On a Generalized Boltzmann Equation for Non-Classical Particle Transport*

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Abstract. We are interested in non-standard transport equations where the description of the scattering events involves an additional "memory variable". We establish the well posedness and investigate the diffusion asymptotics of such models. While the questions we address are quite classical the analysis is original since the usual dissipative properties of collisional transport equations is broken by the introduction of the memory terms.

Key words. Transport equations. Diffusion asymptotics. Radiative transfer. Neutron transport.

AMS Subject classification. 35Q99, 35B25, 82C70

1 Introduction

This paper is devoted to the analysis of the following non-classical transport equation:

$$\begin{aligned} &\partial_s \Psi(s,x,v) + v \cdot \nabla_x \Psi(s,x,v) + \Sigma(s)\Psi(s,x,v) = 0 \\ &\Psi(0,x,v) = c\int_0^\infty \int \sigma(v,v_\star)\Sigma(s)\Psi(s,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s + Q(x,v), \end{aligned} \tag{1.1}$$

where $s \in (0,\infty)$, $x \in \mathbb{R}^N$, $v \in S^{N-1}$ the unit sphere in $\mathbb{R}^N$. It has first appeared in the literature in a paper by Larsen [8] as a model for photon transport in atmospheric clouds. Subsequently, it has been shown that this model accurately describes experimental data for neutron transport in pebble-bed reactors [13]. Independently, a model of similar type has been proposed for production systems [11].

In this equation, $\Psi(s,x,v)$ is the angular particle flux at point x into direction v. What makes this equation a non-classical generalization of linear transport is the additional variable s, which (in

*This work has been done during the semester "Quantum and Kinetic Transport: Analysis, Computations, and New Applications", held at IPAM, a NSF Math Institute at UCLA. We thank the organizers for the invitation and the staff of IPAM for their care.

the nomenclature of [8]) is the path length traveled by the particle since its previous interaction, or, equivalently, it can be thought of as the time elapsed since the previous interaction. The scattering frequency Σ depends explicitly on s. The source term Q models the particles that are injected into the system. The measure on S^{N-1} satisfies

$$\int \mathrm{d}v = 1, \qquad \int v \, \mathrm{d}v = 0. \tag{1.2}$$

(For instance $\mathrm{d}v$ is the normalized Lebesgue measure on S^{N-1}.)

Let us consider a particle at point x that travels over a distance $\mathrm{d}s$. In classical particle transport, the probability that this particle interacts with the background medium is proportional to $\mathrm{d}s$, and the proportionality constant depends on the density of the medium and on the particle's energy. This typically leads to an exponential attenuation law, i.e. the particle flux decreases as an exponential function of the path length (Beer-Lambert law). In cloud physics, however, recent experimental studies point toward a non-exponential attenuation law [10, 9, 3]. It has been suggested that spatial correlations between the water droplets (scattering centers) within a cloud could be responsible for this behavior [5]. This hypothesis has sparked a vivid discussion in the recent literature [1, 6, 12, 2].

Here, we briefly summarize the derivation of the model equation (1.1). For more details on the derivation of this equation we refer the reader to [8]. In a correlated medium, the probability of a scattering event is no longer independent of the previous scattering event, thus the proportionality constant depends on the distance to the last interaction. If there is no interaction, the particles move into direction v, and the distance to the last interaction increases. Due to scattering, there is a loss of particles proportional to the number of particles $\Psi(s, x, v)$ with proportionality constant $\Sigma(s)$. Conversely, there is also a gain of particles at x that are moving into direction v, due to scattering and due to particles being injected into the system. These particles have zero distance to the last scattering event. In terms of linear transport, these processes can be expressed in the balance equation

$$\begin{aligned}\partial_s \Psi(s, x, v) &+ v \cdot \nabla_x \Psi(s, x, v) + \Sigma(s)\Psi(s, x, v) \\ &= c\, \delta(s = 0) \int_0^\infty \int \sigma(v, v_\star)\Sigma(s_\star)\Psi(s_\star, x, v_\star) \, \mathrm{d}v_\star \, \mathrm{d}s_\star + \delta(s = 0)Q(x, v),\end{aligned} \tag{1.3}$$

where $0 < c < 1$ is the probability that a particle is scattered when it experiences a collision. It is easy to see that (1.3) and (1.1) are equivalent. In addition, for Σ independent of s, (1.1) reduces to standard linear transport.

We specify the following mathematical setting:

Hypothesis 1.1

(a) The source term $Q(x, v) \geq 0$ satisfies $Q \in L^1(\mathbb{R}^N \times S^{N-1})$.

(b) The scattering kernel $\sigma(v, v_\star)$ is positive

$$\sigma(v_\star, v) \geq \underline{\sigma} > 0 \tag{1.4}$$

and normalized

$$\int \sigma(v_\star, v) \, \mathrm{d}v_\star = 1, \qquad \int \sigma(v, v_\star) \, \mathrm{d}v_\star = 1. \tag{1.5}$$

(c) The scattering frequency Σ is positive, and satisfies

$$\int_0^\infty \Sigma(s)\,\mathrm{d}s = +\infty.$$

From c), a probability density p for the distance between collisions can be defined. Indeed, we set

$$p(s) = \Sigma(s)\exp\left(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\right). \tag{1.6}$$

and thus we get

$$\int_0^\infty p(s)\,\mathrm{d}s = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\right]\,\mathrm{d}s = 1.$$

From now on we use the following notation

$$\langle\phi\rangle = \int_0^\infty \phi(s)\ p(s)\,\mathrm{d}s$$

and we note that

$$\langle s\rangle = \int_0^\infty s\ p(s)\,\mathrm{d}s = -\int_0^\infty s\frac{\mathrm{d}}{\mathrm{d}s}\left[\exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\right]\,\mathrm{d}s = \int_0^\infty \exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\,\mathrm{d}s. \tag{1.7}$$

As a special case which is physically relevant, we can consider an isotropic medium, where

$$\begin{aligned}&\mathrm{d}v \text{ is the normalized Lebesgue measure on } S^{N-1},\\ &\text{and } \sigma(v,v_\star) = \sigma_0(v\cdot v_\star).\end{aligned} \tag{1.8}$$

In this case, we can define the mean scattering cosine

$$\bar{\mu}_0\ v = \int v_\star\ \sigma_0(v\cdot v_\star)\,\mathrm{d}v_\star,$$

where $\bar{\mu}_0$ is a constant independent of v. If σ_0 is constant, then $\bar{\mu}_0 = 0$. More generally, we always have $|\bar{\mu}_0| < 1$ since for any fixed $v \in S^{N-1}$, we have $|v\cdot v_\star| < 1$ a. e. on S^{N-1}, hence $|\bar{\mu}_0| = |v\cdot\int v_\star\sigma_0(v\cdot v_\star)\,\mathrm{d}v_\star| \leq \int |v\cdot v_\star|\sigma_0(v\cdot v_\star)\,\mathrm{d}v_\star < \int \sigma_0(v\cdot v_\star)\,\mathrm{d}v_\star = 1$.

Our first aim in this paper is to provide existence and uniqueness results for (1.1). This will be done in Section 2 by using a natural iterative process. Larsen [8] performed an asymptotic analysis and obtained a generalized diffusion equation. In Section 3, we review this formal analysis and prove convergence of the solution to the scaled equation to the solution of the diffusion equation.

2 Existence of solutions

Theorem 2.1 *Under the assumptions above, there exists a unique non negative solution $\Psi(s,x,v) \geq 0$ of (1.1) satisfying $\Psi \in L^\infty(0,\infty;L^1(\mathbb{R}^N\times S^{N-1}))$, $\Sigma\Psi \in L^1((0,\infty)\times\mathbb{R}^N\times S^{N-1})$.*

Proof. The existence of solutions is obtained through the following iterative process. We set $\Psi^{(0)} = 0$ and, knowing $\Psi^{(n)}$, we define $\Psi^{(n+1)}$ as the solution of

$$\begin{aligned}
&\partial_s \Psi^{(n+1)} + v \cdot \nabla_x \Psi^{(n+1)} + \Sigma(s)\Psi^{(n+1)} = 0 \\
&\Psi^{(n+1)}(0,x,v) = c\int_0^\infty \int \sigma(v,v_\star)\Sigma(s)\Psi^{(n)}(s,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s + Q(x,v).
\end{aligned} \tag{2.9}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[\Psi(s+t,x+tv,v)\exp\Big(\int_0^t \Sigma(s+\tau)\,\mathrm{d}\tau\Big)\Big] = 0,$$

we get

$$\Psi^{(n+1)}(s,x,v) = \exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\Big[c\int_0^\infty \int \sigma(v,v_\star)\Sigma(s_\star)\Psi^{(n)}(s_\star,x-sv,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star + Q(x-sv,v)\Big].$$

Therefore we observe that

$$\Psi^{(1)}(s,x,v) = \exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)Q(x-sv,v) \geq 0 = \Psi^{(0)}(s,x,v)$$

holds. Furthermore, we can check that

$$\int_0^\infty \int\int \Sigma(s)\Psi^{(1)}(s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s = \|Q\|_{L^1} < \infty.$$

Next, assuming $\Psi^{(n)}(s,x,v) \geq \Psi^{(n-1)}(s,x,v)$, we have $\Psi^{(n+1)}(s,x,v) - \Psi^{(n)}(s,x,v) \geq 0$. The sequence is non decreasing, and it remains to establish a uniform estimate.

To this end, we integrate (2.9) which yields

$$\begin{aligned}
&\int\int \Psi^{(n+1)}(s,x,v)\,\mathrm{d}v\,\mathrm{d}x + \int_0^s \int\int \Sigma(\tau)\ \Psi^{(n+1)}(\tau,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad = c\int\int\int_0^\infty \int \sigma(v,v_\star)\Sigma(s_\star)\Psi^{(n)}(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star\,\mathrm{d}v\,\mathrm{d}x + \int\int Q(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&\quad = c\int_0^\infty \int\int \Sigma(s_\star)\Psi^{(n)}(s_\star,x,v_\star)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s_\star + \int\int Q(x,v)\,\mathrm{d}v\,\mathrm{d}x
\end{aligned} \tag{2.10}$$

by using (1.5). As a first consequence we deduce that

$$\begin{aligned}
\int_0^\infty \int\int \Sigma(\tau)\ \Psi^{(n+1)}(\tau,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau &= \lim_{s\to\infty}\int_0^s \int\int \Sigma(\tau)\ \Psi^{(n+1)}(\tau,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \\
&\leq c\int_0^\infty \int \Sigma(s_\star)\Psi^{(n)}(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star + \int\int Q(x,v)\,\mathrm{d}v\,\mathrm{d}x
\end{aligned}$$

holds. Since $0 \leq \Psi^{(1)} \leq ... \leq \Psi^{(n)} \leq \Psi^{(n+1)}...$ and $0 < c < 1$ we arrive at

$$\int_0^\infty \int\int \Sigma(\tau)\ \Psi^{(n)}(\tau,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau \leq \frac{1}{1-c}\int\int Q(x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

Next, we go back to (2.10) which leads to

$$\sup_{s\geq 0}\int\int \Psi^{(n+1)}(s,x,v)\,\mathrm{d}v\,\mathrm{d}x \leq \frac{1}{1-c}\int\int Q(x,v)\,\mathrm{d}v\,\mathrm{d}x.$$

We show uniqueness by considering the case $Q = 0$. Integration along the characteristics yields

$$\Psi(s,x,v) = \exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\left[c\int_0^\infty \int \sigma(v,v_\star)\Sigma(s)\Psi(s,x-sv,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s\right].$$

Hence, we get

$$\begin{aligned}
\int\int |\Psi(s,x,v)|\,\mathrm{d}v\,\mathrm{d}x &\le c\exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big) \\
&\quad\times \int_0^\infty \int\int\int \sigma(v,v_\star)\Sigma(s_\star)\ |\Psi(s_\star,x-sv,v_\star)|\,\mathrm{d}v_\star\,\mathrm{d}s_\star\,\mathrm{d}v\,\mathrm{d}x \\
&\le c\exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\int_0^\infty \int\int\int \sigma(w,v_\star)\Sigma(s_\star)\ |\Psi(s_\star,y,v_\star)|\,\mathrm{d}v_\star\,\mathrm{d}s_\star\,\mathrm{d}w\,\mathrm{d}y \\
&\le c\exp\Big(-\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\Big)\int_0^\infty \int\int \Sigma(s_\star)\ |\Psi(s_\star,y,v_\star)|\,\mathrm{d}v_\star\,\mathrm{d}s_\star\,\mathrm{d}y.
\end{aligned}$$

Since p is a probability density we obtain

$$\int_0^\infty \int\int \Sigma(s)|\Psi(s,x,v)|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \le c\int_0^\infty \int\int \Sigma(s)|\Psi(s,x,v)|\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.$$

Therefore, $0 < c < 1$ implies that $\Psi(s,x,v) = 0$ a.e. ∎

Remark 2.1 *The iteration* (2.9) *has the natural physical interpretation that $\Psi^{(n)}$ consists of the particles that have been scattered at most n times. It treats s as a pseudo-time and the particles $\psi^{(n)}$ from the n-th generation as an initial condition for the particles of the next generation.*

3 Diffusion asymptotics

3.1 Scaling issues

Let $0 < \varepsilon \ll 1$ be a parameter intended to tend to 0. We assume that in (1.1) the parameters scale as follows

$$Q \to \varepsilon Q, \qquad \Sigma(s) \to \frac{1}{\varepsilon}\Sigma(s/\varepsilon), \qquad c \to 1-\kappa\varepsilon^2.$$

where $\kappa > 0$. This means that the system is optically thick and dominated by scattering that is not forward-peaked, see [7]. We have

$$\begin{aligned}
&\partial_s\Psi + v\cdot\nabla_x\Psi + \frac{1}{\varepsilon}\Sigma(s/\varepsilon)\Psi = 0 \\
&\Psi(0,x,v) = (1-\kappa\varepsilon^2)\int_0^\infty \int \sigma(v,v_\star)\frac{1}{\varepsilon}\Sigma(s/\varepsilon)\Psi(s,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s + \varepsilon Q(x,v).
\end{aligned}$$

Next we change the pseudo-time scale by setting

$$\Psi_\varepsilon(s,x,v) = \Psi(\varepsilon s,x,v).$$

The new unknown satisfies

$$\begin{aligned}
&\partial_s\Psi_\varepsilon + \varepsilon v\cdot\nabla_x\Psi_\varepsilon + \Sigma(s)\Psi_\varepsilon = 0 \\
&\Psi_\varepsilon(0,x,v) = (1-\kappa\varepsilon^2)\int_0^\infty \int \sigma(v,v_\star)\ \Sigma(s)\Psi_\varepsilon(s,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s + \varepsilon Q(x,v).
\end{aligned}$$

As a matter of fact, the estimates obtained in the proof of Theorem 2.1 recasts in this context as follows

$$\int_0^\infty \int \int \Sigma(s)\Psi_\varepsilon(s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \le \frac{C}{\varepsilon}, \qquad \int \int \Psi_\varepsilon(s,x,v)\,\mathrm{d}v\,\mathrm{d}x \le \frac{C}{\varepsilon}$$

where C depends on κ and the L^1 norm of Q, supposed to be finite.

Then, we introduce a new unknown

$$\Phi_\varepsilon(s,x,v) = \varepsilon\ \langle s\rangle\ \exp\left(\int_0^s \Sigma(\tau)\,\mathrm{d}\tau\right)\ \Psi_\varepsilon(s,x,v). \tag{3.11}$$

This quantity satisfies the following equation

$$\begin{aligned} &\partial_s\Phi_\varepsilon + \varepsilon v\cdot\nabla_x\Phi_\varepsilon = 0\\ &\Phi_\varepsilon(0,x,v) = (1-\kappa\varepsilon^2)\int_0^\infty\int \sigma(v,v_\star)\ p(s)\Phi_\varepsilon(s,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s + \varepsilon^2\langle s\rangle Q(x,v). \end{aligned} \tag{3.12}$$

We wish to investigate the behavior as $\varepsilon\to 0$; we shall see that the asymptotics leads to a macroscopic diffusion equation

$$-\nabla_x\cdot(D\nabla_x\Phi) + \kappa\Phi = \langle s\rangle \int Q\,\mathrm{d}v$$

and we obtain that

$$\Phi_\varepsilon(s,x,v)\to\Phi(x).$$

The (non-negative) matrix D depends on the coefficients p, σ and Σ of the original equation (3.12). According to [8], we can come back to the original unknown Ψ and, using (3.11) and (1.7), the result can be interpreted as

$$\int_0^\infty\int \Psi(s,x,v)\,\mathrm{d}v\,\mathrm{d}s \simeq \int_0^\infty\int \frac{1}{\langle s\rangle}\exp\Big(-\int_0^s\Sigma(\tau)\,\mathrm{d}\tau\Big)\Phi(x)\,\mathrm{d}v\,\mathrm{d}s = \Phi(x).$$

Thus Φ is the classical radiation flux up to first order corrections.

3.2 Formal asymptotics

It is convenient to define the following operators: The standard scattering operator $\mathcal{Q} : L^2(S^{N-1}) \to L^2(S^{N-1})$ is given by

$$(\mathcal{Q}F)(v) = \int \sigma(v,v_\star)F(v_\star)\,\mathrm{d}v_\star - F(v).$$

We also define a second operator

$$(\mathscr{Q}f)(s,v) = \int_0^\infty\int p(s_\star)\sigma(v,v_\star)f(s_\star,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star - f(s,v).$$

To define this operator properly, we define the space $L^2_{p(s)}((0,\infty)\times S^{N-1})$, where the subscript $p(s)$ means that the s-integral is weighted with the probability density $p(s)$. We summarize the properties that will be needed as follows (see e. g. [4]):

Proposition 3.1 *Under hypothesis 1.1, the following properties hold*

(a) *The operator $\mathcal{Q}$ is a bounded operator on $L^2(S^{N-1})$ which satisfies*

$$\begin{aligned} -\int \mathcal{Q}(F)F\,\mathrm{d}v &= \frac{1}{2}\int\int \sigma(v,v_\star)\big(F(v)-F(v\star)\big)^2\,\mathrm{d}v_\star\,\mathrm{d}v \\ &\geq \underline{\sigma}\int \left|F(v)-\int F(v_\star)\,\mathrm{d}v_\star\right|^2\mathrm{d}v \geq 0 \end{aligned}$$

Its kernel consists of the functions that are independent of v. Its range consists of all functions F such that $\int F(v)\,\mathrm{d}v = 0$. Thus, $\mathcal{Q}$ and its adjoint $\mathcal{Q}^\star$, given by

$$(\mathcal{Q}^\star F)(v) = \int \sigma(v_\star, v)F(v_\star)\,\mathrm{d}v_\star - F(v),$$

satisfy a Fredholm alternative: Given any $g \in L^2(S^{N-1})$ satisfying the compatibility condition $\int G(v)\,\mathrm{d}v = 0$, there exists a unique F with $\int F(v)\,\mathrm{d}v = 0$ such that $\mathcal{Q}F = G$ (resp. $\mathcal{Q}^\star F = G$).

(b) *Similarly, the operator $\mathscr{Q}$ is a bounded operator on $L^2_{p(s)}((0,\infty)\times S^{N-1})$ which satisfies*

$$\begin{aligned} -\int \mathscr{Q}(f)f\,\mathrm{d}v\; p(s)\,\mathrm{d}s &= \int_0^\infty\int_0^\infty\int\int \sigma(v,v_\star)p(s)p(s_\star)\;\big(f(s,v)-f(s_\star,v_\star)\big)^2\,\mathrm{d}v_\star\,\mathrm{d}v\,\mathrm{d}s_\star\,\mathrm{d}s \\ &\geq \underline{\sigma}\int_0^\infty\int \left|f(s,v)-\int_0^\infty\int f(s_\star,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star\right|^2\mathrm{d}v\,\mathrm{d}s \geq 0. \end{aligned}$$

Its kernel consists of the functions that are independent of v and s. Its range consists of all functions f such that $\int_0^\infty\int p(s)f(s,v)\,\mathrm{d}v\,\mathrm{d}s = 0$. Thus, $\mathscr{Q}$ and its adjoint $\mathscr{Q}^\star$ with respect to the weight $p(s)$, given by

$$(\mathscr{Q}^\star f)(s,v) = \int_0^\infty\int \sigma(v_\star,v)p(s_\star)f(s_\star,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star - f(s,v)$$

satisfy a Fredholm alternative: Given any $g \in L^2_{p(s)}((0,\infty)\times S^{N-1})$ satisfying the compatibility condition $\int_0^\infty\int p(s)g(s,v)\,\mathrm{d}v\,\mathrm{d}s = 0$, there exists a unique $f \in L^2_{p(s)}((0,\infty)\times S^{N-1})$ with $\int_0^\infty\int p(s)f(s,v)\,\mathrm{d}v\,\mathrm{d}s = 0$ such that $\mathscr{Q}f = g$ (resp. $\mathscr{Q}^\star f = g$)

We insert into (3.12) the following Hilbert expansion

$$\Phi_\varepsilon(s,x,v) = \Phi_0(s,x,v) + \varepsilon\Phi_1(s,x,v) + \varepsilon^2\Phi_2(s,x,v) + ...$$

Identifying terms arising with the same power of ε we obtain:

- At leading order $\partial_s\Phi_0 = 0$ shows that Φ_0 does not depend on s. But, then, the condition at $s = 0$ reduces to $\mathcal{Q}(\Phi_0) = 0$ which implies that $\Phi_0 = \Phi_0(x)$ does not depend on v anymore.

- Next, we get $\partial_s\Phi_1 = -v\cdot\nabla_x\Phi_0$, that is $\Phi_1(s,x,v) = \Phi_1(0,x,v) - sv\cdot\nabla_x\Phi_0(x)$ with the boundary condition

  $$\Phi_1(0,x,v) = \int_0^\infty\int p(s_\star)\sigma(v,v_\star)\Phi_1(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star.$$

  In other words, we have

  $$\mathscr{Q}\Phi_1 = sv\cdot\nabla_x\Phi_0(x).$$

  By using the Fredholm alternative and (1.2), we can write

  $$\Phi_1(s,x,v) = \chi(s,v)\cdot\nabla_x\Phi_0(x) + \tilde{\Phi}_1(x),$$

  where $\chi = (\chi_1, ..., \chi_N)$ solves

  $$(\mathscr{Q}\chi)(s,v) = sv.$$

- Finally, we obtain a closed equation for Φ_0 by looking at the equation for Φ_2, that is $\partial_s \Phi_2 = -v \cdot \nabla_x \Phi_1$, together with

$$\begin{aligned}\Phi_2(0,x,v) &= \int_0^\infty \int p(s_\star)\sigma(v,v_\star)\Phi_2(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star \\ &\quad -\kappa \int_0^\infty \int p(s_\star)\sigma(v,v_\star)\Phi_0(x)\,\mathrm{d}v_\star\,\mathrm{d}s_\star + \langle s\rangle Q(x,v) \\ &= \int_0^\infty \int p(s_\star)\sigma(v,v_\star)\Phi_2(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star - \kappa\Phi_0(x) + \langle s\rangle Q(x,v).\end{aligned}$$

The equation can be recast as

$$\mathscr{Q}(\Phi_2) = \int_0^s v\cdot\nabla_x\Phi_1(\tau,x,v)\,\mathrm{d}\tau + \kappa\Phi_0(x) - \langle s\rangle Q(x,v).$$

Using the compatibility condition and the expression derived for Φ_1, we end up with

$$-\nabla_x\cdot(D\nabla_x\Phi_0) + \kappa\Phi_0 = \langle s\rangle \int Q\,\mathrm{d}v \tag{3.13}$$

with the diffusion matrix

$$D = -\int_0^\infty \int p(s)\left(\int_0^s v\otimes\chi(\tau,v)\,\mathrm{d}\tau\right)\mathrm{d}v\,\mathrm{d}s. \tag{3.14}$$

In the case of an isotropic medium (1.8), the auxiliary function can be written as

$$\chi(s,v) = -\left(s + \frac{\bar\mu_0}{1-\bar\mu_0}\langle s\rangle\right)v.$$

Therefore, the diffusion coefficient is

$$D = \frac{1}{N}\left(\frac{\langle s^2\rangle}{2} + \frac{\bar\mu_0}{1-\bar\mu_0}\langle s\rangle^2\right).$$

It is clearly positive since $\bar\mu_0 \in (-1,+1)$. In the general context, we have

Lemma 3.1 *The diffusion matrix satisfies, for any $\xi\in\mathbb{R}^N$*

$$D\xi\cdot\xi = -\int_0^\infty \int p(s)\left(\int_0^s v\cdot\xi\ \chi(\tau,v)\cdot\xi\,\mathrm{d}\tau\right)\mathrm{d}v\,\mathrm{d}s \ge 0.$$

If the matrix $\int v\otimes v\,\mathrm{d}v$ is positive definite then $D\xi\cdot\xi > 0$ for any $\xi\in\mathbb{R}^N\setminus\{0\}$.

Proof. We observe that

$$\begin{aligned}D\xi\cdot\xi &= \int_0^\infty \int p(s)\left(\int_0^s \partial_\tau\big(\chi(\tau,v)\cdot\xi\big)\ \chi(\tau,v)\cdot\xi\,\mathrm{d}\tau\right)\mathrm{d}v\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^\infty \int p(s)\left(\int_0^s \partial_\tau\big|\chi(\tau,v)\cdot\xi\big|^2\,\mathrm{d}\tau\right)\mathrm{d}v\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^\infty \int p(s)\Big(\big|\chi(s,v)\cdot\xi\big|^2 - \big|\chi(0,v)\cdot\xi\big|^2\Big)\mathrm{d}v\,\mathrm{d}s \\ &= \frac{1}{2}\int_0^\infty \int p(s)\Big(\big|\chi(s,v)\cdot\xi\big|^2 \\ &\quad -\Big|\int_0^\infty \int p(s_\star)\sigma(v,v_\star)\chi(s_\star,v_\star)\cdot\xi\,\mathrm{d}v_\star\,\mathrm{d}s_\star\Big|^2\Big)\mathrm{d}v\,\mathrm{d}s.\end{aligned}$$

However, the Cauchy-Schwarz inequality yields, for a. e. $v \in S^{N-1}$,

$$\begin{aligned}
&\Big|\int_0^\infty \int p(s_\star)\sigma(v,v_\star)\chi(s_\star,v_\star)\cdot\xi \,\mathrm{d}v_\star\,\mathrm{d}s_\star\Big|^2 \\
&\quad\le \int_0^\infty \int p(s_\star)\sigma(v,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star \times \int_0^\infty \int p(s_\star)\sigma(v,v_\star)\big|\chi(s_\star,v_\star)\cdot\xi\big|^2\,\mathrm{d}v_\star\,\mathrm{d}s_\star \\
&\quad= \int_0^\infty \int p(s_\star)\sigma(v,v_\star)\big|\chi(s_\star,v_\star)\cdot\xi\big|^2\,\mathrm{d}v_\star\,\mathrm{d}s_\star
\end{aligned}$$

by using (1.5) and the fact that p is a probability density. Therefore, we get

$$\begin{aligned}
&\int_0^\infty \int p(s)\Big|\int_0^\infty \int p(s_\star)\sigma(v,v_\star)\chi(s_\star,v_\star)\cdot\xi\,\mathrm{d}v_\star\,\mathrm{d}s_\star\Big|^2\,\mathrm{d}v\,\mathrm{d}s \\
&\quad\le \int_0^\infty \int p(s_\star)\big|\chi(s_\star,v_\star)\cdot\xi\big|^2\,\mathrm{d}v_\star\,\mathrm{d}s_\star
\end{aligned}$$

by using (1.5) again. This shows that $D\xi\cdot\xi$ is non negative. Assuming that it vanishes means that $\chi(s,v)\cdot\xi$ is constant in s which in turn implies that $Q\chi\cdot\xi = 0 = v\cdot\xi$. If the matrix $\int v\otimes v\,\mathrm{d}v$ is non degenerate, this is possible only for $\xi = 0$. ∎

Remark 3.1 *There is another way to think of these equations. Indeed, let us set*

$$G_1(x,v) = \int_0^\infty p(s)\Phi_1(s,x,v)\,\mathrm{d}s.$$

It satisfies

$$\mathcal{Q}G_1(x,v) = \langle s\rangle\ v\cdot\nabla_x\Phi_0(x).$$

Therefore, with $\bar\chi(v)$ solution of $(\mathcal{Q}\bar\chi)(v) = v$ we get

$$G_1(x,v) = \langle s\rangle\bar\chi(v)\cdot\nabla_x\Phi_0 + \tilde G_1(x).$$

Then, we remark that

$$\Phi_1(0,x,v) = \int \sigma(v,v_\star)G_1(x,v_\star)\,\mathrm{d}v_\star,$$

and thus

$$\Phi_1(s,x,v) = \langle s\rangle \int \sigma(v,v_\star)\bar\chi(x,v_\star)\cdot\nabla_x\Phi_0(x)\,\mathrm{d}v_\star - sv\cdot\nabla_x\Phi_0(x).$$

It follows that

$$\int_0^\infty p(s)\left(\int_0^s v\cdot\nabla_x\Phi_1(s_\star,x,v)\,\mathrm{d}s_\star\right)\mathrm{d}s = \langle s\rangle^2\nabla_x\left[\left(\int \sigma(v,v_\star)v\otimes\bar\chi(v_\star)\,\mathrm{d}v_\star\right)\nabla_x\Phi_0\right] - \frac{\langle s^2\rangle}{2}(v\cdot\nabla_x)^2\Phi_0$$

which yields another expression of the diffusion coefficient.

3.3 Analysis of the diffusion asymptotics

In this Section, we shall prove the following statement.

Theorem 3.1 *Let* (1.2) *and Hypothesis 1.1 be fulfilled. Let us assume*

$$Q \in L^\infty(\mathbb{R}^N \times S^{N-1}) \tag{3.15}$$

Then, the solutions Φ_ε *of* (3.12) *converge weakly-*$\star$ *in* L^∞ *to* $\Phi(x)$*, solution of* (3.13), (3.14).

The proof starts with the derivation of uniform estimates.

Lemma 3.2 *Let the assumptions of Theorem 3.1 be fulfilled. There exists a constant* $C > 0$*, depending on* Q *and* κ *such that*

$$0 \leq \Phi_\varepsilon(s, x, v) \leq C < \infty$$

holds. Furthermore we have $\sup_{0\leq s\leq S} \|\Phi(s, \cdot)\|_{L^1} \leq M(S)$ *with a constant depending on* $0 < S < \infty$.

Proof. The L^1 estimate is a direct consequence of the estimates obtained in Theorem 2.1 and the definition of Φ_ε. For proving the L^∞ estimate we observe that

$$\begin{aligned} \Phi_\varepsilon(s, x, v) &= \Phi_\varepsilon(0, x - \varepsilon s v, v) \\ &= (1 - \kappa\varepsilon^2) \int_0^\infty \int \sigma(v, v_\star) p(s_\star) \Phi_\varepsilon(s_\star, x - \varepsilon s v, v_\star) \,\mathrm{d}s_\star \,\mathrm{d}v_\star + \varepsilon^2 \langle s\rangle Q(x - \varepsilon s v, v) \\ &= (1 - \kappa\varepsilon^2)\|\Phi_\varepsilon\|_\infty + \varepsilon^2 \langle s\rangle \|Q\|_\infty. \end{aligned}$$

We deduce that

$$\|\Phi_\varepsilon\|_\infty \leq \frac{\langle s\rangle \|Q\|_\infty}{\kappa}$$

holds. ∎

Then, we make rigorous the first guessed property deduced from the formal ansatz.

Lemma 3.3 *We can assume that* $\Phi_\varepsilon(0, x, v) \rightharpoonup \Phi(x, v)$ *weakly-*$\star$ *in* L^∞*. Then, for any test function* $\zeta \in C_c^\infty$*, we have*

$$\int\int \Phi_\varepsilon(s, x, v)\zeta(x, v) \,\mathrm{d}v \,\mathrm{d}x \xrightarrow[\varepsilon\to 0]{} \int\int \Phi(x, v)\zeta(x, v) \,\mathrm{d}v \,\mathrm{d}x$$

uniformly on $[0, S]$ *for any* $0 < S < \infty$.

Proof. The proof follows form the identity

$$\int\int \big(\Phi_\varepsilon(s, x, v) - \Phi_\varepsilon(0, x, v)\big)\zeta(x, v) \,\mathrm{d}v \,\mathrm{d}x = \varepsilon \int_0^s \int\int \Phi_\varepsilon(\tau, x, v)\ v \cdot \nabla_x\zeta(x, v) \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}\tau \tag{3.16}$$

where the right hand side is dominated by $\varepsilon\ S\ \|\Phi_\varepsilon\|_\infty \|v \cdot \nabla_x\zeta\|_{L^1}$.

Remark that this convergence property allows us to show that

$$\int_0^\infty \int\int \Phi_\varepsilon(s, x, v)\zeta(s, x, v) \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}s \xrightarrow[\varepsilon\to 0]{} \int_0^\infty \int\int \Phi(x, v)\zeta(s, x, v) \,\mathrm{d}v \,\mathrm{d}x \,\mathrm{d}s$$

holds for any trial function $\zeta \in L^1$. Indeed, we split the integral over $(0, \infty)$ into integrals over $(0, S)$ and (S, ∞). The latter can be made as small as we wish, uniformly with respect to ε, for a large enough S

by using the integrability of ζ and the L^∞ bound in Lemma 3.2, the former passes to the limit owing to Lemma 3.3. ∎

We can also write (3.16) by using the boundary condition; we get

$$\begin{aligned}
&\int\int \Phi_\varepsilon(s,x,v)\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&= (1-\kappa\varepsilon^2)\int_0^\infty\int\int\int \sigma(v,v_\star)p(s_\star)\Phi_\varepsilon(s_\star,x,v_\star)\zeta(x,v)\,\mathrm{d}v_\star\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s_\star \\
&\quad +\varepsilon\int_0^s\int\int \Phi_\varepsilon(\tau,x,v)\ v\cdot\nabla_x\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon^2\langle s\rangle\int\int Q\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x \\
&= (1-\kappa\varepsilon^2)\int_0^\infty\int\int \Phi_\varepsilon(s_\star,x,v_\star)\ p(s_\star)\left(\int\sigma(v,v_\star)\zeta(x,v)\,\mathrm{d}v\right)\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}s_\star \\
&\quad +\varepsilon\int_0^s\int\int \Phi_\varepsilon(\tau,x,v)\ v\cdot\nabla_x\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}\tau + \varepsilon^2\langle s\rangle\int\int Q\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned} \tag{3.17}$$

Using Lemma 3.3, we let ε go to 0 and we obtain

$$\begin{aligned}
&\int\int \Phi(x,v)\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int_0^\infty\int\int \Phi(x,v_\star)\ p(s_\star)\left(\int\sigma(v,v_\star)\zeta(x,v)\,\mathrm{d}v\right)\mathrm{d}v_\star\,\mathrm{d}x\,\mathrm{d}s_\star = 0 \\
&= \int\int \Phi(x,v)\zeta(x,v)\,\mathrm{d}v\,\mathrm{d}x - \int\int \Phi(x,v_\star)\left(\int_0^\infty p(s_\star)\,\mathrm{d}s_\star\times\int\sigma(v,v_\star)\zeta(x,v)\,\mathrm{d}v\right)\mathrm{d}v_\star\,\mathrm{d}x = 0
\end{aligned}$$

It recasts as

$$\int\int \Phi(x,v)\mathscr{Q}^\star(\zeta)(x,v)\,\mathrm{d}v\,\mathrm{d}x = 0.$$

Since it holds for any trial function, we conclude that Φ belongs to the kernel of Q which means that it does not depend on the microscopic variable v.

We obtain the equation satisfied by Φ by multiplying by a test function $\zeta(s,x,v)$ depending also on the s variable. After integration with respect to s we get from (3.17)

$$\begin{aligned}
&\int_0^\infty\int\int \Phi_\varepsilon(s,x,v)\zeta(s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&= (1-\kappa\varepsilon^2)\int_0^\infty\int\int\int_0^\infty\int \Phi_\varepsilon(s,x,v)p(s)\sigma(v_\star,v)\zeta(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad +\varepsilon\int_0^\infty\int\int\int_s^\infty \Phi_\varepsilon(s,x,v)v\cdot\nabla_x\zeta(s_\star,x,v)\,\mathrm{d}s_\star\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\quad +\varepsilon^2\int\int\int \langle s\rangle Q(x,v)\zeta(s,x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{3.18}$$

Now we choose a specific form of the test function; namely we consider

$$\zeta(s,x,v) = p(s)\big(\zeta(x)+\varepsilon\widetilde{\zeta}(s,x,v)\big)$$

where $\widetilde{\zeta}$ will be constructed in a suitable way, depending on the leading term. Since ζ depends only on x the $\mathscr{O}(1)$ term vanishes. Then, the $\mathscr{O}(\varepsilon)$ term reads

$$\begin{aligned}
&\int_0^\infty\int\int \Phi_\varepsilon(s,x,v) \\
&\qquad\times\Big(-v\cdot\nabla_x\zeta(x)\int_s^\infty p(\tau)\,\mathrm{d}\tau + p(s)\widetilde{\zeta}(s,x,v) - p(s)\int_0^\infty\int\sigma(v_\star,v)p(s_\star)\widetilde{\zeta}(s_\star,x,v_\star)\,\mathrm{d}v_\star\,\mathrm{d}s_\star\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&= \int_0^\infty\int\int \Phi_\varepsilon(s,x,v)\Big(-v\cdot\nabla_x\zeta(x)\int_s^\infty p(\tau)\,\mathrm{d}\tau - p(s)\mathscr{Q}^\star(\widetilde{\zeta})(s,x,v)\Big)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned}$$

We can make it vanish by choosing $\widetilde{\zeta}$ as the solution of

$$\mathscr{Q}^\star(\widetilde{\zeta}) = -v \cdot \nabla_x \zeta(x) \ \frac{\displaystyle\int_s^\infty p(\tau)\,\mathrm{d}\tau}{p(s)}$$

According to (1.2) and Proposition 3.1 we can define $\chi^\star(s,v)$ solution of

$$\mathscr{Q}^\star(\chi^\star) = -\frac{\displaystyle\int_s^\infty p(\tau)\,\mathrm{d}\tau}{p(s)}\ v$$

and we set

$$\widetilde{\zeta}(s,x,v) = \chi^\star(s,v)\cdot\nabla_x\zeta(x).$$

Then, we go back to (3.18). Dividing by ε^2 and letting ε go to 0 we are led to

$$\begin{aligned}
&\int_0^\infty \int\int p(s)\zeta(x)\langle s\rangle Q(x,v)\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&\qquad\qquad -\kappa\int_0^\infty\int\int p(s)\Phi(x)\left(\int_0^\infty\int p(s_\star)\sigma(v_\star,v)\zeta(x)\,\mathrm{d}v_\star\,\mathrm{d}s_\star\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&= \int\zeta(x)\left(\langle s\rangle\int Q(x,v)\,\mathrm{d}v - \kappa\Phi(x)\right)\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0}\int_0^\infty\int\int\Phi_\varepsilon(s,x,v)\left(\int_s^\infty p(\tau)v\cdot\nabla_x\widetilde{\zeta}(\tau,x,v)\,\mathrm{d}\tau\right)\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \\
&= -\int\Phi(x)\nabla_x\cdot\left(\int_0^\infty\int_s^\infty\int p(\tau)v\otimes\chi^\star(\tau,v)\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s\ \nabla_x\zeta(x)\right)\mathrm{d}x.
\end{aligned}$$

This is the weak form of the diffusion equation (3.13). Indeed, the matrix

$$\Gamma = \int_0^\infty\int_s^\infty\int p(\tau)v\otimes\chi^\star(\tau,v)\,\mathrm{d}v\,\mathrm{d}\tau\,\mathrm{d}s$$

can be rewritten as follows

$$\begin{aligned}
\Gamma &= \int_0^\infty\int_0^\tau\int p(\tau)v\otimes\chi^\star(\tau,v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\tau = \int_0^\infty\int\tau p(\tau)v\otimes\chi^\star(\tau,v)\,\mathrm{d}v\,\mathrm{d}\tau \\
&= \int_0^\infty\int p(\tau)\mathscr{Q}(\chi)\otimes\chi^\star(\tau,v)\,\mathrm{d}v\,\mathrm{d}\tau = \int_0^\infty\int p(\tau)\chi\otimes\mathscr{Q}^\star(\chi^\star)(\tau,v)\,\mathrm{d}v\,\mathrm{d}\tau \\
&= -\int_0^\infty\int\chi(\tau,v)\left(\int_\tau^\infty p(s)\,\mathrm{d}s\right)v\,\mathrm{d}v\,\mathrm{d}\tau = -\int_0^\infty\int p(s)\left(\int_0^s\chi(\tau,v)\otimes v\,\mathrm{d}\tau\right)\mathrm{d}v\,\mathrm{d}s \\
&= D^T,
\end{aligned}$$

with D defined by (3.14). ∎

Remark 3.2 *It is worth mentioning several extensions of this work, that can be of physical interest:*

- *We can consider cross section σ depending on the space variable as well. In such a case the auxiliary functions χ and $\chi^\star$ might depend on x, and thus the diffusion coefficient can be space-dependent too.*

- *The scaling we have adopted does not exactly fit the one considered in [8] but it is straightforward to extend the analysis to such a case. First the existence theory works by considering a cross section verifying*

$$\int \tilde{\sigma}(v_\star, v)\, \mathrm{d}v_\star \leq 1.$$

Second, in the definition of the scaling $c\tilde{\sigma}(v, v_\star)$ *is replaced by* $\sigma(v, v_\star) - \kappa\varepsilon^2$ *(instead of* $(1 - \kappa\varepsilon^2)\sigma(v, v_\star)$*), which still makes sense since* σ *is bounded from below by a positive constant.*

- *Finally, it is also relevant to deal with a probability* c *that depends on* s*, with* $0 \leq c(s) \leq \bar{c} < 1$*, a typical situation being to consider a non increasing function of* s*. The asymptotic analysis is thus performed by setting* $c(s) \to 1 - \varepsilon^2\kappa(s/\varepsilon)$ *and assuming* $0 < \underline{\kappa} \leq \kappa(s) \leq \bar{\kappa}$*.*

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