Numerical analysis of the DDFV method for the Stokes problem with mixed Neumann/Dirichlet boundary conditions

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Abstract The aim of this work is to analyze "Discrete Duality Finite Volume" schemes (DDFV for short) on general meshes by adapting the theory known for the linear Stokes problem with Dirichlet boundary conditions to the case of Neumann boundary conditions on a fraction of the boundary. We prove well-posedness for stabilized schemes and we derive some error estimates. Finally, we illustrate some numerical results in which we compare stabilized and unstabilized schemes.

Key words: Stokes system, DDFV scheme, Neumann boundary conditions.
MSC2010: 65M08, 76D05, 35Q35.

1 Introduction

Since the early 2000's a new family of numerical methods, of Finite Volume type, has been developed. The DDFV schemes have been first introduced and studied in [6] and [4] to approximate Laplace equation on a large class of 2D meshes including non-conformal and distorted meshes. A way to consider general families of meshes is to add some unknowns to the problem: we require unknowns on both vertices and centers of primal control volumes. In this way it is possible to obtain a full approximation of the gradient. DDFV is a method oriented to this kind of reconstruction and is designed by mimicking at the discrete level the dual properties of the continuous differential operators.

In the previous works of [3], [7] and [1] the DDFV method was studied for Stokes problem with Dirichlet boundary conditions. In the case of [3], well-posedness of

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the scheme was proved in the case of non-stabilized mass equation only for conformal triangle meshes, conformal and non conformal square meshes. Then this result was improved in [7] by adding a stabilization term to the equation of conservation of mass that led to prove existence and uniqueness of the solution on general meshes.

Successively, since it was observed that very accurate approximations could be computed even without stabilization, in [1] Boyer, Krell and Nabet worked on the inf-sup stability condition for the non-stabilized scheme. This condition relies on the well-posedness of the scheme; it holds unconditionally for certain meshes (e.g. conforming acute triangle meshes) or, with some restrictions, for specific mesh geometries. This work aims at extending the theory known for the Stokes problem to the case of Neumann boundary conditions on a fraction of the boundary. Thus the work is concerned with the numerical simulation of the following problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla \mathrm{p} = \mathbf{f} & \text{in} \quad \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on} \quad \Gamma_D, \\ -\dfrac{\partial \mathbf{u}}{\partial \overrightarrow{\mathbf{n}}} + \mathrm{p}\overrightarrow{\mathbf{n}} = \Phi & \text{on} \quad \Gamma_N, \end{cases} \tag{1}$$

where the unknowns are the velocity $\mathbf{u} : \Omega \to \mathbb{R}^2$ and the pressure $\mathrm{p} : \Omega \to \mathbb{R}$. The data are $\mathbf{f} \in (L^2(\Omega))^2$, $\Phi, \mathbf{g} \in (H^{\frac{1}{2}}(\partial\Omega))^2$ and $\overrightarrow{\mathbf{n}}$ is the unitary outer normal. We will consider an open bounded polygonal domain Ω of $\mathbb{R}^2$ with $\partial\Omega = \Gamma_D \cup \Gamma_N$, where $\Gamma_D \neq \emptyset$ is the fraction of domain with Dirichlet boundary conditions and $\Gamma_N \neq \emptyset$ is the fraction of domain with Neumann boundary conditions.

This paper is organized as follows. In Sect. 2, we detail the DDFV framework, by introducing the meshes, the unknowns and the discrete operators. In Sect. 3 we construct the scheme and we state some properties. Finally, in Sect. 4, we discuss some numerical results.

2 The DDFV framework

The meshes: The complete description of the DDFV scheme for the 2D Stokes problem can be found in [7]. A DDFV mesh is a pair $(\mathfrak{T}, \mathfrak{D})$; $\mathfrak{T}$ combines the primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ (whose cells are denoted by κ), and the dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$, (whose cells κ^* are built around the vertices x_{κ^*} of the primal mesh), see Fig.1.

The primal mesh $\mathfrak{M}$ consists of disjoints polygons κ called "primal cells", whose union covers Ω. The symbol $\partial\mathfrak{M}$ denotes the set of edges of primal mesh included in $\partial\Omega$, that are considered as degenerated primal cells. We associate to each κ a point x_κ, called "center". For the cells of the boundary, the point x_κ is situated at the middle point of the edge. For all the neighbors volumes κ and L, we suppose that $\partial\kappa \cap \partial\mathrm{L}$ is a segment that we call $\sigma = \kappa|\mathrm{L}$, edge of the primal mesh $\mathfrak{M}$.

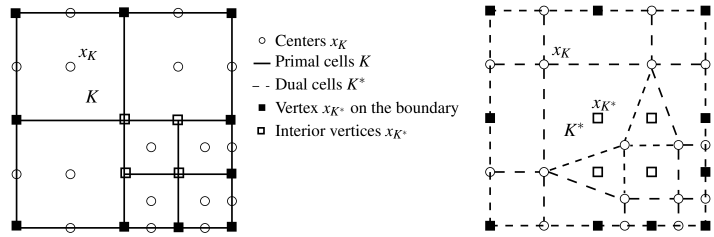

Fig. 1 The mesh $\mathfrak{T}$: primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ (on the *left*), dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ (on the *right*)

From this primal mesh, we build the associated dual mesh. A dual cell $\mathtt{K}^*$ is associated to a vertex $x_{\mathtt{K}^*}$ of the primal mesh. The dual cells are obtained by joining the centers of the primal cells that have $x_{\mathtt{K}^*}$ as vertex. Then, the point $x_{\mathtt{K}^*}$ is called center of $\mathtt{K}^*$. We will distinguish interior dual mesh, for which $x_{\mathtt{K}^*}$ does not belong to $\partial\Omega$, denoted by $\mathfrak{M}^*$ and the boundary dual mesh, for which $x_{\mathtt{K}^*}$ belongs to $\partial\Omega$, denoted by $\partial\mathfrak{M}^*$. We denote with $\sigma^* = \mathtt{K}^*|\mathtt{L}^*$ the edges of the dual mesh.

Next, $\mathfrak{D}$ stands for the diamond mesh, whose cells $\mathtt{D} = \mathtt{D}_{\sigma,\sigma^*}$ are built such that their principal diagonals are a primal edge σ and a dual edge σ^*. Thus a diamond is a quadrilateral with vertices $x_\mathtt{K}, x_\mathtt{L}, x_{\mathtt{K}^*}$ and $x_{\mathtt{L}^*}$. Note that we have $\Omega = \bigcup_{D\in\mathfrak{D}} \mathfrak{D}$. We distinguish the diamonds of the boundary as $\mathfrak{D}_{ext} = \{D_{\sigma,\sigma^*} \in \mathfrak{D},$ such that $\sigma \subset \Omega\}$.

For a diamond cell $\mathtt{D}$ we note by $m_\mathtt{D}$ its measure, m_σ the length of the primal edge σ, m_{σ^*} the length of the dual edge σ^*, $\vec{\mathbf{n}}_{\sigma\mathtt{K}}$ the unit vector normal to σ oriented from $x_\mathtt{K}$ to $x_\mathtt{L}$, $\vec{\mathbf{n}}_{\sigma^*\mathtt{K}^*}$ the unit vector normal to σ^* oriented from $x_{\mathtt{K}^*}$ to $x_{\mathtt{L}^*}$.

Let $size(\mathfrak{T})$ be the maximum of the diameters of the diamonds and $reg(\mathfrak{T})$ be a positive number that measures the regularity of the mesh (see [7] for more details). Finally, we denote by $\mathbf{f}_\mathtt{K}$ (resp. $\mathbf{f}_{\mathtt{K}^*}$) the mean-value of the source term $\mathbf{f}$ on $\mathtt{K} \in \mathfrak{M}$ (resp. on $\mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$) and Φ_σ the mean-value of the Neumann data on $\sigma \in \Gamma_N$.

<u>The unknowns:</u> The DDFV method for Stokes problem uses staggered unknowns. We associate to every $\mathtt{K} \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown $\mathbf{u}_\mathtt{K} \in \mathbb{R}^2$, to every $\mathtt{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown $\mathbf{u}_{\mathtt{K}^*} \in \mathbb{R}^2$ for the velocity and to every $\mathtt{D} \in \mathfrak{D}$ an unknown $p^\mathtt{D} \in \mathbb{R}$ for the pressure. Those unknowns are collected in the families:

$$\mathbf{u}^\mathfrak{T} = \left((\mathbf{u}_\mathtt{K})_{\mathtt{K}\in(\mathfrak{M}\cup\partial\mathfrak{M})}, (\mathbf{u}_{\mathtt{K}^*})_{\mathtt{K}^*\in(\mathfrak{M}^*\cup\partial\mathfrak{M}^*)}\right) \in \left(\mathbb{R}^2\right)^\mathfrak{T} \quad \text{and} \quad p^\mathfrak{D} = \left((p^\mathtt{D})_{\mathtt{D}\in\mathfrak{D}}\right) \in \mathbb{R}^\mathfrak{D}.$$

Since we are considering mixed boundary conditions, we have to define two subspaces of the boundary meshes:

$$\partial\mathfrak{M}_D = \{\mathtt{K} \in \partial\mathfrak{M} : x_\mathtt{K} \in \Gamma_D\}; \quad \partial\mathfrak{M}_N = \{\mathtt{K} \in \partial\mathfrak{M} : x_\mathtt{K} \in \Gamma_N\};$$
$$\partial\mathfrak{M}^*_D = \{\mathtt{K}^* \in \partial\mathfrak{M}^* : x_{\mathtt{K}^*} \in \Gamma_D\}; \quad \partial\mathfrak{M}^*_N = \{\mathtt{K}^* \in \partial\mathfrak{M}^* : x_{\mathtt{K}^*} \in \Gamma_N \backslash \Gamma_D\};$$

and the subspace of $\left(\mathbb{R}^2\right)^\mathfrak{T}$ useful to take into account Dirichlet boundary conditions:

$$\mathbb{E}^D_{m,g} = \{\mathbf{u}^\mathfrak{T} \in \left(\mathbb{R}^2\right)^\mathfrak{T}, \text{s. t. } \forall\mathtt{K} \in \partial\mathfrak{M}_D, \mathbf{u}_\mathtt{K} = (\mathbb{P}^\mathfrak{T}_m g)_\mathtt{K} \text{ and } \forall\mathtt{K}^* \in \partial\mathfrak{M}^*_D, \mathbf{u}_{\mathtt{K}^*} = (\mathbb{P}^\mathfrak{T}_m g)_{\mathtt{K}^*}\},$$

where $\mathbb{P}_m^{\mathfrak{T}}$ is a discrete average projection on the mesh.
The discrete gradient and the discrete divergence: We define a piecewise constant approximation of the gradient operator denoted by $\nabla^{\mathfrak{D}} : (\mathbb{R}^2)^{\mathfrak{T}} \to (\mathtt{M}_2(\mathbb{R}))^{\mathfrak{D}}$,

$$\nabla^{\mathtt{D}}\mathbf{u}^{\mathfrak{T}} := \frac{1}{2m_{\mathtt{D}}}\left[m_\sigma(\mathbf{u}_{\mathtt{L}}-\mathbf{u}_{\mathtt{K}})\otimes \overrightarrow{\mathbf{n}}_{\sigma\mathtt{K}} + m_{\sigma^*}(\mathbf{u}_{\mathtt{L}^*}-\mathbf{u}_{\mathtt{K}^*})\otimes \overrightarrow{\mathbf{n}}_{\sigma^*\mathtt{K}^*}\right],\ \ \forall \mathtt{D}\in\mathfrak{D}.$$

Its discrete dual operator is the approximation of the divergence operator denoted by $\mathbf{div}^{\mathfrak{T}} : (\mathtt{M}_2(\mathbb{R}))^{\mathfrak{D}} \to (\mathbb{R}^2)^{\mathfrak{T}}$, mind the change of the mesh, such that

$$\mathbf{div}^{\mathtt{K}}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}}}\sum_{\sigma\in\partial\mathtt{K}} m_\sigma \xi^D \overrightarrow{\mathbf{n}}_{\sigma\mathtt{K}}, \quad \forall \mathtt{K}\in\mathfrak{M}$$

$$\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}^*}}\sum_{\sigma^*\in\partial\mathtt{K}^*} m_{\sigma^*} \xi^D \overrightarrow{\mathbf{n}}_{\sigma^*\mathtt{K}^*}, \quad \forall \mathtt{K}^*\in\mathfrak{M}^*$$

$$\mathbf{div}^{\mathtt{K}^*}\xi^{\mathfrak{D}} = \frac{1}{m_{\mathtt{K}^*}}\left(\sum_{\sigma^*\in\partial\mathtt{K}^*\setminus\partial\Omega} m_{\sigma^*} \xi^{\mathtt{D}} \overrightarrow{\mathbf{n}}_{\sigma^*\mathtt{K}^*} + \sum_{\sigma\in\partial\mathtt{K}^*\cap\partial\Omega}\frac{m_\sigma}{2}\xi^{\mathtt{D}}\overrightarrow{\mathbf{n}}_{\sigma\mathtt{K}}\right) \quad \forall \mathtt{K}^*\in\partial\mathfrak{M}^*.$$

Those two operators are in *discrete duality* (this is what gives the name to the scheme) since we can prove a discrete Green formula (see Thm. 1 below) that links them. For the proof we refer to [4] and [7]. In order to write this formula, we have to define the *trace operators* and *inner products*.
Trace operators: We define two trace operators. The first one is $\gamma^{\mathfrak{T}} : \mathbf{u}^{\mathfrak{T}} \mapsto \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}) = (\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}))_{\sigma\in\partial\mathfrak{M}} \in (\mathbb{R}^2)^{\mathfrak{T}}$, such that $\gamma_\sigma(\mathbf{u}^{\mathfrak{T}}) = \frac{\mathbf{u}_{\mathtt{K}^*}+2\mathbf{u}_L+\mathbf{u}_{L^*}}{4} \quad \forall \sigma = [x_{\mathtt{K}^*}, x_{L^*}] \in \partial\mathfrak{M}$.
The second operator is $\gamma^{\mathfrak{D}} : \Phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto (\Phi^D)_{D\in\mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$.
Inner products: We define the scalar products on the approximation spaces:

$$[[\mathbf{v}^{\mathfrak{T}},\mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = \frac{1}{2}\left(\sum_{\mathtt{K}\in\mathfrak{M}} m_{\mathtt{K}}\,\mathbf{u}_{\mathtt{K}}\cdot\mathbf{v}_{\mathtt{K}} + \sum_{\mathtt{K}^*\in\mathfrak{M}^*\cup\partial\mathfrak{M}^*} m_{\mathtt{K}^*}\,\mathbf{u}_{\mathtt{K}^*}\cdot\mathbf{v}_{\mathtt{K}^*}\right) \quad \forall \mathbf{u}^{\mathfrak{T}},\mathbf{v}^{\mathfrak{T}}\in(\mathbb{R}^2)^{\mathfrak{T}}$$

$$(\Phi^{\mathfrak{D}},\mathbf{v}^{\mathfrak{T}})_{\partial\Omega} = \sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_{ext}} m_\sigma \Phi^D\cdot\mathbf{v}_\sigma \quad \forall \Phi^{\mathfrak{D}}\in(\mathbb{R}^2)^{\mathfrak{D}_{ext}}, \mathbf{v}^{\mathfrak{T}}\in(\mathbb{R}^2)^{\partial\mathfrak{M}}$$

$$(\xi^{\mathfrak{D}}:\Phi^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{D\in\mathfrak{D}} m_D\big(\xi^D:\Phi^D\big) \quad \forall \xi^{\mathfrak{D}},\Phi^{\mathfrak{D}}\in(\mathscr{M}_2(\mathbb{R}))^{\mathfrak{D}}$$

$$(p^{\mathfrak{D}},q^{\mathfrak{D}})_{\mathfrak{D}} = \sum_{D\in\mathfrak{D}} m_D p^D q^D \quad \forall p^{\mathfrak{D}},q^{\mathfrak{D}}\in\mathbb{R}^{\mathfrak{D}},$$

to which we can associate norms, e.g. $\|\mathbf{u}^\tau\|_2 = [[\mathbf{u}^\tau,\mathbf{u}^\tau]]_\tau^{\frac{1}{2}}$, $\|p^{\mathfrak{D}}\|_2 = (p^{\mathfrak{D}},p^{\mathfrak{D}})_{\mathfrak{D}}^{\frac{1}{2}}$.

Definition 1. (*Bilinear form associated to the scheme* (2))
For all $(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}}),(\widetilde{\mathbf{u}}^{\mathfrak{T}},\widetilde{\mathrm{p}}^{\mathfrak{D}})\in((\mathbb{R}^2)^{\mathfrak{T}}\times\mathbb{R}^{\mathfrak{D}})^2$ we define

$$B(\mathbf{u}^{\mathfrak{T}},\mathrm{p}^{\mathfrak{D}};\widetilde{\mathbf{u}}^{\mathfrak{T}},\widetilde{\mathrm{p}}^{\mathfrak{D}}) := [[\mathbf{div}^{\mathfrak{T}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}+\mathrm{p}^{\mathfrak{D}}Id),\widetilde{\mathbf{u}}^{\mathfrak{T}}]]_{\mathfrak{T}} \\ +(\mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}})+\mu\,\mathrm{size}(\mathfrak{T})\mathrm{p}^{\mathfrak{D}}-\lambda\,\mathrm{d}_{\mathfrak{D}}^2\Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}},\widetilde{\mathrm{p}}^{\mathfrak{D}})_{\mathfrak{D}}.$$

Theorem 1. *(Discrete Green's formula) For all* $\xi^{\mathfrak{D}} \in (\mathscr{M}_2(\mathbb{R}))^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$*:*

$$[[\boldsymbol{div}^{\mathfrak{T}}\xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}})\vec{\boldsymbol{n}}, \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}.$$

We also need a second order stabilization operator $\Delta^{\mathfrak{D}} : \mathbb{R}^{\mathfrak{D}} \to \mathbb{R}^{\mathfrak{D}}$ (see [7]) and the discrete divergence of a vector field $\mathrm{div}^{\mathfrak{D}} : \left(\mathbb{R}^2\right)^{\mathfrak{T}} \to (\mathbb{R}^{\mathfrak{D}})$, $\mathrm{div}^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = \mathrm{Tr}(\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}})$.

3 DDFV schemes for the Stokes equation

In this work we consider a domain in which the boundary is split in two: a fraction with Dirichlet conditions, the other one with Neumann's. We present the scheme with stabilized equation of conservation of mass (through two parameters $\lambda, \mu \geq 0$) and strong boundary conditions (i.e. we impose Dirichlet boundary conditions on $\partial\mathfrak{M}_D \cup \partial\mathfrak{M}_D^*$). The scheme reads: find $\mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_{m,g}^{\mathrm{D}}$ and $\mathrm{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that

$$\begin{cases} \mathbf{div}^{\mathrm{K}}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}} Id) = \mathbf{f}^{\mathrm{K}} & \forall \mathrm{K} \in \mathfrak{M} \\ \mathbf{div}^{\mathrm{K}^*}(-\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathfrak{D}} Id) = \mathbf{f}^{\mathrm{K}^*} & \forall \mathrm{K}^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}_N^* \\ \mathrm{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) + \mu \mathrm{size}(\mathfrak{T})\mathrm{p}^{\mathfrak{D}} - \lambda \mathrm{d}_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}}\mathrm{p}^{\mathfrak{D}} = 0 & \\ (-\nabla^{\mathrm{D}}\mathbf{u}^{\mathfrak{T}} + \mathrm{p}^{\mathrm{D}}\mathrm{Id})\vec{\mathbf{n}}_{\sigma\mathrm{K}} = \Phi_\sigma & \forall \sigma \in \partial\mathfrak{M}_N. \end{cases} \quad (2)$$

Remark that, as the mesh becomes finer, the stabilization terms vanish.

Theorem 2. *(Well-posedness of the scheme) Let* $\lambda + \mu > 0$. *Then the stabilized scheme* (2) *has a unique solution* $(\boldsymbol{u}^{\mathfrak{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$.

Proof. By studying the kernel of the system, we have: $0 = B(\mathbf{u}^{\mathfrak{T}}, \mathrm{p}^{\mathfrak{D}}; \widetilde{\mathbf{u}}^{\mathfrak{T}}, \widetilde{\mathrm{p}}^{\mathfrak{D}})$.
By applying Green's formula and by imposing boundary conditions we end up with:

$$0 = ||\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}||_2^2 + \mu \mathrm{size}(\mathfrak{T})||\mathrm{p}^{\mathfrak{D}}||_2^2 + \lambda |\mathrm{p}^{\mathfrak{D}}|_h^2,$$

with $|\cdot|_h$ a semi-norm (see [7]). This means that $||\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}||_2^2 = 0$, from which we deduce $\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}} = 0$. So $\mathbf{u}^{\mathfrak{T}} = const$ and thanks to Dirichlet boundary conditions we get $\mathbf{u}^{\mathfrak{T}} = 0$. Moreover, if $\mu > 0$, then $||\mathrm{p}^{\mathfrak{D}}||_2^2 = 0$ that implies $\mathrm{p}^{\mathfrak{D}} = 0$; otherwise we have $\lambda > 0$, from which we can deduce $|\mathrm{p}^{\mathfrak{D}}|_h^2 = 0$ that gives $\mathrm{p}^{\mathfrak{D}} = const$. Using Neumann condition we get $\mathrm{p}^{\mathfrak{D}} = 0$. In fact, the well-posedness can be justified also for the unstabilized system, at the price of dealing with specific meshes.
Error estimates: Since we are working with mixed boundary conditions of the type Neumann/Dirichlet, i.e. $\Gamma_N \neq \emptyset$, we need to suppose more regularity (with respect to [7]) for the exact solution $\mathbf{u}$ in order to get a better error estimate.
Thus, we define the space of regularity of the solution as follows:

$$(W^{2,\infty}(\mathfrak{D}))^2 = \left\{\mathbf{u} \in (W^{1,\infty}(\Omega))^2 \ s.t. \ \mathbf{u}|_{\mathrm{D}} \in (W^{2,\infty}(\mathrm{D}))^2, \quad \forall \mathrm{D} \in \mathfrak{D}\right\},$$
$$W^{1,\infty}(\mathfrak{D}) = \left\{\mathrm{p} \in L^\infty(\Omega) \ s.t. \ \mathrm{p}|_{\mathrm{D}} \in W^{1,\infty}(\mathrm{D}), \quad \forall \mathrm{D} \in \mathfrak{D}\right\}.$$

To derive the following error estimates, we have to prove a trace theorem to deal with the new boundary terms that appear due to the Neumann boundary conditions.

Given $\mathbf{u}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$ we associate the approximate solution on the boundary:

$$\mathbf{u}^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*} = \frac{1}{2}\sum_{\mathtt{K}\in\partial\mathfrak{M}} \mathbf{u}_{\mathtt{K}} \mathbb{1}_{\mathtt{K}} + \frac{1}{2}\sum_{\mathtt{K}^*\in\partial\mathfrak{M}^*} \mathbf{u}_{\mathtt{K}^*} \mathbb{1}_{(\bar{\mathtt{K}}^*)\cap\partial\Omega}.$$

Theorem 3. *(Trace theorem) Let $\mathfrak{T}$ be a DDFV mesh associated to Ω. There exists a constant $C > 0$ that depends only on Ω and reg$(\mathfrak{T})$ such that $\forall \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}^D_{m,0}$:*

$$\|\boldsymbol{u}^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*}\|_{2,\partial\Omega} \leq C\|\nabla^{\mathfrak{D}}\mathbf{u}^{\mathfrak{T}}\|_2,$$

where $||\cdot||_{2,\partial\Omega}$ is the L^2 norm on $\partial\Omega$.

The computations of the proof are similar to those present in [5] and [2]. In [5], the proof is given for finite volume methods; in [2], the proof is given for DDFV method but in the case of L^1 norm and with a different definition of $\mathbf{u}^{\partial\mathfrak{M}\cup\partial\mathfrak{M}^*}$. Moreover, our proof has been adapted to the vectorial case.

To get an error estimate of order 1 for the velocity and the pressure we need to consider $\lambda > 0$. The proof will rely on the following stability theorem:

Theorem 4. *(Stability) Suppose $\lambda > 0$. Then $\exists\, C_1, C_2 > 0$, depending only on Ω, λ and reg$(\mathfrak{T})$, such that, for every pair $(\boldsymbol{u}^{\mathfrak{T}}, p^{\mathfrak{D}}) \in \left(\mathbb{R}^2\right)^{\mathfrak{T}} \times \mathbb{R}^{\mathfrak{D}}$ with $\gamma^{\sigma}(\boldsymbol{u}^{\mathfrak{T}}) = 0 \quad \forall \sigma \in \Gamma_D$ and $(-\nabla^D \boldsymbol{u}^{\mathfrak{T}} + p^D Id)\vec{\boldsymbol{n}}_{\sigma,\mathtt{K}} = \Phi_\sigma \quad \forall \sigma \in \Gamma_N$, $\exists\, \widetilde{\boldsymbol{u}}^{\mathfrak{T}} \in \left(\mathbb{R}^2\right)^{\mathfrak{T}}$ with $\gamma^{\sigma}(\widetilde{\boldsymbol{u}}^{\mathfrak{T}}) = 0$ on $\sigma \in \Gamma_D$ and $\widetilde{p}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:*

$$\|\nabla^{\mathfrak{D}}\widetilde{\boldsymbol{u}}^{\mathfrak{T}}\|_2^2 + \|\widetilde{p}^{\mathfrak{D}}\|_2^2 \leq C(\|\nabla^{\mathfrak{D}}\boldsymbol{u}^{\mathfrak{T}}\|_2^2 + \|p^{\mathfrak{D}}\|_2^2)$$

$$\|\nabla^{\mathfrak{D}}\boldsymbol{u}^{\mathfrak{T}}\|_2^2 + \|p^{\mathfrak{D}}\|_2^2 \leq B(\boldsymbol{u}^{\mathfrak{T}}, p^{\mathfrak{D}}; \widetilde{\boldsymbol{u}}^{\mathfrak{T}}, \widetilde{p}^{\mathfrak{D}}) + \left|\sum_{D_{\sigma,\sigma^*}\in\mathfrak{D}_N} m_\sigma \Phi_\sigma \gamma^{\sigma}(\widetilde{\boldsymbol{u}}^{\mathfrak{T}})\right| + C\|\Phi_\sigma\|_2^2.$$

Thanks to Theorem 4, we are able to prove:

Theorem 5. *(Optimal error estimate) We suppose that the solution $(\boldsymbol{u}, p)$ of (1) belongs to $(W^{2,\infty}(\mathfrak{D}))^2 \times W^{1,\infty}(\mathfrak{D})$. Let $\lambda > 0$ and $(\boldsymbol{u}^{\mathfrak{T}}, p^{\mathfrak{D}})$ be the solution of the problem (2). Then $\exists\, C > 0$ that depends on reg$(\mathfrak{T}), \lambda, \|\boldsymbol{u}\|_{W^{2,\infty}}$ and $\|p\|_{W^{1,\infty}}$ s. t.*

$$\|\boldsymbol{u} - \boldsymbol{u}^{\mathfrak{T}}\|_2 + \|\nabla\boldsymbol{u} - \nabla^{\mathfrak{D}}\boldsymbol{u}^{\mathfrak{T}}\|_2 \leq C size(\mathfrak{T}) \quad and \quad \|p - p^{\mathfrak{D}}\|_2 \leq C size(\mathfrak{T}).$$

4 Numerical results

We validate the scheme by showing a few numerical experiments. The computational domain is $\Omega = [0,1]^2$. We studied the error in the case of unstabilized and stabilized mass equation (i.e. with a linear stabilization, $\mu > 0$, or Brezzi-Pitkaranta type stabilization, $\lambda > 0$). In the following discussion, we present results only in the unstabilized case, since we observed that the stabilization terms do not influence the result. For those tests we give the expression of the exact solution (**u**,p), from which we deduce the source term **f**, the Dirichlet boundary condition **g** and the Neumann boundary condition Φ for which (**u**,p) is solution of (1). We will compare the L^2-norm of the error (difference between a centered projection of the exact solution and the approximated solution obtained with DDFV scheme) for the velocity (denoted Ervel), the velocity gradient (Ergradvel) and the pressure (Erpre).

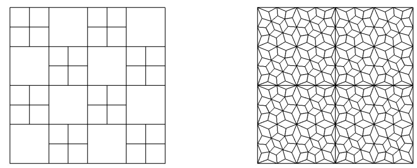

Fig. 2 Family of meshes. On the *left*: non conformal square mesh. On the *right*: quadrangle mesh.

On Tables 1, 2 we give the number of primal cells (NbCell) and the convergence rates (Ratio). We remark that, to discuss the error estimates, a family of meshes (Fig. 2) is obtained by refining successively and uniformly the original mesh.
Green-Taylor vortexes: In this test case, the exact solution is given by:

$$\mathbf{u}(x,y) = \begin{pmatrix} \frac{1}{2}\sin(2\pi x)\cos(2\pi y), \\ -\frac{1}{2}\cos(2\pi x)\sin(2\pi y) \end{pmatrix} \quad \mathrm{p}(x,y) = \frac{1}{8}\cos(2\pi x)\sin(2\pi y).$$

In this example we use the non conformal square mesh of Fig. 2. As we can see in Table 1, we observe super convergence in L^2 norm of the velocity; instead, for the H^1 norm of the velocity and for the L^2 norm of the pressure we get exactly what was expected from Theorem 5. As we mentioned before, an important remark is that the order of convergence does not change whether or not a stabilization is present and this has been observed in all the tests. This underlines the fact that the stabilization term is just a useful tool for the proofs of Theorems 2 and 5, but in practice it doesn't affect the results. Moreover, we tested our schemes on other meshes where we are not able to prove well-posedness of the unstabilized scheme because of their geometry and we numerically observed good behaviour. Remark also that the mesh in this example is non conformal.
Polynomial solutions: The exact solution is given by

$$\mathbf{u}(x,y) = \begin{pmatrix} 2000(x^4-2x^3+x^2)(2y^2-3y^2+y), \\ -2000(y^4-2y^3+y^2)(2x^3-3x^2+x) \end{pmatrix} \quad \mathrm{p}(x,y) = x^2+y^2-1.$$

In this example we use the quadrangle mesh on the right of Fig. 2. Remark that, for this mesh, we have not proved the well-posedness of the unstabilized scheme. However, it is invertible and in Table 2, we observe (as in the first test case) super

Table 1 Green-Taylor vortexes on the non conformal square mesh of Fig. 2

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
64	6.693E-02	-	9.762E-02	-	1.179E+00	-
208	1.665E-02	2.00	4.485E-02	1.12	5.621E-01	1.07
736	4.173E-03	1.99	2.167E-02	1.05	2.770E-01	1.02
2752	1.045E-03	1.99	1.068E-02	1.02	1.380E-01	1.00
10624	2.615E-04	1.99	5.304E-03	1.01	6.895E-02	1.00

convergence in L^2 norm of the velocity and the expected rate for the gradient of the velocity and for the pressure. The order of convergence does not change if we work with or without stabilization. As in the previous case, we tested our schemes on different general meshes, and every time we got good results.

Table 2 Polynomial solutions on the quadrangle mesh of Fig. 2

NbCell	Ervel	Ratio	Ergradvel	Ratio	Erpre	Ratio
400	5.081E-02	-	6.309E-02	-	5.450E+00	-
1536	1.284E-02	1.98	2.796E-02	1.17	2.643E+00	1.04
6016	3.225E-03	1.99	1.346E-02	1.05	1.307E+00	1.01
23808	8.078E-04	1.99	6.660E-03	1.01	6.517E-01	1.00
94720	2.022E-04	1.99	3.320E-03	1.00	3.256E-01	1.00

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The paper is in final form and no similar paper has been or is being submitted elsewhere.