# A staggered scheme for the Euler equations.

Thierry Goudon, Julie Llobell and Sebastian Minjeaud

**Abstract** We extend to the full Euler system the scheme introduced in [Berthelin, Goudon, Minjeaud, Math. Comp. 2014] for solving the barotropic Euler equations. This finite volume scheme is defined on staggered grids with numerical fluxes derived in the spirit of kinetic schemes. The difficulty consists in finding a suitable treatment of the energy equation while density and internal energy on the one hand, and velocity on the other hand, are naturally defined on dual locations. The proposed scheme uses the density, the velocity and the internal energy as computational variables and stability conditions can be identified in order to preserve the positivity of the discrete density and internal energy. Moreover, the introduction of auxiliary variables allows us to define a consistent averaged total energy, which satisfies a *local* conservation equation. Finally, we provide numerical simulations of Riemann problems to illustrate the behaviour of the scheme.

**Key words:** Finite Volumes, Conservation laws, Staggered grids, Euler equations

**MSC** (2010)**:** 65M08, 76M12, 35L65, 35Q31

## 1 Introduction.

This work aims at designing a scheme to numerically solve the 1D-Euler system:

$$
\begin{aligned}
&\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho E u + p u \end{pmatrix} = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\
&E = u^2/2 + e, \quad p = (\gamma - 1)\rho e,
\end{aligned}
\tag{1}
$$

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where $\rho$, $u$, $E$ and $p$ stand for the density, the velocity, the total energy and the pressure, respectively, $u^2/2$ and $e$ are the kinetic and internal energies, and $\gamma > 1$ is the adiabatic exponent.

We wish to extend to (1) the scheme designed in [1] for the barotropic Euler equations. This scheme works on staggered grids – meaning that densities and velocities are not collocated – and this raises a difficulty for the Euler system as the definition of the total energy mixes quantities, namely the velocity and the internal energy, naturally defined on different grids. To address this issue, it is convenient to work with the internal energy, namely

$$\partial_t(\rho e) + \partial_x(\rho e u) = -p\partial_x u, \tag{2}$$

replaces the evolution equation for $\rho E$, since discrete densities, pressures, and internal energies are natural stored at the same locations. This formulation has also the advantage of making more direct the analysis of the positivity of $e$. Unfortunately, as it is well-known, this non conservative formulation is not equivalent to (1) when the solution presents discontinuities. We shall follow the approach discussed in [2] by introducing in (2) correction terms accounting for the kinetic energy balance. Then, the scheme introduced in [2] can be shown: a) to be consistent with (a weak form of) the total energy equation as the space step $\delta x$ goes to zero and b) to conserve the *global* discrete total energy. Our purpose is two-fold. First of all, we shall adapt the scheme of [1] for dealing with (1). Second of all, we shall introduce auxiliary variables, which allows us to define a suitable averaged total energy satisfying a *local* conservation equation. Finally we provide some numerical simulations in Section 4.

## 2 Staggered scheme and stability conditions.

Let $(x_j)_j$ be a subdivision of the 1D computational domain and denote by $\delta x_{j+\frac{1}{2}} = x_{j+1} - x_j$ the size of the cells. The cells centers, $x_{j+\frac{1}{2}} = (x_j + x_{j+1})/2$, define the dual mesh and we set $\delta x_j = (\delta x_{j-\frac{1}{2}} + \delta x_{j+\frac{1}{2}})/2$. The discrete densities $\rho_{j+\frac{1}{2}}$ and the internal energy $e_{j+\frac{1}{2}}$ are stored at the centers $x_{j+\frac{1}{2}}$ whereas the velocities $u_j$ are located at the edges $x_j$. The time dicretization is explicit and we use the convention that, with $q$ the evaluation of a certain quantity at time $t$, $\overline{q}$ stands for its update at time $t + \delta t$.

The density $\rho_{j+\frac{1}{2}}$ is updated using the following discrete mass balance equation:

$$\frac{\overline{\rho}_{j+\frac{1}{2}} - \rho_{j+\frac{1}{2}}}{\delta t} + \frac{\mathscr{F}_{j+1} - \mathscr{F}_j}{\delta x_{j+\frac{1}{2}}} = 0.$$

The mass fluxes are defined by $\mathscr{F}_j = \mathscr{F}_j^+ + \mathscr{F}_j^-$ with $\mathscr{F}_j^+ = \mathscr{F}^+(\rho_{j-\frac{1}{2}}, e_{j-\frac{1}{2}}, u_j)$ and $\mathscr{F}_j^- = \mathscr{F}^-(\rho_{j+\frac{1}{2}}, e_{j+\frac{1}{2}}, u_j)$. The definition of $\mathscr{F}^\pm$ is extracted from [1]

$$\mathscr{F}^+(\rho,e,u) = \begin{cases} 0 & \text{if } u \leqslant -c(e), \\ \dfrac{\rho(u+c(e))^2}{4c(e)} & \text{if } |u| < c(e), \\ \rho u & \text{if } u > c(e), \end{cases} \quad \text{and } \mathscr{F}^-(\rho,e,u) = -\mathscr{F}^+(\rho,e,-u),$$

where $c(e) = \sqrt{(\gamma-1)\gamma e}$ is the speed of sound. In the sequel – except in the numerical tests – we do not use this explicit definition, but only the following two properties: $\forall u \in \mathbb{R}$, $\forall \rho \geqslant 0$,

$$0 \leqslant \mathscr{F}^+(\rho,e,u) \leqslant \rho[\lambda_+(e,u)]^+ \quad \text{and} \quad -\rho[\lambda_-(e,u)]^- \leqslant \mathscr{F}^-(\rho,e,u) \leqslant 0, \quad (3)$$

where $\lambda_\pm(e,u) = u \pm c(e)$ and $[z]^\pm = \frac{1}{2}(|z| \pm z)$.

The velocity $u_j$ is then updated using the following discrete momentum balance equation:

$$\frac{\overline{\rho}_j \overline{u}_j - \rho_j u_j}{\delta t} + \frac{\mathscr{G}_{j+\frac{1}{2}} - \mathscr{G}_{j-\frac{1}{2}}}{\delta x_j} + \frac{\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}}{\delta x_j} = 0. \quad (4)$$

The momentum flux $\mathscr{G}_{j+\frac{1}{2}}$ and the pressure $\Pi_{j+\frac{1}{2}}$ are defined by:

$$\mathscr{G}_{j+\frac{1}{2}} = u_j \mathscr{F}^+_{j+\frac{1}{2}} + u_{j+1} \mathscr{F}^-_{j+\frac{1}{2}} \quad \text{and} \quad \Pi_{j+\frac{1}{2}} = (\gamma-1)\rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}}.$$

The quantities $\rho_j$ and $\mathscr{F}^\pm_{j+\frac{1}{2}}$ are expressed as mean values of $\rho_{j-\frac{1}{2}}$, $\rho_{j+\frac{1}{2}}$ and $\mathscr{F}^\pm_j$, $\mathscr{F}^\pm_{j+1}$:

$$\rho_j = \frac{\delta x_{j+\frac{1}{2}} \rho_{j+\frac{1}{2}} + \delta x_{j-\frac{1}{2}} \rho_{j-\frac{1}{2}}}{2\delta x_j} \quad \text{and} \quad \mathscr{F}^\pm_{j+\frac{1}{2}} = \frac{\mathscr{F}^\pm_{j+1} + \mathscr{F}^\pm_j}{2}. \quad (5)$$

Finally, the internal energy $e_{j+\frac{1}{2}}$ is updated using the following discrete equation:

$$\frac{\overline{\rho}_{j+\frac{1}{2}} \overline{e}_{j+\frac{1}{2}} - \rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}}}{\delta t} + \frac{\mathscr{E}_{j+1} - \mathscr{E}_j}{\delta x_{j+\frac{1}{2}}} + \Pi_{j+\frac{1}{2}} \frac{\overline{u}_{j+1} - \overline{u}_j}{\delta x_{j+\frac{1}{2}}} = \mathscr{R}_{j+\frac{1}{2}}. \quad (6)$$

The internal energy flux $\mathscr{E}_j$ is given by $\mathscr{E}_j = e_{j-\frac{1}{2}} \mathscr{F}^+_j + e_{j+\frac{1}{2}} \mathscr{F}^-_j$. According to [2], the rhs $\mathscr{R}_{j+\frac{1}{2}}$ is designed to account for the source terms that appear in the discrete kinetic energy balance and that do not vanish when $\delta x$ goes to zero. To be more specific, the kinetic energy balance is obtained by multiplying (4) by $\overline{u}_j$; we find, see [1, 2]:

$$\frac{1}{2} \frac{\overline{\rho}_j \overline{u}_j^2 - \rho_j u_j^2}{\delta t} + \frac{\mathscr{H}_{j+\frac{1}{2}} - \mathscr{H}_{j-\frac{1}{2}}}{\delta x_j} + \frac{\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}}{\delta x_j} \overline{u}_j = -R_j,$$

where the kinetic energy flux is given by $\mathscr{H}_{j+\frac{1}{2}} = \left(\frac{1}{2} u_j^2\right) \mathscr{F}^+_{j+\frac{1}{2}} + \left(\frac{1}{2} u_{j+1}^2\right) \mathscr{F}^-_{j+\frac{1}{2}}$ and

$$R_j = \frac{1}{2\delta t}\overline{\rho}_j(\overline{u}_j - u_j)^2 + \frac{1}{\delta x_j}\left(\frac{(u_j-u_{j-1})^2}{2}\mathscr{F}^+_{j-\frac{1}{2}} - \frac{(u_{j+1}-u_j)^2}{2}\mathscr{F}^-_{j+\frac{1}{2}}\right) + \frac{1}{\delta x_j}(\overline{u}_j - u_j)(u_j - u_{j-1})\mathscr{F}^+_{j-\frac{1}{2}} + \frac{1}{\delta x_j}(\overline{u_j} - u_j)(u_{j+1}-u_j)\mathscr{F}^-_{j+\frac{1}{2}}.$$

Therefore, it is quite natural to set $\mathscr{R}_{j+\frac{1}{2}} = (R_{j+1}+R_j)/2$.

The scheme presented above is close to the 1D version of the scheme presented in [3] but the two schemes differs by two points. Firstly, the mass fluxes in [3] are upwinded with respect to the material velocity (in other words, it corresponds to the choice $\mathscr{F}^{\pm}(\rho,e,u) = \pm\rho[u]^{\pm}$). Secondly, the time steppings are different: even if both schemes are explicit, the variables are not updated in the same order. We solve the discrete equations in the following way: $\rho \to u \to e$ whereas [3] proceeds with $\rho \to e \to u$. In particular, here the corrective term $\mathscr{R}_{j+\frac{1}{2}}$ does not need any time shift since the updated velocity $\overline{u}$ is known when solving (6).

We now turn to the study of the stability conditions which ensure the positivity of the density and the internal energy.

**Proposition 1.** *Assuming that* $e_{j+\frac{1}{2}} \geqslant 0$*,* $\rho_{j+\frac{1}{2}} \geqslant 0$*,* $\forall j$ *and the following CFL-like conditions hold for all* $j$

$$\frac{\delta t}{\delta x_{j+\frac{1}{2}}}\left([u_{j+1}]^+ + c(e_{j+\frac{1}{2}}) + [u_j]^- + c(e_{j+\frac{1}{2}})\right) \leqslant \frac{1}{\gamma}, \tag{7}$$

$$\frac{\delta t}{\delta x_{j+\frac{1}{2}}}c(e_{j+\frac{1}{2}}) \leqslant \frac{\gamma-1}{2}\frac{\delta x_{j+\frac{1}{2}}}{\delta x_{j+i}}, \quad \forall i \in \{0,1\}, \tag{8}$$

*then* $\overline{e}_{j+\frac{1}{2}} \geqslant 0$ *and* $\overline{\rho}_{j+\frac{1}{2}} \geqslant 0$.

*Proof.* We assume that $e_{j+\frac{1}{2}} \geqslant 0$, $\rho_{j+\frac{1}{2}} \geqslant 0$ and that (7) and (8) holds for all $j$.

*Positivity of the density:* As proved in [1], the positivity of $\overline{\rho}_{j+\frac{1}{2}}$ comes from the inequality

$$\frac{\delta t}{\delta x_{j+\frac{1}{2}}}\left([\lambda_+(e_{j+\frac{1}{2}}, u_{j+1})]^+ + [\lambda_-(e_{j+\frac{1}{2}}, u_j)]^-\right) \leqslant 1.$$

This latter is directly implied by (7) since $\gamma > 1$ and $[\lambda_{\pm}(e,u)]^{\pm} \leqslant [u]^{\pm} + c(e)$.

*Positivity of the internal energy:* We start by rewriting the terms $(-1)^i\Pi_{j+\frac{1}{2}}\overline{u}_{j+i}$, $i \in \{0,1\}$ involved in (6) by making the discrete time derivative $(\overline{u}_{j+i} - u_{j+i})$ appear. Then, we make use of the Young inequality as follows:

$$(-1)^i\Pi_{j+\frac{1}{2}}\overline{u}_{j+i} = (-1)^i(\gamma-1)\left(\rho_{j+\frac{1}{2}}e_{j+\frac{1}{2}}(\overline{u}_{j+i} - u_{j+i}) + \rho_{j+\frac{1}{2}}e_{j+\frac{1}{2}}u_{j+i}\right)$$
$$\geqslant -\rho_{j+\frac{1}{2}}\left(\frac{c(e_{j+\frac{1}{2}})}{4\gamma}(\overline{u}_{j+i} - u_{j+i})^2 + (\gamma-1)e_{j+\frac{1}{2}}\left(c(e_{j+\frac{1}{2}}) - (-1)^i u_{j+i}\right)\right).$$

Next, we write $\overline{\rho}_{j+\frac{1}{2}}\overline{e}_{j+\frac{1}{2}} \geqslant \mathrm{T}_0+\mathrm{T}_1^0+\mathrm{T}_1^1$ where:

$$\mathrm{T}_0 = \rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}} \left(1 - \frac{\delta t}{\delta x_{j+\frac{1}{2}}}(\gamma-1)(2c(e_{j+\frac{1}{2}})+u_j-u_{j+1})\right) - \delta t \frac{\mathscr{E}_{j+1}-\mathscr{E}_j}{\delta x_{j+\frac{1}{2}}},$$
$$\mathrm{T}_1^i = \frac{\delta t}{2}R_{j+i} - \frac{\delta t}{\delta x_{j+\frac{1}{2}}}\frac{c(e_{j+\frac{1}{2}})}{4\gamma}\rho_{j+\frac{1}{2}}(\overline{u}_{j+i}-u_{j+i})^2.$$

Thus, to guarantee that $\overline{e}_{j+\frac{1}{2}}$ is non negative it is sufficient to ensure that these three terms are non negative. This holds under the assumptions (7) and (8).

Indeed, using the definition of the flux $\mathscr{E}_j$ and owing to (3), we obtain

$$\mathrm{T}_0 \geqslant \rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}} \left(1 - \frac{\delta t}{\delta x_{j+\frac{1}{2}}}(\gamma-1)[u_j]^+ + c(e_{j+\frac{1}{2}}) + [u_{j+1}]^- + c(e_{j+\frac{1}{2}})\right)$$
$$- \frac{\delta t}{\delta x_{j+\frac{1}{2}}}\rho_{j+\frac{1}{2}} e_{j+\frac{1}{2}}\left([\lambda_+(e_{j+\frac{1}{2}},u_{j+1})]^+ + [\lambda_-(e_{j+\frac{1}{2}},u_j)]^-\right)$$

where, due to (7), the rhs is non negative since $[\lambda_\pm(e,u)]^\pm \leqslant [u]^\pm + c(e)$.

Next, we turn to $\mathrm{T}_1^i$. Using twice the Young inequality and bearing in mind the definition of $\overline{\rho}_j$, we observe that

$$\frac{\delta t}{2}R_{j+i} \geqslant \frac{1}{4}(\overline{u}_{j+i}-u_{j+i})^2\left(\rho_{j+i} - \frac{\delta t}{\delta x_{j+i}}(\mathscr{F}^+_{j+i+\frac{1}{2}} - \mathscr{F}^-_{j+i-\frac{1}{2}})\right).$$

Hence, we have

$$\mathrm{T}_1^i \geqslant \frac{1}{4}(\overline{u}_{j+i}-u_{j+i})^2\left(\rho_{j+i} - \frac{\delta t}{\delta x_{j+i}}(\mathscr{F}^+_{j+i+\frac{1}{2}} - \mathscr{F}^-_{j+i-\frac{1}{2}}) - \delta t\frac{\rho_{j+\frac{1}{2}}c(e_{j+\frac{1}{2}})}{\gamma\delta x_{j+\frac{1}{2}}}\right).$$

Coming back to (5), we can write $T_1^i \geqslant \frac{1}{4}(\overline{u}_{j+i}-u_{j+i})^2\left(T_2^{i,0}+T_2^{i,1}\right)$ where, for $k\in\{0,1\}$,

$$\mathrm{T}_2^{i,k} = \frac{\delta x_{j+i+k-\frac{1}{2}}}{2\delta x_{j+i}}\rho_{j+i+k-\frac{1}{2}} - \delta t\frac{\mathscr{F}^+_{j+i+k}-\mathscr{F}^-_{j+i+k-1}}{2\delta x_{j+i}} - \delta t\frac{\rho_{j+i+k-\frac{1}{2}}c(e_{j+\frac{1}{2}})}{\gamma\delta x_{j+i+k-\frac{1}{2}}}.$$

Note that a non negative term has been added to obtain a symmetric formulation in the above inequality. Due to (3) and (7) we get

$$\mathscr{F}^+_{j+i+k} - \mathscr{F}^-_{j+i+k-1} \leqslant \frac{\delta x_{j+i+k-\frac{1}{2}}}{\gamma\delta t}\rho_{j+i+k-\frac{1}{2}},$$

and this allows us to write

$$\mathrm{T}_2^{i,k} \geqslant \rho_{j+i+k-\frac{1}{2}} \left( \frac{\delta x_{j+i+k-\frac{1}{2}}}{2\delta x_{j+i}} \frac{\gamma-1}{\gamma} - \frac{\delta t}{\delta x_{j+i+k-\frac{1}{2}}} \frac{c(e_{j+\frac{1}{2}})}{\gamma} \right).$$

We conclude by observing that this term is non negative by virtue of (8).

## 3 A conservation equation on a cell averaged total energy.

We start by introducing auxiliary variables $\rho_j$, $e_j$ and $u_{j+\frac{1}{2}}$ stored at the dual locations with respect to the numerical unknowns $\rho_{j+\frac{1}{2}}$, $e_{j+\frac{1}{2}}$ and $u_j$ actually used within the scheme. The quantity $\rho_j$ has already been defined in (5) and it satisfies

$$\frac{\overline{\rho}_j - \rho_j}{\delta t} + \frac{\mathscr{F}_{j+\frac{1}{2}} - \mathscr{F}_{j-\frac{1}{2}}}{\delta x_j} = 0,$$

where we have set $\mathscr{F}_{j+\frac{1}{2}} = \mathscr{F}^+_{j+\frac{1}{2}} + \mathscr{F}^-_{j+\frac{1}{2}}$. Denoting $\mathscr{G}_j = u_{j-\frac{1}{2}} \mathscr{F}^+_j + u_{j+\frac{1}{2}} \mathscr{F}^-_j$ and $\Pi_j = (\gamma-1)\rho_j e_j$, we can define an updated velocity $\overline{u}_{j+\frac{1}{2}}$ at the cell centers using the following discrete momentum equation:

$$\frac{\overline{\rho}_{j+\frac{1}{2}} \overline{u}_{j+\frac{1}{2}} - \rho_{j+\frac{1}{2}} u_{j+\frac{1}{2}}}{\delta t} + \frac{\mathscr{G}_{j+1} - \mathscr{G}_j}{\delta x_{j+\frac{1}{2}}} + \frac{\Pi_{j+1} - \Pi_j}{\delta x_{j+\frac{1}{2}}} = 0.$$

Finally, denoting $\mathscr{E}_{j+\frac{1}{2}} = e_j \mathscr{F}^+_{j+\frac{1}{2}} + e_{j+1} \mathscr{F}^-_{j+\frac{1}{2}}$, we define an internal energy $e_j$ at cell edges with the following equation

$$\frac{\overline{\rho}_j \overline{e}_j - \rho_j e_j}{\delta t} + \frac{\mathscr{E}_{j+\frac{1}{2}} - \mathscr{E}_{j-\frac{1}{2}}}{\delta x_j} + \Pi_j \frac{\overline{u}_{j+\frac{1}{2}} - \overline{u}_{j-\frac{1}{2}}}{\delta x_j} = \frac{\delta x_{j+\frac{1}{2}} R_{j+\frac{1}{2}} + \delta x_{j-\frac{1}{2}} R_{j-\frac{1}{2}}}{2\delta x_j},$$

where

$$R_{j+\frac{1}{2}} = \frac{\overline{\rho}_{j+\frac{1}{2}}}{2\delta t} \left( \overline{u}_{j+\frac{1}{2}} - u_{j+\frac{1}{2}} \right)^2 + \frac{(u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}})^2}{2\delta x_{j+\frac{1}{2}}} \mathscr{F}^+_{j-1} - \frac{(u_{j+3/2} - u_{j+\frac{1}{2}})^2}{2\delta x_{j+\frac{1}{2}}} \mathscr{F}^-_j$$
$$+ \frac{\overline{u}_{j+\frac{1}{2}} - u_{j+\frac{1}{2}}}{\delta x_{j+\frac{1}{2}}} \left( (u_{j+\frac{1}{2}} - u_{j-\frac{1}{2}}) \mathscr{F}^+_{j-1} + (u_{j+3/2} - u_{j+\frac{1}{2}}) \mathscr{F}^-_j \right).$$

A complete set of variables is now available at both cell centers and edges. It allows us to naturally define the total energies $E_{j+\frac{1}{2}} = e_{j+\frac{1}{2}} + u^2_{j+\frac{1}{2}}/2$ and $E_j = e_j + u_j^2/2$.

With these auxiliary variables at hand, we can now introduce an average of the discrete total energy on the mesh cells as follows:

$$\mathscr{Q}_{j+\frac{1}{2}} = \frac{1}{4\delta x_{j+\frac{1}{2}}} \left( \delta x_{j+1}\rho_{j+1}E_{j+1} + 2\delta x_{j+\frac{1}{2}}\rho_{j+\frac{1}{2}}E_{j+\frac{1}{2}} + \delta x_j \rho_j E_j \right).$$

This quantity – like its counterpart similarly defined on the dual cells – satisfies a conservation equation.

**Theorem 1.** *The averaged total energy* $\mathscr{Q}_{j+\frac{1}{2}}$ *satisfies the conservation equation:*

$$\frac{\overline{\mathscr{Q}}_{j+\frac{1}{2}} - \mathscr{Q}_{j+\frac{1}{2}}}{\delta t} + \frac{\mathscr{T}_{j+1} - \mathscr{T}_j}{\delta x_{j+\frac{1}{2}}} + \frac{(\Pi * \overline{u})_{j+1} - (\Pi * \overline{u})_j}{\delta x_{j+\frac{1}{2}}} = 0.$$

*where we use the notation* $(s*t)_j = \frac{1}{4}\left( (s_{j+\frac{1}{2}} + s_{j-\frac{1}{2}})t_j + (t_{j+\frac{1}{2}} + t_{j-\frac{1}{2}})s_j \right)$ *and with the flux*

$$\mathscr{T}_j = \frac{1}{4}\left( (\mathscr{E}+\mathscr{H})_{j+\frac{1}{2}} + 2(\mathscr{E}+\mathscr{H})_j + (\mathscr{E}+\mathscr{H})_{j-\frac{1}{2}} \right) - \frac{1}{4}\left( \frac{\delta x_{j+\frac{1}{2}} R_{j+\frac{1}{2}} - \delta x_{j-\frac{1}{2}} R_{j-\frac{1}{2}}}{2} - (\delta x_{j-\frac{1}{2}} - \delta x_j) R_j \right).$$

*Proof.* The result simply comes from the two total energy balance that we sum up properly:

$$\frac{\overline{\rho}_{j+\frac{1}{2}}\overline{E}_{j+\frac{1}{2}} - \rho_{j+\frac{1}{2}}E_{j+\frac{1}{2}}}{\delta t} + \frac{(\mathscr{E}+\mathscr{H})_{j+1} - (\mathscr{E}+\mathscr{H})_j}{\delta x_{j+\frac{1}{2}}} + \frac{\Pi_{j+1} - \Pi_j}{\delta x_{j+\frac{1}{2}}} \overline{u}_{j+\frac{1}{2}} + \Pi_{j+\frac{1}{2}} \frac{\overline{u}_{j+1} - \overline{u}_j}{\delta x_{j+\frac{1}{2}}} = \frac{R_{j+1} + R_j}{2} - R_{j+\frac{1}{2}}$$

and

$$\frac{\overline{\rho}_j \overline{E}_j - \rho_j E_j}{\delta t} + \frac{(\mathscr{E}+\mathscr{H})_{j+\frac{1}{2}} - (\mathscr{E}+\mathscr{H})_{j-\frac{1}{2}}}{\delta x_j} + \frac{\Pi_{j+\frac{1}{2}} - \Pi_{j-\frac{1}{2}}}{\delta x_j} \overline{u}_j + \Pi_j \frac{\overline{u}_{j+\frac{1}{2}} - \overline{u}_{j-\frac{1}{2}}}{\delta x_j} = \frac{\delta x_{j+\frac{1}{2}} R_{j+\frac{1}{2}} + \delta x_{j-\frac{1}{2}} R_{j-\frac{1}{2}}}{2\delta x_j} - R_j.$$

## 4 Numerical simulations of Riemann problems.

We perform the numerical resolutions of some Riemann problems – see [4] – on the computational domain $[0,1]$. The number of grid points is equal to 1000 and the time step is given by $\delta t = \frac{\delta x}{100}$. We take $\gamma = 1.4$. The initial data $\rho$, $u$, $p$ are piecewise constant functions with a discontinuity located at $x_0 = 0.5$, according to the table below. In Figures 1, 2, and 3 we represent the pressure $p_{j+\frac{1}{2}}$, velocity $u_j$ and internal energy $e_{j+\frac{1}{2}}$ at the final time $T$ (also given in the table below).

| Figure | $\rho_l$ | $\rho_r$ | $u_l$ | $u_r$ | $p_l$ | $p_r$ | $T$ |
|---|---|---|---|---|---|---|---|
| Fig 1 | 1 | 0.125 | 0 | 0 | 1 | 0.1 | 0.025 |
| Fig 2 | 1 | 1 | 0 | 0 | 1000 | 0.001 | 0.012 |
| Fig 3 | 5.99924 | 5.99242 | 19.5975 | −6.19633 | 460.894 | 46.0950 | 0.035 |

Test #1, the so-called Sod test problem, is a mild test whose solution consists of a left rarefaction, a contact disconinuity and a right shock. Test #2 is a more severe test problem whose solution contains a left rarefaction, a contact discontinuity and a right shock. Test #3 corresponds to the collision of two strong shocks and consists of a left facing shock (travelling very slowly to the right), a right travelling contact discontinuity and a right travelling shock wave.

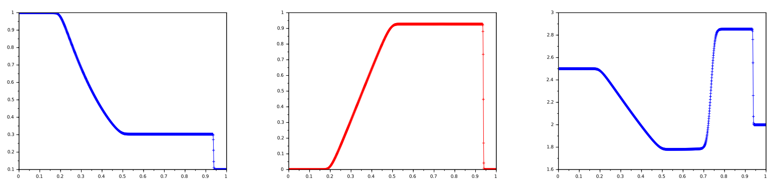

**Fig. 1** Sod test problem: pressure (left), velocity (middle), internal energy (right).

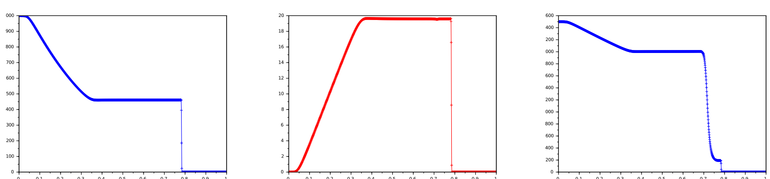

**Fig. 2** Very severe test problem : pressure (left), velocity (middle), internal energy (right).

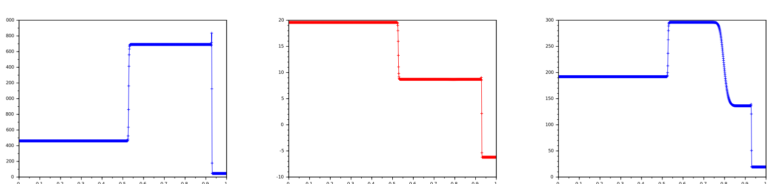

**Fig. 3** Collision of two strong shocks: pressure (left), velocity (middle), internal energy (right).

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