GENERALIZED INVARIANT SETS FOR THE BOLTZMANN EQUATION

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We are interested in the initial value problem for the Boltzmann equation, when the initial data u_0 belongs to a set $B_0 = \{\delta_0 m_1(0,x,v) \le u_0(x,v) \le C_0 m_2(0,x,v)\}$ where m_1, m_2 are traveling Maxwellians. We consider soft or Maxwell's interactions with cut-off ($\frac{7}{3} < s \le 5$) and C_0 smaller than a bound depending on the coefficients of m_2. We obtain global existence of solutions remaining in a "generalized invariant set" $B_t \subset B_\infty$, characterized by these particular states.

1. Introduction

This paper is concerned with the initial value problem for the Boltzmann equation

$$\left\{ \begin{array}{l} (\partial_t + v \cdot \nabla_x) u = Q(u,u) \ \ in \ \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \\ u|_{t=0} = u_0 \ \ in \ \mathbb{R}_x^3 \times \mathbb{R}_v^3. \end{array} \right. \tag{1.1}$$

This equation modelizes the evolution of a rarefied gas and the unknown $u(t,x,v)$ is the kinetic density of gas occupying, at time t, position x with velocity v. The gas is assumed rarefied enough so that collisions between more than two molecules are unprobable. Then, the binary collision dynamic is described by the right hand term of (1.1) which reads as follows

$$Q(u,u) = \int_{\mathbb{R}^3_{v_*} \times S^2_\omega} B(v - v_*, \omega)(u'_* u' - u_* u) d\omega dv_*. \tag{1.2}$$

As usual, in (1.2) we have used the notations $u' = u(t,x,v'), u'_* = u(t,x,v'_*), \ldots$ where

$$v' = v - (v - v_*, \omega)\omega \text{ and } v'_* = v_* + (v - v_*, \omega)\omega$$

are the post collisional velocities of two molecules colliding with velocities v and v_*. The collision kernel $B \ge 0$ is related to the intermolecular potential and we assume in all what follows a sufficient regularity on B in order that the following splitting makes sense

$$\begin{cases} Q(u,u) = Q^+(u,u) - Q^-(u,u) \\ Q^-(u,u) = uLu \; ; Lu = \int_{S^2_\omega} B(.,\omega) d\omega *_v \; u. \end{cases} \tag{1.3}$$

We refer to Refs. 4, 5 for a more detailed description of the Boltzmann equation.

Our aim in this work is, on the one hand, to prove the existence of solutions for particular initial data and, on the other hand, to obtain some informations on solutions u of (1.1), related to assumptions made on the initial data u_0. We assume that u_0 lies in a set B_0 of functions of x, v and we search for informations on the set B_t which contains $u(t)$ at the time t. Defining a relationship between B_t and B_0, we can say that B_t is a "generalized invariant set" for the Boltzmann equation (1.1) (see Refs. 12, 23). Results of this kind are of great interest for a better understanding of the asymptotic trend to equilibrium.

A crucial role is played by the so called "traveling Maxwellians"[5]

$$m(t,x,v) = \exp(-\beta \mid x - tv \mid^2) \exp(-\alpha \mid v \mid^2) \tag{1.4}$$

with $\alpha \geq 0, \beta \geq 0$. These functions are particular solutions of the Boltzmann equation since we have

$$(\partial_t + v.\nabla_x) m = 0 = Q(m,m). \tag{1.5}$$

Hence, Maxwellians (1.4) describe free molecules regimes characterized by the macroscopic quantities

$$\textit{density} \;\; \rho_m(t,x) = \int m(t,x,v) dv,$$

$$\textit{bulk velocity} \;\; \rho_m(t,x) u_m(t,x) = \int v m(t,x,v) dv,$$

$$\textit{temperature} \;\; \rho_m(t,x)(3T_m(t,x) + \mid u_m(t,x) \mid^2) = \frac{1}{2} \int \mid v \mid^2 m(t,x,v) dv.$$

This work is devoted to the existence of solutions u to problem (1.1) when the initial data u_0 satisfies a.e.

$$\delta_0 m_1(0,x,v) \leq u_0(x,v) \leq C_0 m_2(0,x,v) \tag{1.6}$$

where $C_0 \geq \delta_0 \geq 0$ and m_1, m_2 are two traveling Maxwellians (1.4) with coefficients $\beta_1 \geq \beta_2, \alpha_1 \geq \alpha_2$. The "generalized invariant set" B_t is defined by an estimate

$$\delta(t) m_1(t,x,v) \leq u(t,x,v) \leq C(t) m_2(t,x,v) \tag{1.7}$$

where C, δ are functions of time satisfying $C(0) = C_0, \delta(0) = \delta_0$.

The question we address has been treated in Ref. 24 in the particular case $m_1 = m_2$. Existence is obtained in Ref. 24 by a Kaniel-Schinbrot scheme[15] under an additional strong restriction in (1.6), namely, it is assumed that $\delta_0 = K_0 - \varepsilon, C_0 = K_0 + \varepsilon$ with ε small enough. In other words, the initial data is close to a given Maxwellian. We also mention the work Ref. 20 for gases of infinite mass and well known results for rare gas cloud in vacuum (with $\delta_0 = 0$), see Refs. 14, 2, 12, 13, 18. For an extensive discussion, we refer to the survey Ref. 5.

In this paper, we prove global existence of solutions to (1.1) when initial data satisfies (1.6), allowing $m_1 \neq m_2$. Our proof uses rather new and original arguments. Crucial to our approach are, on the one hand, some order preserving properties in transport equations and, on the other hand, the compactness of velocity averages of solutions[9,8,6]. The method of proof provides the required information (1.7) with $B_0 \subset B_t \subset B_\infty$ and generalizes the result of Ref. 24.

This work is organized as follows. In Section 2, we introduce some useful notations and recall basic facts. The statement of the main result of this paper is given at the end of Section 2. Next, in Section 3, we consider a problem with "given l" and the gain term Q^+ of the Boltzmann equation as right hand side. Then, we denote by u_l the corresponding solution to

$$\begin{cases} (\partial_t + v \cdot \nabla_x + l)u_l = Q^+(u_l, u_l) \quad in\ \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \\ u_{l,|t=0} = u_0 \ \ in\ \mathbb{R}_x^3 \times \mathbb{R}_v^3. \end{cases} \tag{1.8}$$

Existence of u_l is deduced from monotonicity and comparison properties. Finally, we show in Section 4 that the mapping $\mathcal{J} : l \longmapsto Lu_l$ has a fixed point which clearly defines a solution to (1.1). We conclude in Section 5 with some remarks on the behaviour of solutions when time becomes large.

2. Preliminaries

In this section, we essentially introduce some useful notations and fundamental assumptions. We consider free molecules regime given by the traveling Maxwellian

$$m(t, x, v) = \exp(-\beta \mid x - tv \mid^2) \exp(-\alpha \mid v \mid^2). \tag{2.1}$$

Density and mass associated with m are defined by

$$\rho_m(t, x) = \int m(t, x, v) dv = (\frac{\pi}{\alpha + \beta t^2})^{\frac{3}{2}} \exp(-\frac{\alpha\beta}{\alpha + \beta t^2} \mid x \mid^2)$$

$$M_m(t) = M_m(0) = (\frac{\pi}{\sqrt{\alpha\beta}})^3. \tag{2.2}$$

In particular, we remark that ρ_m decreases with time like t^{-3}. We introduce two Maxwellians m_1, m_2 defined by (2.1) with

$$\beta_1 \geq \beta_2 \,;\alpha_1 \geq \alpha_2. \tag{2.3}$$

Let $C_0 \geq \delta_0 \geq 0$. We are dealing with functions u_0 enclosed between the Maxwellians m_1, m_2, i.e.

$$\delta_0 m_1(0,x,v) \leq u_0(x,v) \leq C_0 m_2(0,x,v) \tag{2.4}$$

holds a.e. $I\!R^3_x \times I\!R^3_v$. Obviously, we deduce from (2.4) the following estimates

$$u_0 \in L^1 \cap L^\infty(I\!R^3_x \times I\!R^3_v) \tag{2.5}$$

with

$$\| u_0 \|_\infty \leq C_0$$

$$\delta_0 (\frac{\pi}{\sqrt{\alpha_1\beta_1}})^3 \leq \| u_0 \|_1 \leq C_0 (\frac{\pi}{\sqrt{\alpha_2\beta_2}})^3. \tag{2.6}$$

Next, we recall that the loss term of the Boltzmann equation is defined by a convolution operator

$$L\phi(v) = \int_{I\!R^3_{v_*}} \overline{B}(v-v_*)\phi(v_*)dv_*. \tag{2.7}$$

We need some information related to the action of the operator L on the Maxwellian (2.1). We assume that

h1) For all $0 < T, R < \infty$, one has

$$Lm \in L^1((0,T)_t \times I\!R^3_x \times B(0,R)_v)$$

and there exists a function $k_m \in L^1(I\!R^+_t)$ such that

$$0 \leq Lm(t,x,v) \leq k_m(t) \quad a.e.\ I\!R^+_t \times I\!R^3_x \times I\!R^3_v. \tag{2.8}$$

Furthermore, we introduce a continuity assumption

h2) Let ϕ_n be a sequence of functions bounded in $L^1 \cap L^\infty(I\!R^3_v)$ which converges strongly in $L^1(I\!R^3_v)$ to ϕ. Then

$$L\phi_n \longrightarrow L\phi \text{ strongly in } L^1_{loc}(I\!R^3_v).$$

In fact properties *(h1)* and *(h2)* are related to assumptions made on the collision kernel B. For instance, when the force between two molecules separated by a distance r takes the form

$$F(r) = \frac{1}{r^s},\ s > 1$$

then B reads as follows

$$\begin{cases} B(z,\omega) = b(\theta) \mid z \mid^{\gamma} \\ \gamma = \frac{s-5}{s-1} \ ; \cos\theta = \frac{|(z,\omega)|}{|z|}. \end{cases} \tag{2.9}$$

Following Ref. 10, a potential is said "soft" for $1 < s < 5$, and "hard" for $s > 5$. In the limit case, $s = 5$, we are dealing with a gas of Maxwell's molecules. Soft potentials are often considered until $s = 3$[1,3] while the bound $s > \frac{7}{3}$ is discussed in Refs. 12, 13.

The splitting (1.3) is justified under the usual cut-off assumption

$$b \in L^1(S^2)$$

and we set

$$\overline{B}(z) = \int_{S^2_\omega} B(z,\omega)d\omega = \overline{b} \mid z \mid^{\gamma} . \tag{2.10}$$

It is clear that *(h1)* restricts to soft potentials $\frac{7}{3} < s \leq 5$. Moreover, *(h2)* holds for $2 < s \leq 5$. Indeed, in this case, we have

$$0 \leq -\gamma < 3.$$

Let p, r in $]1, +\infty[$ and $\phi \in L^p(I\!R^3), \psi \in L^r(I\!R^3)$. Then, Hardy-Littlewood's inequality[21] yields

$$\int \frac{\phi(v)\psi(v_*)}{\mid v - v_* \mid^{-\gamma}} dv dv_* \leq C \parallel \phi \parallel_{L^p} \parallel \psi \parallel_{L^r}$$

with $\frac{1}{p} + \frac{1}{r} + \frac{-\gamma}{3} = 2$. Hence, choosing $\phi \in L^1 \cap L^\infty \subset L^p$ with $\parallel \phi \parallel_{L^\infty} \leq M$ and $\psi \in \mathcal{C}_0^\infty, supp(\psi) \subset B(0,R)$, one has

$$\int \frac{\phi(v)\psi(v_*)}{\mid v - v_* \mid^{-\gamma}} dv dv_* \leq C \parallel \psi \parallel_{L^\infty} \ \ C_R M^{1-\frac{1}{p}} \parallel \phi \parallel_{L^1}^{\frac{1}{p}} \tag{2.11}$$

which implies *(h2)*.

Let us detail the example of Maxwell's molecules, i.e. $s = 5$, which provides simple computations. In this case $L\phi$ does not depend on v. One has

$$Lm(t,x) = \overline{b}\rho_m(t,x) = \frac{\overline{b}\pi^{\frac{3}{2}}}{(\alpha + \beta t^2)^{\frac{3}{2}}} \exp(-\frac{\alpha\beta}{\alpha + \beta t^2} \mid x \mid^2) \tag{2.12}$$

and

$$0 \leq Lm(t,x) \leq k_m(t) = \frac{\overline{b}\pi^{\frac{3}{2}}}{(\alpha + \beta t^2)^{\frac{3}{2}}} \in L^1(I\!R_t^+).$$

Finally, we set

$$\kappa_m = \frac{1}{\displaystyle\int_0^{+\infty} k_m(s)ds} \tag{2.13}$$

where k_m is the bound of Lm given in (2.8). In particular, κ_m only depends on the parameters α, β. Then, recalling (2.3), we can assume that $\kappa_1 \geq \kappa_2$.

We conclude this section with the statement of our main result.

Theorem 1 *We assume that* (h1) *and* (h2) *hold. Let* $u_0 \in L^1 \cap L^\infty(I\!R^3_x \times I\!R^3_v)$ *satisfying*

$$\delta_0 m_1(0,x,v) \leq u_0(x,v) \leq C_0 m_2(0,x,v) \tag{2.14}$$

with $\beta_1 \geq \beta_2, \alpha_1 \geq \alpha_2$ *and, moreover*

$$0 \leq \delta_0 \leq C_0 < \kappa_2. \tag{2.15}$$

We set

$$C(t) = \frac{1}{\frac{1}{C_0} - \displaystyle\int_0^t k_2(s)ds} \; ; \delta(t) = \frac{\delta_0 C_0}{C(t)}. \tag{2.16}$$

Then, there exists a solution u to the Boltzmann equation (1.1) satisfying

$$\delta(t) m_1(t,x,v) \leq u(t,x,v) \leq C(t) m_2(t,x,v) \tag{2.17}$$

a.e. $I\!R^+_t \times I\!R^3_x \times I\!R^3_v$.

Remark 1 *The inequalities (2.15) allow to define* $C(t)$ *as in (2.16) for all* $t \geq 0$ *since*

$$\frac{1}{C_0} - \int_0^t k_2(s)ds \geq \frac{1}{C_0} - \int_0^\infty k_2(s)ds = \frac{1}{C_0} - \frac{1}{\kappa_2} > 0.$$

Remark 2 *The set*

$$B_t = \{\phi \in L^1 \cap L^\infty(I\!R^3_x \times I\!R^3_v), \delta(t) m_1(t,x,v) \leq \phi \leq C(t) m_2(t,x,v)\}$$

increases with time but remains in the band

$$B_\infty = \{\phi \in L^1 \cap L^\infty(I\!R^3_x \times I\!R^3_v), \delta_\infty m_1(t,x,v) \leq \phi \leq C_\infty m_2(t,x,v)\} \tag{2.18}$$

with

$$C_\infty = \frac{1}{\frac{1}{C_0} - \frac{1}{\kappa_2}} > C_0 \; ; 0 < \delta_\infty = \frac{\delta_0 C_0}{C_\infty} < \delta_0.$$

Moreover, we recover the well known local rarefaction phenomenom since, recalling (2.2), the density $\rho(t,x)$ *decreases like* t^{-3}.

3. Problem with Given l

In this section we deal with the problem

$$\begin{cases} (\partial_t + v \cdot \nabla_x + l)u = Q^+(u,u) & in\ \mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v \\ u_{|t=0} = u_0 & in\ \mathbb{R}^3_x \times \mathbb{R}^3_v \end{cases} \tag{3.1}$$

where l is a given non negative function of t, x, v. We are going to prove

Proposition 1 *Let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ satisfying*

$$\delta_0 m_1(0,x,v) \le u_0(x,v) \le C_0 m_2(0,x,v) \tag{3.2}$$

with $0 \le \delta_0 \le C_0 < \kappa_2$. We set

$$C(t) = \frac{1}{\frac{1}{C_0} - \int_0^t k_2(s)ds}\ ; \delta(t) = \frac{\delta_0 C_0}{C(t)},$$

$$\lambda(t,x,v) = \delta(t) L m_1(t,x,v)\ ; \Lambda(t,x,v) = C(t) L m_2(t,x,v). \tag{3.3}$$

We assume that $l \in L^\infty(\mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$ satisfies a.e.

$$\lambda(t,x,v) \le l(t,x,v) \le \Lambda(t,x,v). \tag{3.4}$$

Then, there exists a solution u_l to the problem (3.1) satisfying a.e. $\mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$

$$\delta(t) m_1(t,x,v) \le u_l(t,x,v) \le C(t) m_2(t,x,v). \tag{3.5}$$

To obtain this result, we introduce the mapping

$$S\ :\ w \in L^1 \cap L^\infty \longmapsto u = S(w) \in L^1 \cap L^\infty \tag{3.6}$$

where $S(w)$ is a solution to (3.1) with $Q^+(u,u)$ replaced by $Q^+(w,w)$. Then, $u = S(w)$ is solution to the following linear problem

$$\begin{cases} (\partial_t + v \cdot \nabla_x + l)u = Q^+(w,w) & in\ \mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v \\ u_{|t=0} = u_0 & in\ \mathbb{R}^3_x \times \mathbb{R}^3_v. \end{cases} \tag{3.7}$$

Proposition 1 follows from existence of a fixed point of S, which is a consequence of the following simple properties

a) S is non decreasing, i.e.

$$w_2 \ge w_1 \text{ a.e. implies } S(w_2) \ge S(w_1) \text{ a.e.}$$

b) There exists h, g in L^∞ such that a.e.

$$h \le S(h)\ ; g \ge S(g).$$

Indeed, we recall the following result (see Ref. 16 or Ref. 22, Part IV, p. 25-26).

Lemma 1 *Let E be an ordered space and S a non decreasing mapping from E to E. We assume that the ordering on E is such that every bounded non empty subset of E has a supremum and an infimum. We also assume that there exists h, g in E such that $h \leq S(h)$ and $g \geq S(g)$. Then, S has at least a fixed point u in the segment $[h, g]$.*

We apply this lemma to the mapping S defined by (3.6-3.7) in $E = L^\infty(\mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3)$ (it is known that L^∞ supplied with a.e. ordering satisfies the property required in Lemma 1[17]). It is clear that S satisfies *(a)*. Indeed, for $w_2 \geq w_1$, we set $U = S(w_2) - S(w_1)$. Then, U satisfies

$$\begin{cases} (\partial_t + v \cdot \nabla_x + l)U = Q^+(w_2, w_2) - Q^+(w_1, w_1) \quad in\ \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3 \\ u_{|t=0} = 0 \quad in\ \mathbb{R}_x^3 \times \mathbb{R}_v^3. \end{cases} \tag{3.8}$$

Since the right hand side of (3.8) is non negative a.e. we deduce that U remains non negative, i.e. $S(w_2) \geq S(w_1)$.

To obtain the property *(b)*, we recall that Maxwellians (2.1) satisfy the following fundamental equality[4]

$$Q^+(m, m) = Q^-(m, m) = mLm. \tag{3.9}$$

Hence, we search for suitable h, g of the form

$$h(t, x, v) = \delta(t)m_1(t, x, v)\,;\, g(t, x, v) = C(t)m_2(t, x, v). \tag{3.10}$$

Expressions (3.3) of δ and C are such that *(b)* holds true. Indeed, by using (3.9), one has

$$(\partial_t + v \cdot \nabla_x + l)(g - S(g)) = m_2(C'(t) + lC(t) - C^2(t)Lm_2)$$

$$(\partial_t + v \cdot \nabla_x + l)(S(h) - h) = m_1(\delta^2(t)Lm_1 - \delta'(t) - l\delta(t)). \tag{3.11}$$

Since (3.2) implies $(g - S(g))_{|t=0} \geq 0$ and $(S(h) - h)_{|t=0} \geq 0$, *(b)* follows from non negativeness of the right hand terms of (3.11). Thus, we need

$$C'(t) + lC(t) - C^2(t)Lm_2 \geq 0\,;\, \delta^2(t)Lm_1 - \delta'(t) - l\delta(t) \geq 0. \tag{3.12}$$

However, (3.3) yields

$$C'(t) = k_2(t)C^2(t)\,;\, \delta'(t) = -\frac{C'(t)}{C^2(t)}\delta_0 C_0 = -\delta(t)C(t)k_2(t). \tag{3.13}$$

Remarking that (3.4) and *(h1)* imply $0 \leq l(t, x, v) \leq C(t)k_2(t)$, we get

$$C'(t) + lC(t) \geq C'(t) = C^2(t)k_2(t) \geq C^2(t)Lm_2$$

$$\delta^2(t)Lm_1 \geq 0 = \delta'(t) + C(t)k_2(t)\delta(t) \geq \delta'(t) + l\delta(t). \tag{3.14}$$

We deduce that $h = \delta(t)m_1, g = C(t)m_2$ define an invariant segment for the mapping S. Hence, Lemma 1 insures existence of a fixed point u_l of S, between h and g. Clearly, u_l is solution to (3.1). This ends the proof of Proposition 1. □

4. Boltzmann Equation

Now, let us define the mapping

$$\mathcal{J} \;:\; l \longmapsto Lu_l \tag{4.1}$$

where u_l is a solution to problem (3.1) given by Proposition 1. This section is devoted to showing the existence of a fixed point of $\mathcal{J}$ in a subset of L^1_{loc}. The argument is a simple application of Schauder's theorem.

We introduce the following non empty convex subset of L^1_{loc}

$$\mathcal{C} = \{l \in L^\infty(\mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v),\ \lambda(t,x,v) \leq l(t,x,v) \leq \Lambda(t,x,v)\} \tag{4.2}$$

with λ, Λ defined in (3.3). By using Proposition 1, for l given in $\mathcal{C}$, there exists a solution u_l to (3.1) and one has

$$\lambda(t,x,v) = Lh(t,x,v) \leq Lu_l(t,x,v) \leq Lg(t,x,v) = \Lambda(t,x,v). \tag{4.3}$$

Hence, we deduce that

$$\mathcal{J}(\mathcal{C}) \subset \mathcal{C}. \tag{4.4}$$

In the following Lemmas, we establish that $\mathcal{J}$ satisfies on $\mathcal{C}$ the conditions of the Schauder fixed point theorem.

Lemma 2 *$\mathcal{J}$ is continuous from $\mathcal{C}$ to $\mathcal{C}$.*

Proof. Let l_n be a sequence of elements of $\mathcal{C}$ which converges to l strongly in $L^1((0,T)_t \times \mathbb{R}^3_x, L^1_{loc}(\mathbb{R}^3_v))$, i.e., for all $\varepsilon > 0, R > 0$ there exists $n_{\varepsilon,R} \in \mathbb{N}$ such that for $n \geq n_{\varepsilon,R}$

$$\| \, l - l_n \, \|_{L^1((0,T)\times \mathbb{R}^3_x \times B(0,R))} \leq \varepsilon. \tag{4.5}$$

We denote by u_n and u the solutions to (3.1) given by Proposition 1, associated with l_n and l respectively. We have

$$\begin{cases} (\partial_t + v\cdot\nabla_x)(u_n - u) = Q^+(u_n,u_n) - Q^+(u,u) + lu - l_n u_n \;\; in\ \mathbb{R}^+_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v \\ (u_n - u)_{|t=0} = 0 \;\; in\ \mathbb{R}^3_x \times \mathbb{R}^3_v. \end{cases} \tag{4.6}$$

Hence, we deduce the following estimate, for all $t \in (0,T)$

$$\| \, (u - u_n)(t) \, \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \leq \int_0^t \| \, (Q^+(u_n,u_n) - Q^+(u,u))(s) \, \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \, ds$$

$$+ \int_0^t \| \, (lu - l_n u_n)(s) \, \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \, ds. \tag{4.7}$$

On the one hand, we recall that

$$Q^+(u_n,u_n) - Q^+(u,u) = Q^+(u_n-u,u_n) + Q^+(u,u_n-u) \tag{4.8}$$

and we estimate each term by using (3.5)

$$\int \mid Q^+(u_n-u,u_n) \mid dv \le \int \mid u_n - u \mid Lu_n dv \le C(s)k_2(s) \int \mid u_n - u \mid dv$$

$$\int \mid Q^+(u,u_n-u) \mid dv \le \int L(\mid u_n - u \mid) g dv \le C(s)k_2(s) \int \mid u_n - u \mid (v_*) dv_*. \tag{4.9}$$

On the other hand, we split the "loss terms" as follows

$$lu - l_n u_n = (l-l_n)u + l_n(u-u_n) \tag{4.10}$$

where, by using (4.3), we have

$$\| l_n(u-u_n)(s) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} \le \int \Lambda \mid u - u_n \mid dvdx$$

$$\le C(s)k_2(s) \| (u-u_n)(s) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} . \tag{4.11}$$

It remains to study

$$\| (l-l_n)u(s) \|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)} = \int_{|v|>R} \mid l - l_n \mid u\, dxdv + \int_{|v|<R} \mid l - l_n \mid u\, dxdv$$

$$\le (\| l \|_{L^\infty_{x,v}} + \sup_n \| l_n \|_{L^\infty_{x,v}}) \int_{|v|>R} u\, dxdv + \| u \|_{L^\infty} \int_{|v|<R} \mid l - l_n \mid dxdv. \tag{4.12}$$

Indeed, by using the definition of the set $\mathcal{C}$, we have for all $\phi \in \mathcal{C}$,

$$0 \le l(s,x,v) \le \Lambda(s,x,v) \le C(s)k_2(s).$$

Then, let $\varepsilon > 0$ fixed. We choose $R = R_\varepsilon > 0$ large enough to insure, by integrability of u, the estimate

$$\int_{|v|>R_\varepsilon} udxdv \le C(s) \int_{|v|>R_\varepsilon} m_2 dxdv \le \frac{\varepsilon e^{-\frac{3C_\infty}{\kappa_2}} C(s)}{2(C_\infty - C_0)}. \tag{4.13}$$

Combining estimates (4.8-4.13), (4.7) becomes

$$\| (u_n-u)(t) \|_{L^1_{x,v}} \le \frac{\varepsilon e^{-\frac{3C_\infty}{\kappa_2}}}{2(C_\infty - C_0)} \int_0^t 2C^2(s)k_2(s)ds$$

$$+3\int_0^t C(s)k_2(s) \parallel (u_n-u)(s) \parallel_{L^1_{x,v}} ds + \int_0^t C(s)\int_{\mathbb{R}^3_x\times B(0,R_\varepsilon)} (l_n-l)(s)dxdvds. \tag{4.14}$$

By using Gronwall's lemma, we get for all $t\in(0,T)$

$$\parallel (u_n-u)(t) \parallel_{L^1_{x,v}} \leq \varepsilon + C_\infty e^{3\frac{C_\infty}{\kappa_2}} \parallel l_n - l \parallel_{L^1((0,T)\times\mathbb{R}^3_x\times B(0,R_\varepsilon))} . \tag{4.15}$$

Finally, recalling (4.5), we choose $n_\varepsilon\in\mathbb{N}$ large enough so that

$$\parallel l - l_n \parallel_{L^1((0,T)\times\mathbb{R}^3_x\times B(0,R_\varepsilon))} \leq \frac{\varepsilon}{C_\infty \exp(\frac{3C_\infty}{\kappa_2})} \tag{4.16}$$

for $n\geq n_\varepsilon$ and we deduce that for all $t\in(0,T)$

$$\parallel (u-u_n)(t) \parallel_{L^1(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \leq 2\varepsilon. \tag{4.17}$$

Next, we conclude thanks to assumption *(h2)* that

$$\mathcal{J}(l_n)-\mathcal{J}(l) = Lu_n - Lu$$

converges strongly to 0 in L^1_{loc}. Indeed, we set in (2.11), $\phi = u_n - u, M = 2C_\infty$ and we get for all $0<T,R<\infty$, and $\psi\in\mathcal{C}^\infty_0, supp(\psi)\subset B(0,R)$,

$$\parallel \psi L(u-u_n) \parallel_{L^1((0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_v)} \leq C_{\psi,R,M,\gamma} \parallel u-u_n \parallel_{L^1((0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_v)} \leq C_{\psi,R,M,\gamma} T\varepsilon. \tag{4.18}$$

when n is large enough. $\square$

Now, we prove

Lemma 3 *$\mathcal{J}$ is a compact mapping from $\mathcal{C}$ to $\mathcal{C}$.*

Proof. This result is a consequence of well-known regularity property of velocity averages for solutions to transport equations[9,8,19]. Let l_n be a sequence of elements of $\mathcal{C}$. We denote by u_n the associated sequence of solutions to (3.1) given by Proposition 1. The definition of $\mathcal{C}$ implies the following bound

$$\parallel l_n \parallel_\infty \leq C_\infty \sup_{s\geq 0} k_2(s) = C_\infty K_2. \tag{4.19}$$

Moreover, by using (3.5), u_n satisfies

$$\parallel u_n \parallel_{L^\infty} \leq C_\infty,$$

$$\parallel u_n \parallel_{L^1((0,T)\times\mathbb{R}^3_x\times\mathbb{R}^3_v)} \leq C_\infty T(\frac{\pi}{\sqrt{\alpha_2\beta_2}})^3. \tag{4.20}$$

Furthermore, we also deduce immediately

$$\parallel l_n u_n \parallel_{L^\infty} \leq C^2_\infty K_2,$$

$$\| l_n u_n \|_{L^1} \leq \int g\Lambda \, dx dv ds \leq (\frac{\pi}{\sqrt{\alpha_2\beta_2}})^3 \int C^2(s) k_2(s) ds = (\frac{\pi}{\sqrt{\alpha_2\beta_2}})^3 (C_\infty - C_0) \tag{4.21}$$

and, for "gain terms" we have

$$\| Q^+(u_n, u_n) \|_{L^\infty} \leq \| Q^+(g,g) \|_\infty \leq C_\infty^2 K_2,$$

$$\| Q^+(u_n, u_n) \|_{L^1} \leq \| Q^+(g,g) \|_{L^1} \leq (\frac{\pi}{\sqrt{\alpha_2\beta_2}})^3 (C_\infty - C_0). \tag{4.22}$$

Hence, u_n is bounded in $L^\infty(I\!R_t^+ \times I\!R_x^3 \times I\!R_v^3) \cap L^1((0,T)_t \times I\!R_x^3 \times I\!R_v^3)$ and satisfies a transport equation

$$\begin{cases} (\partial_t + v \cdot \nabla_x) u_n = Q^+(u_n, u_n) - l_n u_n \ \ in \ I\!R_t^+ \times I\!R_x^3 \times I\!R_v^3 \\ u_{n,|t=0} = u_0 \ \ in \ I\!R_x^3 \times I\!R_v^3 \end{cases} \tag{4.23}$$

where the right hand term is bounded in $L^\infty \cap L^1$. Applying average lemma (see Refs. 9, 18, 6, 8, 19) leads to the required compactness of

$$Lu_n = \int \overline{B}(v - v_*) u_n(v_*) dv_*$$

in $L^1((0,T)_t \times I\!R_x^3, L^1_{loc}(I\!R_v^3))$. □

We deduce that $\mathcal{J}$ has a fixed point $l \in \mathcal{C}$. Then, u_l is solution to

$$\begin{cases} (\partial_t + v \cdot \nabla_x) u_l = Q^+(u_l, u_l) - u_l L u_l \ \ in \ I\!R_t^+ \times I\!R_x^3 \times I\!R_v^3 \\ u_{l,|t=0} = u_0 \ \ in \ I\!R_x^3 \times I\!R_v^3 \end{cases} \tag{4.24}$$

and remains, at the time t, in the generalized invariant set B_t. Hence, the proof of Theorem 1 is complete. □

5. Concluding Remarks

We end this work with some comments on the asymptotic trend to equilibrium. First, we define a wave operator associated with the problem (1.1)[12,13].

We denote by $\{S(t)\}_{t\geq 0}$ the familly of operators which map an initial data u_0 into $u(t)$, solution to (1.1). Then, we introduce the familly $\{S_0(t)\}_{t\geq 0}$ associated with the collisionless equation

$$S_0(t) u_0(x,v) = u_0(x - tv, v) \tag{5.1}$$

solution to

$$(\partial_t + v \cdot \nabla_x) S_0(t) u_0 = 0 \, ; (S_0(t) u_0)_{|t=0} = u_0. \tag{5.2}$$

Hence, integration along the characteristic lines $x + tv$ in (1.1) leads to the following "mild" formulation

$$u(t, x + tv, v) = u_0(x,v) + \int_0^t Q(u,u)(\tau, x + \tau v, v) d\tau$$

$$= S_0(-t)S(t)u_0(x,v) = u^\sharp(t, x+tv, v). \tag{5.3}$$

One has the following

Lemma 4 *We set $W^+(t) = S_0(-t)S(t)$. Then the wave operator $W^+ = \lim_{t\to\infty} W^+(t)$ is defined on $B_0 = \{u_0 \in L^1 \cap L^\infty_{x,v}, \delta_0 m_1(0) \le u_0 \le C_0 m_2(0)\}$.*

Remark 3 *Existence of W^+ means that $u^\sharp(t,x,v)$ converges to a function $g_\infty(x,v)$ when time becomes large. Convergence is uniform with respect to x, v[24]. Inequality (2.17) implies*

$$\delta_\infty e^{-\beta_1 x^2} e^{-\alpha_1 v^2} \le g_\infty(x,v) \le C_\infty e^{-\beta_2 x^2} e^{-\alpha_2 v^2}. \tag{5.4}$$

Proof. We use the estimates obtained on the solutions u and the mild formulation (5.3), then, for $t, s \ge 0$, one has

$$| u^\sharp(t+s) - u^\sharp(s) | \le | \int_s^{t+s} Q^\sharp(u,u)(\tau) d\tau |$$

$$\le 2\int_s^{t+s} (gLg)^\sharp(\tau)d\tau \le 2\int_s^{t+s} C^2(\tau)k_2(\tau)d\tau = 2(C(t+s) - C(s)). \tag{5.5}$$

One deduces that

$$\lim_{s\to\infty} \sup_t \| W^+(t+s)u_0 - W^+(s)u_0 \|_{L^\infty_{x,v}} \le 2 \lim_{s\to\infty} (C_\infty - C(s)) \tag{5.6}$$

which tends to 0 when s goes to infinity. □

Hence, there exists g_∞, satisfying (5.4), such that $u^\sharp$ converges uniformly to g_∞ when t goes to ∞. We set $u_\infty(t,x,v) = g_\infty(x - tv, v)$, then, for t large, $u(t)$ behaves like u_∞. Next, we remark that the estimates (2.17) imply that $u \ln(u)$ is integrable. Then, one deduces an H-theorem satisfied by u. In particular, one obtains for $t \ge 0$

$$\int u(t)\ln(u)(t) dxdv + \int_0^t \int H(u)(\tau) dxdvd\tau = \int u_0 \ln(u_0) dxdv \tag{5.7}$$

where $H(u)$ is the (non negative) entropy production term and u_0 satisfies $u_0 \ln(u_0) \in L^1_{x,v}$. By using classical arguments[7,18], one finds that the limit u_∞ satisfies $H(u_\infty) = 0$ which implies

$$u_\infty u_{\infty,*} = u'_\infty u'_{\infty,*} \tag{5.8}$$

and u_∞ is a local Maxwellian. We have proved

Proposition 2 *Let u be a solution to (1.1) defined in Theorem 1. When time becomes large, u behaves like*

$$u_\infty(t,x,v) = g_\infty(x - tv, v) = d_\infty e^{-\beta_\infty |x-tv|^2} e^{-\alpha_\infty |v|^2} \tag{5.9}$$

which satisfies (5.4).

Appendix A. Computation of Lm

In what follows, we assume that the collision kernel satisfies

$$\overline{B}(z) = \overline{b} \mid z \mid^{\gamma} \ ; \gamma = \frac{s-5}{s-1}. \tag{A.1}$$

Our aim is to compute

$$Lm(t,x,v) = \overline{b} \int \mid v - v_* \mid^{\gamma} \exp(-\alpha \mid v_* \mid^2) \exp(-\beta \mid x - tv_* \mid^2) dv_*$$

$$= \overline{b} \exp(-\frac{\alpha\beta \mid x \mid^2}{\alpha + \beta t^2}) \int \mid v - v_* \mid^{\gamma} \exp(-(x\sqrt{\beta - \frac{\alpha\beta}{\alpha+\beta t^2}} - v_* \sqrt{\alpha + \beta t^2})^2) dv_*. \tag{A.2}$$

We introduce the following notations

$$K(t,x) = \overline{b} \exp(-\frac{\alpha\beta}{\alpha+\beta t^2} \mid x \mid^2),$$

$$A(t) = \sqrt{\beta - \frac{\alpha\beta}{\alpha+\beta t^2}} \ ; B(t) = \sqrt{\alpha+\beta t^2} \text{ and } u = A(t)x - B(t)v. \tag{A.3}$$

Then, we define the following change of variables

$$v_* \in \mathbb{R}^3 \to w = v - v_* \in \mathbb{R}^3 \to (\rho, \Sigma) \in \mathbb{R}^+ \times S^2 \to (\rho, \theta, \psi) \in \mathbb{R}^+ \times (0, 2\pi) \times (0, \pi),$$

with $\cos(\psi) = \frac{u.\Sigma}{|u|}$ and

$$dv_* = dw = \rho^2 d\rho d\Sigma = \rho^2 d\rho d\theta \sin(\psi) d\psi.$$

Hence, one has

$$\begin{aligned} Lm(t,x,v) &= K(t,x) \int \mid w \mid^{\gamma} \exp(-(A(t)x - B(t)v + B(t)w)^2)\, dw \\ &= K(t,x) \int \rho^{\gamma+2} e^{-(u+B(t)\rho\Sigma)^2} d\rho d\Sigma \\ &= K(t,x) \int \rho^{\gamma+2} e^{-u^2} e^{-B^2\rho^2} e^{-2B\rho u.\Sigma} d\Sigma d\rho \end{aligned}$$

$$= K(t,x) \int_0^{\infty} \rho^{\gamma+1} e^{-u^2} e^{-B^2\rho^2} \int_0^{2\pi} d\theta \int_0^{\pi} \rho \sin(\psi) e^{-2B|u|\rho\cos(\psi)} d\psi d\rho. \tag{A.4}$$

It is obvious that

$$\int_0^{\pi} \rho \sin(\psi) e^{-2B|u|\rho\cos(\psi)} d\psi = \int_0^{\pi} \frac{d}{d\psi} (\frac{e^{-2B|u|\rho\cos(\psi)}}{2B \mid u \mid}) d\psi$$

$$= \frac{1}{B \mid u \mid} \frac{e^{2\rho B|u|} - e^{-2\rho B|u|}}{2}.$$

Hence, we get

$$Lm(t,x,v) = \frac{K(t,x)e^{-u^2}}{B \mid u \mid}.2\pi. \int_0^\infty \rho^{\gamma+1} e^{-B^2\rho^2} \sinh(2B \mid u \mid \rho)\, d\rho. \tag{A.5}$$

Let $c > 0, \mu > -\frac{1}{2}$ and $a \in I\!R$. Then, it is known (see Ref. 11, p. 365) that

$$\int_0^\infty s^{2\mu-1} e^{-cs^2} \sinh(as) ds = \frac{\Gamma(2\mu)}{2}.(2c)^{-\mu}.e^{\frac{a^2}{8c}}.(D_{-2\mu}(-\frac{a}{\sqrt{2c}}) - D_{-2\mu}(+\frac{a}{\sqrt{2c}}))$$

where $D_r(\theta)$ is a parabolic cylinder function. We apply this formula with $c = B^2, a = 2B \mid u \mid$ and $2\mu - 1 = \gamma + 1$, i.e.

$$\mu = \frac{1}{2}(3 - \frac{4}{s-1}).$$

Then, the condition $\mu > -\frac{1}{2}$ is satisfied for $s > 2$.

We obtain

$$Lm = \frac{K(t,x)e^{-u^2}}{B \mid u \mid}.\frac{2\pi}{2}.\Gamma(3 - \frac{4}{s-1}).(2B^2)^{-\frac{1}{2}(3-\frac{4}{s-1})}.e^{\frac{4B^2u^2}{8B^2}}$$

$$.(D_{-3+\frac{4}{s-1}}(-\frac{2 \mid u \mid}{\sqrt{2}}) - D_{-3+\frac{4}{s-1}}(+\frac{2 \mid u \mid}{\sqrt{2}})). \tag{A.6}$$

We recall that (see Ref. 11, p. 1064)

$$D_r(\theta) = 2^{\frac{r}{2}}.e^{-\frac{\theta^2}{4}}(\frac{\sqrt{\pi}}{\Gamma(\frac{1-r}{2})}\Phi(-\frac{r}{2}, \frac{1}{2}, \frac{\theta^2}{2}) - \frac{\theta\sqrt{2\pi}}{\Gamma(-\frac{r}{2})}\Phi(\frac{1-r}{2}, \frac{3}{2}, \frac{\theta^2}{2}))$$

thus, finally

$$D_r(-\theta) - D_r(\theta) = 2.2^{\frac{r}{2}}.e^{-\frac{\theta^2}{4}}.\frac{\sqrt{2\pi}}{\Gamma(-\frac{r}{2})}\theta\Phi(\frac{1-r}{2}, \frac{3}{2}, \frac{\theta^2}{2})$$

where Φ is a degenerate hypergeometric function (see Ref. 11, p. 1058). It follows that

$$Lm(t,x,v) = \frac{K(t,x)e^{-u^2}}{\mid u \mid B^{4-\frac{4}{s-1}}}.2\pi^{\frac{3}{2}}.\frac{\Gamma(3 - \frac{4}{s-1})}{\Gamma(\frac{3}{2} - \frac{2}{s-1})}. \mid u \mid$$

$$.2^{2(-\frac{3}{2}+\frac{2}{s-1})}.e^{-\frac{u^2}{2}}.e^{+\frac{u^2}{2}}.\sqrt{2}.\Phi(\frac{1}{2} + \frac{3}{2} - \frac{2}{s-1}, \frac{3}{2}, u^2). \tag{A.7}$$

We set

$$d(u) = \frac{\Gamma(3 - \frac{4}{s-1})}{\Gamma(\frac{3}{2} - \frac{2}{s-1})}.\pi^{\frac{3}{2}}.2^{\frac{4}{s-1}-1}.\Phi(2(1 - \frac{1}{s-1}), \frac{3}{2}, u^2) = d'\Phi(2(1 - \frac{1}{s-1}), \frac{3}{2}, u^2).$$

Hence, for $s \leq 5$, one has $d(u) \leq d' \exp(u^2)$ and we find

$$Lm(t,x,v) = \frac{\overline{b}d(u)}{(\alpha+\beta t^2)^{2-\frac{2}{s-1}}}.\exp(-\frac{\alpha\beta \mid x \mid^2}{\alpha+\beta t^2}).\exp(-u^2)$$

$$\leq \frac{\overline{b}d'}{(\alpha+\beta t^2)^{2-\frac{2}{s-1}}}.\exp(-\frac{\alpha\beta \mid x \mid^2}{\alpha+\beta t^2}). \tag{A.8}$$

Then, we set

$$k(t) = \frac{\overline{b}d'}{(\alpha+\beta t^2)^{2-\frac{2}{s-1}}} \tag{A.9}$$

which is integrable on $I\!R_t^+$ for $s > \frac{7}{3}$. (In particular, for Maxwell's molecules, $s = 5$, one has $k(t) = \frac{\overline{b}d'}{(\alpha+\beta t^2)^{\frac{3}{2}}}$.)

Appendix B. On Lemma 1 with $E = L^\infty$

We consider a measured space $(X, \mathcal{T}, \mu)$. For $1 \leq p < \infty$ (resp. $p = \infty$) we denote by $L^p = L^p(X, d\mu)$ the space of $\mu-$measurable functions u from X to $I\!R$ such that

$$\int_X \mid u \mid^p d\mu < \infty$$

(resp. $\mid u \mid$ is essentially bounded). The ordering on L^p is defined by

$$u \geq v \textit{ if and only if } u(x) \geq v(x) \textit{ a.e. } x \in X.$$

The conditions required in Lemma 1 are satisfied for $E = L^p$, for all $1 \leq p \leq \infty$[17]. For the sake of completness, we give a proof for the case $E = L^\infty$ which is adapted from Ref. 22.

We consider a non empty familly $\{u_\alpha, \alpha \in \mathcal{A}\}$ of elements in $E = L^\infty$, where $\mathcal{A}$ denotes a set of indices. We assume, without loss of generality, that there exists $g \in E$ such that for all $\alpha \in \mathcal{A}$,

$$0 \leq u_\alpha \leq g \tag{B.1}$$

holds a.e.. Then, we are going to prove that the familly $\{u_\alpha\}$ has a least upper bound (the proof for the existence of a greatest lower bound follows the same lines).

We set, for $\phi \in L^1, \phi \geq 0$

$$l(\phi) = \sup_{N \in I\!N} \{\sum_{i=1}^N \langle u_{\alpha_i}, \phi_i \rangle_{L^\infty, L^1},\ \alpha_i \in \mathcal{A}, \phi_i \geq 0, \sum_{i=1}^N \phi_i \leq \phi\}. \tag{B.2}$$

First, we show that, for $\phi \geq 0, \psi \geq 0$ in L^1 one has

$$l(\phi) + l(\psi) = l(\phi + \psi). \tag{B.3}$$

Let $\varepsilon > 0$ be fixed. There exist $N, P \in I\!N$ and N, P-uplets $\{(\alpha_1, \phi_1), \ldots, (\alpha_N, \phi_N)\}$, $\{(\beta_1, \psi_1), \ldots, (\beta_P, \psi_P)\}$ such that

$$\sum_{i=1}^{N} \phi_i \leq \phi \, ; \sum_{j=1}^{P} \psi_j \leq \psi$$

$$l(\phi) \leq \sum_{i=1}^{N} \langle u_{\alpha_i}, \phi_i \rangle + \varepsilon \, ; l(\psi) \leq \sum_{j=1}^{P} \langle u_{\beta_j}, \psi_j \rangle + \varepsilon. \tag{B.4}$$

Then, we define $Q = N + P$ and we set $\gamma_k = \alpha_i$ and $\theta_k = \phi_i$ if $k, i \in \{1, \ldots, N\}$ and $\gamma_k = \beta_j$ and $\theta_k = \psi_j$ if $k \in \{N+1, \ldots, N+P\}, j \in \{1, \ldots, P\}$ in such a way that

$$\sum_{k=1}^{Q} \theta_k \leq \sum_{j=1}^{N} \phi_i + \sum_{j=1}^{P} \psi_j \leq \phi + \psi. \tag{B.5}$$

Hence, by using (B.4), we deduce

$$l(\phi) + l(\psi) \leq \sum_{k=1}^{Q} \langle u_{\gamma_k}, \theta_k \rangle + 2\varepsilon \leq l(\phi + \psi) + 2\varepsilon. \tag{B.6}$$

Since this inequality holds for all $\varepsilon > 0$, it follows that

$$l(\phi) + l(\psi) \leq l(\phi + \psi). \tag{B.7}$$

Next, one has

$$l(\phi + \psi) \leq \sum_{k=1}^{Q} \langle u_{\gamma_k}, \theta_k \rangle + \varepsilon \tag{B.8}$$

for a suitable choice of $Q \in I\!N, \{(\gamma_1, \theta_1), \ldots (\gamma_Q, \theta_Q)\} \in \mathcal{A} \times L^1$ with

$$\sum_{k=1}^{Q} \theta_k \leq \phi + \psi. \tag{B.9}$$

Then, we set

$$\begin{cases} \phi_k = \begin{cases} \frac{\theta_k}{\phi+\psi}\phi \;\; if \; \phi + \psi \neq 0 \\ 0 \;\; otherwise \end{cases} \\ \\ \psi_k = \begin{cases} \frac{\theta_k}{\phi+\psi}\psi \;\; if \; \phi + \psi \neq 0 \\ 0 \;\; otherwise. \end{cases} \end{cases} \tag{B.10}$$

Obviously, $\theta_k = \phi_k + \psi_k$ and $\displaystyle\sum_{k=1}^{Q} \phi_k \leq \phi, \sum_{k=1}^{Q} \psi_k \leq \psi$ and we get immediately

$$l(\phi + \psi) \leq \sum_{k=1}^{Q} (\langle u_{\gamma_k}, \phi_k \rangle + \langle u_{\gamma_k}, \psi_k \rangle) + \varepsilon \leq l(\phi) + l(\psi) + \varepsilon \tag{B.11}$$

for every $\varepsilon > 0$. Hence (B.11) implies

$$l(\phi + \psi) \leq l(\phi) + l(\psi). \tag{B.12}$$

Combining (B.7) and (B.12) gives (B.3).

We extend l on L^1, setting

$$l(\phi) = l(\max(\phi, 0)) - l(\max(-\phi, 0)). \tag{B.13}$$

Thus, l is linear and satisfies, for all $\phi \in L^1$

$$| \, l(\phi) \, | \leq \| \, g \, \|_{\infty} \| \, \phi \, \|_{1} \, . \tag{B.14}$$

Since $L^\infty = (L^1)'$, one deduces that there exists $\overline{u} \in L^\infty$ such that

$$l(\phi) = \langle \overline{u}, \phi \rangle_{\infty,1} \tag{B.15}$$

for all $\phi \in L^1$.

Now, we show that $\overline{u}$ is the supremum of $\{u_\alpha\}$. Indeed, for all $\alpha \in \mathcal{A}$ and $\phi \in L^1, \phi \geq 0$, we have

$$\langle u_\alpha, \phi \rangle \leq l(\phi) = \langle \overline{u}, \phi \rangle \, ; l(\phi) \leq \langle g, \phi \rangle. \tag{B.16}$$

One deduces that

$$u_\alpha \leq \overline{u} \leq g \;\; a.e. \tag{B.17}$$

Finally, let $w \in L^\infty$ such that for all $\alpha \in \mathcal{A}$, $u_\alpha \leq w$. Then, we have, for $\phi \in L^1, \phi \geq 0$

$$\langle w, \phi \rangle \geq l(\phi) = \langle \overline{u}, \phi \rangle \tag{B.18}$$

so, $\overline{u} \leq w$ holds a.e. and the proof is complete. $\square$

References

1. K. Asano and S. Ukai, *On the Cauchy problem for the Boltzmann equation with a soft potential, Publ. RIMS* **18** (1982) 57-99.
2. N. Bellomo and G. Toscani, *On the Cauchy problem for the nonlinear Boltzmann equation. Global existence, uniqueness and asymptotic stability, J. Math. Phys.* **26** (1985) 334-338.
3. R. Caflish, *The Boltzmann equation with a soft potential, Comm. Math. Phys.* **74** (1980) 71-95.
4. C. Cercignani, **The Boltzmann equation and its applications** (Springer-Verlag, 1988).
5. C. Cercignani, R. Illner and M. Pulvirenti, **The mathematical theory of dilute gases** (Springer-Verlag, 1994).
6. R. Di Perna and P. L. Lions, *On the Cauchy problem for Boltzmann equation: global existence and weak stability, Ann. Math.* **130** (1989) 321-366.
7. R. Di Perna and P. L. Lions, *Global solutions of Boltzmann's equation and entropy inequality, Arch. Rat. Mech. Anal.* **114** (1991) 47-55.

8. R. Di Perna, P. L. Lions and Y. Meyer, *L^p regularity of velocity averages, Ann. IHP. An. Non Linéaire* **8**, 3-4 (1991) 271-287.
9. F. Golse, P. L. Lions, B. Perthame and R. Sentis, *Regularity of the moments of the solution of a transport equation, J. Funct. Anal.* **76** (1988) 110-125.
10. H. Grad, *Asymptotic theory of the Boltzmann equation* in *Rarefied Gas Dynamic, Third Symposium, Paris* (1962), pp. 26-59.
11. I. S. Gradshteyn and I. M. Ryzhik, **Table of integrals, series and products** (Academic Press, 1980).
12. K. Hamdache, *Sur l'existence globale et le comportement asymptotique de quelques solutions de l'équation de Boltzmann*, Thèse Doctorat d'Etat, Paris VI (1986).
13. K. Hamdache, *Existence in the large and asymptotic behavior for the Boltzmann equation, Japan J. Appl. Math.* **2** (1985) 1-15.
14. R. Illner and M. Shinbrot, *The Boltzmann equation : global existence for a rare gas in an infinite vacuum, Comm. Math. Phys.* **95** (1984) 217-226.
15. S. Kaniel and M. Shinbrot, *The Boltzmann equation I : uniqueness and local existence, Comm. Math. Phys.***58** (1978) 65-84.
16. A. Krasnoselskii, **Positive solutions of operator equations** (P. Noordhoff Ltd., 1954).
17. J. Lindenstrauss and L. Tzafriri, **Classical Banach spaces II, function spaces** (Springer, 1979).
18. P. L. Lions, *Compactness in Boltzmann equation via Fourier integral operators and applications. Part I, II and III, J. Math. Kyoto Univ.* **34**, 2 (1994) 391-461, and *J. Math. Kyoto Univ.* **34**, 3 (1994) 539-584.
19. P. L. Lions, *Global solutions of kinetic models and related questions*, in **Nonequilibrium problems in many-particle systems, Montecatini 1992**, eds. C. Cercignani and M. Pulvirenti, Lect. Notes in Math. (Springer, 1993) vol. 1551, pp. 58-86.
20. S. Mischler and B. Perthame, *Boltzmann equation with infinite energy : renormalized solutions, distributional solutions for small initial data and initial data close to a Maxwellian*, preprint (1995).
21. M. Reed and B. Simon, **Methods of modern mathematical physics**, Vol. 2 (Academic Press, 1970).
22. L. Tartar, *Topics in nonlinear analysis*, Publications de l'Université d'Orsay (1978).
23. L. Tartar, *Oscillations and asymptotic behaviour for two semi-linear hyperbolic systems*, in **Dynamics of infinite dimensional systems**, NATO ASI Series, Vol. F37, eds. F. N. Chow and F. K. Hale, (Springer Verlag, 1987), pp. 341-356.
24. G. Toscani, *Global solution of the initial value problem for the Boltzmann equation near a local Maxwellian, Arch. Rat. Mech. Anal.* **102** (1988) 231-241.