Diffusion regime with a high and oscillating field

Thierry Goudon*

Univ. Côte d'Azur, Inria, CNRS, LJAD

Abstract

We investigate the behavior of solutions of simple linear kinetic equations, subjected to a strong and fast time-oscillating force field, in the diffusion regime. We consider quasi-periodic oscillations. The asymptotic behavior is governed by a convection-diffusion equation, for which we can identify the effective coefficients.

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1 Introduction

We are interested in the behavior as ε tends to 0 of the solution f_ε of the following PDE

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon^2} \mathcal{E}(t/\varepsilon^2, x) \cdot \nabla_v f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon), \tag{1}$$

where $t \geq 0$, $x \in \mathbb{R}^N$, $v \in \mathbb{R}^N$. In (1), the right hand side is either the linear Boltzmann operator

$$Q(f) = \langle f \rangle M(v) - f, \tag{2}$$

where $\langle f \rangle = \int_{\mathbb{R}^N} f \, \mathrm{d}v$ stands for the velocity average, or the Fokker-Planck operator

$$Q(f) = \nabla_v \cdot (vf + \nabla_v f) = \nabla_v \cdot \Big(M \nabla_v \frac{f}{M} \Big). \tag{3}$$

In both cases,

$$v \longmapsto M(v) = \frac{e^{-v^2/2}}{(2\pi)^{N/2}}$$

stands for the normalized Maxwellian. This PDE, completed by an initial data

$$f_\varepsilon\big|_{t=0} = f_{0,\varepsilon} \geq 0,$$

is a standard model from statistical physics (with applications in plasma physics, gas dynamics, laden-flows...). The unknown $f_\varepsilon(t, x, v) \geq 0$ is the distribution function in phase space of a large set of particles subjected to both collisional mechanisms, embodied into the operator Q, and an acceleration field $(t, x) \mapsto \frac{1}{\varepsilon^2} \mathcal{E}(t/\varepsilon^2, x)$. Here, the small parameter $0 < \varepsilon \ll 1$ describes the interaction between various physical scales of the problem: there are many collision events per time units, the force field is "strong" and it time-oscillates very fast, and the phenomena is considered on a

*thierry.goudon@inria.fr

large time scale of observation.

The question of the behavior of solutions of highly-collisional kinetic equations with *space-oscillating* force fields has been addressed in [3, 26, 27, 35], which have revealed some unexpected effect in the homogenization process. However, in these contributions the scaling differs from the scaling in (1): in [3, 26, 27, 35], only the collision term $Q(f_\varepsilon)$ scales like $1/\varepsilon^2$, the strength of the force field is of the lower order $1/\varepsilon$. The present situation instead relies on *high-field regimes*, and the balance between the acceleration term and the collision term modifies the shape of the equilibrium states that govern the asymptotic behavior, a phenomena brought out, for non oscillating fields, in [30] and [9], and further investigated in [4, 18, 29]. The question of strong and oscillating force fields in collisionless models appears naturally in plasma physics, with specific motivations related to the mathematical modeling of tokamaks [5, 14, 15]. We also refer the reader to [31] where different oscillations mechanisms, of stochastic nature, asymptotically lead to effective collisional models.

The homogenization problem (1) is addressed in [11] in the *stochastic framework*, $\mathcal{E}$ being a random quantity. In particular, [11] offers an instructive overview of the interaction between arguments from probability theory and PDE analysis. We wish to revisit this question when dealing with deterministic oscillations, which might be complementary to the analysis performed in [11]. We distinguish the following settings, inspired from [7]:

- ⓐ Periodic oscillations: we simply have $\mathcal{E} = E$ where $\tau \mapsto E(\tau, x)$ is 1-periodic,
- ⓑ Finite number of modes: $\mathcal{E}(\tau, x) = E(\omega\tau, x)$ where $\omega \in \mathbb{R}^r \setminus \{0\}$ has rationally independent components (which means that $k \cdot \omega = 0$ holds with $k \in \mathbb{Q}^r$ iff $k = 0$), and $\theta \in \mathbb{R}^r \mapsto E(\theta, x) = \sum_{|k|\leq K} \widehat{E}(k, x)e^{2i\pi k\cdot\theta}$ for a certain finite $K \in \mathbb{N} \setminus \{0\}$,
- ⓒ Quasi-periodic oscillations: $\mathcal{E}(\tau, x) = E(\omega\tau, x)$ where $\omega \in \mathbb{R}^r \setminus \{0\}$ has rationally independent components and $\theta \mapsto E(\theta, x)$ is $\mathbb{Y} = (0, 1)^r$-periodic. In such a case, we will also assume the following Diophantine condition

$$\begin{aligned} &\text{there exists } C, \gamma > 0 \text{ such that } |\omega \cdot k| \geq \frac{C}{|k|^\gamma} \text{ for any } k \in \mathbb{Z}^r, \\ &\text{and, moreover, } |k|^\gamma \widehat{E}(k) \in \ell^1(\mathbb{Z}^r). \end{aligned} \tag{4}$$

  Observe that such a condition holds for almost every ω.

Clearly, ⓐ is a sub-case of ⓑ, which itself is contained in ⓒ. As it will be detailed below, generalizing the framework induces difficulties and slightly different statements. Besides, throughout the paper, we assume that E has enough regularity with respect to the space variable

$$(\theta, x) \mapsto \partial_x^\alpha E(\theta, x) \in L^\infty(\mathbb{Y} \times \mathbb{R}^N) \text{ for } |\alpha| \leq 2, \tag{5}$$

and that its mean vanishes

$$\int_{\mathbb{Y}} E(\theta, x)\,\mathrm{d}\theta = 0 \text{ for a. e. } x, \tag{6}$$

(which can be equivalently cast as an assumption on the zeroth Fourier coefficient $\widehat{E}(0, x) = 0$). This (quasi-)periodic setting differs from the stochastic framework. On the one hand, it leads to difficulties for solving the “cell-problems” that define the effective coefficients. The stochastic framework induces some hidden structure, so that the resolvant of the cell-problems is well-defined, a property intensively used in [11]. In the determistic case, we face small divisors issues, which make the analysis more intricate, see [7] for related discussions. On the other hand, the asymptotic analysis of the deterministic problem in itself is far simpler, since we can make use of the quite systematic approach based on double-scale analysis [1, 28]. It leads to more complete results; in particular, we completely identify the leading expression of the particle distribution function for small ε's (in [11] the main results are only concerned with the convergence of the macroscopic density $\langle f_\varepsilon \rangle$). It also makes easier the treatment of cases where E depends on the space variable, and stronger the statement when E is space homogeneous. The main result of the paper states as follows.

Theorem 1.1 *Suppose that the sequence of initial data is such that $(1 + |v|^k)f_{0,\varepsilon}$ is bounded in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$, for some $k > 2$. Then, possibly at the price of extracting a subsequence, f_ε double-scale converges to $\rho(t, x)\Phi(\theta, x, v)$, where*

i) Φ is the positive and normalized solution of $(\omega \cdot \nabla_\theta + E \cdot \nabla_v - Q)\Phi = 0$,

ii) $\rho_\varepsilon = \int_{\mathbb{R}^N} f_\varepsilon \, \mathrm{d}v$ converges to ρ in $C^0([0, T]; \mathscr{M}^1(\mathbb{R}^N) - \text{vaguely})$,

iii) ρ is the solution of the convection-diffusion problem

$$\partial_t \rho + \nabla_x \cdot (\rho U - D\nabla_x \rho) = 0, \tag{7}$$

with effective coefficients U and D defined by auxilliary equations that involve E, and initial data $\rho\big|_{t=0} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} f_{0,\varepsilon} \, \mathrm{d}v$ (in the vague sense of measures on $\mathbb{R}^N$).

The suitable notions of compactness and double scale convergence are detailed in Section 4, see in particular Proposition 4.3 and Proposition 4.5. The definition of the effective coefficients U and D, which are space-dependent vector-valued and matrix-valued functions, respectively, is given in Section 2, see formula (8), (9), (10). In particular, D has a positive symmetric part.

The paper is organized as follows. In Section 2 we guess the asymptotic behavior by using a formal double scale ansatz. This discussion makes the cell equations appear, which are discussed in details in Section 3. The analysis relies a/ on the identification of the eigenfunction Φ which is possible for the simple collision operators considered here (we also give some hints on possible generalizations when such explicit form is not available); b/ on the identification of a dissipation structure of the cell equations. We also investigate the properties of the diffusion matrix D. Section 4 is devoted to the asymptotic analysis and the proof of Theorem 1.1. In Section 5, we go back to the specific situation where the force field E is space-homogeneous, a case where we are able to establish the strong convergence of (a modified version of) the macroscopic density, and to rigorously identify the first order corrector.

2 Formal asymptotics

In order to guess the asymptotic behavior of the solution f_ε for small ε's, we insert the following formal double-scale ansatz

$$f_\varepsilon(t, x, v) = \sum_{k=0}^{\infty} \varepsilon^k F^{(k)}(\omega t/\varepsilon^2, t, x, v)$$

into the equations, where the functions $F^{(k)}$ are assumed to be $\mathbb{Y}$-periodic with respect to the first variable. It makes the following operator

$$\mathscr{T}_E = \omega \cdot \nabla_\theta + E(\theta, x) \cdot \nabla_v - Q$$

appear. In what follows, we shall denote $\mathscr{T}_E^\star$ the adjoint operator, for the standard L^2 inner product. We identify terms which appear with the same order of magnitude with respect to ε. We are led to

a) Leading order $\mathscr{O}(1/\varepsilon^2)$: $\mathscr{T}_E F^{(0)} = 0$,

b) Order $\mathscr{O}(1/\varepsilon)$: $\mathscr{T}_E F^{(1)} = -v \cdot \nabla_x F^{(0)}$,

c) Order $\mathscr{O}(1/\varepsilon^0)$: $\mathscr{T}_E F^{(2)} = -\partial_t F^{(0)} - v \cdot \nabla_x F^{(1)}$, etc.

Then, the analysis relies on the following two key ingredients:

- We need to identify the leading term $F^{(0)}$, which means to characterize the kernel of the operator $\mathscr{T}_E$. We shall see that we can exhibit a positive function $\Phi(\theta, x, v)$, periodic with respect to the first variable, such that

$$\mathscr{T}_E \Phi = 0, \qquad \iint_{\mathbb{Y}\times\mathbb{R}^N} \Phi \, \mathrm{d}v \, \mathrm{d}\theta = 1$$

  and in fact

$$\mathrm{Ker}(\mathscr{T}_E) = \mathrm{Span}\{\Phi\}.$$

  The variable x plays only the role of a parameter in the equation; for further purposes, it will be necessary to study the regularity of Φ with respect to x, as a consequence of the regularity assumption (5) on the coefficients.

- We observe that

$$\mathrm{Ran}(\mathscr{T}_E) \subset \Big\{ G : \mathbb{Y} \times \mathbb{R}^N, \ \iint_{\mathbb{Y}\times\mathbb{R}^N} G \, \mathrm{d}v \, \mathrm{d}\theta = 0 \Big\}.$$

  Hence, $\iint_{\mathbb{Y}\times\mathbb{R}^N} G \, \mathrm{d}v \, \mathrm{d}\theta = 0$ is a necessary condition for the solvability of the equation $\mathscr{T}_E F = G$. We expect it is also sufficient so that the equation determines a unique function F with the selection criterion $\iint_{\mathbb{Y}\times\mathbb{R}^N} F \, \mathrm{d}v \, \mathrm{d}\theta = 0$.

Based on these properties, we guess that

- At leading order $F^{(0)}(\theta, t, x, v) = \rho(t, x)\Phi(\theta, x, v)$, and the asymptotic behavior will be described through an evolution equation for the macroscopic density ρ.

- The equation for $F^{(1)}$ can be recast as $\mathscr{T}_E F^{(1)} = -v\Phi \cdot \nabla_x \rho - \rho v \cdot \nabla_x \Phi$. The equation makes sense under the condition

$$\iint_{\mathbb{Y}\times\mathbb{R}^N} v\Phi \, \mathrm{d}v \, \mathrm{d}\theta = 0$$

  that needs to be fulfilled by the function Φ. This will be obtained as a consequence of (6): the fact that the ballistic velocity vanishes is crucial for the analysis. It allows us to introduce the auxilliary functions $\chi = (\chi_1, ..., \chi_N)$ and β defined on $\mathbb{Y} \times \mathbb{R}^N \times \mathbb{R}^N$ solution of

$$\mathscr{T}_E \chi_j = v_j \Phi, \qquad \mathscr{T}_E \beta = v \cdot \nabla_x \Phi. \tag{8}$$

  Then, we get

$$F^{(1)}(\theta, t, x, v) = -\chi(\theta, x, v) \cdot \nabla_x \rho(t, x) - \beta(\theta, x, v)\rho(t, x).$$

- The compatibility condition for $\mathscr{T}_E F^{(2)} = -\partial_t F^{(0)} - v \cdot \nabla_x F^{(1)}$ yields

$$\partial_t \iint_{\mathbb{Y}\times\mathbb{R}^N} F^{(0)} \, \mathrm{d}v \, \mathrm{d}\theta + \nabla_x \cdot \iint_{\mathbb{Y}\times\mathbb{R}^N} vF^{(1)} \, \mathrm{d}v \, \mathrm{d}\theta = 0,$$

  which, in view of the expressions obtained for $F^{(0)}$ and $F^{(1)}$ becomes the convection-diffusion equation (7) for ρ, with the effective coefficients

$$D(x) = \iint_{\mathbb{Y}\times\mathbb{R}^N} v \otimes \chi(\theta, x, v) \, \mathrm{d}v \, \mathrm{d}\theta, \tag{9}$$

  and

$$U(x) = -\iint_{\mathbb{Y}\times\mathbb{R}^N} v\beta(\theta, x, v) \, \mathrm{d}v \, \mathrm{d}\theta. \tag{10}$$

3 Cell equations

This Section is concerned with the analysis of the cell equation $\mathscr{T}_E F = G$. Since the space variable appears only as a parameter, we disregard it during the discussion, bearing in mind that the solutions we are going to discuss are also parametrized by x. In particular, we shall establish estimates which will be uniform with respect to the space variable x.

3.1 Equilibrium states

We search for solutions of $\mathscr{T}_E \Phi = 0$. It turns out that, for the simple collision operators dealt with here, we can find explicit solutions (see also [11, Theorem 3.1]). Of course, the presence of the acceleration field modifies substantially the shape of the equilibrium function, compared to the case where $E = 0$. This is reminiscent to phenomena pointed out in [30].

3.1.1 Boltzmann operator

We seek the equilibrium function by imposing the strengthened normalization condition

$$\int_{\mathbb{R}^N} \Phi(\theta, v)\,\mathrm{d}v = 1,$$

which amounts to consider the equation

$$\omega \cdot \nabla_\theta \Phi + E \cdot \nabla_v \Phi + \Phi = M.$$

We define the characteristics curves $(\Theta, \mathscr{V})$, solution of the ODE system

$$\frac{\mathrm{d}}{\mathrm{d}s}\Theta = \omega, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\mathscr{V} = E(\Theta).$$

Then, the problem can be cast as

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[e^s \Phi(\Theta(s), \mathscr{V}(s))\Big] = e^s M(\mathscr{V}(s)).$$

In fact, we simply have

$$\Theta(s) = \theta + \omega s, \qquad \mathscr{V}(s) = v + \int_0^s E(\theta + \omega\varsigma)\,\mathrm{d}\varsigma.$$

These considerations yield the formula

$$\Phi(\theta, v) = \int_{-\infty}^0 e^\sigma M\Big(v + \int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\Big)\,\mathrm{d}\sigma,$$

which defines a positive and bounded (by $(2\pi)^{-N/2}$) function, periodic with respect to θ, and such that $\big\langle \Phi(\theta, \cdot)\big\rangle = 1$. Observe that in these manipulations, we did not use that much the specific form of the function M. However, the qualitative properties of M becomes important in order to identify the functional framework. For instance, for further purposes we shall need the following claim.

Lemma 3.1 *The function* $(\theta, v) \mapsto (1 + v^2)\frac{\Phi^2(\theta,v)}{M(v)}$ *lies in* $L^\infty(\mathbb{Y}; L^1(\mathbb{R}^N))$*, and we shall denote* $\Xi = \sup_{\theta\in\mathbb{Y}} \int_{\mathbb{R}^N} (1 + v^2)\frac{\Phi^2(\theta,v)}{M(v)}\,\mathrm{d}v$.

Proof. Let us introduce the shorthand notation

$$\mathcal{U}(\sigma,\theta) = \int_0^\sigma E(\theta+\omega s)\,\mathrm{d}s.$$

By using the Cauchy-Schwarz inequality, we obtain

$$\frac{\Phi^2(\theta,v)}{M(v)} \le \frac{1}{(2\pi)^{N/2}}\int_{-\infty}^0 e^\sigma e^{-|v+\mathcal{U}(\sigma,\theta)|^2+v^2/2}\,\mathrm{d}\sigma = \frac{1}{(2\pi)^{N/2}}\int_{-\infty}^0 e^\sigma e^{|\mathcal{U}(\sigma,\theta)|^2} e^{-|v+2\mathcal{U}(\sigma,\theta)|^2/2}\,\mathrm{d}\sigma.$$

It follows that

$$\int_{\mathbb{R}^N}(1+v^2)\frac{\Phi^2(\theta,v)}{M(v)}\,\mathrm{d}v \le \int_{-\infty}^0 (1+N+4|\mathcal{U}(\sigma,\theta)|^2)e^\sigma e^{|\mathcal{U}(\sigma,\theta)|^2}\,\mathrm{d}\sigma.$$

We conclude by proving that $(\sigma,\theta)\mapsto\mathcal{U}(\sigma,\theta)$ lies in L^∞. We distinguish the following three cases:

ⓐ *Periodic case.* Since the average of the force field vanishes (see (6)), we get

$$\mathcal{U}(\sigma,\theta) = \sum_{n=0}^{\lfloor\sigma\rfloor}\int_n^{n+1} E(\theta+s)\,\mathrm{d}s + \int_{\lfloor\sigma\rfloor}^\sigma E(\theta+s)\,\mathrm{d}s = \int_{\lfloor\sigma\rfloor}^\sigma E(\theta+s)\,\mathrm{d}s$$

which yields

$$|\mathcal{U}(\sigma,\theta)| \le \|E\|_{L^\infty}.$$

ⓑ *Finite number of harmonics.* We expand E by means of its Fourier series, bearing in mind that $\widehat{E}(0) = \int E(\theta)\,\mathrm{d}\theta = 0$, and we get

$$\mathcal{U}(\sigma,\theta) = \int_0^\sigma \sum_{k\in\mathbb{Z}^N\setminus\{0\}} \widehat{E}(k)\ e^{2i\pi k\cdot(\theta+\omega s)}\,\mathrm{d}s = \sum_{k\in\mathbb{Z}^N\setminus\{0\}} \frac{\widehat{E}(k)}{2\pi\omega\cdot k} e^{2i\pi k\cdot\theta} e^{2i\pi k\cdot\omega\sigma/2} 2\sin(\pi k\cdot\omega\sigma).$$

Assuming that E has a finite number of harmonics means that $\widehat{E}(k)=0$ when $|k|>K$ for a certain integer K. Therefore, we get

$$|\mathcal{U}(\sigma,\theta)| \le \frac{1}{\pi}\sup_{|k|\le K}\frac{|\widehat{E}(k)|}{|\omega\cdot k|} \le \frac{1}{\pi}\|E\|_{L^1(\mathbb{Y})}\sup_{|k|\le K}\frac{1}{|\omega\cdot k|}.$$

ⓒ *Quasi-periodic case.* We proceed similarly, up to the final estimate which now relies on the Diophantine condition (4) and the regularity assumption $|k|^\gamma\widehat{E}(k)\in\ell^1(\mathbb{Z}^N)$. We get

$$|\mathcal{U}(\sigma,\theta)| \le C\||k|^\gamma\widehat{E}(k)\|_{\ell^1(\mathbb{Z}^N)}.$$

□

3.1.2 Fokker-Planck operator

For the Fokker-Planck equation, we set

$$\Phi(\theta,v) = M\Big(v - \int_{-\infty}^0 e^\sigma E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big),$$

which defines a positive function, periodic with respect to θ, and such that $\big\langle\Phi(\theta,\cdot)\big\rangle = 1$. We check that, on the one hand,

$$\begin{aligned}\omega\cdot\nabla_\theta\Phi(\theta,v) &= (\nabla_v M)\Big(v-\int_{-\infty}^0 e^\sigma E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big)\cdot\Big(-\int_{-\infty}^0 e^\sigma(\omega\cdot\nabla_\theta)E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big)\\ &= (\nabla_v M)\Big(v-\int_{-\infty}^0 e^\sigma E(\theta+\omega\sigma)\Big)\cdot\Big(-\int_{-\infty}^0 e^\sigma\frac{\mathrm{d}}{\mathrm{d}\sigma}\Big[E(\theta+\omega\sigma)\Big]\,\mathrm{d}\sigma\Big)\\ &= (\nabla_v M)\Big(v-\int_{-\infty}^0 e^\sigma E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big)\cdot\Big(-E(\theta)+\int_{-\infty}^0 e^\sigma E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big),\end{aligned}$$

and, on the other hand,

$$\nabla_v \cdot (v\Phi(\theta,v) + \nabla_v \Phi(\theta,v)) = 0 + \int_{-\infty}^{0} e^{\sigma} E(\theta+\omega\sigma)\,\mathrm{d}\sigma \cdot (\nabla_v M)\Big(v - \int_{-\infty}^{0} e^{\sigma} E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big).$$

Therefore, we get $\omega \cdot \nabla_\theta \Phi - Q(\Phi) = -E \cdot \nabla_v \Phi$.

3.2 Dissipative structure

That the kernel of $\mathscr{T}_E$ is one-dimensional relies on a dissipative structure of the operator $\mathscr{T}_E$; this property will be also useful for proving the solvability of the cell equations.

Proposition 3.2 *The following property holds*

$$\iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E f \ \frac{f}{\Phi} \,\mathrm{d}v\,\mathrm{d}y = D(f|\Phi)$$

where $D(f|\Phi) \geq 0$ *vanishes iff* $f \in \mathrm{Span}\{\Phi\}$. *Consequently,* $\mathrm{Ker}(\mathscr{T}_E) = \mathrm{Span}\{\Phi\}$.

We start by computing

$$\begin{aligned}
&\iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E f \ \frac{f}{\Phi} \,\mathrm{d}v\,\mathrm{d}\theta \\
&\quad = \iint_{\mathbb{Y}\times\mathbb{R}^N} (\omega\cdot\nabla_\theta - E\cdot\nabla_v)\Big(\frac{f^2}{2}\Big)\ \frac{1}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta - \iint_{\mathbb{Y}\times\mathbb{R}^N} Q(f)\ \frac{f}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta \\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \Big(\frac{f^2}{2\Phi^2}\Big)\ (\omega\cdot\nabla_\theta - E\cdot\nabla_v)\Phi\,\mathrm{d}v\,\mathrm{d}\theta - \iint_{\mathbb{Y}\times\mathbb{R}^N} Q(f)\ \frac{f}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta \\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \Big(\frac{f^2}{2\Phi^2}\Big)\ Q(\Phi)\,\mathrm{d}v\,\mathrm{d}\theta - \iint_{\mathbb{Y}\times\mathbb{R}^N} Q(f)\ \frac{f}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta.
\end{aligned}$$

The remaining of the proof depends on the specific form of the collision operator.

3.2.1 Boltzmann operator

For the linear Boltzmann operator, using $\int_{\mathbb{R}^N} M(v)\,\mathrm{d}v = 1$, we get

$$\begin{aligned}
&\iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E f \ \frac{f}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta \\
&= \iiint_{\mathbb{Y}\times\mathbb{R}^N\times\mathbb{R}^N} M(v)\Phi(\theta,v') \left(\frac{f(\theta,v)^2}{2\Phi(\theta,v)^2} - \frac{f(\theta,v')^2}{2\Phi(\theta,v')^2} - \frac{f(\theta,v')}{\Phi(\theta,v')}\frac{f(\theta,v)}{\Phi(\theta,v)} + \frac{f(\theta,v')^2}{\Phi(\theta,v')^2}\right)\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}\theta \\
&= \frac{1}{2}\iiint_{\mathbb{Y}\times\mathbb{R}^N\times\mathbb{R}^N} M(v)\Phi(\theta,v') \left(\frac{f(\theta,v)}{\Phi(\theta,v)} - \frac{f(\theta,v')}{\Phi(\theta,v')}\right)^2 \mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}\theta = D(f|\Phi) \geq 0.
\end{aligned}$$

In particular, having $\mathscr{T}_E f = 0$ imposes that $\frac{f}{\Phi}(\theta,v) = \alpha(\theta)$ does not depend on v, or, in other words, $f(\theta,v) = \alpha(\theta)\Phi(\theta,v)$. Since $\int \Phi(\theta,v)\,\mathrm{d}v = 1$, it follows that $\alpha(\theta) = \int f(\theta,v)\,\mathrm{d}v$. Coming back to $\mathscr{T}_E f = 0$, we get $\omega\cdot\nabla_\theta\langle f\rangle = 0 = \omega\cdot\nabla_\theta\alpha(\theta)$, and thus α does not depend on θ. The condition $\iint f\,\mathrm{d}v\,\mathrm{d}\theta = 0$ leads to $\alpha = 0$.

Finally, for our purposes it is important to observe that, for a positive function $(\theta,v)\mapsto w(\theta,v)$, we have

$$
\begin{aligned}
&\iint_{\mathbb{Y}\times\mathbb{R}^N} w(\theta,v)|f(\theta,v)-\rho(\theta)\Phi(\theta,v)|^2\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad=\iint_{\mathbb{Y}\times\mathbb{R}^N} w(\theta,v)\left|\int_{\mathbb{R}^N}\Big(\Phi(\theta,v')f(\theta,v)-f(\theta,v')\Phi(\theta,v)\Big)\,\mathrm{d}v'\right|^2\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad=\iint_{\mathbb{Y}\times\mathbb{R}^N} w(\theta,v)\left|\int_{\mathbb{R}^N}\Phi(\theta,v')\Phi(\theta,v)\Big(\frac{f}{\Phi}(\theta,v)-\frac{f}{\Phi}(\theta,v')\Big)\,\mathrm{d}v'\right|^2\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad\leq\iint_{\mathbb{Y}\times\mathbb{R}^N} w(\theta,v)\Phi^2(\theta,v)\underbrace{\int_{\mathbb{R}^N}\Phi(\theta,v')\,\mathrm{d}v'}_{=1}\int_{\mathbb{R}^N}\Phi(\theta,v')\Big(\frac{f}{\Phi}(\theta,v)-\frac{f}{\Phi}(\theta,v')\Big)^2\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}\theta.
\end{aligned}
$$

Therefore, defining the weight w by

$$w(\theta,v)=\frac{M(v)}{\Phi^2(\theta,v)},$$

we arrive at the coercivity estimate

$$\iint_{\mathbb{Y}\times\mathbb{R}^N}\frac{M(v)}{\Phi^2(\theta,v)}\;|f(\theta,v)-\rho(\theta)\Phi(\theta,v)|^2\,\mathrm{d}v\,\mathrm{d}\theta\leq 2D(f|\Phi). \tag{11}$$

3.2.2 Fokker-Planck operator

Similarly, for the Fokker-Planck operator, we get

$$
\begin{aligned}
&\iint_{\mathbb{Y}\times\mathbb{R}^N}\mathscr{T}_E f\;\frac{f}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad=\iint_{\mathbb{Y}\times\mathbb{R}^N}\frac{f^2}{2\Phi^2}\nabla_v\cdot\Big(M\nabla_v\frac{\Phi}{M}\Big)-\iint_{\mathbb{Y}\times\mathbb{R}^N}\nabla_v\cdot\Big(M\nabla_v\Big(\frac{f}{\Phi}\frac{\Phi}{M}\Big)\Big)\;\frac{f}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad=\iint_{\mathbb{Y}\times\mathbb{R}^N}\frac{f^2}{2\Phi^2}\nabla_v\cdot\Big(M\nabla_v\frac{\Phi}{M}\Big)+\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi\Big|\nabla_v\frac{f}{\Phi}\Big|^2\,\mathrm{d}v\,\mathrm{d}\theta\\
&\qquad+\iint_{\mathbb{Y}\times\mathbb{R}^N}\Big(M\nabla_v\frac{\Phi}{M}\Big)\cdot\Big(\frac{f}{\Phi}\nabla_v\frac{f}{\Phi}\Big)\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad=\iint_{\mathbb{Y}\times\mathbb{R}^N}\frac{f^2}{2\Phi^2}\nabla_v\cdot\Big(M\nabla_v\frac{\Phi}{M}\Big)+\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi\Big|\nabla_v\frac{f}{\Phi}\Big|^2\,\mathrm{d}v\,\mathrm{d}\theta\\
&\qquad-\iint_{\mathbb{Y}\times\mathbb{R}^N}\nabla_v\cdot\Big(M\nabla_v\frac{\Phi}{M}\Big)\frac{1}{2}\Big(\frac{f}{\Phi}\Big)^2\,\mathrm{d}v\,\mathrm{d}\theta\\
&\quad=\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi\Big|\nabla_v\frac{f}{\Phi}\Big|^2\,\mathrm{d}v\,\mathrm{d}\theta=D(f|\Phi)\geq 0.
\end{aligned}
$$

If $\mathscr{T}_E f=0$, we deduce that $\frac{f}{\Phi}(\theta,v)=\alpha(\theta)$ does not depend on v, and actually $\alpha(\theta)=\int f(\theta,v)\,\mathrm{d}v$. Integrating the equation $\mathscr{T}_E f=0$ with respect to v leads to $\omega\cdot\nabla_\theta\alpha=0$, and thus $\alpha(\theta)=\alpha$ is constant. The condition $\iint f\,\mathrm{d}v\,\mathrm{d}\theta=0$ yields $\alpha=0$.

Here, we bear in mind that the following Sobolev inequality holds

$$\iint_{\mathbb{Y}\times\mathbb{R}^N}|f(\theta,v)-\rho(\theta)\Phi(\theta,v)|^2\frac{\mathrm{d}v\,\mathrm{d}\theta}{\Phi(\theta,v)}\leq\Lambda D(f|\Phi), \tag{12}$$

for a certain constant $\Lambda>0$, see [2, sp. Corollary 2.18] (note that $v\mapsto\big|v-\int_{-\infty}^0 e^\sigma E(\theta+\omega\sigma)\,\mathrm{d}\sigma\big|^2$ satisfies the convexity estimate [2, (A2)], uniformly with respect to θ).

3.3 Resolution of the cell equations

The previous computations make the Hilbert spaces

$$
\begin{aligned}
H &= \Big\{ f : \mathbb{Y} \times \mathbb{R}^N,\ \|f\|^2 = \iint_{\mathbb{Y}\times\mathbb{R}^N} |f(\theta,v)|^2 \frac{M(v)\,\mathrm{d}v\,\mathrm{d}\theta}{\Phi^2(\theta,v)} < \infty \Big\},\\
H_\star &= \Big\{ \phi : \mathbb{Y} \times \mathbb{R}^N,\ \|\phi\|_\star^2 = \iint_{\mathbb{Y}\times\mathbb{R}^N} |\phi(\theta,v)|^2 \frac{\mathrm{d}v\,\mathrm{d}\theta}{M(v)} < \infty \Big\},
\end{aligned}
$$

for the Boltzmann case and

$$
\begin{aligned}
&H = \Big\{ f : \mathbb{Y} \times \mathbb{R}^N,\ \|f\|^2 = \iint_{\mathbb{Y}\times\mathbb{R}^N} |f(\theta,v)|^2 \frac{\mathrm{d}v\,\mathrm{d}\theta}{\Phi(\theta,v)} < \infty \Big\},\\
&H_\star = H,
\end{aligned}
$$

for the Fokker-Planck case, an adapted functional framework for analyzing the cell equations. We denote by $\|\cdot\|$ (resp. $\|\cdot\|_\star$) the weighted L^2 norm on H (resp. $H_\star$) associated to these definitions. Note that in both cases we have

$$
\left| \iint_{\mathbb{Y}\times\mathbb{R}^N} f\phi \frac{\mathrm{d}v\,\mathrm{d}\theta}{\Phi} \right| \le \|f\| \|\phi\|_\star, \qquad \|\phi\|_{L^1} \le \|\phi\|_\star,
$$

as direct consequences of the Cauchy-Schwarz inequality. For the Fokker-Planck case, we thus also have $H \subset L^1$. For the linear Boltzmann case, the embedding $H \subset L^1$ equally holds, still by virtue of the Cauchy-Schwarz inequality, combined to Lemma 3.1.

Proposition 3.3 *Let $h \in H_\star$. We suppose that E and h fulfill one of the following conditions:*

a) they are both purely periodic,

b) they both have finitely many harmonics,

c) ω satisfies the Diophantine condition (4) and $h \in H_\star$ satisfies

$$
\|\langle h \rangle\|_{H^\gamma}^2 = \sum_{k\in\mathbb{Z}^r} |k|^{2\gamma} |\widehat{\langle h \rangle}(k)|^2 < \infty.
$$

Then, the problem $\mathscr{T}_E f = h$ has a solution $f \in H$ iff h satisfies the compatibility condition $\iint_{\mathbb{Y}\times\mathbb{R}^N} h\,\mathrm{d}v\,\mathrm{d}\theta = 0$. The solution is unique when imposing the additional constraint $\iint_{\mathbb{Y}\times\mathbb{R}^N} f\,\mathrm{d}v\,\mathrm{d}\theta = 0$.

Proof. For $\lambda > 0$, we consider

$$
\lambda f_\lambda + \mathscr{T}_E f_\lambda = h \in H_\star. \tag{13}
$$

We shall detail in Appendix A that this equation is indeed well-posed for any positive λ. If $\big(f_\lambda\big)_{\lambda>0}$ is bounded in H, possibly at the price of extracting a subsequence we can assume that it converges weakly to some f as $\lambda \to 0$. Then, letting $\lambda \to 0$ in (14), we justify the existence of a solution to $\mathscr{T}_E f = h$. Uniqueness under the constraint $\iint f\,\mathrm{d}v\,\mathrm{d}\theta = 0$ follows from Proposition 3.2.

In order to obtain the bound on f_λ, we argue by contradiction. Suppose $\lim_{\lambda\to 0} \|f_\lambda\| = \infty$. Set $\tilde{f}_\lambda = f_\lambda / \|f_\lambda\|$. It satisfies $\|\tilde{f}_\lambda\| = 1$, and

$$
\lambda \tilde{f}_\lambda + \mathscr{T}_E \tilde{f}_\lambda = h_\lambda \tag{14}
$$

where

$$
\lim_{\lambda\to 0} \|h_\lambda\|_\star = 0.
$$

Integrating (14) and using the assumption that $\iint h\,\mathrm{d}v\,\mathrm{d}\theta = 0$, we observe that

$$
\iint_{\mathbb{Y}\times\mathbb{R}^N} \tilde{f}_\lambda\,\mathrm{d}v\,\mathrm{d}\theta = 0.
$$

Let

$$\rho_\lambda(\theta) = \int_{\mathbb{R}^N} \tilde{f}_\lambda(\theta, v)\,\mathrm{d}v.$$

Note that

$$\int_{\mathbb{Y}} |\rho_\lambda(\theta)|^2\,\mathrm{d}\theta \leq \begin{cases} \displaystyle\iint_{\mathbb{Y}\times\mathbb{R}^N} \frac{|\tilde{f}_\lambda(\theta,v)|^2}{\Phi(\theta,v)}\,\mathrm{d}v\,\mathrm{d}\theta = \|\tilde{f}_\lambda\|^2 = 1 & \text{(Fokker-Planck case)}, \\ \displaystyle\Big\| \int_{\mathbb{R}^N} \frac{\Phi^2}{M}\,\mathrm{d}v \Big\|_{L^\infty(\mathbb{Y})} \iint_{\mathbb{Y}\times\mathbb{R}^N} |\tilde{f}_\lambda(\theta,v)|^2 \frac{M(v)}{\Phi^2(\theta,v)}\,\mathrm{d}v\,\mathrm{d}\theta \leq \Xi & \text{(Boltzmann case)} \end{cases}$$

is also bounded (we used Lemma 3.1 for the Boltzmann case). We can assume that $\tilde{f}_\lambda \rightharpoonup f$ in H, and $\rho_\lambda \rightharpoonup \rho = \langle f \rangle$ in $L^2(\mathbb{Y})$. Letting λ go to 0 in (14), we obtain $\mathscr{T}_E f = 0$, so that f is proportional to Φ: $f(\theta, v) = \rho\Phi(\theta, v)$, with ρ a constant. However, $\iint \tilde{f}_\lambda\,\mathrm{d}v\,\mathrm{d}\theta = 0$ implies $\iint f\,\mathrm{d}v\,\mathrm{d}\theta = \rho = 0$, and thus $f = 0$. We will get a contradiction by showing that $\tilde{f}_\lambda$ converges strongly to 0 in H. Multiplying (14) by $\frac{\tilde{f}_\lambda}{\Phi}$, we get

$$\lambda \iint_{\mathbb{Y}\times\mathbb{R}^N} \frac{|\tilde{f}_\lambda|^2}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta + D(\tilde{f}_\lambda|\Phi) = \iint_{\mathbb{Y}\times\mathbb{R}^N} h_\lambda \frac{\tilde{f}_\lambda}{\Phi}\,\mathrm{d}v\,\mathrm{d}\theta \leq \|h_\lambda\|_\star \|\tilde{f}_\lambda\| = \|h_\lambda\|_\star \xrightarrow[\lambda\to 0]{} 0.$$

As a consequence of (11) or (12), we deduce that

$$\|\tilde{f}_\lambda - \rho_\lambda \Phi\| \xrightarrow[\lambda\to 0]{} 0,$$

while ρ_λ satisfies

$$\omega \cdot \nabla_\theta \rho_\lambda = \langle h_\lambda \rangle \xrightarrow[\lambda\to 0]{} 0 \text{ in } L^2(\mathbb{Y}). \tag{15}$$

We distinguish the following three cases:

ⓐ Periodic case. In this case, $\omega \cdot \nabla_\theta$ becomes ∂_τ; both ρ_λ and $\partial_\tau \rho_\lambda$ are bounded in $L^2(\mathbb{Y})$. Therefore, ρ_λ is relatively compact in $L^2(\mathbb{Y})$.

ⓑ Finite number of harmonics. We consider the relation on the Fourier coefficients deduced from (15): for $k \neq 0$,

$$\widehat{\rho}_\lambda(k) = \frac{\widehat{\langle h_\lambda \rangle}(k)}{2i\pi\omega \cdot k}. \tag{16}$$

By assumption on the oscillations, $\omega \cdot k \neq 0$ for any $k \in \mathbb{Z}^N \setminus \{0\}$. We are led to

$$\|\widehat{\rho}_\lambda\|_{\ell^2} \leq \frac{1}{2\pi} \left(\sup_{|k|\leq K} \frac{1}{|\omega \cdot k|} \right) \|\widehat{\langle h_\lambda \rangle}\|_{\ell^2}.$$

ⓒ Quasi-periodic case with Diophantine condition. We suppose (4). Then (16) leads to

$$\|\rho_\lambda\|_{L^2} \leq C \|\langle h_\lambda \rangle\|_{H^\gamma}.$$

Therefore, in all cases ρ_λ tends to 0 in $L^2(\mathbb{Y})$ as $\lambda \to 0$. We are thus led to the conclusion that $\tilde{f}_\lambda = (\tilde{f}_\lambda - \rho_\lambda\Phi) + \rho_\lambda\Phi$ tends to 0 in H as $\lambda \to 0$, a contradiction. □

Remark 3.4 *A similar analysis can be performed for the adjoint operator $\mathscr{T}_E^\star$. In particular $\mathscr{K}(\mathscr{T}_E^\star)$ is spanned by constants, and the problem $\mathscr{T}_E^\star \phi = \kappa$ is well-posed as far as $\iint \kappa\Phi\,\mathrm{d}v\,\mathrm{d}\theta = 0$.*

For both the Boltzmann operator and the Fokker-Planck operator, the compatibility condition

$$\iint_{\mathbb{Y}\times\mathbb{R}^N} v\Phi(\theta,v)\,\mathrm{d}v\,\mathrm{d}\theta = 0$$

is satisfied, as a consequence of the assumption (6). Indeed, for the Boltzmann operator we have

$$\begin{aligned}
&\iint_{\mathbb{Y}\times\mathbb{R}^N} v\Phi(\theta,v)\,\mathrm{d}v\,\mathrm{d}\theta \\
&= \int_{-\infty}^{0} e^{\sigma} \underbrace{\left(\iint_{\mathbb{Y}\times\mathbb{R}^N} \Big(v + \int_0^{\sigma} E(\theta+\omega s)\,\mathrm{d}s\Big) M\Big(v + \int_0^{\sigma} E(\theta+\omega s)\,\mathrm{d}s\Big)\,\mathrm{d}v\,\mathrm{d}\theta\right)}_{=0}\,\mathrm{d}\sigma \\
&\qquad - \int_{-\infty}^{0}\int_{\mathbb{Y}} e^{\sigma}\left(\int_0^{\sigma} E(\theta+\omega s)\,\mathrm{d}s \underbrace{\int_{\mathbb{R}^N} M\Big(v + \int_0^{\sigma} E(\theta+\omega s)\,\mathrm{d}s\Big)\,\mathrm{d}v}_{=1}\right)\mathrm{d}\theta\,\mathrm{d}\sigma \\
&= 0 - \int_{-\infty}^{0} e^{\sigma}\left(\int_0^{\sigma} \underbrace{\Big(\int_{\mathbb{Y}} E(\theta+\omega s)\,\mathrm{d}\theta\Big)}_{=0}\,\mathrm{d}s\right)\mathrm{d}\sigma = 0,
\end{aligned}$$

and for the Fokker-Planck operator

$$\begin{aligned}
&\iint_{\mathbb{Y}\times\mathbb{R}^N} v\Phi(\theta,v)\,\mathrm{d}v\,\mathrm{d}\theta \\
&= \int_{\mathbb{Y}} \underbrace{\left(\int_{\mathbb{R}^N} \Big(v - \int_{-\infty}^{0} e^{\sigma}E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big) M\Big(v - \int_{-\infty}^{0} e^{\sigma}E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}v\right)}_{=0}\,\mathrm{d}\theta \\
&\qquad + \int_{\mathbb{Y}}\left(\int_{-\infty}^{0} e^{\sigma}E(\theta+\omega\sigma)\,\mathrm{d}\sigma \underbrace{\int_{\mathbb{R}^N} M\Big(v - \int_{-\infty}^{0} e^{\sigma}E(\theta+\omega\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}v}_{=1}\right)\mathrm{d}\theta \\
&= 0 + \int_{-\infty}^{0} e^{\sigma} \underbrace{\Big(\int_{\mathbb{Y}} E(\theta+\omega\sigma)\,\mathrm{d}\theta\Big)}_{=0}\,\mathrm{d}\sigma = 0.
\end{aligned}$$

Therefore, in view of Proposition 3.3, it makes sense to consider χ and β solutions of

$$\mathscr{T}_E\chi = v\Phi, \qquad \mathscr{T}_E\beta = v\cdot\nabla_x\Phi,$$

and then to define the effective coefficients in (9) and (10).

Corollary 3.5 *The matrix* $\frac{1}{2}\big(D(x)+D(x)^{\intercal}\big)$ *is positive definite.*

Proof. This is an immediate consequence of Proposition 3.2. Indeed, for any $\xi\in\mathbb{R}^N\setminus\{0\}$, we have

$$\begin{aligned}
D(x)\xi\cdot\xi &= \iint_{\mathbb{Y}\times\mathbb{R}^N} v\cdot\xi\ \chi\cdot\xi\,\mathrm{d}v\,\mathrm{d}\theta = \iint_{\mathbb{Y}\times\mathbb{R}^N} v\Phi\cdot\xi\ \chi\cdot\xi\frac{\mathrm{d}v\,\mathrm{d}\theta}{\Phi} \\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E(\chi\cdot\xi)\ \chi\cdot\xi\frac{\mathrm{d}v\,\mathrm{d}\theta}{\Phi} = D(\chi\cdot\xi|\Phi)\ge 0.
\end{aligned}$$

This quantity vanishes iff $\chi(\theta,v)\cdot\xi = \alpha\Phi(\theta,v)$ for a certain $\alpha\in\mathbb{R}$. But, integrating this relation yields $\alpha=0$. Going back to the equation $\mathscr{T}_E(\chi\cdot\xi)=0=(v\cdot\xi)\Phi$, we deduce that $v\cdot\xi=0$ for a. e. $v\in\mathbb{R}^N$, which holds only for $\xi=0$. □

In fact, it is convenient to express the diffusion matrix by means of the adjoint equation: with $\chi^\star$ solution of $\mathscr{T}_E^\star \chi^\star = v$, we get

$$D = \iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E^\star \chi^\star \otimes \chi \,\mathrm{d}v\,\mathrm{d}\theta = \iint_{\mathbb{Y}\times\mathbb{R}^N} \chi^\star \otimes v\Phi \,\mathrm{d}v\,\mathrm{d}\theta.$$

As remarked in [11, Section 5.3], the solution of this adjoint equation can be searched for under the form

$$\chi^\star(\theta, v) = v + \psi^\star(\theta),$$

which leads to

$$\omega \cdot \nabla_\theta \psi^\star(\theta) = -E(\theta), \tag{17}$$

for both the Boltzmann and the Fokker-Planck cases (since $v \mapsto v$ is an eigenfunction for $Q^\star$). As already noticed this equation can be solved in the three cases: ⓐ purely periodic oscillations, ⓑ quasi-periodic oscillations with a finite number of harmonics, ⓒ quasi-periodic oscillations with a Diophantine conditions and a loss of regularity (namely E in H^γ yields a solution $\psi^\star$ in L^2). With these observations at hand, we obtain

- *for the Boltzmann operator*

$$D \quad = \quad \int_{-\infty}^0 \iint_{\mathbb{Y}\times\mathbb{R}^N} e^\sigma (v + \psi^\star(\theta)) \otimes vM\Big(v + \mathcal{U}(\sigma,\theta)\Big)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}\sigma$$

where we still denote $\mathcal{U}(\sigma,\theta) = \int_0^\sigma E(\theta+\omega s)\,\mathrm{d}s$. By using the remarkable identities

$$\int_{\mathbb{R}^N} (v + \mathcal{U}(\sigma,\theta))\; M(v+\mathcal{U}(\sigma,\theta))\,\mathrm{d}v = 0,$$

and

$$\int_{\mathbb{R}^N} (v + \mathcal{U}(\sigma,\theta)) \otimes (v+\mathcal{U}(\sigma,\theta)) M(v+\mathcal{U}(\sigma,\theta))\,\mathrm{d}v = \mathbb{I},$$

D can be recast as

$$D = \mathbb{I} - \int_{-\infty}^0 e^\sigma \left(\int_{\mathbb{Y}} \big(\psi^\star(\theta) - \mathcal{U}(\sigma,\theta)\big) \otimes \mathcal{U}(\sigma,\theta) \underbrace{\left(\int_{\mathbb{R}^N} M(v+\mathcal{U}(\sigma,\theta))\,\mathrm{d}v\right)}_{=1} \mathrm{d}\theta \right) \mathrm{d}\sigma.$$

However from

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[\psi^\star(\theta+\omega s)\big] = \omega\cdot\nabla_\theta \psi^\star(\theta+\omega s) = -E(\theta+\omega s)$$

we can write

$$\psi^\star(\theta+\omega\sigma) - \psi^\star(\theta) = -\mathcal{U}(\sigma,\theta)$$

so that

$$D = \mathbb{I} + \underbrace{\int_{-\infty}^0 e^\sigma \left(\int_{\mathbb{Y}} \psi^\star(\theta+\omega\sigma) \otimes \big(\psi^\star(\theta+\omega\sigma) - \psi^\star(\theta)\big)\,\mathrm{d}\theta \right) \mathrm{d}\sigma}_{\widetilde{D}}\,.$$

We remark that

$$
\begin{aligned}
&\int_{-\infty}^{0}\int_{\mathbb{Y}} e^{\sigma}\psi^\star(\theta+\omega\sigma)\otimes\psi^\star(\theta)\,\mathrm{d}\theta\,\mathrm{d}\sigma\\
&\quad = 2\int_{-\infty}^{0}\int_{-\infty}^{0}\int_{\mathbb{Y}} e^{2s}e^{\sigma}\psi^\star(\theta+\omega\sigma)\otimes\psi^\star(\theta)\,\mathrm{d}\theta\,\mathrm{d}\sigma\,\mathrm{d}s\\
&\quad = 2\int_{-\infty}^{0}\int_{-\infty}^{s}\int_{\mathbb{Y}} e^{\sigma'+s}\psi^\star(\theta-\omega s+\omega\sigma')\otimes\psi^\star(\theta)\,\mathrm{d}\theta\,\mathrm{d}\sigma'\,\mathrm{d}s\\
&\qquad \text{with the change of variable } \sigma+s=\sigma'\\
&\quad = 2\int_{-\infty}^{0}\int_{-\infty}^{s}\int_{\mathbb{Y}} e^{\sigma'}\psi^\star(\theta'+\omega\sigma')\otimes e^{s}\psi^\star(\theta'+\omega s)\,\mathrm{d}\theta\,\mathrm{d}\sigma'\,\mathrm{d}s\\
&\qquad \text{with the change of variable } \theta-\omega s=\theta' \text{ and using } \mathbb{Y}\text{-periodicity.}
\end{aligned}
$$

Consequently, for any $\xi\in\mathbb{R}^N$, we have

$$
\begin{aligned}
\widetilde{D}\xi\cdot\xi &= \int_{\mathbb{Y}}\int_{-\infty}^{0} e^{\sigma}\big(\psi^\star(\theta+\omega\sigma)\cdot\xi\big)^2\,\mathrm{d}\sigma\,\mathrm{d}\theta\\
&\quad -\int_{\mathbb{Y}}\int_{-\infty}^{0}\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{-\infty}^{s} e^{\sigma'}\psi^\star(\theta+\omega\sigma')\cdot\xi\,\mathrm{d}\sigma'\right)^2\mathrm{d}s\,\mathrm{d}\theta\\
&= \int_{\mathbb{Y}}\left(\int_{-\infty}^{0} e^{\sigma}\big(\psi^\star(\theta+\omega\sigma)\cdot\xi\big)^2\,\mathrm{d}\sigma-\left|\int_{-\infty}^{0} e^{s}\psi^\star(\theta+\omega s)\cdot\xi\,\mathrm{d}s\right|^2\right)\mathrm{d}\theta\geq 0,
\end{aligned}
$$

by virtue of the Cauchy-Schwarz inequality.

- *for the Fokker-Planck operator*

$$
D = \iint_{\mathbb{Y}\times\mathbb{R}^N}(v+\psi^\star(\theta))\otimes vM\Big(v-\mathcal{U}(\theta)\Big)\,\mathrm{d}v\,\mathrm{d}\theta
$$

where now $\mathcal{U}(\theta)=\int_{-\infty}^{0}e^{s}E(\theta+\omega s)\,\mathrm{d}s$. It becomes

$$
\begin{aligned}
D &= \int_{\mathbb{Y}}\left(\int_{\mathbb{R}_N} v\otimes v+\psi^\star(\theta)\otimes(v-\mathcal{U}(\theta))+\psi^\star(\theta)\otimes\mathcal{U}(\theta)\Big)M(v-\mathcal{U}(\theta))\,\mathrm{d}v\right)\mathrm{d}\theta\\
&= \mathbb{I}+\underbrace{\int_{\mathbb{Y}}(\mathcal{U}(\theta)+\psi^\star(\theta))\otimes\mathcal{U}(\theta)\,\mathrm{d}\theta}_{=\widetilde{D}}.
\end{aligned}
$$

Going back to (17), we infer

$$
\psi^\star(\theta)-\int_{-\infty}^{0}e^{s}\psi^\star(\theta+\omega s)\,\mathrm{d}s=-\mathcal{U}(\theta),
$$

which leads to

$$
\widetilde{D}=\int_{\mathbb{Y}}\left\{\left(\int_{-\infty}^{0}e^{s}\psi^\star(\theta+\omega s)\,\mathrm{d}s\otimes\int_{-\infty}^{0}e^{s}\psi^\star(\theta+\omega s)\,\mathrm{d}s\right)-\int_{-\infty}^{0}e^{s}\psi^\star(\theta+\omega s)\,\mathrm{d}s\otimes\psi^\star(\theta)\right\}\mathrm{d}\theta.
$$

Hence, by repeating the same manipulations as for the Boltzmann case, it follows that

$$
\widetilde{D}\xi\cdot\xi=0.
$$

These computations provide further information about the effective coefficients (compare to [11, Section 5.4]).

Lemma 3.6 *For both the Boltzmann and the Fokker-Planck cases, we get*

$$D = \mathbb{I} + \widetilde{D},$$

where $\widetilde{D}$ *is the difference of a symmetric matrix and*

$$A = \int_{\mathbb{Y}} \int_{-\infty}^{0} e^s \psi^\star(\theta + \omega s) \otimes \psi^\star(\theta)\,\mathrm{d}\theta\,\mathrm{d}s.$$

If for all components of the force field $\theta \mapsto E_j(\theta)$ *is odd (or even), then,* A*, and thus* $\widetilde{D}$*, is symmetric. Otherwise the effective diffusion matrix might contain a skew-symmetric component. Anyway, we have* $\widetilde{D} + \widetilde{D}^\intercal \geq 0$ *for the Boltzmann operator, and* $\widetilde{D} + \widetilde{D}^\intercal = 0$ *for the Fokker-Planck operator.*

Proof. We have already identified the structure of D and $\widetilde{D}$. If all the components of E are odd or even, then $\psi^\star$ has the opposite property. In turn, the product $(\theta, s) \mapsto \psi_j^\star(\theta + \omega s)\psi_k^\star(\theta)$ is even for any $j, k \in \{1, ..., N\}$. Therefore, by using the change of variable $\theta + \omega s = -\theta'$, we get

$$\begin{aligned}
A_{jk} &= \int_{-\infty}^{0} e^s \left(\int_{\mathbb{Y}} \psi_j^\star(\theta + \omega s)\psi_k^\star(\theta)\,\mathrm{d}\theta \right) \mathrm{d}s \\
&= \int_{-\infty}^{0} e^s \left(\int_{\mathbb{Y}} \psi_j^\star(-\theta')\psi_k^\star(-\theta' - \omega s)\,\mathrm{d}\theta' \right) \mathrm{d}s \\
&= \int_{-\infty}^{0} e^s \left(\int_{\mathbb{Y}} \psi_j^\star(\theta')\psi_k^\star(\theta' + \omega s)\,\mathrm{d}\theta' \right) \mathrm{d}s = A_{kj}.
\end{aligned}$$

□

Example 3.7 *It is worth detailing the expression of the effective coefficients for the simple 2D examples with periodic oscillations:*

$$E^{(1)}(t) = \Upsilon \cos(2\pi t), \qquad E^{(2)}(t) = \upsilon \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix},$$

with Υ *a fixed vector in* $\mathbb{R}^2$ *(or, in the space dependent case, a given vector field), and* υ *a scalar quantity (which, again, can be space-dependent). In this periodic setting, we have*

$$\psi^\star(t) = -\int_0^t E(s)\,\mathrm{d}s,$$

and thus

$$\psi^{\star,(1)}(t) = -\frac{\Upsilon}{2\pi}\sin(2\pi t), \qquad \psi^{\star,(2)}(t) = \frac{\upsilon}{2\pi}\begin{pmatrix} -\sin(2\pi t) \\ \cos(2\pi t) - 1 \end{pmatrix}.$$

It follows that

$$\begin{aligned}
A^{(1)} &= \frac{\Upsilon \otimes \Upsilon}{4\pi^2} \int_0^1 \sin(2\pi t)\mathrm{Im}\Big(\frac{e^{2i\pi t}}{1 + 2i\pi}\Big)\,\mathrm{d}t \\
&= \frac{\Upsilon \otimes \Upsilon}{(1 + 4\pi^2)4\pi^2} \int_0^1 \Big(\frac{1 - \cos(4\pi t)}{2} - \pi \sin(4\pi t)\Big)\,\mathrm{d}t = \frac{\Upsilon \otimes \Upsilon}{(1 + 4\pi^2)8\pi^2}
\end{aligned}$$

is symmetric, while

$$\begin{aligned}
A_{12}^{(2)} &= -\frac{\upsilon^2}{4\pi^2} \int_0^1 (\cos(2\pi t) - 1)\mathrm{Im}\Big(\frac{e^{2i\pi t}}{1 + 2i\pi}\Big)\,\mathrm{d}t \\
&= \frac{\upsilon^2}{(1 + 4\pi^2)4\pi^2} \int_0^1 \Big(\pi + \pi\cos(4\pi t) - \frac{\sin(4\pi t)}{2}\Big)\,\mathrm{d}t = \frac{\upsilon^2}{(1 + 4\pi^2)4\pi}
\end{aligned}$$

and

$$\begin{aligned} A^{(2)}_{21} &= -\frac{\upsilon^2}{4\pi^2}\int_0^1 \sin(2\pi t)\mathrm{Re}\Big(\frac{e^{2i\pi t}}{1+2i\pi}\Big)\,\mathrm{d}t \\ &= -\frac{\upsilon^2}{(1+4\pi^2)4\pi^2}\int_0^1 \Big(\pi - \pi\cos(4\pi t) + \frac{\sin(4\pi t)}{2}\Big)\,\mathrm{d}t = -\frac{\upsilon^2}{(1+4\pi^2)4\pi} = -A^{(2)}_{12}. \end{aligned}$$

The presence of such, somehow surprising, skew-symmetric components of the diffusion matrix has been reported for instance in oceanographic flows [16, 21].

Note however that, when the force field is space homogeneous, the drift coefficient vanishes and the diffusion coefficient is space homogeneous. In this specific situation, the possible skew-symmetric part does not play any role since, A being a constant skew-symmetric matrix, we have $\nabla_x\cdot(A\nabla_x\rho) = \sum_{i,j=1}^N \partial_{x_i}\big(A_{ij}\partial_{x_j}\rho\big) = \sum_{i,j=1}^N A_{ij}\partial^2_{x_ix_j}\rho = \sum_{j,i=1}^N A_{ji}\partial^2_{x_jx_i}\rho = -\sum_{j,i=1}^N A_{ji}\partial^2_{x_jx_i}\rho = 0.$

As said above, we consider force fields $(\theta,x)\mapsto E(\theta,x)$. Thus the auxilliary functions $\chi,\chi^\star,\phi^\star$... depend on the space variable, which implies that U and D are functions of x, too. The regularity of E is, roughly speaking, transferred, to the solutions of the cell equations, as it can be understood by taking space derivatives of these equations. Owing to (5), we can check that U and D belong to $W^{2,\infty}(\mathbb{R}^N)$.

3.4 Generalization

The linear Boltzmann operator might be considered as too simple. It has the advantage of leading to explicit formula for the equilibrium function Φ. However, it can be interesting for the applications to consider more general linear collision operators, having the form

$$Q(f) = K(f) - \Sigma f,$$

where

$$K(f) = \int_{\mathbb{R}^N} k(v,v')f(v')\,\mathrm{d}v', \qquad \Sigma(v) = \int_{\mathbb{R}^N} k(v',v)\,\mathrm{d}v'.$$

The definition of Σ guarantees the mass conservation property: we always have $\int Q(f)\,\mathrm{d}v = 0$. Accordingly, with $\mathscr{T}_E = \omega\cdot\nabla_\theta + E\cdot\nabla_v - Q$, the kernel of the adjoint $\mathscr{T}_E^\star$ still contains the constant functions. We refer the reader to [12] for the analysis of the diffusion asymptotics in such a generality. The existence of a positive equilibrium function, which thus spans $\mathrm{Ker}(Q)$, follows from compactness arguments and the application of the Krein-Rutman theorem [22]. We are going to adapt this approach. To this end, let us introduce

$$\mathscr{A} = \omega\cdot\nabla_\theta + E\cdot\nabla_v + \Sigma.$$

In what follows we assume that

$$\begin{aligned} &0<\kappa_\star\eta(v)\le k(v,v')\le\kappa^\star\eta(v) \text{ with } 0<\kappa_\star<\kappa^\star, \\ &\text{and } \eta\in C^0\cap L^1(\mathbb{R}^N),\ \eta \text{ positive}, \int_{\mathbb{R}^N}\eta(v)\,\mathrm{d}v = 1,\ \lim_{|v|\to\infty}\eta(v)=0, \end{aligned} \tag{18}$$

holds. (For the relaxation operator considered so far, we simply have $k(v,v') = M(v)\mathbf{1}(v')$.) We further assume

$$|\nabla_v k(v,v')|\le\kappa^\star, \qquad |\nabla_{v'}k(v,v')|\le\kappa^\star\eta(v). \tag{19}$$

We wish to solve the equation

$$\mathscr{A}f = g.$$

We make use of the characteristics

$$\frac{\mathrm{d}}{\mathrm{d}s}\Theta = \omega, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\mathscr{V} = E(\Theta),$$

with $(\Theta, \mathscr{V})(0) = (\theta, v)$ so that the problem can be cast as

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big[f(\Theta(s), \mathscr{V}(s)) \exp\Big(\int_0^s \Sigma(\mathscr{V}(\sigma)\,\mathrm{d}\sigma\Big)\Big] = \exp\Big(\int_0^s \Sigma(\mathscr{V}(\sigma))\,\mathrm{d}\sigma\Big)\ g(\Theta(s), \mathscr{V}(s)).$$

It leads to

$$f(\theta, v) = \int_{-\infty}^0 \exp\Big(\int_0^s \Sigma(\mathscr{V}(\sigma))\,\mathrm{d}\sigma\Big)\ g(\Theta(s), \mathscr{V}(s))\,\mathrm{d}s$$

with

$$\Theta(s) = \theta + \omega s, \qquad \mathscr{V}(s) = v + \int_0^s E(\theta + \omega\sigma)\,\mathrm{d}\sigma.$$

This formula defines the operator $\mathscr{A}^{-1}$ on L^p spaces.

Lemma 3.8 *In the purely periodic setting, the operator $K \circ \mathscr{A}^{-1}$ is compact and positive.*

Proof. We study the operator which associate to a function g the function

$$K \circ \mathscr{A}^{-1} g(\theta, v) = \int_{-\infty}^0 \int_{\mathbb{R}^N} k(v, v') \exp\Big(\int_0^s \Sigma\Big(v + \int_0^\sigma E(\theta + \varsigma)\,\mathrm{d}\varsigma\Big)\,\mathrm{d}\sigma\Big)$$
$$\times g\Big(\theta + s, v' + \int_0^s E(\theta + \varsigma)\,\mathrm{d}\varsigma\Big)\,\mathrm{d}v'\,\mathrm{d}s.$$

With g and E being $1-$periodic with respect to the variable θ, $K \circ \mathscr{A}^{-1} g$ is periodic too.

Let g be a non negative function, non identically 0. In particular, we have $\iint g\,\mathrm{d}v\,\mathrm{d}\tau > 0$. By using (18), we get

$$\begin{aligned}
K \circ \mathscr{A}^{-1} g(\theta, v) &\geq \int_{-\infty}^0 \int_{\mathbb{R}^N} e^{\kappa^\star s} k\Big(v, w - \int_0^s E(\theta + \varsigma)\,\mathrm{d}\varsigma\Big) g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&\geq \kappa_\star \eta(v) \int_{-\infty}^0 \int_{\mathbb{R}^N} e^{\kappa^\star s} g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&\geq \kappa_\star \eta(v) \sum_{n\in\mathbb{N}} \int_{-(n+1)}^{-n} \int_{\mathbb{R}^N} e^{\kappa^\star s} g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&\geq \kappa_\star \eta(v) \sum_{n\in\mathbb{N}} e^{-\kappa^\star (n+1)} \int_{-(n+1)}^{-n} \int_{\mathbb{R}^N} g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&\geq \kappa_\star \eta(v) \sum_{n\in\mathbb{N}} e^{-\kappa^\star (n+1)} \|g\|_{L^1((0,1)\times\mathbb{R}^N)} = \eta(v) \frac{\kappa_\star e^{-\kappa^\star}}{1 - e^{-\kappa^\star}} \|g\|_{L^1((0,1)\times\mathbb{R}^N)} > 0.
\end{aligned}$$

We work on functions g that are both bounded and integrable: $g \in C^0 \cap L^1((0,1) \times \mathbb{R}^N)$, with

$$\|g\|_{L^\infty} \leq R, \qquad \|g\|_{L^1} \leq R. \tag{20}$$

As a matter of fact, we remark that

$$\begin{aligned}
|K \circ \mathscr{A}^{-1} g(\theta, v)| &\leq \kappa^\star \eta(v) \int_{-\infty}^0 e^{\kappa_\star s} \int_{\mathbb{R}^N} g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&\leq \kappa^\star \eta(v) \sum_{n=0}^\infty \int_{-(n+1)}^{-n} e^{\kappa_\star s} \int_{\mathbb{R}^N} g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&\leq \kappa^\star \eta(v) \sum_{n=0}^\infty e^{-\kappa_\star n} \int_{-(n+1)}^{-n} \int_{\mathbb{R}^N} g(\theta + s, w)\,\mathrm{d}w\,\mathrm{d}s \\
&= \frac{\kappa^\star}{1 - e^{-\kappa_\star}} \eta(v) \|g\|_{L^1} \leq \frac{R\kappa^\star}{1 - e^{-\kappa_\star}} \eta(v).
\end{aligned}$$

It proves that the range by $K\circ\mathscr{A}^{-1}$ of the set defined by (20) is equibounded, and it uniformly vanishes at infinity.

Similar manipulations show that $\nabla_v K\circ\mathscr{A}^{-1}g(\theta,v)$ is also uniformly bounded for any g verifying (20), by virtue of (19). Finally, let $f=\mathscr{A}^{-1}g$, which satisfies $(\partial_\tau+E\cdot\nabla_v+\Sigma)f=g$. Then, we compute

$$\partial_\tau\int_{\mathbb{R}^N}k(v,v')f(\tau,v')\,\mathrm{d}v'=\int_{\mathbb{R}^N}k(v,v')\big(g-\Sigma f\big)(\tau,v')\,\mathrm{d}v'+\int_{\mathbb{R}^N}\nabla_v k(v,v')Ef(\tau,v')\,\mathrm{d}v'$$

which, again, is uniformly bounded for any g verifying (20). The Arzela-Ascoli theorem implies that $Kf=K\circ\mathscr{A}^{-1}g$ lies in a compact set of $C^0((0,1)\times\mathbb{R}^N)$.

Corollary 3.9 *There exists a continuous and positive function $(\tau,v)\mapsto\Phi(\tau,v)$ such that*

$$\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi\,\mathrm{d}\theta\,\mathrm{d}v=1\qquad\textit{and}\qquad\mathscr{T}_E\Phi=0.$$

Moreover, we have $\mathrm{Ker}(\mathscr{T}_E)=\mathrm{Span}(\Phi)$.

Proof. Applying the Krein-Rutman theorem [22], we deduce that we can find a non negative function $G\in C^0\cap L^1$, $G\neq 0$, such that $K\circ\mathscr{A}^{-1}G=\lambda G$, with $\lambda>0$ the spectral radius of $K\circ\mathscr{A}^{-1}$. This relation shows that actually $G>0$, since $K\circ\mathscr{A}^{-1}$ is a positive operator. Then, $\Phi=\mathscr{A}^{-1}G>0$ and it satisfies $\lambda\mathscr{A}\Phi=K\Phi$. Integrating this equation shows that $\lambda=1$, by definition of Σ. Dissipative properties, as discussed for the linear Boltzmann operator imply that the kernel of $\mathscr{T}_E$ is spanned by Φ. □

The reasoning for the linear Boltzmann operator can be adapted to this situation, likely at the price of a more intricate functional framework, see [12] for some hints in this direction. Similarly, the Fokker-Planck operator can be generalized into $Q(f)=\nabla_v\cdot(\nabla_v W(v)f+\nabla_v f)$, for a certain energy function W, such that $v\mapsto e^{-W(v)}\in L^1(\mathbb{R}^N)$.

Another possible generalization consists in extending the analysis to force fields of KBM-type (for Krylov, Bogolyubov and Mitropolski), that is dealing with $\mathcal{E}$ that admits a mean value

$$\langle\mathcal{E}\rangle(x)=\lim_{T\to\infty}\frac{1}{T}\int_0^T\mathcal{E}(s,x)\,\mathrm{d}s.$$

We refer the reader to [32] for a thorough introduction to such functions. Such an almost-periodic field can still be represented by means of a Fourier series

$$\mathcal{E}(t,x)=\sum_{\lambda\in\Lambda}a_\lambda e^{2i\pi\lambda t}$$

where the set Λ is the spectrum of the function $\mathcal{E}$, see for instance [24]. In turn, the cell equations can be solved up to a condition like $\sum_{\lambda\in\Lambda}\frac{|a_\lambda|}{\lambda}<\infty$. The notion of double-scale convergence that we shall use in the forthcoming sections can be extended to this framework as well, see [6]. However, the discussion on the compactness properties necessary for the asymptotic analysis can be more delicate, see e. g. [8], and are beyond the objectives of the present paper.

4 Double-scale analysis

The formal analysis indicates that

$$f_\varepsilon(t,x,v)\simeq\rho(t,x)\Phi(t/\varepsilon^2,x,v).$$

In particular the asymptotic regime keeps fast time-oscillations, which prevents from obtaining a strong convergence statement. We shall use the framework of double-scale convergence, as introduced in [1, 28]. We refer to [14, 20] for specific statements dealing with time-oscillations. We shall deal with sequences of measure-valued functions $\mu_n : [0,\infty) \to \mathscr{M}^1(\mathbb{R}^D)$. We start by recalling the following definitions and a standard compactness statement [20].

Definition 4.1 *Let I be an interval of $\mathbb{R}$. A family $\{t \in I \mapsto \mu(t)\}$ of Radon measures on $\mathbb{R}^D$ is said to be* vaguely continuous *iff*

$$\forall \varphi \in C_c^0(\mathbb{R}^D), \qquad t \longmapsto \int_{\mathbb{R}^D} \varphi(x)\ \mu(t,x)\,\mathrm{d}x \ \text{ is a continuous function on } I.$$

Definition 4.2 *A sequence $\{t \in I \mapsto \mu_n(t),\ n \in \mathbb{N}\}$ is said to be equibounded and vaguely equicontinuous on I iff*

(i) *there exists $M > 0$ such that $\sup_{t\in I,\, n\in\mathbb{N}} |\mu_n(t)|(\mathbb{R}^D) \le M$;*

(ii) *for any $\varphi \in C_c^0(\mathbb{R}^D)$, the sequence of functions $\big(t \longmapsto \int_{\mathbb{R}^D} \varphi(x)\ \mu_n(t,x)\,\mathrm{d}x\big)_{n\in\mathbb{N}}$ is equicontinuous on I.*

Proposition 4.3 *Let I be an interval of $\mathbb{R}$. Let $\big(\mu_n(t)\big)_{n\in\mathbb{N}}$ be a sequence of Radon measures on $\mathbb{R}^D$, equibounded and vaguely equicontinuous on I. Then there exist a measure $\mu(t)$ vaguely continuous on I and a subsequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$ such that*

$$\forall \varphi \in C_c^0(I \times \mathbb{R}^D), \quad \int_{\mathbb{R}^D} \varphi(t,x)\ \mu_{n_k}(t,x)\,\mathrm{d}x \xrightarrow[k\to\infty]{} \int_{\mathbb{R}^D} \varphi(t,x)\ \mu(t,x)\,\mathrm{d}x,$$

uniformly with respect to $t \in I$. We say that sequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$ converges vaguely to $\mu(t)$ locally, uniformly on I.

Next, we give a version of the double-scale compactness, adapted to our purposes from [1, 28]. There are two main difficulties. One the one hand, we are lacking useful estimates: we are dealing with sequences which are only bounded in L^1, or in $\mathscr{M}^1$, and we do not have any direct estimates on the macroscopic current which arises in the mass conservation equation. On the other hand, we consider quasi-periodic oscillations, which is a source of technical difficulties. For the extension of the double scale convergence to measure-valued functions, we refer the reader for instance to [20]. The extension to the quasi-periodic framework of the analysis proposed in [1, 28], for which we refer the reader to [7, Proposition 5.2], relies on a variant of the Birkhoff theorem [13] which involves the ergodic condition "ω has rationally independent components".

Proposition 4.4 *Let f_ε be a bounded sequence in $L^2(\mathbb{R})$. Let $\omega \in \mathbb{R}^r$ with rationally independent components. Then, there exists a subsequence, still labelled by ε, and a function $F \in L^2_\#(\mathbb{R} \times \mathbb{Y})$ such that for any test function $\psi \in L^2(\mathbb{R}; C^0_\#(\mathbb{Y}))$,[1] we have*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} f_\varepsilon(t)\ \psi(\omega t/\varepsilon^2, t)\,\mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{Y}} F(\theta,t)\ \psi(\theta,t)\,\mathrm{d}\theta\,\mathrm{d}t.$$

Then we also have the existence of a double-scale limit, in the spirit of [1, 28].

Proposition 4.5 *Let $\big(\varepsilon_n\big)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0. Let $\big(\mu_n(t)\big)_{n\in\mathbb{N}}$ be a sequence of measures on $\mathbb{R}^D$, equibounded on an interval $I \subset \mathbb{R}$. Then there exist a subsequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$ and a measure M on $I \times \mathbb{Y} \times \mathbb{R}^D$ such that for any $\varphi \in C^0_{c,\#}(I \times \mathbb{R}^r \times \mathbb{R}^D)$, we have*

$$\int_I \int_{\mathbb{R}^D} \varphi(\omega t/\varepsilon_{n_k}^2, t, x)\ \mu_{n_k}(t,x)\,\mathrm{d}x\,\mathrm{d}t \xrightarrow[k\to\infty]{} \int_I \int_{\mathbb{R}^D} \int_{\mathbb{Y}} \varphi(\theta,t,x)\ M(\theta,t,x)\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t.$$

[1] Hereafter, the symbol # indicates $\mathbb{Y}-$periodicity with respect to the variable $\theta \in R^r$. Referring to [1, Section 5], $L^2(\mathbb{R}; C^0_\#(\mathbb{Y}))$ is the class of functions $\psi : \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}$ which are measurable and square integrable with respect to the variable $t \in \mathbb{R}$, with values in the Banach space of continuous and $\mathbb{Y}-$periodic functions.

We say that the measure M is the double-scale limit of the sequence $\big(\mu_{n_k}(t)\big)_{k\in\mathbb{N}}$.

The underlying operator $\mathscr{T}_E$ has a dissipative role that pushes the particles distribution function towards the equilibrium Φ which spans $\mathrm{Ker}(\mathscr{T}_E)$. However, the usual entropy inequalities associated to this dissipative structure do not provide useful estimates, due to the stiff acceleration term. For this reason, we shall work with L^1 estimates only, and thus in the framework of bounded Radon measures.

Proposition 4.6 *Let $f_{0,\varepsilon} \geq 0$ be the initial data for* (1). *We suppose that*

$$(1+|v|^k) f_{0,\varepsilon} \text{ is bounded in } L^1(\mathbb{R}^N \times \mathbb{R}^N),$$

for some $k > 2$. Then, for any $0 < T < \infty$,

$$(1+|v|^k) f_\varepsilon \text{ is bounded in } L^\infty(0,T; L^1(\mathbb{R}^N \times \mathbb{R}^N)),$$

and, up to a subsequence, we can suppose that $t \mapsto \rho_\varepsilon(t,x) = \int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\,\mathrm{d}v \in \mathscr{M}^1(\mathbb{R}^N)$ converges vaguely, uniformly on $[0,T]$.

Proof. The only immediate a priori estimate for the problem (1) is mass conservation which directly implies that

$$f_\varepsilon \text{ is bounded in } L^\infty(0,\infty; L^1(\mathbb{R}^N \times \mathbb{R}^N)).$$

Consequently,

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\,\mathrm{d}v \text{ is bounded in } L^\infty(0,\infty; L^1(\mathbb{R}^N)).$$

Thus, it will be possible to apply Proposition 4.3 and Proposition 4.5 (with $D = 2N$ or $D = N$, respectively), to these sequences. Next, let us consider the evolution of the kinetic energy. Owing to integration by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{v^2}{2} f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x = \frac{1}{\varepsilon^2}\left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{E}(t/\varepsilon^2, x)\cdot v f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{v^2}{2} Q(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x \right).$$

The last term is nothing but

$$\begin{aligned} &\frac{Nm}{2} - \frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} v^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x && \text{for the Boltzmann operator,}\\ &Nm - \iint_{\mathbb{R}^N\times\mathbb{R}^N} v^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x && \text{for the Fokker-Planck operator} \end{aligned}$$

where $m = \iint f_{0,\varepsilon}\,\mathrm{d}v\,\mathrm{d}x$ stands for the total mass. The Cauchy-Schwarz inequality yields

$$\left| \iint_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{E}(t/\varepsilon^2, x)\cdot v f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \right| \leq \frac{\delta}{2} m \|\mathcal{E}\|^2_{L^\infty} + \frac{1}{2\delta}\iint_{\mathbb{R}^N\times\mathbb{R}^N} v^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x$$

where we choose $\delta = 2$ for the Boltzmann case and $\delta = 1$ for the Fokker-Planck case. Then, we conclude that

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} v^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \leq e^{-t/(\delta\varepsilon^2)} \iint_{\mathbb{R}^N\times\mathbb{R}^N} v^2 f_{0,\varepsilon} \,\mathrm{d}v\,\mathrm{d}x + (1 - e^{-t/(\delta\varepsilon^2)}) C(m, N, \|\mathcal{E}\|_{L^\infty}).$$

Consequently,

$$v^2 f_\varepsilon \text{ is bounded in } L^\infty(0,\infty; L^1(\mathbb{R}^N \times \mathbb{R}^N)).$$

Using the Cauchy-Schwarz inequality

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |v| f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x \leq \sqrt{m}\sqrt{\iint_{\mathbb{R}^N\times\mathbb{R}^N} v^2 f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x}$$

is however not enough to deduce a useful estimate on the *rescaled* current

$$J_\varepsilon(t,x) = \int_{\mathbb{R}^N} \frac{v}{\varepsilon} f_\varepsilon(t,x,v)\,\mathrm{d}v$$

which arises in the local mass conservation relation

$$\partial_t \rho_\varepsilon + \nabla_x \cdot J_\varepsilon = 0.$$

For technical purposes, we shall need further moment estimates. Namely, we study the time evolution of

$$M_k(t) = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |v|^k f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x,$$

for some $k > 2$. We start from

$$\frac{\mathrm{d}}{\mathrm{d}t} M_k = \frac{1}{\varepsilon^2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{E}(t/\varepsilon^2)\cdot \frac{v}{|v|}\; k|v|^{k-1} f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon^2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} |v|^k Q(f_\varepsilon)\,\mathrm{d}v\,\mathrm{d}x,$$

where the contribution with the collision term reads

$-kM_k + (N + (k-2))M_{k-2}$	for the Fokker-Planck operator,
$-M_k + mC(k)$	for the Boltzmann operator,

with $C(k)$ the moment of order k of the Maxwellian. By using Cauchy-Schwartz and Young inequalities to estimate the acceleration term, we arrive at a differential inequation, and we conclude like with the kinetic energy:

$$|v|^k f_\varepsilon \text{ is bounded in } L^\infty(0,\infty; L^1(\mathbb{R}^N\times\mathbb{R}^N)).$$

Proposition 4.3 cannot be applied directly to ρ_ε; that the compactness property holds will be detailed in the forthcoming discussion. □

As said above, we can suppose that

$$f_\varepsilon(t,x,v) \rightharpoonup F(\theta,t,x,v), \qquad \rho_\varepsilon(t,x) \rightharpoonup R(\theta,t,x)$$

in the double scale sense of Proposition 4.5 (with $D = 2N$ or $D = N$, respectively). The bound on the velocity moments can be translated into a tightness criterion, which allows us to deal with test functions $\phi(\theta,t,x,v)$, which do not necessarily vanish at infinity. Namely, the double scale convergence still applies as soon as the test function verifies

$$\lim_{|v|\to\infty} \frac{\phi(\theta,t,x,v)|}{|v|^k} = 0.$$

In particular, we deduce that

$$R(\theta,t,x) = \int_{\mathbb{R}^N} F(\theta,t,x,v)\,\mathrm{d}v,$$

and, next (using test functions that do not depend on θ), that

$$\rho(t,x) = \int_{\mathbb{Y}} R(\theta,t,x)\,\mathrm{d}\theta = \iint_{\mathbb{Y}\times\mathbb{R}^N} F(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta.$$

We start by rewriting the following weak form of (1), working with oscillating test functions

$$\int_0^\infty \iint_{\mathbb{R}^N\times\mathbb{R}^N} f_\varepsilon(t,x,v) \times \left(\frac{1}{\varepsilon^2}\omega\cdot\nabla_\theta + \partial_t + \frac{1}{\varepsilon} v\cdot\nabla_x + \frac{1}{\varepsilon^2}E\cdot\nabla_v + \frac{1}{\varepsilon^2}Q^\star\right)\phi(\omega t/\varepsilon^2, t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{21}$$

Mutiplying by ε^2 and letting ε go to 0, we obtain

$$\int_0^\infty \iint_{\mathbb{Y}\times\mathbb{R}^N\times\mathbb{R}^N} F\mathscr{T}_E^\star\phi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t = 0.$$

This will imply

$$F(\theta,t,x,v) = \rho(t,x)\Phi(\theta,x,v),$$

but we cannot conclude directly due to the lack of regularity of the limit F which is only known to be measure-valued. Let us pick a smooth function Γ such that $\iint_{\mathbb{Y}\times\mathbb{R}^N}\Gamma\Phi\,\mathrm{d}v\,\mathrm{d}\theta = 0$. According to Remark 3.4, we can find γ such that $\mathscr{T}_E^\star\gamma = \Gamma$. We thus have $\iint_{\mathbb{Y}\times\mathbb{R}^N}\zeta\Gamma F\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t = 0$ for any $\zeta\in C_c^\infty((0,\infty)\times\mathbb{R}^N)$. Let now consider $\phi\in C_c^\infty(\mathbb{Y}\times(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$. We expand

$$\begin{cases} \phi(\theta,t,x,v) = \langle\!\langle\phi\rangle\!\rangle(t,x)\Phi(\theta,x,v) + (\phi(\theta,t,x,v) - \langle\!\langle\phi\rangle\!\rangle(t,x)\Phi(\theta,x,v)),\\ \langle\!\langle\phi\rangle\!\rangle(t,x) = \displaystyle\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\phi(t,\theta,x,v)\Phi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\Big(\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\Phi^2(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\Big)^{-1}. \end{cases}$$

By definition $\int_{\mathbb{Y}}\int_{\mathbb{R}^N}(\phi - \langle\!\langle\phi\rangle\!\rangle\Phi)\Phi\,\mathrm{d}v\,\mathrm{d}\theta = 0$, and we have

$$\begin{aligned} &\int_0^\infty\int_{\mathbb{R}^N}\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\phi(\theta,t,x,v)F(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad= \int_0^\infty\int_{\mathbb{R}^N}\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\langle\!\langle\phi\rangle\!\rangle(t,x)\Phi(\theta,x,v)F(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad= \int_0^\infty\int_{\mathbb{R}^N}\Big(\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\phi(\theta',t,x,w)\Phi(\theta',x,w)\,\mathrm{d}w\,\mathrm{d}\theta'\Big)\overline{\rho}(t,x)\,\mathrm{d}x\,\mathrm{d}t \end{aligned}$$

where the (nonnegative) measure $\overline{\rho}\in\mathcal{M}^1((0,\infty)\times\mathbb{R}^N)$ is defined by

$$\overline{\rho}(t,x) = \Big(\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\Phi^2(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\Big)^{-1}\int_{\mathbb{Y}}\int_{\mathbb{R}^N}\Phi F(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t.$$

It follows that $F(\theta,t,x,v) = \Phi(\theta,x,v)\overline{\rho}(t,x)$. Moreover, dealing with ϕ depending only on t,x, the previous computation yields $\overline{\rho} = \rho$.

Remark 4.7 *Of course, we can also assume that f_ε admits a weak limit f, say in $\mathscr{M}^1((0,T)\times\mathbb{R}^N\times\mathbb{R}^N)$. We can check (just working with trial functions that do not depend on the variable θ) that this quantity is nothing but the cell-average of the double scale limit, namely we have*

$$f(t,x,v) = \rho(t,x)\int_{\mathbb{Y}}\Phi(\theta,x,v)\,\mathrm{d}\theta.$$

Quite interestingly, we observe that $f(t,x,v)\neq\rho(t,x)M(v)$. Note also that time-oscillations of the equilibrium function is an obstruction to the strong convergence of the particle distribution function, which usually holds in diffusion regimes, see [17] for similar comments.

Next, pick $\varphi\in C_c^\infty((0,\infty)\times\mathbb{R}^N)$, which thus belongs to the kernel of the operator $\mathscr{T}_E^\star$. We associate to φ, the auxilliary function defined by

$$\mathscr{T}_E^\star\Psi = v\cdot\nabla_x\varphi.$$

In fact, the computations made in the previous section, when identifying the effective coefficients, allow us to detail the expression of this quantity: we have

$$\Psi(\theta,t,x,v) = \chi^\star(\theta,x,v)\cdot\nabla_x\varphi(t,x) = (v+\psi^\star(\theta,x))\cdot\nabla_x\varphi(t,x),$$

where we remind the reader that $\omega\cdot\nabla_\theta\psi^\star = -E$. We now use as test function

$$\phi(\theta,t,x,v) = \varphi(t,x) + \varepsilon\Psi(\theta,t,x,v),$$

so that the singular terms in (21) disappear. To be specific, we have (in the sense of distributions on $[0,\infty)$)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_\varepsilon(t,x)\ \Big(\varphi(t,x)+\varepsilon\Psi(t/\varepsilon^2,t,x,v)\Big)\,\mathrm{d}v\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\Big(\partial_t\varphi(t,x)+\varepsilon(\partial_t\Psi)(t/\varepsilon^2,t,x,v)\Big)\,\mathrm{d}v\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\ v\cdot\nabla_x\Psi(t/\varepsilon^2,t,x,v)\,\mathrm{d}v\,\mathrm{d}x,
\end{aligned}\tag{22}$$

where the last integral can be cast as [2]

$$\begin{aligned}
-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_\varepsilon(t,x,v)\Big(&v\otimes\chi^\star(t/\varepsilon^2,x,v):D_x^2\Psi(t,x)\\
&+(D_x\chi^\star)^\intercal(t/\varepsilon^2,x,v)v\cdot\nabla_x\Psi(t,x)\Big)\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

The right hand side of (22) is bounded, uniformly with respect to ε. We deduce that, for any given function $\varphi\in C_c^2([0,\infty)\times\mathbb{R}^N)$, the family

$$t\longmapsto\iint_{\mathbb{R}^N\times\mathbb{R}^N} f_\varepsilon(t,x)\ \Big(\varphi(t,x)+\varepsilon\Psi(t/\varepsilon^2,t,x,v)\Big)\,\mathrm{d}v\,\mathrm{d}x$$

is equicontinuous. Since $\|\varphi-(\varphi+\varepsilon\Psi)\|_{L^\infty}$ is of the order $\mathscr{O}(\varepsilon)$, the family

$$t\longmapsto\iint_{\mathbb{R}^N\times\mathbb{R}^N} f_\varepsilon\ \varphi(t,x)\,\mathrm{d}v\,\mathrm{d}x=\int_{\mathbb{R}^N}\rho_\varepsilon\ \varphi(t,x)\,\mathrm{d}x$$

is equicontinuous too. By density of $C_c^2(\mathbb{R}^N)$ in $C_c^0(\mathbb{R}^N)$, and with a separability argument, we conclude that the family of nonnegative measures $\big(\rho^\varepsilon(t)\big)_{\varepsilon>0}$ is vaguely equicontinuous on $[0,\infty)$.

Letting ε go to 0, we are led to

$$\begin{aligned}
&\int_0^\infty\iiint_{\mathbb{Y}\times\mathbb{R}^N\times\mathbb{R}^N} F(\theta,t,x,v)\,(\partial_t\varphi+v\cdot\nabla_x\Psi)\,(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t=0\\
&=\int_0^\infty\int_{\mathbb{R}^N}\rho(t,x)\partial_t\varphi(t,x)\underbrace{\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta}_{=1}\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\int_0^\infty\int_{\mathbb{R}^N}\rho(t,x)\left(\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi\ v\cdot\nabla_x\Psi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\right)\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^\infty\int_{\mathbb{R}^N}\rho(t,x)\partial_t\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t+\int_0^\infty\int_{\mathbb{R}^N}\rho(t,x)\nabla_x\cdot\left(\iint_{\mathbb{Y}\times\mathbb{R}^N}\Phi\ v\Psi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\right)\mathrm{d}x\,\mathrm{d}t\\
&\qquad-\int_0^\infty\int_{\mathbb{R}^N}\rho(t,x)\left(\iint_{\mathbb{Y}\times\mathbb{R}^N}\Psi\ v\cdot\nabla_x\Phi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\right)\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We only have to check that $v\cdot\nabla_x\Psi(\theta,t,x,v)=v\cdot\nabla_x\big((v+\psi^\star(\theta,x))\cdot\nabla_x\varphi(t,x)\big)=v\otimes(v+\psi^\star(\theta,x)):D^2\varphi(t,x)+v\cdot(\nabla_x\psi^\star(\theta,x)\nabla_x\varphi(t,x))$ is an admissible test function. This is indeed the case, since the regularity of E with respect to x is transfered to $\psi^\star$ and $v\cdot\nabla_x\Psi$ can be dominated by $(1+v^2)$ so that we can appeal to the the extra velocity moment with order higher than 2. Coming back to

[2] For a vector valued function $g:\mathbb{R}^N\to\mathbb{R}^N$, D_xg stands for the Jacobian matrix $\partial_j g_i$. For two $N\times N$ matrices A,B, we denote $A:B=(AB^\intercal)=\sum_{ij=1}^N A_{ij}B_{ij}$.

the notation of Section 2, we write, on the one hand,

$$\begin{aligned}
\iint_{\mathbb{Y}\times\mathbb{R}^N} \Phi\ v\Psi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta &= \iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E\chi\ \Psi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E\chi\ \Psi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \chi\ \mathscr{T}_E^\star\Psi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \chi\otimes v(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\nabla_x\varphi(t,x)\\
&= D(x)^\intercal\nabla_x\varphi(t,x)
\end{aligned}$$

and, on the other hand,

$$\begin{aligned}
\iint_{\mathbb{Y}\times\mathbb{R}^N} \Psi\ v\cdot\nabla_x\Phi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta &= \iint_{\mathbb{Y}\times\mathbb{R}^N} \Psi\mathscr{T}_E\beta(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} \mathscr{T}_E^\star\Psi\ \beta(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\\
&= \iint_{\mathbb{Y}\times\mathbb{R}^N} v\beta(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\nabla_x\varphi(t,x) = -U(x)\cdot\nabla_x\varphi(t,x).
\end{aligned}$$

We conclude that

$$\int_0^\infty \iint_{\mathbb{R}^N} \rho\Big(\partial_t\varphi + \nabla_x\cdot(D^\intercal\nabla_x\varphi) + U\nabla_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

which corresponds to the weak formulation of the convection-diffusion equation (7).

Remark 4.8 *In view of the regularity assumptions on E, the effective coefficients are smooth. In particular, the limit equation admits a unique solution, even in the framework of bounded measure-valued solutions, see [36, Theorem A.7] and the references therein.*

Remark 4.9 *The asymptotic analysis can be applied to the case where the force field depend on both the fast variable t/ε^2 and the slow variable t, namely working with a force field that reads $\frac{1}{\varepsilon^2}\mathcal{E}(t/\varepsilon^2,t,x)$. Details are left to the reader. Note that in this situation the effective coefficients D and U also depend on the time variable. It can be also relevant to consider space oscillations scaling like x/ε; however, this situation would require different arguments and the effective coefficients would be more intricate.*

5 Strengthened estimates for space-homogeneous fields

We are going to see that further estimates can be derived in the specific case where the oscillating field does not depend on the space variable. To this end, we shall use remarkable identities satisfied by the collision operators. These relations hold *pointwise* (they do not rely on integration by parts of inversion of integrals) and they provide additional dissipation properties.

Proposition 5.1 (Linear Boltzmann case) *Consider the linear Boltzmann operator $Q(f)(v) = \int k(v,v')f(v')\,\mathrm{d}v' - f(v)\int k(v',v)\,\mathrm{d}v'$. Let f,g be any function in $C^\infty(\mathbb{R}^N)$, with $g>0$; then, the following* pointwise *equality holds*

$$\frac{f}{g}Q(f)(v) = \frac{1}{2}Q\Big(\frac{f^2}{g}\Big)(v) + \frac{1}{2}\Big(\frac{f}{g}\Big)^2 Qg(v) - \frac{1}{2}\int_{\mathbb{R}^N} k(v,v')g(v')\Big|\frac{f}{g}(v)-\frac{f}{g}(v')\Big|^2\,\mathrm{d}v'.$$

Proof. We observe that

$$\begin{aligned}
\frac{f}{g}Q(f)(v) &= \frac{1}{2}\frac{f}{g}(v)\int_{\mathbb{R}^N} k(v,v')g(v')\Big(\frac{f}{g}(v')-\frac{f}{g}(v)\Big)\,\mathrm{d}v' \\
&\quad +\frac{1}{2}\Big(\frac{f}{g}\Big)^2(v)\left(\int_{\mathbb{R}^N} k(v,v')g(v')\,\mathrm{d}v' - g(v)\int_{\mathbb{R}^N} k(v',v)\,\mathrm{d}v'\right) + \frac{1}{2}\frac{f}{g}Q(f)(v) \\
&= \frac{1}{2}\frac{f}{g}(v)\int_{\mathbb{R}^N} k(v,v')g(v')\Big(\frac{f}{g}(v')-\frac{f}{g}(v)\Big)\,\mathrm{d}v' + \frac{1}{2}\Big(\frac{f}{g}\Big)^2(v)Q(g)(v) \\
&\quad +\frac{1}{2}\frac{f}{g}(v)\left(\int_{\mathbb{R}^N} k(v,v')f(v')\,\mathrm{d}v' - f(v)\int_{\mathbb{R}^N} k(v',v)\,\mathrm{d}v'\right).
\end{aligned}$$

The first term in the right hand side can be recast as

$$\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^N} k(v,v')g(v')\Big(\Big(\frac{f}{g}\Big)^2(v)+\Big(\frac{f}{g}\Big)^2(v')-2\frac{f}{g}(v)\frac{f}{g}(v')\Big)\,\mathrm{d}v' \\
&\quad +\frac{1}{2}\int_{\mathbb{R}^N} k(v,v')g(v')\frac{f}{g}(v')\Big(\frac{f}{g}(v')-\frac{f}{g}(v)\Big)\,\mathrm{d}v' \\
&= -\frac{1}{2}\int_{\mathbb{R}^N} k(v,v')g(v')\Big(\frac{f}{g}(v)-\frac{f}{g}(v')\Big)^2\,\mathrm{d}v' \\
&\quad +\frac{1}{2}Q\Big(\frac{f^2}{g}\Big) - \frac{1}{2}\frac{f}{g}(v)\int_{\mathbb{R}^N} k(v,v')f(v')\,\mathrm{d}v' + \frac{1}{2}\frac{f^2}{g}(v)\int_{\mathbb{R}^N} k(v',v)\,\mathrm{d}v'
\end{aligned}$$

Plugging this information in the expression obtained for $\frac{f}{g}Q(f)(v)$ leads to the announced result.
□

Proposition 5.2 (Fokker-Planck case) *Consider the Fokker-Planck operator* $Q(f) = \nabla_v\cdot\big(M\nabla_v \frac{f}{M}\big)$. *Let* f, g *be any function in* $C^\infty(\mathbb{R}^N)$, *with* $g > 0$; *then, the following* pointwise *equality holds*

$$\frac{f}{g}Q(f) = \frac{1}{2}Q\Big(\frac{f^2}{g}\Big) + \frac{1}{2}\Big(\frac{f}{g}\Big)^2 Qg - g\Big|\nabla_v\frac{f}{g}\Big|^2.$$

Remark 5.3 *The statement does not use the definition of the Maxwellian. In particular, it holds for any operator of the form*

$$Q(f)(v) = \Delta_v f(v) + \nabla_v\cdot(f(v)\nabla_v W(v))$$

where $W : \mathbb{R}^N \to \mathbb{R}$ *is a certain "energy" function such that* $Z = \int_{\mathbb{R}^N} e^{-W(v)}\,\mathrm{d}v < \infty$. *Indeed such an operator can be recast in the asserted form with* $M(v) = \frac{e^{-W(v)}}{Z}$.

Proof. We start by computing

$$\begin{aligned}
\frac{1}{2}Q\Big(\frac{f^2}{g}\Big) &= \frac{1}{2}\nabla_v\cdot\Big(M\nabla_v\Big(\frac{f^2}{g^2}\times\frac{g}{M}\Big)\Big) = \frac{1}{2}\nabla_v\cdot\Big(g\nabla_v\Big(\frac{f^2}{g^2}\Big)\Big) + \frac{1}{2}\nabla_v\cdot\Big(M\frac{f^2}{g^2}\nabla_v\Big(\frac{g}{M}\Big)\Big) \\
&= \nabla_v\cdot\Big(g\frac{f}{g}\nabla_v\Big(\frac{f}{g}\Big)\Big) + \frac{1}{2}\frac{f^2}{g^2}\nabla_v\cdot\Big(M\nabla_v\Big(\frac{g}{M}\Big)\Big) + M\frac{f}{g}\nabla_v\frac{g}{M}\cdot\nabla_v\frac{f}{g} \\
&= g\Big|\nabla_v\frac{f}{g}\Big|^2 + \frac{1}{2}\frac{f^2}{g^2}Q(g) + \frac{f}{g}\Big\{\nabla_v\cdot\Big(g\nabla_v\frac{f}{g}\Big) + M\nabla_v\frac{g}{M}\cdot\nabla_v\frac{f}{g}\Big\}.
\end{aligned}$$

We compare this expression to

$$\begin{aligned}
\frac{f}{g}Q(f) &= \frac{f}{g}\nabla_v\cdot\Big(M\nabla_v\Big(\frac{f}{g}\times\frac{g}{M}\Big)\Big) = \frac{f}{g}\nabla_v\cdot\Big(M\frac{f}{g}\nabla_v\Big(\frac{g}{M}\Big) + g\nabla_v\Big(\frac{f}{g}\Big)\Big) \\
&= \frac{f}{g}M\nabla_v\Big(\frac{g}{M}\Big)\cdot\nabla_v\Big(\frac{f}{g}\Big) + \Big(\frac{f}{g}\Big)^2\underbrace{\nabla_v\cdot\Big(M\nabla_v\Big(\frac{g}{M}\Big)\Big)}_{=Q(g)} + \frac{f}{g}\nabla_v\cdot\Big(g\nabla_v\Big(\frac{f}{g}\Big)\Big).
\end{aligned}$$

We arrive at

$$\frac{f}{g}Q(f) = \frac{1}{2}Q\Big(\frac{f^2}{g}\Big) - g\Big|\nabla_v \frac{f}{g}\Big|^2 \underbrace{-\frac{1}{2}\frac{f^2}{g^2}Q(g) + \frac{f^2}{g^2}Q(g)}_{=+\frac{1}{2}\frac{f^2}{g^2}Q(g)},$$

which is the asserted equality. □

We are going to use these identities to derive further estimates, in weighted L^2 spaces, at least when the oscillating field E is space homogenous. In this specific case, it is possible to establish directly the strong compactness in L^1 of the macroscopic density ρ_ε, when assuming the strong compactness in L^1 of the initial data, see Appendix B. The analysis strengthens the result in the following directions:

- the first order ansatz of the solution can be justified, with a rigorous identification of the corrector,
- strong convergence properties can be established, with relaxed estimates on the sequence of initial data, namely assuming only finite moments and finite "entropy".

Proposition 5.4 *We suppose that the field E is space homogeneous. Accordingly, the associated equilibrium function Φ does not depend on the space variable and we denote $\Phi_\varepsilon(t,v) = \Phi(\omega t/\varepsilon^2, v)$. The following estimate holds*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint \frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} \mathscr{D}_\varepsilon\,\mathrm{d}x \le 0$$

where

$$\mathscr{D}_\varepsilon = \begin{cases} \dfrac{1}{2}\displaystyle\iint k(v,v')\Phi_\varepsilon(v')\Big(\frac{f_\varepsilon}{\Phi_\varepsilon}(v') - \frac{f_\varepsilon}{\Phi_\varepsilon}(v)\Big)^2\,\mathrm{d}v'\,\mathrm{d}v & \textit{for the Boltzmann operator,}\\ \displaystyle\int_{\mathbb{R}^N}\Phi_\varepsilon\Big|\nabla_v\frac{f_\varepsilon}{\Phi_\varepsilon}\Big|^2\,\mathrm{d}v & \textit{for the Fokker-Planck operator.}\end{cases}$$

Difficulties for extending this property to non homogeneous fields are related to the fact that $\nabla_x\cdot\int v\Phi\,\mathrm{d}v \ne 0$ induces a stiff term in the estimate that will be discussed below, see also Appendix B. In order to get rid of this term, it would be possible to adapt the computations replacing Φ by $\Phi+\varepsilon\beta$, see [17, 20]; however this is useless if this quantity does not remain positive, at least for small enough ε's. Indeed, in the context addressed here, the equilibrium function Φ is positive but it is not bounded from below.

Proof. We have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x &= \iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f_\varepsilon}{\Phi_\varepsilon}\Big(-\frac{v}{\varepsilon}\cdot\nabla_x f_\varepsilon - \frac{1}{\varepsilon^2}\mathscr{E}(t/\varepsilon^2)\cdot\nabla_v f_\varepsilon + \frac{1}{\varepsilon^2}Q(f_\varepsilon)\Big)\,\mathrm{d}v\,\mathrm{d}x\\ &\quad -\frac{1}{2\varepsilon^2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f_\varepsilon^2}{\Phi_\varepsilon^2}\;\omega\cdot\nabla_\theta\Phi(\omega t/\varepsilon^2, v)\,\mathrm{d}v\,\mathrm{d}x\\ &= 0 + \frac{1}{\varepsilon^2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\Big\{-\frac{f_\varepsilon^2}{2\Phi_\varepsilon^2}(\omega\cdot\nabla_\theta + E\cdot\nabla_v)\Phi(\omega t/\varepsilon^2, v) + \frac{f_\varepsilon}{\Phi_\varepsilon}Q(f_\varepsilon)\Big\}\,\mathrm{d}v\,\mathrm{d}x\end{aligned}$$

by using integration by parts and the space homogeneity of Φ. Next, Proposition 5.1 or Proposition 5.2 yield (since Q is conservative)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N}\mathscr{D}_\varepsilon\,\mathrm{d}x - \frac{1}{\varepsilon^2}\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{f_\varepsilon^2}{2\Phi_\varepsilon^2}\underbrace{(\omega\cdot\nabla_\theta + E\cdot\nabla_v - Q)\Phi(\omega t/\varepsilon^2, v)}_{=0}\,\mathrm{d}v\,\mathrm{d}x.$$

□

As a consequence we can work in a functional framework which is less singular than the set of bounded measures.

Corollary 5.5 *We suppose that*

$$\sup_{\varepsilon>0} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f_{0,\varepsilon}^2}{\Phi(0,v)}\,\mathrm{d}v\,\mathrm{d}x = \mu < \infty.$$

Then, the sequence ρ_ε is bounded in $L^\infty(0,\infty;L^2(\mathbb{R}^N))$.

Proof. We remind the reader that the equilibrium function satisfies $\int_{\mathbb{R}^N}\Phi(\theta,v)\,\mathrm{d}v = 1$. Then, the estimate is a direct consequence of the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^N} \rho_\varepsilon^2\,\mathrm{d}x = \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} \frac{f_\varepsilon}{\sqrt{\Phi_\varepsilon}}\ \sqrt{\Phi_\varepsilon}\,\mathrm{d}v\right|^2\mathrm{d}x \le \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x \le \mu,$$

by virtue of Proposition 5.4. □

The new estimate suggests to expand

$$f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)\Phi(t/\varepsilon^2,v) + \varepsilon g_\varepsilon(t,x,v), \qquad \int_{\mathbb{R}^N} g_\varepsilon(t,x,v)\,\mathrm{d}v = 0.$$

Corollary 5.6 *For the Fokker-Planck case, the sequence $\frac{g_\varepsilon}{\sqrt{\Phi_\varepsilon}}$ is bounded in $L^2((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$; for the Boltzmann case, the sequence $\frac{g_\varepsilon\sqrt{M}}{\Phi_\varepsilon}$ is bounded in $L^2((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$.*

Proof. This is a direct consequence of the fact that

$$0 \le \frac{1}{\varepsilon^2}\int_0^\infty\int_{\mathbb{R}^N} \mathscr{D}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t \le \mu,$$

once remarking that

$$\mathscr{D}_\varepsilon = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Phi_\varepsilon\Big|\nabla_v \frac{g_\varepsilon}{\Phi_\varepsilon}\Big|^2\,\mathrm{d}v & \text{for the Fokker-Planck operator,}\\ \displaystyle\frac{1}{2}\iint_{\mathbb{R}^N\times\mathbb{R}^N} M(v)\Phi_\varepsilon(v')\Big(\frac{g_\varepsilon}{\Phi_\varepsilon}(v') - \frac{g_\varepsilon}{\Phi_\varepsilon}(v)\Big)^2\,\mathrm{d}v'\,\mathrm{d}v & \text{for the Boltzmann operator.}\end{cases}$$

For the former case, we simply appeal to the Poincaré inequality (12). For the latter case, since $\int_{\mathbb{R}^N}\Phi_\varepsilon\,\mathrm{d}v = 1$ and $\int_{\mathbb{R}^N} g\,\mathrm{d}v = 0$, we conclude by using

$$\begin{aligned}
\int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N} M\frac{g_\varepsilon^2}{\Phi_\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t &= \int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}\Big|\int_{\mathbb{R}^N}(g_\varepsilon(v)\Phi_\varepsilon(v') - g_\varepsilon(v')\Phi_\varepsilon(v))\,\mathrm{d}v'\Big|^2\frac{M}{\Phi_\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&= \int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}\Big|\int_{\mathbb{R}^N}\Big(\frac{g_\varepsilon}{\Phi_\varepsilon}(v) - \frac{g_\varepsilon}{\Phi_\varepsilon}(v')\Big)\Phi_\varepsilon(v)\Phi_\varepsilon(v')\,\mathrm{d}v'\Big|^2\frac{M}{\Phi_\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}\Big(\int_{\mathbb{R}^N}\Big(\frac{g_\varepsilon}{\Phi_\varepsilon}(v) - \frac{g_\varepsilon}{\Phi_\varepsilon}(v')\Big)^2\Phi_\varepsilon(v')\,\mathrm{d}v'\Big)\\
&\qquad\times\underbrace{\Big(\int_{\mathbb{R}^N}\Phi_\varepsilon(v')\,\mathrm{d}v'\Big)}_{=1} M(v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\
&\le \int_0^\infty\iiint_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N}\Big(\frac{g_\varepsilon}{\Phi_\varepsilon}(v) - \frac{g_\varepsilon}{\Phi_\varepsilon}(v')\Big)^2 M(v)\Phi_\varepsilon(v')\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The argument can be readily adapted to deal with the general situation devised in Section 3.4. □

In particular, we have the following expression for the current

$$J_\varepsilon(t,x) = \int_{\mathbb{R}^N} \frac{v}{\varepsilon} f_\varepsilon(t,x,v)\,\mathrm{d}v = \frac{\rho_\varepsilon(t,x)}{\varepsilon}\overline{\mathcal{U}}(\omega t/\varepsilon^2) + \int_{\mathbb{R}^N} v g_\varepsilon(t,x,v)\,\mathrm{d}v,$$

where

$$\overline{\mathcal{U}}(\theta) = \int_{\mathbb{R}^N} v\Phi(\theta,v)\,\mathrm{d}v.$$

For both the Boltzmann and the Fokker-Planck cases (using an integration by parts for the former case), we find the same expression

$$\overline{\mathcal{U}}(\theta) = \int_{-\infty}^{0} e^{\sigma} E(\theta+\omega\sigma)\,\mathrm{d}\sigma.$$

We can use the Fourier series expansion, bearing in mind that the zeroth coefficient of E vanishes (see (6)); it yields

$$\frac{1}{\varepsilon}\overline{\mathcal{U}}(\omega t/\varepsilon^2) = \frac{1}{\varepsilon}\sum_{k\neq 0} \frac{\widehat{E}(k)}{1+2i\pi\omega\cdot k} e^{2i\pi\omega\cdot kt/\varepsilon^2}.$$

It shows that the sequence of functions $\big(t\mapsto \overline{\mathcal{U}}(\omega t/\varepsilon^2)\big)_{\varepsilon>0}$ is bounded in $L^\infty(\mathbb{R})$, but $\frac{1}{\varepsilon}\overline{\mathcal{U}}(\omega t/\varepsilon^2)$ can blow up. Consequently the mass conservation equation contains a stiff term since it reads

$$\partial_t\rho_\varepsilon + \frac{1}{\varepsilon}\overline{\mathcal{U}}(\omega t/\varepsilon^2)\cdot\nabla_x\rho_\varepsilon + \nabla_x\cdot\int_{\mathbb{R}^N} v g_\varepsilon\,\mathrm{d}v = 0.$$

This stiff term can be an obstacle to establish strong convergence properties. Before starting further analysis, it is worth remarking that

$$\overline{\mathcal{U}} + \omega\cdot\nabla_\theta\overline{\mathcal{U}} = E,$$

a relation which comes by taking the first order moment of the equation that defines Φ.

Now, we introduce the characteristics associated to the oscillating velocity field: the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}X_\varepsilon = \frac{1}{\varepsilon}\overline{\mathcal{U}}(\omega t/\varepsilon^2), \qquad X_\varepsilon(0) = x$$

is simply given by

$$X_\varepsilon(t) = x + \frac{1}{\varepsilon}\int_0^t \overline{\mathcal{U}}(\omega s/\varepsilon^2)\,\mathrm{d}s = x + \varepsilon V(t/\varepsilon^2)$$

where

$$V(s) = \int_0^s \overline{\mathcal{U}}(\omega\tau)\,\mathrm{d}\tau = \int_{-\infty}^0 e^{\sigma}\left(\int_0^s E(\omega(\sigma+\tau))\,\mathrm{d}\tau\right)\mathrm{d}\sigma = \sum_{k\neq 0}\frac{\widehat{E}(k)}{1+2i\pi\omega\cdot k}\,\frac{e^{2i\pi k\cdot\omega s}-1}{2i\pi\omega\cdot k}.$$

As a matter of fact, we observe that

$$|V(s)| \leq C\||k|^{\gamma}\widehat{E}(k)\|_{\ell^1(\mathbb{Z}^N)}. \tag{23}$$

Let us introduce the auxilliary quantity

$$\widetilde{f}_\varepsilon(t,x,v) = f_\varepsilon(t,x+\varepsilon V(t/\varepsilon^2),v).$$

We define similarly

$$\widetilde{\rho}_\varepsilon(t,x) = \rho_\varepsilon(t,x+\varepsilon V(t/\varepsilon^2)) = \int_{\mathbb{R}^N}\widetilde{f}_\varepsilon(t,x,v)\,\mathrm{d}v \quad\text{and}\quad \widetilde{g}_\varepsilon(t,x,v) = \frac{\widetilde{f}(t,x,v)-\widetilde{\rho}_\varepsilon(t,x)\Phi(\omega t/\varepsilon^2,v)}{\varepsilon}.$$

The modified distribution function satisfies

$$\partial_t \widetilde{f}_\varepsilon + \frac{v - \overline{\mathscr{U}}(\omega t/\varepsilon^2)}{\varepsilon} \cdot \nabla_x \widetilde{f}_\varepsilon + \frac{1}{\varepsilon^2} \mathscr{E}_\varepsilon \cdot \nabla_v \widetilde{f}_\varepsilon = \frac{1}{\varepsilon^2} Q(\widetilde{f}_\varepsilon). \tag{24}$$

Moreover, we readily check that $\widetilde{f}_\varepsilon$, $\widetilde{\rho}_\varepsilon$ and $\widetilde{g}_\varepsilon$ satisfy the same uniform estimates as the original quantities. In particular, we can still expand $\widetilde{f}_\varepsilon(t,x,v) = \widetilde{\rho}_\varepsilon(t,x)\Phi(\omega t/\varepsilon^2, v) + \varepsilon \widetilde{g}_\varepsilon(t,x,v)$ and we can extract weakly convergent sequences and the macroscopic densities coincide as $\varepsilon \to 0$. As we shall see below, the remarkable fact is that we will be able to justify the *strong* convergence of $\widetilde{\rho}_\varepsilon$.

Theorem 5.7 *Let $E \in W^{1,\infty}(\mathbb{Y})$. Suppose that the sequence of initial data is such that $(1+|v|^k) f_{0,\varepsilon}$ is bounded in $L^1(\mathbb{R}^N \times \mathbb{R}^N)$, for some $k > 2$ and $\sup_\varepsilon \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|f_{0,\varepsilon}|^2}{\Phi(0,v)} \,\mathrm{d}v\,\mathrm{d}x < \infty$. Then, possibly at the price of extracting a subsequence, f_ε and $\widetilde{f}_\varepsilon = \widetilde{\rho}_\varepsilon \Phi_\varepsilon + \varepsilon \widetilde{g}_\varepsilon$ double-scale converge to $\rho(t,x)\Phi(\theta,v)$, where*

i) $\widetilde{\rho}_\varepsilon = \int_{\mathbb{R}^N} \widetilde{f}_\varepsilon \,\mathrm{d}v$ converges to ρ in $C^0([0,T]; L^2(\mathbb{R}^N) - weak)$, and strongly in $L^2_{\mathrm{loc}}((0,T) \times \mathbb{R}^N)$,

ii) ρ is the solution of the diffusion problem

$$\partial_t \rho - \nabla_x \cdot (\widetilde{\mathcal{D}} \nabla_x \rho) = 0,$$

with an effective coefficient $\widetilde{\mathcal{D}}$ defined by auxilliary equations that involve E (see Lemma 5.9), and initial data $\rho\big|_{t=0} = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} f_{0,\varepsilon} \,\mathrm{d}v$ (weakly in $L^2(\mathbb{R}^N)$),

iii) $\widetilde{g}_\varepsilon$ converges in the double scale sense to $\widetilde{G}(\theta,t,x,v) = -\widetilde{\chi}(\theta,v) \cdot \nabla_x \rho(t,x)$, where $\mathscr{T}_E \widetilde{\chi} = (v - \overline{\mathscr{U}})\Phi$.

We start with the following observation.

Lemma 5.8 *If $\rho_\varepsilon \rightharpoonup \rho$ weakly in the sense of distributions, then, $\widetilde{\rho}_\varepsilon \rightharpoonup \rho$ too.*

Proof. The proof is just matter of a change of variables: $y = x + \varepsilon V(t/\varepsilon^2)$ yields

$$\int_0^\infty \int_{\mathbb{R}^N} (\rho_\varepsilon - \widetilde{\rho}_\varepsilon)(t,x)\varphi(t,x) \,\mathrm{d}x\,\mathrm{d}t = \int_0^\infty \int_{\mathbb{R}^N} \rho_\varepsilon(t,y) \underbrace{(\varphi(t,y) - \varphi(t, y - \varepsilon V(t/\varepsilon^2)))}_{=\mathscr{O}(\varepsilon)} \,\mathrm{d}y\,\mathrm{d}t$$

by virtue of (23). □

The double-scale analysis can be reproduced for the problem (24). Details are left to the reader, but it is worth explaining formally the adaptations. At leading order, as it is already clear from the ansatz $\widetilde{f}_\varepsilon(t,x,v) = \widetilde{\rho}_\varepsilon(t,x)\Phi(\omega t/\varepsilon^2, v) + \varepsilon \widetilde{g}_\varepsilon(t,x,v)$, we infer that $\widetilde{f}_\varepsilon(t,x,v)$ converges in the double-scale sense towards $\widetilde{\rho}(t,x)\Phi(\theta,v) = \widetilde{F^{(0)}}(\theta,t,x,v)$. What changes notably is the definition of the corrector: it becomes $\widetilde{F^{(1)}}(\theta,t,x,v) = -\widetilde{\chi}(\theta,v) \cdot \nabla_x \widetilde{\rho}(t,x)$, where $\widetilde{\chi}$ is now solution of the cell equation

$$\mathscr{T}_E \widetilde{\chi}(\theta,v) = (v - \overline{\mathscr{U}}(\theta))\Phi(\theta,v).$$

(Note that the right hand side still satisfies the compatibility condition $\iint_{\mathbb{Y} \times \mathbb{R}^N} (v - \overline{\mathscr{U}}(\theta))\Phi(\theta,v) \,\mathrm{d}v\,\mathrm{d}\theta = 0$.) We obtain the limiting equation by integration of the $\mathscr{O}(\varepsilon)$ equation:

$$\begin{aligned} \partial_t \iint_{\mathbb{Y} \times \mathbb{R}^N} \widetilde{F^{(0)}} \,\mathrm{d}v\,\mathrm{d}\theta + \nabla_x \cdot \iint_{\mathbb{Y} \times \mathbb{R}^N} (v - \overline{\mathscr{U}}(\theta)) \widetilde{F^{(1)}} \,\mathrm{d}v\,\mathrm{d}\theta \\ = \partial_t \rho - \nabla_x \cdot \Big(\iint_{\mathbb{Y} \times \mathbb{R}^N} (v - \overline{\mathscr{U}}(\theta)) \otimes \widetilde{\chi}(\theta,v) \,\mathrm{d}v\,\mathrm{d}\theta \ \nabla_x \widetilde{\rho} \Big) = 0. \end{aligned}$$

We recover a diffusion equation, with the effective coefficient

$$\widetilde{\mathcal{D}} = \iint_{\mathbb{Y} \times \mathbb{R}^N} (v - \overline{\mathscr{U}}(\theta)) \otimes \widetilde{\chi}(\theta,v) \,\mathrm{d}v\,\mathrm{d}\theta.$$

As seen above, it is equally possible to express the effective coefficient through the dual cell equation. It leads to a more explicit formula. Indeed, we get

$$\widetilde{\mathcal{D}} = \iint_{\mathbb{Y}\times\mathbb{R}^N} \widetilde{\chi}^\star(\theta, v) \otimes (v - \overline{\mathcal{U}}(\theta))\Phi(\theta, v)\,\mathrm{d}v\,\mathrm{d}\theta,$$

with $\widetilde{\chi}^\star$ solution of $\mathscr{T}_E^\star \widetilde{\chi}^\star = (v - \overline{\mathcal{U}})$. Searching $\widetilde{\chi}^\star$ on the form $v + \widetilde{\psi}^\star(\theta)$, we arrive at $\omega \cdot \nabla_\theta \widetilde{\psi}^\star = \overline{\mathcal{U}} - E$. Finally we get $\widetilde{\psi}^\star(\theta) = -\overline{\mathcal{U}}(\theta)$ and

$$\widetilde{\chi}^\star(\theta, v) = v - \overline{\mathcal{U}}(\theta),$$

so that

$$\widetilde{\mathcal{D}} = \iint_{\mathbb{Y}\times\mathbb{R}^N} (v - \overline{\mathcal{U}}(\theta)) \otimes (v - \overline{\mathcal{U}}(\theta))\Phi(\theta, v)\,\mathrm{d}v\,\mathrm{d}\theta.$$

Let us denote

$$\mathcal{M}(\theta) = \int_{\mathbb{R}^N} (v - \overline{\mathcal{U}}(\theta)) \otimes (v - \overline{\mathcal{U}}(\theta))\Phi(\theta, v)\,\mathrm{d}v = \int_{\mathbb{R}^N} v \otimes v\Phi(\theta, v)\,\mathrm{d}v - \overline{\mathcal{U}}(\theta) \otimes \overline{\mathcal{U}}(\theta)$$

where

$$\begin{aligned}
\int_{\mathbb{R}^N} v \otimes v\Phi(\theta, v)\,\mathrm{d}v &= \begin{cases} \displaystyle\int_{\mathbb{R}^N}\int_{-\infty}^0 e^\sigma v \otimes v M\Big(v + \int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\Big)\,\mathrm{d}\sigma\,\mathrm{d}v \\ \qquad\text{for the Boltzmann operator,} \\ \displaystyle\int_{\mathbb{R}^N} v \otimes v M\Big(v - \int_{-\infty}^0 e^\sigma E(\theta + \omega\sigma)\,\mathrm{d}\sigma\Big)\,\mathrm{d}v \\ \qquad\text{for the Fokker-Planck operator,} \end{cases} \\
&= \mathbb{I} + \begin{cases} \displaystyle\int_{-\infty}^0 e^\sigma \left(\int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s \otimes \int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\right)\,\mathrm{d}\sigma \\ \qquad\text{for the Boltzmann operator,} \\ \displaystyle\int_{-\infty}^0 e^\sigma E(\theta + \omega\sigma)\,\mathrm{d}\sigma \otimes \int_{-\infty}^0 e^\sigma E(\theta + \omega\sigma)\,\mathrm{d}\sigma = \overline{\mathcal{U}}(\theta) \otimes \overline{\mathcal{U}}(\theta) \\ \qquad\text{for the Fokker-Planck operator.} \end{cases}
\end{aligned}$$

Therefore, for the Fokker-Planck case $\mathcal{M}(\theta)$ is nothing but the identity matrix, which simplifies the analysis. Let us detail the argument for the Boltzmann case. The matrix $\mathcal{M}(\theta)$ is symmetric and it always satisfies $\mathcal{M}(\theta) \geq \mathbb{I}$ (as a consequence of the Cauchy-Schwarz inequality); in particular $\mathcal{M}(\theta)$ is invertible. This also allows us to identify the effective diffusion coefficient $\widetilde{\mathcal{D}} = \int_{\mathbb{Y}} \mathcal{M}(\theta)\,\mathrm{d}\theta$. We recap these findings as follows.

Lemma 5.9 *For the Fokker-Planck case, we have $\mathcal{M}(\theta) = \mathbb{I}$, and the diffusion matrix is $\widetilde{\mathcal{D}} = \mathbb{I}$; for the linear Boltzmann case, we have*

$$\begin{aligned}
\mathcal{M}(\theta) &= \mathbb{I} + \int_{-\infty}^0 e^\sigma \left(\int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s \otimes \int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\right)\,\mathrm{d}\sigma \\
&\qquad - \int_{-\infty}^0 e^\sigma \left(\int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\right)\,\mathrm{d}\sigma \otimes \int_{-\infty}^0 e^\sigma \left(\int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\right)\,\mathrm{d}\sigma
\end{aligned}$$

and thus

$$\widetilde{\mathcal{D}} = \mathbb{I} + \int_{-\infty}^0 \int_{\mathbb{Y}} e^\sigma \left(\int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\right) \otimes \left(\int_{-\infty}^0 e^\sigma E(\theta + \omega s)\,\mathrm{d}s\right)\,\mathrm{d}\sigma\,\mathrm{d}\theta - \int_{\mathbb{Y}} \overline{\mathcal{U}}(\theta) \otimes \overline{\mathcal{U}}(\theta)\,\mathrm{d}\theta.$$

We also note that, assuming (for the Boltzmann case) $E \in W^{1,\infty}(\mathbb{Y})$, that the coefficients of the matrix $\mathcal{M}(\theta)$ belong to $W^{1,\infty}(\mathbb{Y})$ too. Accordingly, $\mathcal{M}(\theta) \geq \mathbb{I}$ is invertible, and the inverse matrix $\mathcal{M}(\theta)^{-1}$ lies in $W^{1,\infty}(\mathbb{Y})$.

That the coefficients $\mathcal{M}_{ij}(\theta)$ are elements of $W^{1,\infty}(\mathbb{Y})$ follows by direct inspection of the formula, using the assumption on the Fourier coefficients of E that implies $E \in W^{1,\infty}(\mathbb{Y})$.

The strengthened compactness properties is then obtained by considering the moments equations. The mass conservation has a more advantageous form, which leads immediately to the time-compactness of the modified macroscopic density.

Lemma 5.10 *We have*

$$\partial_t \widetilde{\rho}_\varepsilon + \nabla_x \cdot \widetilde{J}_\varepsilon = 0,$$

where

$$\widetilde{J}_\varepsilon(t,x) = \int_{\mathbb{R}^N} \frac{v - \overline{\mathcal{U}}(\omega t/\varepsilon^2)}{\varepsilon} \widetilde{f}_\varepsilon(t,x,v)\,\mathrm{d}v = \int_{\mathbb{R}^N} (v - \overline{\mathcal{U}}(\omega t/\varepsilon^2))\widetilde{g}_\varepsilon(t,x,v)\,\mathrm{d}v$$

is bounded in $L^2((0,\infty)\times\mathbb{R}^N)$. Accordingly, $\widetilde{\rho}_\varepsilon$ is compact in $C^0([0,T];L^2(\mathbb{R}^N)-\text{weak})$.

Proof. By definition, we have

$$\int_{\mathbb{R}^N} (v - \overline{\mathcal{U}}(\omega t/\varepsilon^2))\Phi(\omega t/\varepsilon^2, v)\,\mathrm{d}v = 0,$$

and the mass conservation relation holds just by integrating (24). For proving the estimate, we observe that $\int_{\mathbb{R}^N} \widetilde{g}_\varepsilon(t,x,v)\,\mathrm{d}v = 0$ implies $\int (v - \overline{\mathcal{U}}(\omega t/\varepsilon^2))\widetilde{g}_\varepsilon(t,x,v)\,\mathrm{d}v = \int_{\mathbb{R}^N} v\widetilde{g}_\varepsilon(t,x,v)\,\mathrm{d}v$, and we conclude by using the estimate of Corollary 5.6 and the Cauchy-Schwarz inequality as follows. For the Fokker-Planck case, we have

$$\begin{aligned}
\int_0^\infty \int \Big| \int_{\mathbb{R}^N} v\widetilde{g}_\varepsilon(t,x,v)\,\mathrm{d}v\Big|^2 \mathrm{d}x\,\mathrm{d}t &= \int_0^\infty \int \Big| \int_{\mathbb{R}^N} v\sqrt{\Phi_\varepsilon}\frac{\widetilde{g}_\varepsilon}{\sqrt{\Phi_\varepsilon}}\,\mathrm{d}v\Big|^2 \mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^\infty \int \left(\int_{\mathbb{R}^N} v^2 \Phi_\varepsilon\,\mathrm{d}v\right)\left(\int_{\mathbb{R}^N} \frac{|\widetilde{g}_\varepsilon|^2}{\Phi_\varepsilon}\,\mathrm{d}v\right)\mathrm{d}x\,\mathrm{d}t \\
&\le \sup_\theta \int_{\mathbb{R}^N} v^2\Phi(\theta,v)\,\mathrm{d}v \times \int_0^\infty \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\widetilde{g}_\varepsilon|^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \le C.
\end{aligned}$$

For the Boltzmann case, we get

$$\begin{aligned}
\int_0^\infty \int \Big| \int_{\mathbb{R}^N} v\widetilde{g}_\varepsilon(t,x,v)\,\mathrm{d}v\Big|^2 \mathrm{d}x\,\mathrm{d}t &= \int_0^\infty \int \Big| \int_{\mathbb{R}^N} v\frac{\Phi_\varepsilon}{\sqrt{M}}\frac{\widetilde{g}_\varepsilon\sqrt{M}}{\Phi_\varepsilon}\,\mathrm{d}v\Big|^2 \mathrm{d}x\,\mathrm{d}t \\
&\le \int_0^\infty \int \left(\int_{\mathbb{R}^N} v^2 \frac{\Phi_\varepsilon^2}{M}\,\mathrm{d}v\right)\left(\int |\widetilde{g}_\varepsilon|^2 \frac{M}{\Phi_\varepsilon^2}\,\mathrm{d}v\right)\mathrm{d}x\,\mathrm{d}t \\
&\le \sup_\theta \int_{\mathbb{R}^N} v^2\frac{\Phi(\theta,v)^2}{M(v)}\,\mathrm{d}v \times \int_0^\infty \iint_{\mathbb{R}^N\times\mathbb{R}^N} |\widetilde{g}_\varepsilon|^2 \frac{M}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \le C,
\end{aligned}$$

by virtue of Lemma 3.1. □

The stiffest terms in the modified momentum equation cancel out, which leads to a formulation with a structure helpful for proving the strong compactness. Indeed, we have

$$\begin{aligned}
\varepsilon^2\partial_t \widetilde{J}_\varepsilon &= -\frac{1}{\varepsilon}\omega\cdot\nabla_\theta\overline{\mathcal{U}}(\omega t/\varepsilon^2)\int_{\mathbb{R}^N}\widetilde{f}_\varepsilon\,\mathrm{d}v + \frac{1}{\varepsilon}\mathscr{E}_\varepsilon\int_{\mathbb{R}^N}\widetilde{f}_\varepsilon\,\mathrm{d}v \\
&\quad -\nabla_x\cdot\left(\int_{\mathbb{R}^N}(v-\overline{\mathcal{U}}(\omega t/\varepsilon^2))\otimes(v-\overline{\mathcal{U}}(\omega t/\varepsilon^2))\widetilde{f}_\varepsilon\,\mathrm{d}v\right) + \int_{\mathbb{R}^N}\frac{v-\overline{\mathcal{U}}(\omega t/\varepsilon^2)}{\varepsilon}Q(\widetilde{f}_\varepsilon)\,\mathrm{d}v \\
&= \frac{\widetilde{\rho}_\varepsilon}{\varepsilon}\Big(-\omega\cdot\nabla_\theta\overline{\mathcal{U}}(\omega t/\varepsilon^2) + \mathscr{E}_\varepsilon\Big) \\
&\quad -\nabla_x\cdot\left(\int_{\mathbb{R}^N}(v-\overline{\mathcal{U}}(\omega t/\varepsilon^2))\otimes(v-\overline{\mathcal{U}}(\omega t/\varepsilon^2))\Phi(\omega t/\varepsilon^2,v)\,\mathrm{d}v\ \widetilde{\rho}_\varepsilon\right) \\
&\quad -\varepsilon\nabla_x\cdot\left(\int_{\mathbb{R}^N}(v-\overline{\mathcal{U}}(\omega t/\varepsilon^2))\otimes(v-\overline{\mathcal{U}}(\omega t/\varepsilon^2))\widetilde{g}_\varepsilon\,\mathrm{d}v\right) - \widetilde{J}_\varepsilon - \frac{1}{\varepsilon}\overline{\mathcal{U}}(\omega t/\varepsilon^2)\widetilde{\rho}_\varepsilon.
\end{aligned}$$

We arrive at

$$\varepsilon^2\partial_t\widetilde{J}_\varepsilon + \mathcal{M}(\omega t/\varepsilon^2)\nabla_x\widetilde{\rho}_\varepsilon + \varepsilon\nabla_x\int_{\mathbb{R}^N}(v-\overline{\mathscr{U}}(\omega t/\varepsilon^2))\otimes(v-\overline{\mathscr{U}}(\omega t/\varepsilon^2))\widetilde{g}_\varepsilon\,\mathrm{d}v = -\widetilde{J}_\varepsilon,$$

which recasts as

$$\begin{aligned}\nabla_x\widetilde{\rho}_\varepsilon &= -\big((\mathcal{M}^{-1}(\omega t/\varepsilon^2)-(\omega\cdot\nabla_\theta\mathcal{M}^{-1})(\omega t/\varepsilon^2)\big)\widetilde{J}_\varepsilon - \varepsilon^2\partial_t(\mathcal{M}^{-1}(\omega t/\varepsilon^2)\widetilde{J}_\varepsilon)\\ &\quad -\varepsilon\nabla_x\left(\mathcal{M}^{-1}(\omega t/\varepsilon^2)\int_{\mathbb{R}^N}(v-\overline{\mathscr{U}}(\omega t/\varepsilon^2))\otimes(v-\overline{\mathscr{U}}(\omega t/\varepsilon^2))\widetilde{g}_\varepsilon\,\mathrm{d}v\right).\end{aligned}\tag{25}$$

We can now deduce the strong compactness of $\widetilde{\rho}_\varepsilon$ in L^2_{loc}, appealing to a compensated compactness argument. We refer the reader to [12, 17, 19, 23] for various applications of this technique to handle diffusion regime of kinetic models. Let us introduce the vector field (with value in $\mathbb{R}^{N+1}$), $\mathscr{V}_\varepsilon = (\widetilde{\rho}_\varepsilon, \widetilde{J}_\varepsilon)$ and $\mathscr{W}_\varepsilon = (\widetilde{\rho}_\varepsilon, 0, ..., 0)$. The moments equations become

$$\mathrm{div}_{t,x}\mathscr{V}_\varepsilon = 0, \qquad \mathrm{curl}_{t,x}\mathscr{W}_\varepsilon = \left(\begin{array}{c|c} 0 & -\nabla_x\widetilde{\rho}_\varepsilon^{\intercal} \\ \hline \nabla_x\widetilde{\rho}_\varepsilon & 0\end{array}\right) = \mathscr{Z}_\varepsilon$$

where $\mathscr{Z}_\varepsilon$ lies in a compact set of $H^{-1}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$ (note the loss of one derivative with respect to both time and space) and we can assume that $\mathscr{V}_\varepsilon \rightharpoonup (\rho, \widetilde{J})$ $\mathscr{W}_\varepsilon \rightharpoonup (\rho, 0)$. The div-curl lemma [33, 34] tells us that the inner product $\mathscr{V}_\varepsilon\cdot\mathscr{W}_\varepsilon = \widetilde{\rho}_\varepsilon^2 \rightharpoonup \mathscr{V}\cdot\mathscr{W} = \rho^2$ passes to the limit, at least in the sense of distributions. It follows that $\widetilde{\rho}_\varepsilon$ converges to ρ strongly in $L^2_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$. Note also that letting ε go to 0 in (25) shows that $\nabla_x\rho \in L^2((0,\infty)\times\mathbb{R}^N)$.

Finally, the identification of the corrector is obtained by coming back to (21): using the expansion $\widetilde{f}_\varepsilon = \widetilde{\rho}_\varepsilon\Phi_\varepsilon + \epsilon\widetilde{g}_\varepsilon$, it becomes

$$\begin{aligned}&\int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}\widetilde{\rho}_\varepsilon\Phi_\varepsilon\Big(\partial_t + \frac{v-\overline{\mathscr{U}}}{\varepsilon}\cdot\nabla_x\Big)\phi(\omega t/\varepsilon^2, t, x, v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}\widetilde{g}_\varepsilon\Big(-\frac{1}{\varepsilon}\mathscr{T}_E^\star + \varepsilon\partial_t + (v-\overline{\mathscr{U}})\cdot\nabla_x\Big)\phi(\omega t/\varepsilon^2, t, x, v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

By Corollary 5.6, g_ε is bounded also in L^1, and we can assume it admits a double scale limit G. Multiply by ε and let ε go to 0. It yields

$$\int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho_\varepsilon\Phi\;(v-\overline{\mathscr{U}})\cdot\nabla_x\phi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}\theta\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty\iint_{\mathbb{R}^N\times\mathbb{R}^N}G\mathscr{T}_E^\star\phi(\theta,t,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Moreover, the estimates in Corollary 5.6 also show that $G \in L^2((0,\infty)\times\mathbb{R}^N; H)$. Hence, we are reduced to solve $\mathscr{T}_E G = -(v-\overline{\mathscr{U}})\Phi\cdot\nabla_x\rho$, which leads to $G = -\widetilde{\chi}\cdot\nabla_x\rho$.

A Well-posedness of the regularized problem

We go back here to equation (14), for positive λ's. The treatment differs depending on the collision operator. For the Fokker-Planck, the problem can be explicitely solved by means of Fourier transform with respect to the velocity variable. Indeed, with the Fourier transform

$$\widehat{f}(\theta,\xi) = \int_{\mathbb{R}^N} f(\theta, v)e^{-i\xi\cdot v}\,\mathrm{d}v,$$

the equation becomes

$$\lambda\widehat{f}_\lambda + \omega\cdot\nabla_\theta\widehat{f}_\lambda + i\xi\cdot E\widehat{f}_\lambda + \xi\cdot\nabla_\xi\widehat{f}_\lambda + \xi^2\widehat{f}_\lambda = \widehat{h},$$

which can now be handled by using the characteristics, solutions of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}s}\Theta = \omega, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\Xi = \Xi.$$

Namely, we have

$$\Theta(s) = \theta + \omega s, \qquad \Xi(x) = \xi e^s.$$

Therefore, we can recast the equation as follows

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\Big(\lambda s + \int_0^s \big(i\xi\cdot E(\theta+\omega\tau) + \xi^2 e^{2\tau}\big)\,\mathrm{d}\tau\Big)\widehat{f}_\lambda(\theta+\omega s, \xi e^s)\right]$$
$$= \exp\Big(\lambda s + \int_0^s \big(i\xi\cdot E(\theta+\omega\tau) + \xi^2 e^{2\tau}\big)\,\mathrm{d}\tau\Big)\widehat{h}_\lambda(\theta+\omega s, \xi e^s).$$

Integrating between $s = -\infty$ and $s = 0$ leads to

$$\widehat{f}_\lambda(\theta,\xi) = \int_{-\infty}^0 \exp\Big(\lambda\sigma + \int_0^\sigma \big(i\xi\cdot E(\theta+\omega\tau) + \xi^2 e^{2\tau}\big)\,\mathrm{d}\tau\Big)\widehat{h}_\lambda(\theta+\omega\sigma, \xi e^\sigma)\,\mathrm{d}\sigma.$$

We now go back to the physical variable

$$\begin{aligned} f_\lambda(\theta,v) &= \int_{-\infty}^0\int_{\mathbb{R}^N} \exp\Big(\lambda\sigma + \int_0^\sigma \big(i\xi\cdot E(\theta+\omega\tau) + \xi^2 e^{2\tau}\big)\,\mathrm{d}\tau\Big)\widehat{h}_\lambda(\theta+\omega\sigma, \xi e^\sigma)\frac{e^{i\xi\cdot v}}{(2\pi)^N}\,\mathrm{d}\xi\,\mathrm{d}\sigma \\ &= \int_{-\infty}^0\int_{\mathbb{R}^N} h(\theta+\omega\sigma, w)e^{\lambda\sigma} \\ &\quad\times\Big(\int_{\mathbb{R}^N}\exp\Big(i\xi\cdot(v-e^\sigma w) + \int_0^\sigma \big(i\xi\cdot E(\theta+\omega\tau)+\xi^2 e^{2\tau}\big)\,\mathrm{d}\tau\Big)\frac{\mathrm{d}\xi}{(2\pi)^N}\Big)\,\mathrm{d}\sigma\,\mathrm{d}w. \end{aligned}$$

This formula can be re-arranged as follows

$$\begin{aligned} f_\lambda(\theta,v) &= \int_{-\infty}^0\int_{\mathbb{R}^N} h(\theta+\omega\sigma, w)e^{\lambda\sigma} \\ &\quad\times\Big(\int_{\mathbb{R}^N}\exp\Big(i\xi\cdot\Big(v-e^\sigma w + \int_0^\sigma E(\theta+\omega\tau)\,\mathrm{d}\tau\Big)\Big)e^{-\xi^2(1-e^2\sigma)/2}\frac{\mathrm{d}\xi}{(2\pi)^N}\Big)\,\mathrm{d}\sigma\,\mathrm{d}w \\ &= \int_{-\infty}^0\int_{\mathbb{R}^N} h(\theta+\omega\sigma, w)e^{\lambda\sigma} \\ &\quad\times\left(\int_{\mathbb{R}^N}\exp\left(-i\zeta\cdot\frac{\Big(v-e^\sigma w+\int_0^\sigma E(\theta+\omega\tau)\,\mathrm{d}\tau\Big)}{\sqrt{1-e^{2\sigma}}}\right)e^{-\zeta^2/2}\frac{\mathrm{d}\zeta}{(2\pi\sqrt{1-e^{2\sigma}})^N}\right)\,\mathrm{d}\sigma\,\mathrm{d}w \end{aligned}$$

where we recognize the Fourier transform of the Gaussian. Therefore, we obtain

$$f_\lambda(\theta,v) = \int_{-\infty}^0\int_{\mathbb{R}^N} h(\theta+\omega\sigma, w)e^{\lambda\sigma}\mathscr{G}(\sigma,\theta,v,w)\,\mathrm{d}\sigma\,\mathrm{d}w$$

with the kernel

$$\mathscr{G}(\sigma,\theta,v,w) = \left(\frac{1}{2\pi(1-e^{2\sigma})}\right)^{N/2}\exp\left(-\frac{\Big|v-e^\sigma w+\int_0^\sigma E(\theta+\omega\tau)\,\mathrm{d}\tau\Big|^2}{2(1-e^{2\sigma})}\right).$$

(Note that $\mathscr{G}$ is non negative and satisfies $\int\mathscr{G}(\sigma,\theta,v,w)\,\mathrm{d}v = 1$.) This formula provides the solution of (14) for the Fokker-Planck case.

For the Boltzmann operator, we consider the framework addressed in Section 3.4. The linear equation

$$\lambda u + \omega \cdot \nabla_\theta u + E \cdot \nabla_v u + \Sigma u = \widetilde{h},$$

can be simply solved by means of characteristics, for any $\lambda > 0$. Then, we consider the iteration scheme, starting from $f_0 = 0$,

$$\lambda f_{n+1} + \omega \cdot \nabla_\theta f_{n+1} + E \cdot \nabla_v f_{n+1} + \Sigma f_{n+1} = h + K(f_n).$$

Let $\delta_n = f_n - f_{n-1}$. We have $(\lambda + \omega \cdot \nabla_\theta + E \cdot \nabla_v + \Sigma)\delta_{n+1} = K(\delta_n)$, and it follows that

$$\iint_{\mathbb{Y}\times\mathbb{R}^N} (\lambda+\Sigma)|\delta_n| \,\mathrm{d}v\,\mathrm{d}\theta \le \iiint_{\mathbb{Y}\times\mathbb{R}^N\times\mathbb{R}^N} k(v,v')|\delta_{n-1}(v')|\,\mathrm{d}v'\,\mathrm{d}v\,\mathrm{d}\theta = \iint_{\mathbb{Y}\times\mathbb{R}^N} \Sigma(v')|\delta_{n-1}(v')|\,\mathrm{d}v'\,\mathrm{d}\theta.$$

The left hand side is bounded from below by

$$\left(1 + \frac{\lambda}{\kappa^\star}\right) \iint_{\mathbb{Y}\times\mathbb{R}^N} \Sigma(v)|\delta_n(v)|\,\mathrm{d}v\,\mathrm{d}\theta.$$

Therefore, the scheme is contractive for the norm $L^1(\mathbb{Y}\times\mathbb{R}^N;\Sigma(v)\,\mathrm{d}v\,\mathrm{d}\theta)$. The limit f as $n\to\infty$ satisfies $\lambda f + \omega\cdot\nabla_\theta f = h + K(f) - \Sigma f$.

B Further comments on the case when E is space homogeneous

As far as we are interested in the strong compactness of the macroscopic density for the simple models (2) or (3), we can conclude with a simple argument, up to strengthened assumptions on the data. We have already observed that the solution of (1) satisfies $\iint_{\mathbb{R}^N\times\mathbb{R}^N} f_\varepsilon \,\mathrm{d}v\,\mathrm{d}x = \iint_{\mathbb{R}^N\times\mathbb{R}^N} f_{0,\varepsilon}\,\mathrm{d}v\,\mathrm{d}x$, and $f_\varepsilon \ge 0$ when $f_{0,\varepsilon}\ge 0$. Consequently, applying this result to $f^2_\varepsilon - f^1_\varepsilon$ associated to $f^2_{0,\varepsilon} - f^1_{0,\varepsilon} \ge 0$, we deduce that the solution-operator $S_\varepsilon[t] : f_0 \mapsto f_\varepsilon(t)$, solution of (1) is both order-preserving and integral-preserving. The Crandall-Tartar lemma [10] implies the contraction property

$$\|f^2_\varepsilon(t) - f^1_\varepsilon(t)\|_{L^1} \le \|f^2_{0,\varepsilon} - f^1_{0,\varepsilon}\|_{L^1}. \tag{26}$$

Let $\big(f_{0,\varepsilon}\big)_{\varepsilon>0}$ be a sequence which is *strongly* compact in $L^1(\mathbb{R}^N\times\mathbb{R}^N)$. Note that this is a substantial restriction compared to the assumptions in Theorem 5.7. Let $\big(\zeta^\delta\big)_{\delta>0}$ be a sequence of mollifiers: $\zeta^\delta \in C^\infty_c(\mathbb{R}^N)$, $\int \zeta^\delta\,\mathrm{d}x = 1$, $0\le\zeta^\delta\le 1$. By virtue of the Weil-Kolmogorov-Fréchet criterion, we know that

$$\sup_\varepsilon \|f_{0,\varepsilon} - \zeta^\delta \star f_{0,\varepsilon}\|_{L^1} \xrightarrow[\delta\to 0]{} 0.$$

Therefore, denoting $f^\delta_\varepsilon(t) = S_\varepsilon[t](\zeta^\delta \star f_{0,\varepsilon})$, we have

$$\sup_{\varepsilon>0} \|f_\varepsilon(t) - f^\delta_\varepsilon(t)\|_{L^1} \xrightarrow[\delta\to 0]{} 0.$$

Hence, it order to establish that $\big(f_\varepsilon\big)_{\varepsilon>0}$ is compact in $C^0([0,T];L^1(B(0,R))$ for any $0<T,R<\infty$, it suffices to establish the compactness of $\big(f^\delta_\varepsilon\big)_{\varepsilon>0}$ for a fixed δ.

The advantage is that we are now working with smooth data. This is what we assume now, dropping the superscript δ. In particular, since the force field is space homogeneous, $\partial_{x_j} f_\varepsilon$ satisfies (1) too, and (26) yields

$$\sup_{\varepsilon>0} \|\partial_{x_j} f_\varepsilon(t)\|_{L^1} \le \sup_{\varepsilon>0} \|\partial_{x_j} f_{0,\varepsilon}\|_{L^1} < \infty.$$

Hence $\partial_{x_j}\rho_\varepsilon(t,\cdot) = \int \partial_{x_j} f_\varepsilon(t,\cdot)\,\mathrm{d}v$ lies in a bounded set of $L^1(\mathbb{R}^N)$. It follows that, for any non negative $\eta \in C_c^\infty(\mathbb{R}^N)$,

$$\sup_{0\le t\le T}\sup_{\varepsilon>0}\int_{\mathbb{R}^N} |\rho_\varepsilon(t,x+h)-\rho_\varepsilon(t,x)|\ \eta(x)\,\mathrm{d}x \xrightarrow[h\to 0]{} 0. \tag{27}$$

Coming back to the proof of Theorem 1.1, we also know that

$$\sup_{0\le t\le T}\sup_{\varepsilon>0}\left(\int_{\mathbb{R}^N}\rho_\varepsilon(t+\tau,x)\eta(x)\,\mathrm{d}x - \int_{\mathbb{R}^N}\rho_\varepsilon(t,x)\eta(x)\,\mathrm{d}x\right) \xrightarrow[\tau\to 0]{} 0,$$

holds for any such $\eta \in C_c^\infty(\mathbb{R}^N)$. Now, we write

$$\begin{aligned}\|(\rho_\varepsilon(t+\tau,\cdot)-\rho_\varepsilon(t,\cdot))\eta\|_{L^1} \quad\le\quad & \|(\rho_\varepsilon(t+\tau,\cdot)-\zeta^\delta\star\rho_\varepsilon(t+\tau,\cdot))\eta\|_{L^1} + \|(\rho_\varepsilon(t,\cdot)-\zeta^\delta\star\rho_\varepsilon(t,\cdot))\eta\|_{L^1}\\ & + \|(\zeta^\delta\star\rho_\varepsilon(t+\tau,\cdot)-\zeta^\delta\star\rho_\varepsilon(t,\cdot))\eta\|_{L^1}.\end{aligned}$$

The first two terms are dominated by

$$\sup_{0\le t\le T}\sup_{\varepsilon>0}\int_{\mathbb{R}^N}\zeta^\delta(y)\left(\int_{\mathbb{R}^N}|\rho_\varepsilon(t,x-y)-\rho_\varepsilon(t,x)|\eta(x)\,\mathrm{d}x\right)\mathrm{d}y$$

which tends to 0 as $\delta \to 0$, by virtue of (27). Next, the third quantity is dominated by

$$\sup_{0\le t\le T}\sup_{\varepsilon>0}\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\rho_\varepsilon(t+\tau,y)\zeta^\delta(x-y)\,\mathrm{d}y - \int_{\mathbb{R}^N}\rho_\varepsilon(t,y)\zeta^\delta(x-y)\,\mathrm{d}y\right|\mathrm{d}x$$

which tends to 0 as $\tau \to 0$, for any fixed $\delta > 0$. Gathering these estimates we conclude that $\big(\eta\rho_\varepsilon\big)_{\varepsilon>0}$ is equicontinuous in L^1, and, by the Arzela-Ascoli theorem, $\big(\eta\rho_\varepsilon\big)_{\varepsilon>0}$ is compact in $C^0([0,T];L^1(\mathbb{R}^N))$.

This argument is very specific, while it is likely that most of the analysis of the previous sections can be adapted to more intricate collision operators, possibly with non detailed balance and space-dependent kernels, at the price of adapting the functional framework as in [12]. Let us describe a simple example in this direction. We consider the kinetic model

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon}v\partial_x f_\varepsilon + \frac{1}{\varepsilon^2}E(\omega t/\varepsilon^2)\partial_v f_\varepsilon = \frac{1}{\varepsilon^2}(\langle f_\varepsilon\rangle M - f_\varepsilon) \tag{28}$$

where now $x \in \mathbb{R}$, $v \in (-1,+1)$ and, given $\kappa > 0$,

$$M(v) = \frac{1}{4\pi I_0(\kappa)\sqrt{1-v^2}}\sinh(\kappa\sqrt{1-v^2}),$$

with I_0 the zeroth modified Bessel function, see Fig. 1. In particular, we notice that

$$M(v) \ge M(1) = M(-1) = \frac{\kappa}{2\pi I_0(\kappa)} > 0.$$

Equation (28) is endowed with periodic boundary condition with respect to v, and from now on we extend M over $\mathbb{R}$ by 2-periodicity. This equilibrium state can be understood by setting $v = \sin(\theta)$: we get

$$M(v)\,\mathrm{d}v = \frac{e^{\kappa\cos(\theta)}}{2\pi I_0(\kappa)}\,\mathrm{d}\theta,$$

where we recognize the Von Mises-Fischer distribution on the circle [25].

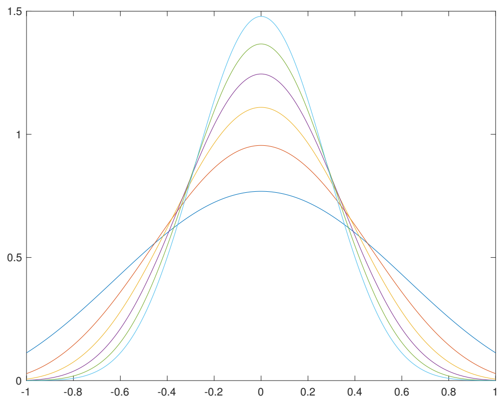

Figure 1: Representation of the function $v \mapsto M(v)$, for $\kappa \in \{4, 6, 8, 10, 12, 14\}$

As a matter of fact, we check that

$$\int_{-1}^{+1} M(v)\,\mathrm{d}v = 1, \qquad \int_{-1}^{+1} vM(v)\,\mathrm{d}v = 0.$$

Working on a bounded domain for the variable v simplifies the analysis and provides easier estimates that do not require to deal with weighted norms. In what follows, we consider the situation where $\kappa : \mathbb{R} \to (0, \infty)$ is a space-dependent quantity. This function is supposed to be smooth, bounded from above and below by positive constants.

The analysis of the leading term of the aysmptotics does not use the precise form of the equilibrium function M: the solution of

$$\omega \cdot \nabla_\theta \Phi + E\partial_v \Phi + \Phi = M$$

is still given by

$$\Phi(\theta, v) = \int_{-\infty}^{0} e^\sigma M\Big(v + \int_0^\sigma E(\theta + \omega s)\,\mathrm{d}s\Big)\,\mathrm{d}\sigma.$$

(Note that $\langle \Phi \rangle = 1$.) Pay attention that Φ is also parametrized by the space variable, since M depends on the function κ. With the assumptions on κ, Φ is bounded from above and below, and $\partial_x \Phi$ lies in $L^\infty(\mathbb{Y} \times \mathbb{R} \times (-1, 1))$ too. Moreover, $\overline{\mathcal{U}}(\theta) = \int_{-1}^{+1} v\Phi(\theta, x, v)\,\mathrm{d}v$ satisfies

$$\overline{\mathcal{U}} + \omega \cdot \nabla_\theta \overline{\mathcal{U}} = E.$$

Accordingly, while Φ can be space dependent, $\overline{\mathcal{U}}$ does not depend on x if E is constant with respect to x. This allows us to apply the strategy devised in Section 5, while the regularization argument detailed above does not work here, due to the stiff terms involving $\partial_x \kappa$. The proof follows exactly the same lines. Note that in this situation, the diffusion coefficient remains space dependent, and there is also a drift term. To be specific, we have

$$D(x) = \int_{-1}^{+1} \int_{\mathbb{Y}} \chi^\star(\theta, v) v\Phi(\theta, x, v)\,\mathrm{d}v\,\mathrm{d}\theta = \int_{-1}^{+1} \int_{\mathbb{Y}} v^2 \Phi(\theta, x, v)\,\mathrm{d}v\,\mathrm{d}\theta + \int_{-1}^{+1} \int_{\mathbb{Y}} \psi^\star(\theta) v\Phi(\theta, x, v)\,\mathrm{d}v\,\mathrm{d}\theta$$

with $\chi^\star$ solution of $\mathscr{T}_E^\star \chi^\star = v$, which can still be written as $v + \psi^\star(\theta)$, with $\omega \cdot \nabla_\theta \psi^\star = -E$, and

$$U(x) = -\int_{-1}^{+1} \int_{\mathbb{Y}} v\beta(\theta, x, v)\,\mathrm{d}v\,\mathrm{d}\theta.$$

with $\mathscr{T}_E\beta = v\partial_x\Phi$. The latter can be recast as follows

$$\begin{aligned} U(x) &= -\int_{-1}^{+1}\int_{\mathbb{Y}} \chi^\star(\theta,v) v\partial_x\Phi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta \\ &= -\int_{\mathbb{Y}}\Big(\psi^\star(\theta)\underbrace{\partial_x\int_{-1}^{+1} v\Phi(\theta,x,v)\,\mathrm{d}v}_{=\partial_x\overline{\mathcal{U}}=0} + \int_{-1}^{+1} v^2\partial_x\Phi(\theta,x,v)\,\mathrm{d}v\Big)\,\mathrm{d}\theta \\ &= -\partial_x\int_{\mathbb{Y}}\int_{-1}^{+1} v^2\Phi(\theta,x,v)\,\mathrm{d}v\,\mathrm{d}\theta. \end{aligned}$$

It is worth detailing on this simple one-dimensional example that the expressions obtained on the shifted unknown coincide: with the same notation as in Section 5, we have

$$\begin{aligned} \widetilde{\mathcal{D}}(x) &= \int_{\mathbb{Y}}\int_{-1}^{1}(v-\overline{\mathcal{U}})\widetilde{\chi}(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v = \int_{\mathbb{Y}}\int_{-1}^{1}(v-\overline{\mathcal{U}})\Phi(\theta,x,v)\widetilde{\chi}^\star(\theta,v)\,\mathrm{d}\theta\,\mathrm{d}v \\ &= \int_{\mathbb{Y}}\int_{-1}^{1}(v-\overline{\mathcal{U}})^2\Phi(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v \qquad \text{since } \widetilde{\chi}^\star(\theta,v) = v-\overline{\mathcal{U}}(\theta), \\ &= \int_{\mathbb{Y}}\int_{-1}^{1} v^2\Phi(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v - \int_{\mathbb{Y}}\overline{\mathcal{U}}^2\,\mathrm{d}\theta. \end{aligned}$$

This has to be compared to

$$D(x) = \int_{\mathbb{Y}}\int_{-1}^{1} v^2\Phi(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v + \int_{\mathbb{Y}}\overline{\mathcal{U}}\psi^\star\,\mathrm{d}\theta.$$

However, we have

$$\int_{\mathbb{Y}}\overline{\mathcal{U}}^2\,\mathrm{d}\theta + \int_{\mathbb{Y}}\overline{\mathcal{U}}\psi^\star\,\mathrm{d}\theta = \int_{\mathbb{Y}}\overline{\mathcal{U}}(\overline{\mathcal{U}}+\psi^\star)\,\mathrm{d}\theta = \int_{\mathbb{Y}}\omega\cdot\nabla_\theta(\overline{\mathcal{U}}+\psi^\star)\;(\overline{\mathcal{U}}+\psi^\star)\,\mathrm{d}\theta = 0,$$

which shows that $\widetilde{\mathcal{D}} = D$. Similarly, for the drift coefficient, we get

$$\begin{aligned} \widetilde{U}(x) &= -\partial_x\int_{\mathbb{Y}}\int_{-1}^{1} v(v-\overline{\mathcal{U}})\Phi(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v = -\partial_x\int_{\mathbb{Y}}\int_{-1}^{1}(v-\overline{\mathcal{U}})^2\Phi(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v \\ &= -\partial_x\left(\int_{\mathbb{Y}}\int_{-1}^{1} v^2\Phi(\theta,x,v)\,\mathrm{d}\theta\,\mathrm{d}v - \int_{\mathbb{Y}}\overline{\mathcal{U}}^2\,\mathrm{d}\theta\right) = U(x) \end{aligned}$$

since $\overline{\mathcal{U}}$ does not depend on x.

We go back to the proof of Proposition 5.4 and Corollary 5.6 which needs to be adapted. Indeed, when we compute the time derivative of $\int_{\mathbb{R}}\int_{-1}^{1}\frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x$, we now have to consider

$$-\frac{1}{\varepsilon}\int_{\mathbb{R}}\int_{-1}^{1}\frac{f_\varepsilon}{\Phi_\varepsilon}v\partial_x f_\varepsilon\,\mathrm{d}v\,\mathrm{d}x = -\frac{1}{2\varepsilon}\int_{\mathbb{R}}\int_{-1}^{1}\Big(\frac{f_\varepsilon}{\Phi_\varepsilon}\Big)^2 v\partial_x\Phi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x$$

which does not vanish. By using the expansion $f_\varepsilon = \rho_\varepsilon\Phi_\varepsilon + \varepsilon g_\varepsilon$, it becomes

$$-\frac{1}{2\varepsilon}\int_{\mathbb{R}}\rho_\varepsilon^2\underbrace{\partial_x\left(\int_{-1}^{1}v\Phi_\varepsilon\,\mathrm{d}v\right)}_{=\partial_x\overline{\mathcal{U}}(\omega t/\varepsilon^2)=0}\mathrm{d}x - \int_{\mathbb{R}}\int_{-1}^{1}\rho_\varepsilon\frac{g_\varepsilon}{\Phi_\varepsilon}v\partial_x\Phi_\varepsilon - \frac{\varepsilon}{2}\int_{\mathbb{R}}\int_{-1}^{1}\Big(\frac{g_\varepsilon}{\Phi_\varepsilon}\Big)^2 v\partial_x\Phi_\varepsilon\,\mathrm{d}v\,\mathrm{d}x.$$

We conclude by using the Cauchy-Schwarz and Young inequalities that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\int_{-1}^{1}\frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x + \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\mathscr{D}_\varepsilon\,\mathrm{d}x \\ &\qquad\le \left\|\frac{v\partial_x\Phi}{M}\right\|_{L^\infty(\mathbb{Y}\times\mathbb{R}\times(-1,+1))}\left(\frac{1}{2\nu}\int_{\mathbb{R}}\int_{-1}^{1}\frac{f_\varepsilon^2}{\Phi_\varepsilon}\,\mathrm{d}v\,\mathrm{d}x + \frac{\varepsilon+\nu}{2}\int_{\mathbb{R}}\int_{-1}^{1}g_\varepsilon^2\frac{M}{\Phi_\varepsilon^2}\,\mathrm{d}v\,\mathrm{d}x\right) \end{aligned}$$

holds for any $0 < \nu \ll 1$. We choose ν small enough (but independent of ε) in order to absorb the last integral by the dissipation term, owing to the coercivity estimate. Finally, we apply the Grönwall lemma, and we deduce the following estimates: for any $0 < T < \infty$,

- $\frac{f_\varepsilon}{\sqrt{\Phi_\varepsilon}}$ is bounded in $L^\infty(0, T; L^2(\mathbb{R} \times (-1, 1))$,
- ρ_ε is bounded in $L^\infty(0, T; L^2(\mathbb{R}))$,
- $g_\varepsilon \frac{M}{\sqrt{\Phi_\varepsilon}}$ is bounded in $L^2((0, T) \times \mathbb{R} \times (-1, 1))$.

Similar conclusions apply to the shifted quantities. The convergence proof then works exactly as in Section 5. The properties of the matrix

$$\mathcal{M}(\theta, x) = \int_{-1}^{1} (v - \overline{\mathcal{U}}(\theta))^2 \Phi(\theta, x, v)\, \mathrm{d}v$$

play a crucial role within the proof. We can check that

- $\widetilde{\mathcal{D}}(x) = \int_{\mathbb{Y}} \mathcal{M}(\theta, x)\, \mathrm{d}\theta$,
- $\mathcal{M}(\theta, x)$ is bounded form above and below, and $\mathcal{M}^{-1}(\theta, x)$, $\nabla_\theta \mathcal{M}^{-1}(\theta, x)$, $\partial_x \mathcal{M}^{-1}(\theta, x)$ all belong to $L^\infty(\mathbb{Y} \times \mathbb{R})$.

Due to the space dependence of Φ, and thus $\mathcal{M}$, the analogue of (25) involves additionally

$$\varepsilon \partial_x \mathcal{M}^{-1}(\omega t/\varepsilon^2, x) \times \int_{\mathbb{R}} (v - \overline{\mathcal{U}}(\theta))^2 g_\varepsilon(t, x, v)\, \mathrm{d}v,$$

which still converges to 0 in $L^2(0, T; H^{-1}(\mathbb{R}))$.

Eventually, we conclude as follows. Let $V(s) = \int_0^s \overline{\mathcal{U}}(\omega\tau)\, \mathrm{d}\tau$ and

$$\widetilde{f_\varepsilon}(t, x, v) = f_\varepsilon(t, x + \varepsilon V(t/\varepsilon^2), v) = \widetilde{\rho_\varepsilon}(t, x)\Phi_\varepsilon(t, x, v) + \varepsilon \widetilde{g_\varepsilon}(t, x, v).$$

Then, both f_ε and $\widetilde{f_\varepsilon}$ double-scale converge to $\rho(t, x)\Phi(\theta, x, v)$, where

i) $\widetilde{\rho_\varepsilon} = \int_{\mathbb{R}^N} \widetilde{f_\varepsilon}\, \mathrm{d}v$ converges to ρ in $C^0([0, T]; L^2(\mathbb{R}) - \text{weak})$, and strongly in $L^2_{\text{loc}}((0, T) \times \mathbb{R})$,

ii) ρ is the solution of the convection-diffusion problem (7) with initial data $\rho\big|_{t=0} = \lim_{\varepsilon \to 0} \int_{-1}^{1} f_{0,\varepsilon}\, \mathrm{d}v$ (weakly in $L^2(\mathbb{R})$),

iii) $\widetilde{g_\varepsilon}$ converges in the double scale sense to $\widetilde{G}(\theta, t, x, v) = -\widetilde{\chi}(\theta, v)\partial_x \rho(t, x) - \widetilde{\beta}(\theta, v)\rho(t, x)$, where $\mathscr{T}_E \widetilde{\chi} = (v - \overline{\mathcal{U}})\Phi$ and $\mathscr{T}_E \widetilde{\beta} = (v - \overline{\mathcal{U}})\partial_x \Phi$.

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