0}\left(\int_{\{|u\varphi|>M\}}(\rho^\varepsilon-\rho)\ u\varphi\,\mathrm{d}x\,\mathrm{d}t\right)\xrightarrow[M\to\infty]{}0,\\ \displaystyle\sup_{\varepsilon>0}\left(\int_{\{\rho^\varepsilon>M\}}\rho^\varepsilon(u^\varepsilon-u)\ \varphi\,\mathrm{d}x\,\mathrm{d}t\right)\xrightarrow[M\to\infty]{}0\end{cases}$$

holds.

To this end, we use the estimate of Lemma 2. Indeed, the Cauchy-Schwarz inequality yields

$$
\begin{aligned}
&\left|\int_{\{\rho^\varepsilon > M\}} \rho^\varepsilon (u^\varepsilon - u)\ \varphi \,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad \le \left(\int_{\{\rho^\varepsilon > M\}} \rho^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^T\!\!\int_{\mathbb{T}^2} \rho^\varepsilon\ |(u^\varepsilon - u)\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\le \frac{1}{\sqrt{\ln(M)}} \left(\int_0^T\!\!\int_{\mathbb{T}^2} \rho^\varepsilon |\ln(\rho^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\qquad\qquad \times \left(C\Big(1 + \sup_{0\le t\le T} \int_{\mathbb{T}^2} \rho^\varepsilon |\ln(\rho^\varepsilon)| \,\mathrm{d}x\,\mathrm{d}t\Big)\right)^{1/2} \|(u^\varepsilon - u)\varphi\|_{L^2(0,T;H^1(\mathbb{T}^2))} \\
&\le \frac{C}{\sqrt{\ln(M)}}.
\end{aligned}
$$

Eventually, we have

$$
\begin{aligned}
&\left|\int_{\{|u\varphi| > M\}} (\rho^\varepsilon - \rho)\ u\varphi \,\mathrm{d}x\,\mathrm{d}t\right| \le \int_{\{|u\varphi| > M\}} (\rho^\varepsilon + \rho)\ |u\varphi| \,\mathrm{d}x\,\mathrm{d}t \\
&\le \left(\int_{\{|u\varphi| > M\}} (\rho^\varepsilon + \rho) \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^T\!\!\int_{\mathbb{T}^2} (\rho^\varepsilon + \rho)|u\varphi|^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\le \left(\int_{\{|u\varphi| > M\}} (\rho^\varepsilon + \rho) \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2} \times \left(C\Big(2 + \sup_{0\le t\le T} \int_{\mathbb{T}^2} \rho^\varepsilon |\ln(\rho^\varepsilon)| \,\mathrm{d}x\right. \\
&\qquad \left. + \sup_{0\le t\le T} \int_{\mathbb{T}^2} \rho |\ln(\rho)| \,\mathrm{d}x\Big)\right)^{1/2} \|u\varphi\|_{L^2(0,T;H^1(\mathbb{T}^2))} \\
&\le C \left(\int_{\{|u\varphi| > M\}} (\rho^\varepsilon + \rho) \,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
\end{aligned}
$$

However, $\mathrm{meas}\big(\{|u\varphi| > M\}\big) \to 0$ as $M \to 0$, so that, by using the Dunford-Pettis theorem (see [8] Th. 4.21.2 p. 274), the equi-integrability of ρ^ε, and the integrability of ρ lead to

$$
\left(\sup_{\varepsilon > 0} \int_{\{|u\varphi| > M\}} \rho^\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \int_{\{|u\varphi| > M\}} \rho \,\mathrm{d}x\,\mathrm{d}t\right) \xrightarrow[M\to\infty]{} 0.
$$

This ends the proof. ∎

Step 5- Uniqueness

We can complete the result by investigating the uniqueness of the solution of the limit problem. The velocity field u is solution in the two-dimensional torus of the incompressible Navier-Stokes equation with the regularity $u \in L^\infty(0,T;L^2(\mathbb{T}^2)) \cap L^2(0,T;H^1(\mathbb{T}^2))$, associated to the initial data $u_0 \in L^2(\mathbb{T}^2)$ (in the sense that $u \in C^0([0,T];L^2(\mathbb{T}^2) - weak)$). This is a well-known fact that the incompressible Navier-Stokes equation has a unique solution in this class and actually $u \in C^0([0,T];L^2(\mathbb{T}^2))$, see e.g. [28].

On the other hand, we know that $\rho \in L^\infty(0,T;L^1(\mathbb{T}^2))$, with $\sup_{0\le t\le T} \int_{\mathbb{T}^2} \rho|\ln(\rho)|\,\mathrm{d}x < \infty$ and $\rho \in C^0([0,T];L^1(\mathbb{T}^2) - weak)$. Furthermore, coming back to (4.6), we notice that $\nabla_x\rho$ is the limit in $\mathcal{D}'((0,T)\times\mathbb{T}^2)$ of $\rho^\varepsilon u^\varepsilon - J^\varepsilon$. Since this sequence is bounded in $L^2(0,T;L^1(\mathbb{T}^2))$ (see Lemma 1-ii)), we deduce that ρ belongs to $L^2(0,T;BV(\mathbb{T}^2))$. (Such a gain of regularity for the limit of the macroscopic density is usual in diffusion asymptotics.) According to the Sobolev embedding $BV \subset L^{N/(N-1)}$, see [11], we thus have $\rho \in L^2((0,T)\times\mathbb{T}^2)$. Then, we show the following result.

Lemma 5 *Let u be a divergence free vector field in $L^2(0,T;H^1(\mathbb{T}^2))$. Let $\rho \in L^\infty(0,T;L^1(\mathbb{T}^2)) \cap L^2((0,T)\times\mathbb{T}^2)$ solution of*

$$\partial_t\rho + \mathrm{div}_x(u\rho - \nabla_x\rho) = 0$$

with $\rho \in C^0([0,T];L^1(\mathbb{T}^2) - weak)$, associated to the initial data $\rho_0 \in L^1(\mathbb{T}^2)$. Then, ρ belongs to $C^0([0,T];L^1(\mathbb{T}^2))$. If $\rho_0 = 0$ then $\rho = 0$.

Proof. Our arguments are quite close to the analysis of transport equations in [5]. First, let us show the uniqueness statement. Since the problem is linear with respect to ρ we are led to prove that ρ, the solution corresponding to the identically 0 initial data (with the regularity discussed above), vanishes. Let $\zeta_\varepsilon(x) = \varepsilon^{-2}\zeta(x/\varepsilon)$ be a sequence of mollifiers. Set $\rho_\varepsilon(t,x) = \big(\rho(t,\cdot) * \zeta_\varepsilon\big)(x)$. It verifies

$$\partial_t\rho_\varepsilon + \mathrm{div}_x(u\rho_\varepsilon - \nabla_x\rho_\varepsilon) = r^\varepsilon$$

where

$$r^\varepsilon = \mathrm{div}_x(u(\rho * \zeta_\varepsilon) - (u\rho) * \zeta_\varepsilon).$$

Reproducing the arguments in [5], we check that $r_\varepsilon \to 0$ in $L^1((0,T)\times\mathbb{T}^2)$ as ε goes to 0 (by using that ρ and $\nabla_x u$ are both in $L^2((0,T)\times\mathbb{T}^2)$). Let us set

$$Z_\eta(s) = \int_0^s \frac{z}{\sqrt{\eta + z^2}}\,\mathrm{d}z$$

which approaches the absolute value, and verifies $|Z'_\eta(s)| \le 1$. We have (using that u is divergence free)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2} Z_\eta(\rho_\varepsilon)\,\mathrm{d}x + \int_{\mathbb{T}^2} Z''_\eta(\rho_\varepsilon)\ |\nabla_x \rho_\varepsilon|^2\,\mathrm{d}x = \int_{\mathbb{T}^2} Z'_\eta(\rho_\varepsilon)\ r_\varepsilon\,\mathrm{d}x.$$

Neglecting the dissipative term $Z''_\eta(\rho_\varepsilon)\ |\nabla_x \rho_\varepsilon|^2 \ge 0$, we get after integration with respect to time

$$\int_{\mathbb{T}^2} Z_\eta(\rho_\varepsilon)(t)\,\mathrm{d}x \le \int_{\mathbb{T}^2} Z_\eta(\rho_\varepsilon)(0)\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^2} |r_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s.$$

Since $\rho_{\varepsilon|t=0} = 0$, passing to the limit $\eta \to 0$ (by using the monotone convergence theorem), we obtain

$$\int_{\mathbb{T}^2} |\rho_\varepsilon(t)|\,\mathrm{d}x \le \int_0^t\int_{\mathbb{T}^2} |r_\varepsilon|\,\mathrm{d}x\,\mathrm{d}s.$$

Letting $\varepsilon \to 0$ yields $\rho = 0$.

Second, let us justify the continuity with respect to time. We restrict to the continuity at $t = 0$. We recall that $\{\rho(t),\, t \in [0, T]\}$ lies in a weakly compact set of $L^1(\mathbb{T}^2)$. Let $M > 0$ and write

$$\begin{aligned}
\int_{\mathbb{T}^2} |\rho(t) - \rho_0|\,\mathrm{d}x \ &\le\ \int_{\mathbb{T}^2} |\rho(t) - \rho_0| \chi_{\{|\rho(t)|\le M\}}\chi_{\{|\rho_0|\le M\}}\,\mathrm{d}x \\
&\quad + \int_{\mathbb{T}^2} |\rho(t) - \rho_0| (\chi_{\{|\rho(t)|\ge M\}} + \chi_{\{|\rho_0|\ge M\}})\,\mathrm{d}x \\
&\le\ \left(\int_{\mathbb{T}^2} |\rho(t) - \rho_0|^2 \chi_{\{|\rho(t)|\le M\}}\chi_{\{|\rho_0|\le M\}}\,\mathrm{d}x\right)^{1/2} |\mathbb{T}^2|^{1/2} \\
&\quad + \int_{|\rho(t)|\ge M} \big(|\rho(t)| + |\rho_0|\big)\,\mathrm{d}x + \int_{|\rho_0|\ge M} \big(|\rho(t)| + |\rho_0|\big)\,\mathrm{d}x.
\end{aligned}$$

We notice that $\sup_{0\le t\le T} \mathrm{meas}(\{|\rho(t)| \ge M\}) \le \sup_{0\le t\le T} \big(\int_{\mathbb{T}^2} |\rho(t)|\,\mathrm{d}x\big)/M \le C/M$, and $\mathrm{meas}(\{|\rho_0| \ge M\}) \le \big(\int_{\mathbb{T}^2} |\rho_0|\,\mathrm{d}x\big)/M \le C/M$. Hence, by using the Dunford-Pettis theorem, integrability of ρ_0 and equi-integrability of $\{\rho(t),\, 0 \le t \le T\}$ imply that

$$\sup_{0\le t\le T}\left(\int_{|\rho(t)|\ge M} \big(|\rho(t)| + |\rho_0|\big)\,\mathrm{d}x + \int_{|\rho_0|\ge M} \big(|\rho(t)| + |\rho_0|\big)\,\mathrm{d}x\right) \xrightarrow[M\to\infty]{} 0.$$

We are thus left with the task of proving that, for $M > 0$ fixed,

$$\int_{\mathbb{T}^2} |\rho(t) - \rho_0|^2 \chi_{\{|\rho(t)| \le M\}} \chi_{\{|\rho_0| \le M\}} \,\mathrm{d}x \xrightarrow[t \to 0]{} 0.$$

Let $Z : \mathbb{R} \to \mathbb{R}$ be a strictly convex function, with Z' bounded. Since Z" is bounded from below on compact sets, we have

$$\begin{aligned}
&\int_{\mathbb{T}^2} \big(Z(\rho(t)) - Z(\rho_0) - Z'(\rho_0)(\rho(t) - \rho_0)\big) \,\mathrm{d}x \\
&\qquad = \int_{\mathbb{T}^2} \int_0^1 (1-\theta)\ Z\text{"}(\rho_0 + \theta(\rho(t) - \rho_0))\ (\rho(t) - \rho_0)^2 \,\mathrm{d}\theta \,\mathrm{d}x \\
&\qquad \ge \alpha_M \int_{\mathbb{T}^2} |\rho(t) - \rho_0|^2 \chi_{\{|\rho(t)| \le M\}} \chi_{\{|\rho_0| \le M\}} \,\mathrm{d}x
\end{aligned}$$

for a certain constant $\alpha_M > 0$. Since $\rho(t) \rightharpoonup \rho_0$ weakly in $L^1(\mathbb{T}^2)$ as $t \to 0$, and $Z'(\rho_0)$ belongs to $L^\infty(\mathbb{T}^2)$ we have $\int_{\mathbb{T}^2} Z'(\rho_0)(\rho(t) - \rho_0) \,\mathrm{d}x \to 0$ as $t \to 0$. Hence, the conclusion follows if we are able to prove that $\int_{\mathbb{T}^2} Z(\rho(t)) \,\mathrm{d}x \to \int_{\mathbb{T}^2} Z(\rho_0) \,\mathrm{d}x$ as $t \to 0$.
First, we immediately notice that

$$\begin{aligned}
\liminf_{t \to 0} \int_{\mathbb{T}^2} Z(\rho)(t) \,\mathrm{d}x &\ge \int_{\mathbb{T}^2} Z(\rho_0) \,\mathrm{d}x + \lim_{t \to 0} \int_{\mathbb{T}^2} Z'(\rho_0)(\rho(t) - \rho_0) \,\mathrm{d}x \\
&\ge \int_{\mathbb{T}^2} Z(\rho_0) \,\mathrm{d}x.
\end{aligned}$$

Next, consider the regularization $\rho_\varepsilon(t, x) = \big(\rho(t, \cdot) * \zeta_\varepsilon\big)(x)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^2} Z(\rho_\varepsilon) \,\mathrm{d}x + \int_{\mathbb{T}^2} Z''(\rho_\varepsilon)\ |\nabla_x \rho_\varepsilon|^2 \,\mathrm{d}x = \int_{\mathbb{T}^2} Z'(\rho_\varepsilon)\ r_\varepsilon \,\mathrm{d}x.$$

Neglecting the dissipative term $Z''(\rho_\varepsilon)\ |\nabla_x \rho_\varepsilon|^2 \ge 0$, we get after integration with respect to time

$$\int_{\mathbb{T}^2} Z(\rho_\varepsilon)(t) \,\mathrm{d}x \le \int_{\mathbb{T}^2} Z(\rho_\varepsilon)(0) \,\mathrm{d}x + C \int_0^t \int_{\mathbb{T}^2} |r_\varepsilon| \,\mathrm{d}x \,\mathrm{d}s.$$

(We used $Z'(s) \le C$ again.) Letting $\varepsilon \to 0$ yields

$$\int_{\mathbb{T}^2} Z(\rho)(t) \,\mathrm{d}x \le \int_{\mathbb{T}^2} Z(\rho_0) \,\mathrm{d}x,$$

so that

$$\limsup_{t \to 0} \int_{\mathbb{T}^2} Z(\rho)(t) \,\mathrm{d}x \le \int_{\mathbb{T}^2} Z(\rho_0) \,\mathrm{d}x.$$

Combining these inequalities ends the proof. ∎

References

[1] Bardos C., Santos R., Sentis R., *Diffusion approximation and computation of the critical size*, Trans. Amer. Math. Soc., 284 (1984) 617–649.

[2] Berthonnaud P. *Limites fluides pour des modèles cinétiques de brouillards de gouttes monodispersés*, C. R. Acad. Sci., 331 (2000) 651–654.

[3] Caflisch R., Papanicolaou G., *Dynamic theory of suspensions with Brownian effects*, SIAM J. Appl. Math., 43 (1983) 885–906.

[4] Clouet J.F., Domelevo K., *Solutions of a kinetic stochastic equation modeling a spray in a turbulent gas flow*, Math. Models Methods Appl. Sci., 7 (1997) 239–263.

[5] Di Perna R., Lions P.-L., *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math., 98 (1989) 511–547.

[6] Domelevo K., Roquejoffre J. M., *Existence and stability of traveling wave solutions in a kinetic model of two-phase flows*, Comm. PDE, 24 (1999) 61–108.

[7] Domelevo K., Vignal M.-H., *Limites visqueuses pour des systèmes de type Fokker-Planck-Burgers unidimensionnels*, C. R. Acad. Sci., 332 (2001) 863–868.

[8] Edwards R. E. **Functional analysis. Theory and applications** (Holt, Rinehart and Winston, 1965).

[9] Gavrilyuck S., Teshukhov V., *Kinetic model for the motion of compressible bubbles in a perfect fluid*, Eur. J. Mech. B/Fluids, 21 (2002) 469–491.

[10] Gilbarg D., Trudinger N., **Elliptic PDE of second order**, (Springer, 1983).

[11] Giusti E. **Minimal surfaces and functions of bounded variation**, Monographs in mathematics 080 (Birkhauser, 1984).

[12] Golse F. *From kinetic to macroscopic models* in **Kinetic equations and asymptotic theory**, B. Perthame, L. Desvillettes Eds., Series in Appl. Math., 4 (Gauthier-Villars, 2000) pp. 41-126.

[13] Goudon T., *Asymptotic problems for a kinetic model of two-phase flow*, Proc. Royal Soc. Edimburgh. 131, (2001) 1371–1384.

[14] Goudon T., Jabin P.-E., Vasseur A., *Hydrodymamic limit for the Vlasov-Navier-Stokes equations: Fine particles regime*, Preprint.

[15] Hamdache K., *Global existence and large time behaviour of solutions for the Vlasov-Stokes equations*, Japan J. Industrial and Appl. Math., 15 (1998) 51–74.

[16] Hamdache K., Unpublished work, Personal Communication.

[17] Herrero H., Lucquin-Desreux B., Perthame B., *On the motion of dispersed balls in a potential flow: a kinetic description of the added mass effect*, SIAM J. Appl. Math., 60 (1999) 61–83.

[18] Jabin P.E., *Large time concentrations for solutions to kinetic equations with energy dissipation*, Comm. PDE., 25 (2000) 541–557.

[19] Jabin P.E., *Macroscopic limit of Vlasov type equations with friction*, Ann. IHP Anal. Non Linéaire, 17 (2000) 651–672.

[20] Jabin P.E., Perthame B., *Notes on mathematical problems on the dynamics of dispersed particles interacting through a fluid* in **Modeling in applied sciences, a kinetic theory approach**, N. Bellomo, M. Pulvirenti Eds. (Birkhäuser, 2000), pp. 111–147.

[21] Lions P.L., *Compactness in Boltzmann's equation via Fourier integral operators and applications. I, II, III*, J. Math. Kyoto Univ. 34 (1994) 391–461, 1994 and 34 (1994) 539–584.

[22] Masmoudi N., Saint-Raymond L., *From the Boltzmann equation to the Stokes-Fourier system in a bounded domain*, Comm. Pure Appl. Math. 56 (2003) 1263–1293.

[23] Nieto J., Poupaud F., Soler J., *High field limit for the VPFP system*, Arch. Rat. Mech. Anal., 158 (2001) 29–59.

[24] Poupaud F., *Diffusion approximation of the linear semiconductor Boltzmann equation: analysis of boundary layers*, Asymptotic Anal. 4 (1991) 293–317.

[25] POUPAUD F., SOLER J., *Parabolic limit and stability of the Vlasov-Poisson-Fokker-Planck system*, Math. Models Methods Appl. Sci., 10 (2000) 1027–1045.

[26] RUSSO G., SMEREKA P., *Kinetic theory for bubbly flows I, II*, SIAM J. Appl. Math., 56 (1996) 327–371.

[27] SIMON J., *Compact sets in* $L^p(0,T;B)$, Ann. Mat. Pura. Appl. IV, 146 (1987) 65–96.

[28] TARTAR L., **Topics in nonlinear analysis**, Publications Mathématiques d'Orsay 78.13 (1982).

[29] WILLIAMS F. A., **Combustion theory** (Benjamin Cummings Publ., 2nd ed., 1985).