Diffusion Approximation and Entropy-based Moment Closure for Kinetic Equations

Jean-François COULOMBEL[1], François GOLSE[2], Thierry GOUDON[1]

[1] CNRS & Université Lille 1, Laboratoire Paul Painlevé, UMR CNRS 8524
Cité scientifique, 59655 VILLENEUVE D'ASCQ Cedex, France

[2] Laboratoire Jacques-Louis Lions, UMR CNRS 7598
Université Paris 7, 175 rue du Chevaleret, 75013 PARIS, France

E-mails: `jfcoulom@math.univ-lille1.fr`, `golse@ann.jussieu.fr`, `thierry.goudon@math.univ-lille1.fr`

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Abstract

It is a well-known fact that, in small mean free path regimes, kinetic equations can lead to diffusion equations. Besides, kinetic equations can be approached by a closed system of moments equations. In this paper, we are interested in a special closure based on an entropy minimization principle, as introduced earlier by Levermore. We investigate the behavior of the resulting nonlinear hyperbolic system in the diffusive scaling. We first establish various fundamental facts on this system, then we show that the hyperbolic system admits global smooth solutions, and is consistent with the diffusion limit. Similar features are also discussed for a simpler limited flux equation.

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1 Introduction

This quite long Introduction is organized as follows. First, we describe the general problem we are interested in, which relies on approximate models for kinetic equations in small mean free path (denoted ε) regimes. We consider situations in which this asymptotics leads to a diffusion equation. Then, we focus on *reduced models* that are intended to describe intermediate regimes, for small but nonzero $\varepsilon > 0$. It is tempting to try to approach the kinetic equation by a finite number of moments equations, where we get rid of the velocity variable. This requires a closure method that defines, in a suitable way, the $(k+1)$-th moment by means of the k preceeding ones. We pay special attention to the hyperbolic system that comes from a closure based on an entropy minimization principle, in the spirit of Levermore's strategy [22, 23, 25]. The discussion is completed by numerous examples and comments. At the end of this introduction, we present our main results.

1.1 The kinetic equation

We are interested in possible approximations of the solution f_ε to the following kinetic equation

$$\varepsilon\, \partial_t f_\varepsilon + v\, \partial_x f_\varepsilon = \frac{1}{\varepsilon}\, Q(f_\varepsilon)\,. \tag{1}$$

The unknown $f_\varepsilon(t,x,v)$ depends on the time and space variables $(t,x) \in [0,\infty) \times \mathbb{R}$, and on a velocity variable v that lies in some measured set (V,μ), $V \subset \mathbb{R}$. It can be interpreted as a density of particles in phase space, meaning that the integral

$$\int_\Omega \int_{\mathcal{V}} f_\varepsilon(t,x,v)\, d\mu(v)\, dx$$

gives the number of particles occupying, at time t, a position in $\Omega \subset \mathbb{R}$ and having a velocity in $\mathcal{V} \subset V$. The parameter $\varepsilon > 0$ is related to the notion of mean free path, that is the average distance that a particle may travel between two scattering events. The dynamics of these scattering events is embodied in the (linear) collision operator Q. It is an integral operator with respect to the variable v, but local with respect to time and space: collisions are localized phenomena which only modify the velocity variable. We shall make the following assumption:

(C1)
- The measure μ is a probability measure on V, that satifies
$$0 < \int_V v^2 d\mu(v) = d < \infty\,.$$
- The collision operator satisfies

Conservation condition: $Q^\star(\mathbb{1}) = 0\,,$

Equilibrium condition: $Q(\mathbb{1}) = 0\,.$

The conservation condition means that collisions only produce a change of the velocity of the particle but do not induce a gain or a loss of particles. In turn, the macroscopic quantities

$$\rho_\varepsilon(t,x) := \int_V f_\varepsilon(t,x,v)\, d\mu(v)\,, \qquad J_\varepsilon(t,x) := \int_V \frac{v}{\varepsilon}\, f_\varepsilon(t,x,v)\, d\mu(v)$$

satisfy the following local conservation law

$$\partial_t \rho_\varepsilon + \partial_x J_\varepsilon = 0\,. \tag{2}$$

As a consequence, the total number of particles is preserved by the equation. Concerning the equilibrium condition, it would be more natural to assume the existence of a positive function $F(v)$ in the kernel of Q, but we can easily reduce to the case $F = \mathbb{1}$ (by changing the unknown $f \to f/F$). As a matter of fact, in most of the applications, the collision operator splits into a gain term, that is a global operator, and a loss term, that is purely local. Namely, we have

$$\text{(C2)} \qquad \begin{cases} Q(f) = \displaystyle\int_V b(v,v') f(v')\, d\mu(v') - \nu(v) f(v)\,, \\ 0 < \beta \le b(v,v') \le B < \infty\,, \qquad 0 < \beta \le \nu(v) \le B < \infty\,. \end{cases}$$

In particular, let us point out that this structure leads to a physically natural maximum principle: starting with nonnegative initial data, the solution f_ε remains nonnegative.

We readily check that (C1) implies

$$\nu(v) = \int_V b(v,v')\, d\mu(v') = \int_V b(v',v)\, d\mu(v')\,.$$

Then, the crucial observation relies on the following dissipation property

$$
\begin{aligned}
-\int_V Q(f) f \, d\mu(v) &= \frac{1}{2} \int_V \int_V b(v,v') \, |f(v') - f(v)|^2 \, d\mu(v') \, d\mu(v) \\
&\geq \frac{\beta}{2} \int_V \int_V |f(v') - f(v)|^2 \, d\mu(v') \, d\mu(v) \geq 0 \,. \quad (3)
\end{aligned}
$$

We can summarize the useful properties of the collision operator as follows:

Lemma 1. *Assume that* (C1) *and* (C2) *hold. Then, Q is a bounded operator on $L^2(V, d\mu)$. The kernel of Q is the one-dimensional subspace of constant functions, and there holds*

$$
-\int_V Q(f) f \, d\mu(v) \geq \beta \int_V |f(v) - \rho_f|^2 \, d\mu(v) \,, \quad \rho_f := \int_V f \, d\mu(v) \,.
$$

Furthermore, Q (resp. the adjoint $Q^\star$) satisfies a Fredholm alternative: for any $g \in L^2(V, d\mu)$ such that $\int_V g(v) \, d\mu(v) = 0$, there exists a unique $h \in L^2(V, \ d\mu)$ such that $Q(h) = g$, (resp. $Q^\star(h) = g$) and $\int_V h(v) \ d\mu(v) = 0$.

The dissipation property is strengthened by assuming that b is symmetric; then, the operator admits many dissipated entropies.

Lemma 2. *Assume that* (C1) *and* (C2) *hold with*

$$
b(v, v') = b(v', v) \,. \tag{4}
$$

Then, for any convex function $\phi : \mathbb{R}^+ \to \mathbb{R}$, we have

$$
-\int_V Q(f) \ \phi'(f) d\mu(v) \geq \beta \int_V b(v,v') \ \big(f(v') - f(v)\big) \ \big(\phi'(f(v')) - \phi'(f(v))\big) \, d\mu(v') \, d\mu(v) \geq 0 \,. \tag{5}
$$

1.2 Diffusion Asymptotics

Coming back to the evolution problem (1), the relation (3) translates into an entropy dissipation:

$$
\frac{d}{dt} \left(\int_{\mathbb{R}} \int_V f_\varepsilon^2 \, d\mu(v) \ dx \right) + \frac{\beta}{\varepsilon^2} \int_{\mathbb{R}} \int_V (f_\varepsilon - \rho_\varepsilon)^2 \, d\mu(v) \ dx \leq 0 \,.
$$

It indicates the $f_\varepsilon(t, x, v)$ behaves essentially like the macroscopic quantity $\rho_\varepsilon(t, x)$ for small values of ε. Similarly, (4) implies the dissipation of $\int_{\mathbb{R}} \int_V \phi(f_\varepsilon) \, d\mu(v) \ dx$ for all convex functions ϕ. The diffusion asymptotics relies crucially on the additional following assumption:

$$
\text{(C3)} \qquad \int_V v \ d\mu(v) = 0 \,.
$$

This means that equilibrium functions have a null flux. In turn, Lemma 1 yields the following claim:

Lemma 3. *Assume that* (C1), (C2), *and* (C3) *hold. Then, there exists a unique $\chi \in L^2(V, d\mu)$ (resp. $\chi^\star \in L^2(V, d\mu)$) such that $Q(\chi) = v$ (resp. $Q^\star(\chi^\star) = v$), and $\int_V \chi(v) \ d\mu(v) = 0$ (resp. $\int_V \chi^\star(v) \ d\mu(v) = 0$).*

It is quite easy to guess the behavior of the f_ε's as ε goes to 0, by inserting formally in (1) the following Hilbert expansion:

$$
f_\varepsilon = f^{(0)} + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \cdots \,,
$$

and by identifying terms having the same power of ε. We compute

$$\varepsilon^{-1} \text{ term :} \qquad Q(f^{(0)}) = 0\,.$$

This implies that the leading term $f^{(0)}$ belongs to the kernel of the collision operator Q:

$$f^{(0)}(t,x,v) = \rho(t,x)\,.$$

Next, we get

$$\varepsilon^{0} \text{ term :} \qquad Q(f^{(1)}) = v\,\partial_x f^{(0)} = v\,\partial_x \rho(t,x)\,.$$

Applying the Fredholm alternative, we can solve this equation and we readily obtain

$$f^{(1)}(t,x,v) = \chi(v)\;\partial_x \rho(t,x)\,.$$

Eventually, we look at

$$\varepsilon^{1} \text{ term :} \qquad Q(f^{(2)}) = \partial_t f^{(0)} + v\,\partial_x f^{(1)}\,.$$

Inserting the expressions obtained above for $f^{(0)}$ and $f^{(1)}$, the solvability condition

$$\int_V Q(f^{(2)})\;d\mu(v) = 0$$

leads to

$$\partial_t \rho - D\,\partial^2_{xx}\rho = 0\,, \qquad D = -\int_V v\,\chi(v)\;d\mu(v) = -\int_V Q(\chi)\,\chi(v)\;d\mu(v) > 0\,. \tag{6}$$

In other words, the limit ρ should satisfy the heat equation with the diffusion coefficient D.

Remark 1. *In the definition of $f^{(1)}$ we can of course add any element of* $\mathrm{Ker}(Q)$*; but by virtue of (C3), it does not contribute anymore to the limit equation satisfied by ρ.*

Another way to obtain this limit equation uses the conservation law (2). By using Lemma 3, we compute the current as follows:

$$J_\varepsilon(t,x) = \int_V \frac{v}{\varepsilon}\,f_\varepsilon\;d\mu(v) = \int_V \chi^\star(\varepsilon\partial_t f_\varepsilon + v\partial_x f_\varepsilon)\,d\mu(v)\,.$$

Inserting this expression into (2), we get

$$\partial_t \rho_\varepsilon + \partial^2_{xx}\left(\int_V v\chi^\star f_\varepsilon\;d\mu(v)\right) + \varepsilon\,\partial^2_{tx}\left(\int_V \chi^\star f_\varepsilon\;d\mu(v)\right) = 0\,. \tag{7}$$

Then, motivated by (3), we replace f_ε by ρ and we get rid of the $\mathcal{O}(\varepsilon)$ term. We obtain the heat equation (6) by noting that

$$D = -\int_V Q^\star(\chi^\star)\,\chi\,d\mu(v) = -\int_V \chi^\star\,v\,d\mu(v) > 0\,.$$

Remark 2. *The diffusion asymptotics relies crucially on* (C3)*. It also explains why the time variable has been rescaled: replacing $\varepsilon\partial_t$ by ∂_t would lead to the uninteresting limit problem $\partial_t\rho = 0$. Effects are sensible only on a large time scale, of order $\mathcal{O}(1/\varepsilon)$. We refer for comments on this aspect to [12, 13, 15].*

The formal derivation above can be rigorously justified. We refer for instance to [2, 3, 12, 26, 14] etc. for proofs of the following claim, which is part of the folk in kinetic theory.

Theorem 1. *Assume that (C1), (C2) and (C3) hold. Let $\overline{\rho} \geq 0$ be a constant, and let $f_0 : \mathbb{R} \times V \to \mathbb{R}$ belong to $\overline{\rho} + L^2(\mathbb{R} \times V)$.*

i) Then, as ε goes to 0, f_ε and ρ_ε converge to ρ strongly in $L^2_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$, and ρ_ε converges to ρ in $\mathcal{C}([0,T]; L^2(\mathbb{R}) - weak)$, where ρ is the solution to the heat equation (6) with initial datum $\rho_{|t=0} = \int_V f_0(x,v)\, d\mu(v)$.

ii) If the initial datum is close to a smooth macroscopic state, say e.g. $\|f_0 - \rho_0\|_{L^2(\mathbb{R}\times V)} \leq \varepsilon$, with $(\rho_0 - \overline{\rho}) \in H^3(\mathbb{R})$, then one has $\|f_\varepsilon - \rho\|_{L^2((0,T)\times\mathbb{R}\times V)} \leq C_T\, \varepsilon$.

We are interested in approximations of the kinetic unknown $f_\varepsilon(t,x,v)$. Of course, the solution ρ to (6) provides a rough approximation. A first attempt to go further in the approximation of f_ε would be to use the formal ansatz described above. This leads to the approximation

$$f_\varepsilon \simeq \rho(t,x) + \varepsilon\, \chi(v)\, \partial_x \rho(t,x)\,. \tag{8}$$

This is often referred to as the $\mathbb{P}_1$ approximation. However, this approximation suffers from severe drawbacks on physical viewpoints:

• The heat equation propagates information with infinite speed. Therefore it cannot describe with accuracy intermediate regimes, the characteristic speed in (1) being $\|v\|_{L^\infty(V)}/\varepsilon$, large but finite when considering bounded velocities.

• The $\mathbb{P}_1$ approximation does not always preserve nonnegativity: while f_ε is naturally nonnegative, in view of its physical meaning, the approximation (8) can be negative at some points since the condition

$$\varepsilon\, |\chi\, \partial_x \rho| \leq \rho$$

is certainly not fulfilled for any (t,x,v).
Actually the two difficulties are related. Indeed, since ρ_ε and J_ε are moments of a nonnegative quantity, they naturally satisfy the following limited flux condition

$$|J_\varepsilon| = \left| \int_V \frac{v}{\varepsilon}\, f_\varepsilon\, d\mu(v) \right| \leq \frac{\|v\|_{L^\infty(V)}}{\varepsilon}\, \rho_\varepsilon\,. \tag{9}$$

In what follows, we shall describe some remedies to treat the difficulties mentionned above. First, we can use an expansion that is (formally) close to (8), up to $\mathcal{O}(\varepsilon^2)$ terms, but preserves nonnegativity. The second strategy consists in using hyperbolic equations (this ensures finite speed of propagation), that are based on a closure of the equations satisfied by the moments of f_ε.

In order to describe the approximate models, we shall use an additional assumption on the collision operator:

$$\text{(C4)} \qquad Q(v) = -\gamma v = Q^\star(v) \text{ for some constant } \gamma > 0.$$

The fact that γ is positive comes from $-\int_V Q(v) v\, d\mu(v) = \gamma \int_V v^2\, d\mu(v) > 0$. The condition (C4) actually means that $\chi(v) = \chi^\star(v) = -v/\gamma$, and thus we have $D = d/\gamma$. With (C4), (7) can be recast as follows

$$\partial_t \rho_\varepsilon - \partial^2_{xx} \left(\int_V \frac{v^2}{\gamma}\, f_\varepsilon\, d\mu(v) \right) - \varepsilon\, \partial^2_{tx} \left(\int_V \frac{v}{\gamma}\, f_\varepsilon\, d\mu(v) \right) = 0\,.$$

However, the last term is nothing but

$$-\frac{\varepsilon^2}{\gamma}\, \partial_t (\partial_x J_\varepsilon) = \frac{\varepsilon^2}{\gamma}\, \partial^2_{tt} \rho_\varepsilon\,.$$

Therefore, we get

$$\partial_t \rho_\varepsilon - \partial^2_{xx} \left(\int_V \frac{v^2}{\gamma}\, f_\varepsilon\, d\mu(v) \right) + \frac{\varepsilon^2}{\gamma}\, \partial^2_{tt} \rho_\varepsilon = 0\,. \tag{10}$$

1.3 Approximate Models

To go further, we need to specify more precisely our framework. The assumptions on the velocity set V and the measure μ are crucial. Throughout the rest of this paper we assume the following:

Assumption 1. *$V \subset \mathbb{R}$ is compact, symmetric with respect to 0, (for instance, $V = [-1, +1]$), and μ is a probability measure on V: $\int_V d\mu(v) = 1$.*

Assumption 2. *For any continuous and odd function $h : V \to \mathbb{R}$, we have $\int_V h(v)\, d\mu(v) = 0$.*

Assumption 3. *$0 < \int_V v^2 d\mu(v) = d < \infty$.*

Assumption 2 strengthens (C3); it means that velocities are equally distributed with respect to the origin. Assumption 3 means that we exclude the Dirac mass at the origin. (Note that in the case of the Dirac mass, the only velocity is 0, and (1) is trivial.) It is convenient to introduce now the following definition and basic properties (the proof of which is postponed to Appendix A).

Lemma 4. *Consider the Laplace transform of the measure μ:*

$$\begin{array}{rccl} \mathbb{F}: & \mathbb{R} & \longrightarrow & \mathbb{R}^+ \\ & \beta & \longmapsto & \displaystyle\int_V \exp(\beta v)\, d\mu(v) > 0\,. \end{array} \tag{11}$$

Assume that the support of μ is a subset of $[-1, 1]$, and that

$$\min\ \mathrm{Supp}\ \mu = -1\,, \quad \max\ \mathrm{Supp}\ \mu = 1\,.$$

Then, the following properties hold:

- *The function $\mathbb{F}$ is $\mathcal{C}^\infty$, and even.*
- *The function $\mathbb{G}(\beta) := \mathbb{F}'(\beta)/\mathbb{F}(\beta)$ is a $\mathcal{C}^\infty$ (increasing) diffeomorphism on $(-1, +1)$. We denote its inverse by $\mathbb{G}^{(-1)}$. Both $\mathbb{G}$ and $\mathbb{G}^{(-1)}$ are odd functions.*

Given a measure μ that satisfies the assumptions of Lemma 4, we define the following $\mathcal{C}^\infty$, and even function ψ:

$$\begin{array}{rccl} \psi: & (-1, 1) & \longrightarrow & \mathbb{R}^+ \\ & u & \longmapsto & u^2 + \mathbb{G}' \circ \mathbb{G}^{(-1)}(u) = \dfrac{\mathbb{F}''}{\mathbb{F}}\big(\mathbb{G}^{(-1)}(u)\big)\,. \end{array} \tag{12}$$

Throughout the paper, we shall often use the following relations:

$$\mathbb{F}(0) = 1\,, \quad \mathbb{F}'(0) = 0\,, \quad \mathbb{G}(0) = 0\,, \quad \psi(0) = \mathbb{G}'(0) = d\,, \quad \psi'(0) = 0\,.$$

For the collision operator, it is worth having in mind the following – definitely oversimplified – operator:

$$Q(f_\varepsilon) := \rho_\varepsilon - f_\varepsilon, \qquad \rho_\varepsilon(t, x) = \int_V f_\varepsilon(t, x, v)\, d\mu(v). \tag{13}$$

Namely, the collision operator reduces to the relaxation operator to the mean value over velocities of the unknown, where the measure μ can be chosen among the following examples:

Example 1. *$V = [-1, +1]$ endowed with the (normalized) Lebesgue measure $d\mu(v) = dv/2$.*

Example 2. *$V = [-1, +1]$ endowed with the discrete measure $d\mu(v) = \big(\delta_{-1} + \delta_{+1}\big)/2$.*

Example 3. *A variant with unbounded velocities is given by $V = \mathbb{R}$ endowed with the Gaussian measure $d\mu(v) = \exp(-v^2/2)\, dv/\sqrt{2\pi}$.*

The collision operator (13) obviously verifies (C1-C4) in these situations. The value of the diffusion coefficient D changes with the measure μ: we have $D = 1/3$ for Example 1, $D = 1$ for Examples 2 and 3.

The closure problem consists in defining a system of equations for some macroscopic quantities $\widehat{\rho}_\varepsilon$ (or $\widehat{\rho}_\varepsilon$, and $\widehat{J}_\varepsilon$), that depend only on the time and space variables, with the following two-fold objective:

1. We expect that the resulting system is easier to solve than (1); for instance, it can be solved with a reduced numerical cost;

2. It provides a "good"approximation of the evolution of the true quantities $(\rho_\varepsilon, J_\varepsilon)$ associated with the solution to (1).

Therefore, we aim at approaching solutions of a linear kinetic equation by solving (possibly) nonlinear equations, where we get rid of the velocity variable v.

1.3.1 Zeroth Order Closure

At the zeroth order, we can close the mass conservation relation (2) by considering a Fick relation between the current and the density:

$$\widehat{J}_\varepsilon = -\mathfrak{D}\, \partial_x \widehat{\rho}_\varepsilon,$$

where the "diffusion"coefficient $\mathfrak{D}$ is chosen in such a way that (9) is fulfilled. In turn, $\mathfrak{D}$ might depend on $\widehat{\rho}_\varepsilon, \partial_x \widehat{\rho}_\varepsilon$... There are a lot of possible definitions for the coefficient $\mathfrak{D}$; we refer among others to [24, 21, 30, 32, 4]... The corresponding approached equation can also be (formally) justified by coming back to a more microscopic level: we define an approximate density of particles $\widehat{f}_\varepsilon$ as a function of its zeroth moment $\widehat{\rho}_\varepsilon$: $\widehat{f}_\varepsilon(t, x, v) = F(\widehat{\rho}_\varepsilon(t, x), v, \varepsilon)$. Then, we use this expression to close the mass conservation relation (2). For instance, using the $\mathbb{P}_1$ like approximation $\widehat{\rho}_\varepsilon - \varepsilon v \partial_x \widehat{\rho}_\varepsilon / \gamma$ leads again to the heat equation. As mentioned above, this approximation is not satisfactory since it is not, in general, nonnegative, and it does not fulfill the physical flux limitation (9). Since (9) is a consequence of the nonnegativity of f_ε, let us seek an approximation that will be, by construction, nonnegative. A very simple strategy consists in modifying the Hilbert expansion procedure by considering

$$f_\varepsilon = \exp(a^{(0)} + \varepsilon a^{(1)} + \varepsilon^2 a^{(2)} + \dots).$$

We readily obtain the following conclusion for the first two terms

$$\begin{aligned} &a^{(0)} \quad \text{depends only on } t \text{ and } x, \\ &a^{(1)}(t, x, v) = -\frac{v}{\gamma}\, \partial_x a^{(0)}(t, x)\,. \end{aligned}$$

Truncation at the first order yields

$$\widehat{f}_\varepsilon(t, x, v) = \exp\Big(a^{(0)}(t, x) - \varepsilon\, \frac{v}{\gamma}\, \partial_x a^{(0)}(t, x)\Big) \geq 0\,, \tag{14}$$

the moments of which are required to satisfy (2). We get the following system

$$\begin{cases} \widehat{\rho}_\varepsilon(t, x) = \exp(a^{(0)})\, \mathbb{F}\big(\varepsilon\, \partial_x a^{(0)}/\gamma\big)\,, \\ \partial_t \widehat{\rho}_\varepsilon - \partial_x \Big(\dfrac{\widehat{\rho}_\varepsilon}{\varepsilon}\, \mathbb{G}\Big(\varepsilon \dfrac{\partial_x a^{(0)}}{\gamma}\Big)\Big) = 0\,. \end{cases}$$

Next, we note that

$$\frac{\partial_x \widehat{\rho}_\varepsilon}{\widehat{\rho}_\varepsilon} = \partial_x a^{(0)} + \varepsilon\, \mathbb{G}\Big(\varepsilon \frac{\partial_x a^{(0)}}{\gamma}\Big)\, \frac{\partial^2_{xx} a^{(0)}}{\gamma}\,.$$

Then, under the assumption that $\partial^2_{xx} a^{(0)}$ remains bounded uniformly with respect to ε, we neglect the last term, and we are eventually led to the simple equation

$$\partial_t \widehat{\rho}_\varepsilon - \partial_x \left(\frac{\widehat{\rho}_\varepsilon}{\varepsilon}\, \mathbb{G}\Big(\frac{\varepsilon\, \partial_x \widehat{\rho}_\varepsilon}{\gamma \widehat{\rho}_\varepsilon}\Big)\right) = 0\,. \tag{15}$$

For the Lebesgue measure, Example 1, we have

$$\mathbb{F}(\beta) = \frac{\sinh(\beta)}{\beta}\,, \qquad \mathbb{G}(\beta) = \coth(\beta) - \frac{1}{\beta}\,.$$

Therefore, we recover the limited flux model introduced by [21] and [24] which reads as follows

$$\partial_t \widehat{\rho}_\varepsilon - \partial_x \left(\frac{\widehat{\rho}_\varepsilon}{\varepsilon} \left(\coth\big(\frac{\varepsilon \partial_x \widehat{\rho}_\varepsilon}{\gamma \widehat{\rho}_\varepsilon}\big) - \frac{\gamma \widehat{\rho}_\varepsilon}{\varepsilon \partial_x \widehat{\rho}_\varepsilon}\right)\right) = 0\,. \tag{16}$$

For the discrete measure, Example 2, we get

$$\partial_t \widehat{\rho}_\varepsilon - \partial_x \left(\frac{\widehat{\rho}_\varepsilon}{\varepsilon}\, \tanh\big(\frac{\varepsilon \partial_x \widehat{\rho}_\varepsilon}{\gamma \widehat{\rho}_\varepsilon}\big)\right) = 0\,,$$

while, for the Gaussian measure, Example 3, we are simply led to the heat equation.

1.3.2 First Order Closure

We can also complete (2) by looking at the evolution equation for the current $J_\varepsilon(t,x)$. We set

$$\mathbb{P}_\varepsilon(t,x) := \int_V v^2\, f_\varepsilon\; d\mu(v)\,.$$

Integrating (1) with respect to the velocity variable yields, by using (C4)

$$\varepsilon^2\, \partial_t J_\varepsilon + \partial_x \mathbb{P}_\varepsilon = \int_V \frac{v}{\varepsilon}\, Q(f_\varepsilon)\, d\mu(v) = \int_V vQ\left(\frac{f_\varepsilon - \rho_\varepsilon}{\varepsilon}\right) d\mu(v) = -\gamma\, J_\varepsilon\,. \tag{17}$$

The second equality, combined to (3), shows that the right-hand side is actually a $\mathcal{O}(1)$ quantity. Of course, (2) and (17) do not form a closed system of equations, since there is no reason why $\mathbb{P}_\varepsilon$ should be defined directly by means of ρ_ε and J_ε.

Observe that

$$\mathbb{P}_\varepsilon = d\, \rho_\varepsilon + \varepsilon\, \mathbb{K}_\varepsilon\,, \tag{18}$$

where $\mathbb{K}_\varepsilon(t,x) = \varepsilon^{-1} \int_V v^2\, (f_\varepsilon - \rho_\varepsilon)\, d\mu(v)$ is $\mathcal{O}(1)$ thanks to (3). Hence, as ε tends to 0, we recover (6) as follows:

$$\partial_t \rho + \partial_x J = 0\,, \qquad \gamma J = -d\, \partial_x \rho\,.$$

It is quite usual to close the system by imposing an expression for the kinetic pressure like

$$0 \le \widehat{\mathbb{P}_\varepsilon} := \Xi\; \widehat{\rho}_\varepsilon \le \|v\|^2_{L^\infty}\; \widehat{\rho}_\varepsilon\,,$$

which involves the so-called Eddington factor Ξ. For instance, we can go back to (18) and get rid of the $\mathcal{O}(\varepsilon)$ remainder. The corresponding system reads

$$\begin{cases} \partial_t \widehat{\rho}_\varepsilon + \partial_x \widehat{J}_\varepsilon = 0\,, \\ \varepsilon^2\, \partial_t \widehat{J}_\varepsilon + d\, \partial_x \widehat{\rho}_\varepsilon = -\gamma \widehat{J}_\varepsilon\,. \end{cases} \tag{19}$$

(This system is sometimes called the $\mathbb{P}_1$ approximation.) The approximation (19) can also be justified through an expansion on Legendre polynomials of f_ε; however, such an approximation is known to be satisfactory only under strong isotropy assumptions, see [4]. Therefore, a huge variety of Eddington factors, possibly depending on $\widehat{\rho}_\varepsilon, \partial_x \widehat{\rho}_\varepsilon$, and so on, has been proposed. We refer, among other works, to [30, 24, 32, 4], and for a recent overview of numerical methods for such problems, we refer to [5]. These approximations can be obtained by postulating that the approximate density is a given function of its two first moments $\widehat{\rho}_\varepsilon$ and $\widehat{J}_\varepsilon$, say

$$\widehat{f}_\varepsilon(t,x,v) = \Theta(\widehat{\rho}_\varepsilon(t,x), \widehat{J}_\varepsilon(t,x), v)\,,$$

so that the second moment of $\widehat{f}_\varepsilon$ can be computed in terms of $\widehat{\rho}_\varepsilon$ and $\widehat{J}_\varepsilon$. We shall now present such a closure, based on an entropy minimization principle.

1.3.3 Entropy Minimization Principle

Closing the moments system by using an entropy minimization principle has been first introduced by Levermore [22, 23, 25]. It has also been used successfully in radiative transfer theory, see [10, 38]. Let us now describe precisely this closure method. We define the (convex) entropy functional

$$H(f) := f\ln(f) - f + 1 \geq 0\,. \tag{20}$$

Then, for $\widehat{\rho}_\varepsilon$ and $\widehat{J}_\varepsilon$ given, we wish to minimize the quantity

$$\int_V H(f)\; d\mu(v)\,,$$

under the constraints

$$\int_V f\; d\mu(v) = \widehat{\rho}_\varepsilon\,, \qquad \int_V \frac{v}{\varepsilon}\; f\; d\mu(v) = \widehat{J}_\varepsilon\,.$$

The minimizer obtained in this way will be our approximate density of particles. A quick calculation yields the following expression for the minimizer

$$\widehat{f}_\varepsilon(v) = e^{\lambda_0 + \lambda_1 v/\varepsilon} > 0\,, \tag{21}$$

where the Lagrange multipliers λ_0, λ_1 are determined by the following system:

$$\begin{cases} \widehat{\rho}_\varepsilon = e^{\lambda_0}\; \mathbb{F}\left(\dfrac{\lambda_1}{\varepsilon}\right)\,, \\ \widehat{J}_\varepsilon = \dfrac{1}{\varepsilon}\; e^{\lambda_0}\, \mathbb{F}'\left(\dfrac{\lambda_1}{\varepsilon}\right) = \dfrac{\widehat{\rho}_\varepsilon}{\varepsilon}\, \mathbb{G}\left(\dfrac{\lambda_1}{\varepsilon}\right)\,. \end{cases} \tag{22}$$

We note that λ_1 is uniquely determined provided that we have $\varepsilon\,|\widehat{J}_\varepsilon| < \widehat{\rho}_\varepsilon$, see Lemma 4. This constraint is nothing but the limited flux condition (9).

Now, we use the previous formulae to close the moment system. Using (21) and (22), we compute

$$\int_V v^2\, \widehat{f}_\varepsilon\; d\mu(v) = e^{\lambda_0}\, \mathbb{F}''\left(\frac{\lambda_1}{\varepsilon}\right) = \widehat{\rho}_\varepsilon\, \frac{\mathbb{F}''}{\mathbb{F}} \circ \mathbb{G}^{(-1)}\left(\varepsilon \frac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}\right) = \widehat{\rho}_\varepsilon\; \psi\left(\varepsilon \frac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}\right)\,.$$

We are thus led to the following system of equations

$$\begin{cases} \partial_t \widehat{\rho}_\varepsilon + \partial_x \widehat{J}_\varepsilon = 0\,, \\ \varepsilon^2\, \partial_t \widehat{J}_\varepsilon + \partial_x \left(\widehat{\rho}_\varepsilon\; \psi\Big(\varepsilon \dfrac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}\Big)\right) = -\gamma \widehat{J}_\varepsilon\,. \end{cases} \tag{23}$$

In the next section, we shall detail some properties of system (23) such as hyperbolicity, and the existence of a convex entropy. Following the approach of [16], we shall also show that (23) admits global smooth solutions, that satisfy some uniform bounds with respect to ε.

We can expect that the system (23) is consistent with the diffusion limit since, expanding formally $\varepsilon^2\partial_t\widehat{J}_\varepsilon+\partial_x(\widehat{\rho}_\varepsilon\ \psi(\varepsilon\widehat{J}_\varepsilon/\widehat{\rho}_\varepsilon))+\gamma\widehat{J}_\varepsilon \simeq d\,\partial_x\widehat{\rho}_\varepsilon+\gamma\widehat{J}_\varepsilon$, we recover (6). Besides, the approximate microscopic density is given by

$$\widehat{f}_\varepsilon(t,x,v) = \widehat{\rho}_\varepsilon\,\frac{\exp\Big(v\ \mathbb{G}^{(-1)}\big(\varepsilon\widehat{J}_\varepsilon/\widehat{\rho}_\varepsilon\big)\Big)}{\mathbb{F}\circ\mathbb{G}^{(-1)}\big(\varepsilon\widehat{J}_\varepsilon/\widehat{\rho}_\varepsilon\big)} \geq 0\,. \tag{24}$$

Performing straightforward Taylor expansions leads to

$$\widehat{f}_\varepsilon \simeq \widehat{\rho}_\varepsilon\left(1+\frac{v\,\varepsilon\,\widehat{J}_\varepsilon}{d\,\widehat{\rho}_\varepsilon}\right) \simeq \widehat{\rho}_\varepsilon\left(1-\varepsilon\,v\frac{\partial_x\widehat{\rho}_\varepsilon}{\gamma\widehat{\rho}_\varepsilon}\right)\,,$$

as well as

$$\widehat{f}_\varepsilon(t,x,v) \simeq \widehat{\rho}_\varepsilon\ \exp\left(-\varepsilon v\,\frac{\partial_x\widehat{\rho}_\varepsilon}{\gamma\widehat{\rho}_\varepsilon}\right)\,,$$

so that (24) formally looks like (8) or (14).

The example of (13) endowed with the discrete measure (Example 2), is particularly illuminating. We have only two velocities $+1$ and -1, and (1) can be recast as

$$\partial_t f_\pm \pm \frac{1}{\varepsilon}\partial_x f_\pm = \pm\frac{f_- - f_+}{2\varepsilon^2}\,,$$

where the unknowns $f_\pm(t,x)$ stand for $f(t,x,\pm 1)$. This is the so-called Goldstein-Taylor model, see [11, 36]. The two first moments $\rho_\varepsilon = \frac{1}{2}(f_+ + f_-)$, $J_\varepsilon = \frac{1}{2\varepsilon}(f_+ - f_-)$, are solutions to (19), with $d = 1$. A remarkable feature is that (19) can be recast as a damped wave equation (see also (10))

$$\varepsilon^2\,\partial_{tt}\widehat{\rho}_\varepsilon + \partial_t\widehat{\rho}_\varepsilon - \partial^2_{xx}\widehat{\rho}_\varepsilon = 0\,.$$

The solution of this damped wave equation can be explicitely computed by using a stochastic argument, as shown in [19]. This probabilistic approach can be used to deduce in a very elegant way the convergence of $\widehat{\rho}_\varepsilon$ towards the solution to (6), see [27]. For the discrete measure, we compute

$$\mathbb{F}(\beta) = \cosh(\beta)\,, \qquad \mathbb{G}(\beta) = \tanh(\beta)\,, \qquad \psi(u) = 1 \quad u \in (-1,1)\,.$$

Hence, we realize that (23) is nothing but (19): the entropy minimization closure actually provides a very good approximation of the original equation (1), since it gives the exact solution! This example shows that it could be dangerous to remove the (formally) smallest term in (19). If we get rid of $\varepsilon^2\partial_t\widehat{J}_\varepsilon$, we are simply led to

$$\partial_t\widehat{\rho}_\varepsilon + \partial_x\widehat{J}_\varepsilon = 0\,, \qquad \widehat{J}_\varepsilon = -\partial_x\widehat{\rho}_\varepsilon\,,$$

and we recover the heat equation (6) that propagates at infinite speed. This is clearly not consistent with the approximation procedure and we cannot go back to the kinetic approximation. Indeed, the relation $\widehat{J}_\varepsilon = -\partial_x\widehat{\rho}_\varepsilon = \frac{\widehat{\rho}_\varepsilon}{\varepsilon}\,\mathbb{G}(\lambda_1/\varepsilon) = \frac{\widehat{\rho}_\varepsilon}{\varepsilon}\,\tanh(\lambda_1/\varepsilon)$ makes sense only under the limited flux condition $\varepsilon|\partial_x\widehat{\rho}_\varepsilon| < \widehat{\rho}_\varepsilon$.

In the case of the Gaussian measure (Example 3), we have

$$\mathbb{F}(\beta) = \exp(\beta^2/2)\,, \qquad \mathbb{G}(\beta) = \beta\,, \qquad \psi(u) = 1 + u^2 \quad u \in \mathbb{R}\,.$$

In this case, system (23) is nothing but the rescaled isothermal Euler system:

$$\begin{cases} \partial_t \widehat{\rho}_\varepsilon + \partial_x \widehat{J}_\varepsilon = 0\,, \\ \varepsilon^2 \left(\partial_t \widehat{J}_\varepsilon + \partial_x \dfrac{(\widehat{J}_\varepsilon)^2}{\widehat{\rho}_\varepsilon} \right) + \partial_x \widehat{\rho}_\varepsilon = -\gamma\, \widehat{J}_\varepsilon\,. \end{cases} \tag{25}$$

A detailed analysis of the convergence to the heat equation has been performed by Junca and Rascle [17] for BV initial data; we shall use some of their arguments when studying the convergence of solutions to (23). It is also worth mentioning that in this case, the approximate density of particles, given by (24), reads

$$\widehat{f}_\varepsilon(t,x,v)\; d\mu(v) = \widehat{\rho}_\varepsilon(t,x)\, \exp\left[-\frac{(v - u_\varepsilon(t,x))^2}{2} \right] \frac{dv}{\sqrt{2\pi}}\,,$$

with $u_\varepsilon = \varepsilon \widehat{J}_\varepsilon / \widehat{\rho}_\varepsilon$ and dv the Lebesgue measure on $\mathbb{R}$. We recover the classical Maxwellian distribution.

Remark 3. *We can use any convex function to define the entropy functional instead of using those based on the function $s \mapsto s\ln(s)$. For instance, if we use the function $s \mapsto s^2/2$, and perform the same minimization procedure, we would obtain $\widehat{f}_\varepsilon(t,x,v) = \widehat{\rho}_\varepsilon(t,x) + \varepsilon\, v\, \widehat{J}_\varepsilon(t,x)$, with $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon)$ solution to (19). Neglecting $\varepsilon^2 \partial_t \widehat{J}_\varepsilon$, we recover (8). The entropy (20) has the advantage to ensure the nonnegativity of the approximate density. There are also strong physical arguments for this expression, see [6, 22, 23, 25].*

Remark 4. *It has been noticed that the entropy minimization approach proposed by Levermore may suffer from some severe drawbacks, at least when one tries to approximate the full gas dynamics Boltzmann equation or a kinetic equation with unbounded velocities. We refer for instance to [18] and [9]. Besides, for numerical purposes, other choices of moment closures, which do not necessarily guarantee nonnegativity, can provide sharp results for a still reasonable numerical cost, and regions of negative density remain negligeable. We refer for approaches based on Hermite polynomials expansions to [9, 34, 33]. Note, however, that the entropy closure has been used successfully for applications in radiative transfer theory, see [10] and [38]. For an analysis of discrete velocity models and applications to gas dynamics, we refer to [28, 29].*

1.4 Main results

Our objective is to investigate the solutions to (23), and to establish the convergence, as ε tends to 0, of both $\widehat{\rho}_\varepsilon$ and $\widehat{f}_\varepsilon$ towards the solution to the heat equation. We shall prove in this way the consistency of the entropy minimization model in the diffusion regime. Here we focus on smooth solutions that are bounded away from vacuum. We refer to [7] for the study of weak solutions that include vacuum regions. Our first main result is stated as follows:

Theorem 2. *Assume that Assumptions 1, 2 and 3 are satisfied. Let $\overline{\rho} > 0$. There exist two constants $\delta > 0$ and $C > 0$ such that, for any $\varepsilon \in]0,1]$, and for any (ρ_0, J_0) with $\|\rho_0 - \overline{\rho}\|_{H^2(\mathbb{R})} \leq \delta$ and $\|\varepsilon\, J_0\|_{H^2(\mathbb{R})} \leq \delta$, there exists a unique global solution $(\widehat{\rho}_\varepsilon, \widehat{J}_\varepsilon)$ to (23) with initial data (ρ_0, J_0), and that satisfies $(\widehat{\rho}_\varepsilon - \overline{\rho}, \widehat{J}_\varepsilon) \in \mathcal{C}(\mathbb{R}^+; H^2(\mathbb{R})) \cap \mathcal{C}^1(\mathbb{R}^+; H^1(\mathbb{R}))$. Furthermore, denoting $\widehat{u_\varepsilon} = \varepsilon\; \widehat{J}_\varepsilon / \widehat{\rho}_\varepsilon$, this global smooth solution satisfies the estimate*

$$\sup_{t\in\mathbb{R}} \left(\|\widehat{\rho_\varepsilon}(t) - \overline{\rho}\|^2_{H^2(\mathbb{R})} + \|\widehat{u_\varepsilon}(t)\|^2_{H^2(\mathbb{R})} \right) + \frac{1}{\varepsilon^2} \int_0^{+\infty} \|\widehat{u_\varepsilon}(t)\|^2_{H^2(\mathbb{R})}\, dt \\ \leq C \left(\|\rho_0 - \overline{\rho}\|^2_{H^2(\mathbb{R})} + \|u_0\|^2_{H^2(\mathbb{R})} \right).$$

Furthermore, we can show the convergence of $\widehat{\rho_\varepsilon}$ towards the solution to the heat equation by using arguments quite close to those of Junca and Rascle [17]. For simplicity, we consider initial data ρ_0 and u_0 that are independent of ε. Then we obtain the following result:

Theorem 3. *Assume that the assumptions of Theorem 1-ii) and Theorem 2 are fulifilled. Let r be the solution to the heat equation* (6) *with initial data ρ_0. Then, there exists a constant $C > 0$ such that*

$$\|\widehat{\rho_\varepsilon} - r\|_{L^2(\mathbb{R}^+\times\mathbb{R})} \leq C\,\varepsilon\,.$$

Furthermore, let $\widehat{f_\varepsilon}(t,x,v)$ be defined by (24)*. Then, we have*

$$\|\widehat{f_\varepsilon} - f_\varepsilon\|_{L^2((0,T)\times\mathbb{R}\times V)} \leq C_T\,\varepsilon\,.$$

Remark 5. *The result of Theorem 2 also holds when the measure μ is the Gaussian measure (Example 3). In this case, we have seen that the system* (23) *is the rescaled isothermal Euler system* (25)*. Theorems 2 and 3 thus give an analogous result to the convergence result of [17], but for global smooth solutions.*

The proof also adapts to a multidimensional framework, in the spirit of [39]. We refer to [8] for the treatment of the isothermal Euler equations.

We can also deal with nonlinear collision models. For instance, the linear collision operator (C2) *can be replaced by $\sigma(\rho)(\rho - f)$, for a smooth function $\sigma : \mathbb{R}^+ \to (0, B)$. This model arises in radiative transfer theory, see e.g. [2]. The right-hand side in* (23) *becomes $-\sigma(\rho)J$.*

Our second convergence result is concerned with the simpler conservation equation (15). We obtain a result in the same spirit, at least locally in time.

Theorem 4. *Assume that Assumptions 1, 2 and 3 are satisfied. Let $\overline{\rho} > 0$. There exist three constants $\delta > 0$, $C > 0$ and $T_* > 0$ such that, for any $\varepsilon \in]0,1]$, and for any ρ_0 with $\|\rho_0 - \overline{\rho}\|_{H^4(\mathbb{R})} \leq \delta$, there exists a unique solution $\widehat{\rho_\varepsilon} \in \mathcal{C}([0,T_*]; \overline{\rho} + H^4(\mathbb{R}))$ to*

$$\begin{cases} \partial_t \widehat{\rho_\varepsilon} - \partial_x \left(\mathbb{D}\Big(\dfrac{\varepsilon\,\partial_x \widehat{\rho_\varepsilon}}{\gamma\,\widehat{\rho_\varepsilon}}\Big)\, \dfrac{\partial_x \widehat{\rho_\varepsilon}}{\gamma} \right) = 0\,, \\ \widehat{\rho_\varepsilon}_{|t=0} = \rho_0\,, \end{cases} \tag{26}$$

where we have set $\mathbb{D}(\beta) := \mathbb{G}(\beta)/\beta$. Moreover, the solution $\widehat{\rho_\varepsilon}$ satisfies the estimates

$$\sup_{t\in[0,T_*]} \|\widehat{\rho_\varepsilon}(t) - \overline{\rho}\|_{H^4(\mathbb{R})} \leq C\,, \qquad \sup_{t\in[0,T_*]} \|\widehat{\rho_\varepsilon}(t) - \overline{\rho}\|_{W^{1,\infty}(\mathbb{R})} \leq \overline{\rho}/2.$$

Let r be the solution to the heat equation (6) *with initial data ρ_0. Then, we also have*

$$\sup_{t\in[0,T_*]} \|\widehat{\rho_\varepsilon}(t) - r(t)\|_{L^2(\mathbb{R})} \leq C\ \varepsilon\,.$$

Eventually, if we set

$$\widehat{f_\varepsilon}(t,x,v) := \widehat{\rho_\varepsilon}(t,x)\ \exp\left(-\frac{\varepsilon v\,\partial_x \widehat{\rho_\varepsilon}(t,x)}{\gamma\,\widehat{\rho_\varepsilon}(t,x)} \right) \geq 0\,,$$

then $\|\widehat{f_\varepsilon} - f_\varepsilon\|_{L^2((0,T_*)\times\mathbb{R}\times V)}$ tends to 0 as $\varepsilon \to 0$, with rate $\mathcal{O}(\varepsilon)$ under the assumptions of Theorem 1-ii).

Theorems 2, 3 and 4 indicate that the equations (23) and (16) provide some good approximations of the solution to the original problem (1) for small values of the parameter ε. These results need smallness assumptions on the initial data for which, in some sense, the diffusion driven regime applies. Note however that the smallness hypothesis is independent of ε.

The remainder of the paper is organized as follows. We first deal with the system (23). The first task is to prove that (23) is a 2×2 symmetrizable hyperbolic system of balance laws. Then we shall show that (23) satisfies the so-called Kawashima condition (see below and [16, 35] for the definition); for fixed $\varepsilon > 0$, this property yields the global existence of smooth solutions for small initial data, by applying the result of [16]. However, a careful analysis of the energy estimates of [16] for the system (23) leads to uniform conclusions with respect to ε, and allows us to prove Theorems 2 and 3. (In the general case, it is not clear whether the result of [16] can be made uniform with respect to ε). The proof of Theorems 2 and 3 is detailed in section 2.

Eventually, we turn to the proof of Theorem 4 in section 3, which relies on a suitable regularization technique and on standard energy estimates for quasilinear parabolic equations. The key point is to check that the estimates are uniform with respect to ε.

2 Proof of Theorem 2

For hyperbolic systems of balance laws that are partially dissipative and are endowed with a strictly convex entropy, a general theory of global existence of smooth solutions (with small initial data) has been developed by Hanouzet and Natalini [16]. This global existence result relies on the so-called Kawashima condition (see e.g. [35] and the references in [16]). In this section, we are going to show that the system (23) satisfies all the assumptions of [16], and thus admits global smooth solutions. However, it is not clear whether the bounds obtained in [16] are uniform with respect to the small parameter ε. For the particular system (23) that we study here, we give a detailed proof of this *uniformity* with respect to ε. The method is more or less the one developed in [16], but some refined estimates are needed. To simplify, we shall consider that $\gamma = 1$ in (23). The general case is similar.

2.1 General facts on the hyperbolic system

In this paragraph, we establish some basic properties of the system (23). We always consider a measure μ that satisfies Assumptions 1-2-3, as well as the assumptions of Lemma 4. We also keep the notation $\mathbb{F}$ for the Laplace transform, and the notation $\mathbb{G} = \mathbb{F}'/\mathbb{F}$ (see Lemma 4). The function ψ is defined by (12).

We focus on the system without source term (the unknowns are now denoted ρ and J for convenience):

$$\begin{cases} \partial_t \rho + \partial_x J = 0\,, \\ \varepsilon^2\, \partial_t J + \partial_x \left(\rho\, \psi(\varepsilon \dfrac{J}{\rho}) \right) = 0\,. \end{cases} \tag{27}$$

In all what follows, we use the notation $u := \varepsilon J/\rho$ to denote the (rescaled) velocity.

System (27) meets the classical hyperbolicity properties, as shown in the following:

Proposition 1. *The system* (27) *is stricly hyperbolic in the open set* $\{(\rho, J)/\rho > 0\,, \varepsilon\,|J| < \rho\}$. *Its characteristic speeds* $\lambda^\varepsilon_{1,2}$ *are given by*

$$\lambda^\varepsilon_{1,2}(\rho, J) := \frac{1}{\varepsilon}\, \boldsymbol{\lambda}_{1,2}(\frac{\varepsilon J}{\rho})\,, \quad \textit{with} \quad \boldsymbol{\lambda}_{1,2}(u) := \frac{\psi'(u) \mp \sqrt{\psi'(u)^2 - 4u\psi'(u) + 4\psi(u)}}{2}\,. \tag{28}$$

Moreover, the function

$$\mathbb{H}(\rho, J) := \rho\, \ln \rho - \rho\, \ln \left[\mathbb{F} \circ \mathbb{G}^{(-1)}(\frac{\varepsilon J}{\rho}) \right] + \varepsilon J\, \mathbb{G}^{(-1)}(\frac{\varepsilon J}{\rho})\,, \tag{29}$$

is a strictly convex entropy for (27)*. The corresponding flux is given by*

$$\mathcal{F}(\rho, J) := J \ln\rho - J \ln\left[\mathbb{F}\circ\mathbb{G}^{(-1)}(\frac{\varepsilon J}{\rho})\right] + \frac{1}{\varepsilon}\,\rho\,\psi(\frac{\varepsilon J}{\rho})\,\mathbb{G}^{(-1)}(\frac{\varepsilon J}{\rho})\,. \tag{30}$$

Proof. If we write system (27) under the compact form

$$\partial_t \begin{pmatrix}\rho\\ J\end{pmatrix} + \partial_x \mathcal{A}_\varepsilon(\rho, J) = 0\,,$$

we compute the Jacobian matrix

$$D\mathcal{A}_\varepsilon(\rho, J) = \begin{pmatrix} 0 & 1\\ \dfrac{1}{\varepsilon^2}(\psi(u) - u\psi'(u)) & \dfrac{1}{\varepsilon}\psi'(u)\end{pmatrix}\,.$$

The reader will then check that the eigenvalues of $D\mathcal{A}_\varepsilon(\rho, J)$ are real, distinct, and given by (28). The discriminant of the characteristic polynomial is positive since

$$\psi'(u)^2 - 4u\psi'(u) + 4\psi(u) = (\psi'(u) - 2u)^2 + 4(\psi(u) - u^2) \geq 4\,\mathbb{G}'\circ\mathbb{G}^{(-1)}(u) > 0\,.$$

Thus, the system (27) is strictly hyperbolic. Eigenvectors of the Jacobian $D\mathcal{A}_\varepsilon(\rho, J)$ are given by

$$\boldsymbol{r}^\varepsilon_{1,2}(\rho, J) = \begin{pmatrix} 1\\ \lambda^\varepsilon_{1,2}(\rho, J)\end{pmatrix}\,. \tag{31}$$

That $\mathbb{H}$ is an entropy for (27) with flux $\mathcal{F}$ is a simple application of the chain rule. The calculations are omitted. The Hessian matrix of $\mathbb{H}$ is

$$D^2\mathbb{H}(\rho, J) = \frac{1}{\rho\,\mathbb{G}'\circ\mathbb{G}^{(-1)}(u)}\begin{pmatrix}\psi(u) & -\varepsilon\,u\\ -\varepsilon\,u & \varepsilon^2\end{pmatrix}\,,$$

therefore $\mathbb{H}$ is a strictly convex function of the conservative variables (ρ, J). □

Remark 6. *The expression of the entropy might look complicated, though it is very natural. As a matter of fact, the entropy* $\mathbb{H}(\rho, J)$ *is nothing but*

$$\mathbb{H}(\rho, J) = \int_V \widehat{f}_\varepsilon\,\ln\widehat{f}_\varepsilon\,d\mu(v)\,,$$

where $\widehat{f}_\varepsilon$ *is the minimizer obtained in* (21)-(22)*. That* $\mathbb{H}$ *is an entropy was already noted in [22].*

Observe that the characteristic speeds of (27) only depend on the velocity u. This situation looks very much like the isothermal Euler system (25).

2.2 Preliminary transformations

Recall that system (23) reads (dropping in this section the $\widehat{\cdot}$ symbols):

$$\begin{cases}\partial_t\rho + \partial_x J = 0\,,\\ \partial_t J + \dfrac{1}{\varepsilon^2}\,\partial_x\left(\rho\,\psi(\dfrac{\varepsilon J}{\rho})\right) = -\dfrac{1}{\varepsilon^2}\,J\,,\end{cases} \tag{32}$$

and thus has the classical structure described by Equation (6) in [16]. The set of equilibrium points is the half-line $\{\rho > 0, J = 0\}$. Furthermore, we have seen that a strictly convex entropy for (32) is given by the following formula (see Proposition 1):

$$\mathbb{H}(\rho, J) := \rho\,\ln\rho - \rho\,\ln\left[\mathbb{F}\circ\mathbb{G}^{(-1)}(\frac{\varepsilon J}{\rho})\right] + \varepsilon J\,\mathbb{G}^{(-1)}(\frac{\varepsilon J}{\rho})\,.$$

Using the velocity $u := \varepsilon J/\rho$, (32) reads

$$\begin{cases} \partial_t\rho + \dfrac{1}{\varepsilon}\,\partial_x(\rho u) = 0\,, \\ \partial_t(\rho u) + \dfrac{1}{\varepsilon}\,\partial_x\left(\rho\,\psi(u)\right) = -\dfrac{1}{\varepsilon^2}\,\rho u\,. \end{cases}$$

In order to simplify some computations, it is convenient to rescale the time variable, and we introduce the new unknowns

$$\widetilde{\rho}(t,x) := \rho(\varepsilon t, x)\,, \quad \widetilde{u}(t,x) := u(\varepsilon t, x)\,. \tag{33}$$

Then $(\rho, J = \rho u/\varepsilon)$ is a global smooth solution to (32) if, and only if, $(\widetilde{\rho}, \widetilde{u})$ is a global smooth solution to

$$\begin{cases} \partial_t\widetilde{\rho} + \partial_x(\widetilde{\rho}\,\widetilde{u}) = 0\,, \\ \partial_t(\widetilde{\rho}\,\widetilde{u}) + \partial_x\left(\widetilde{\rho}\,\psi(\widetilde{u})\right) = -\dfrac{1}{\varepsilon}\,\widetilde{\rho}\,\widetilde{u}\,. \end{cases} \tag{34}$$

A strictly convex entropy for (34) is given by

$$\widetilde{\mathbb{H}}(\widetilde{\rho}, \widetilde{J}) := \widetilde{\rho}\,\ln\widetilde{\rho} - \widetilde{\rho}\,\ln\left[\mathbb{F}\circ\mathbb{G}^{(-1)}\left(\frac{\widetilde{J}}{\widetilde{\rho}}\right)\right] + \widetilde{J}\,\mathbb{G}^{(-1)}\left(\frac{\widetilde{J}}{\widetilde{\rho}}\right).$$

Here, we have used the notation $\widetilde{J} = \widetilde{\rho}\,\widetilde{u}$.

We are going to construct global smooth solutions to (34) that are close to a given equilibrium point, and we shall thus obtain global smooth solutions to (32).

We consider a fixed equilibrium point $(\overline{\rho}, 0)$ for (34), $\overline{\rho} > 0$, and following [16], we define the entropic variables:

$$W = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} := \nabla\widetilde{\mathbb{H}}(\widetilde{\rho}, \widetilde{J}) - \nabla\widetilde{\mathbb{H}}(\overline{\rho}, 0) = \begin{pmatrix} \ln(\widetilde{\rho}/\overline{\rho}) - \ln\mathbb{F}\circ\mathbb{G}^{(-1)}(\widetilde{u}) \\ \mathbb{G}^{(-1)}(\widetilde{u}) \end{pmatrix}, \tag{35}$$

in order to use Godunov's symmetrization. Note that this is a change of variables from the set $\{(\widetilde{\rho}, \widetilde{J})/\widetilde{\rho} > 0, |\widetilde{J}| < \widetilde{\rho}\}$ to the whole plane $\mathbb{R}^2$. After a few simplifications, one shows that for smooth solutions $(\widetilde{\rho}, \widetilde{J})$ away from vacuum, (34) is equivalent to a quasilinear symmetric hyperbolic system for $W = (W_1, W_2)$:

$$A_0(W_2)\,\partial_t W + A_1(W_2)\,\partial_x W = -\frac{1}{\varepsilon}\begin{pmatrix} 0 \\ \mathbb{G}(W_2) \end{pmatrix}, \tag{36}$$

where $A_0(W_2)$ is a symmetric positive definite matrix, $A_1(W_2)$ is symmetric, and are defined as follows:

$$A_0(W_2) := \begin{pmatrix} 1 & \mathbb{G}(W_2) \\ \mathbb{G}(W_2) & \psi\circ\mathbb{G}(W_2) \end{pmatrix}, \tag{37}$$

$$A_1(W_2) := \begin{pmatrix} \mathbb{G}(W_2) & \psi\circ\mathbb{G}(W_2) \\ \psi\circ\mathbb{G}(W_2) & \Phi\circ\mathbb{G}(W_2) \end{pmatrix}, \quad \Phi(u) := (\psi(u) - u\psi'(u))u + \psi(u)\psi'(u)\,. \tag{38}$$

Thanks to our new time scaling, both A_0 and A_1 are independent of ε. Note that they depend only on the second component of the vector W. This will be extensively used in what follows.

In the entropic variables, the set of equilibrium points is the line $\{W_2 = 0\}$. Since $\mathbb{G}$ is an odd function that is increasing, we can write $\mathbb{G}(W_2) = W_2\,\Gamma(W_2)$, where $\Gamma(W_2) > 0$, and $\Gamma \in \mathcal{C}^\infty(\mathbb{R})$. Therefore (36) is *strictly entropy dissipative* in the sense of [16, definition 2].

We now check the Kawashima condition, using Lemma 2 in [16]. In the entropic variables, the equilibrium point $(\overline{\rho}, 0)$ becomes the origin, see (35). Let $\kappa \in \mathbb{R}$, and let $X \in \mathbb{R} \setminus \{0\}$. Then the Kawashima condition is equivalent to

$$[\kappa A_0(0) + A_1(0)] \begin{pmatrix} X \\ 0 \end{pmatrix} \neq 0\,.$$

Using (37)-(38), we compute

$$[\kappa A_0(0) + A_1(0)] \begin{pmatrix} X \\ 0 \end{pmatrix} = X \begin{pmatrix} \kappa \\ d \end{pmatrix} \neq 0\,.$$

The Kawashima condition is thus satisfied, and we can apply the global existence result of [16]: there exists $\delta = \delta(\overline{\rho}, \varepsilon) > 0$ such that, for any $W_0 \in H^2(\mathbb{R})$ with $\|W_0\|_{H^2(\mathbb{R})} \leq \delta$, (36) has a unique global smooth solution with initial data W_0. (Here, smooth means $\mathcal{C}([0,+\infty[; H^2(\mathbb{R})) \cap \mathcal{C}^1([0,+\infty[; H^1(\mathbb{R})))$. In the next paragraph, we are going to show that the radius $\delta(\overline{\rho}, \varepsilon)$ can be chosen independent of ε.

2.3 Energy estimates

We first introduce some classical notations. The Sobolev space $H^k(\mathbb{R})$ $(k = 0, 1, 2)$, is equipped with the usual norm

$$\|f\|_k^2 := \sum_{j=0}^{k} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 \, dx\,.$$

Given any positive time $T > 0$, and any function $W = (W_1, W_2) \in \mathcal{C}([0,T]; H^2(\mathbb{R}))$, we introduce the energy functional

$$N_\varepsilon(T)^2 := \sup_{0 \leq t \leq T} \|W(t)\|_2^2 + \frac{1}{\varepsilon} \int_0^T \|W_2(\tau)\|_2^2 \, d\tau + \varepsilon \int_0^T \|\partial_x W_1(\tau)\|_1^2 \, d\tau\,. \tag{39}$$

Let us remark that the classical Sobolev imbeddings yield the following useful inequalities:

$$\|W\|_{L^\infty([0,T]; W^{1,\infty}(\mathbb{R}))} \leq \underline{C}\, N_\varepsilon(T)\,, \qquad \|W_2\|_{L^2([0,T]; W^{1,\infty}(\mathbb{R}))} \leq \underline{C}\, \sqrt{\varepsilon}\, N_\varepsilon(T)\,,$$

$$\|\partial_x W\|_{L^2([0,T]; L^\infty(\mathbb{R}))} \leq \underline{C}\, \frac{N_\varepsilon(T)}{\sqrt{\varepsilon}}\,,$$

for some numerical constant $\underline{C}$.

Note that (36) has solutions in the space $\mathcal{C}([0,T]; H^2(\mathbb{R})) \cap \mathcal{C}^1([0,T]; H^1(\mathbb{R}))$, at least for a small enough time T, thanks to Kato's result [20]. We are going to prove the following:

Proposition 2. *Let $T > 0$, and assume that $W \in \mathcal{C}([0,T]; H^2(\mathbb{R})) \cap \mathcal{C}^1([0,T]; H^1(\mathbb{R}))$ is a solution to* (36). *There exists an increasing function $C : \mathbb{R}^+ \to \mathbb{R}^+$, that is independent of T, ε, and W, such that the following inequality holds:*

$$N_\varepsilon(T)^2 \leq C(\|W\|_{L^\infty([0,T] \times \mathbb{R})}) \left\{ N_\varepsilon(0)^2 + N_\varepsilon(T)^3 (1 + N_\varepsilon(T)) \right\}.$$

The proof splits into several steps. One first estimates the $L^\infty(H^2)$ norm of W and the $L^2(H^2)$ norm of the component W_2, by using the classical procedure of [20]. However, special attention is needed when dealing with the second order derivatives, in order to derive uniform bounds. Eventually, one recovers the $L^2(H^1)$ estimate of $\partial_x W_1$ by using the Kawashima condition. (This final step was already achieved in [16], but it is crucial to check the independence of the constants with respect to ε.)

2.3.1 The $L^\infty(L^2)$ estimate of W

Let $W \in \mathcal{C}([0,T]; H^2(\mathbb{R})) \cap \mathcal{C}^1([0,T]; H^1(\mathbb{R}))$ be a solution to (36). In the original variables, it corresponds to a solution $(\widetilde{\rho}, \widetilde{J}) \in \mathcal{C}([0,T]; \overline{\rho} + H^2(\mathbb{R})) \times \mathcal{C}([0,T]; H^2(\mathbb{R}))$ to (34), that is bounded away from vacuum.

To obtain the $L^\infty(L^2)$ estimate, we slightly modify the entropy $\widetilde{\mathbb{H}}$, and define

$$\eta(\widetilde{\rho}, \widetilde{J}) := \widetilde{\mathbb{H}}(\widetilde{\rho}, \widetilde{J}) - \widetilde{\mathbb{H}}(\overline{\rho}, 0) - [\nabla \widetilde{\mathbb{H}}(\overline{\rho}, 0)] \cdot (\widetilde{\rho}, \widetilde{J}),$$

which is still, of course, a strictly convex entropy for the system (34). Its flux is denoted $q(\widetilde{\rho}, \widetilde{J})$. Moreover, the entropy η satisfies

$$\eta(\overline{\rho}, 0) = 0, \quad \nabla \eta(\overline{\rho}, 0) = 0.$$

For the smooth solution $(\widetilde{\rho}, \widetilde{J}) \in \mathcal{C}([0,T]; \overline{\rho} + H^2(\mathbb{R})) \times \mathcal{C}([0,T]; H^2(\mathbb{R}))$ to (34), we integrate the balance law

$$\partial_t \eta + \partial_x q = -\frac{1}{\varepsilon} \widetilde{J} \, \mathbb{G}^{(-1)} \left(\frac{\widetilde{J}}{\widetilde{\rho}} \right),$$

over the strip $[0,t] \times \mathbb{R}^d$, and we obtain

$$\int_{\mathbb{R}} \eta \, dx \Big|_0^t + \frac{1}{\varepsilon} \int_0^t \!\! \int_{\mathbb{R}} \widetilde{J} \, \mathbb{G}^{(-1)} \left(\frac{\widetilde{J}}{\widetilde{\rho}} \right) dx \, ds = 0.$$

Using the convexity properties of η, we get

$$\frac{1}{C} \left(|\widetilde{\rho} - \overline{\rho}|^2 + |\widetilde{J}|^2 \right) \le \eta(\widetilde{\rho}, \widetilde{J}) \le C \left(|\widetilde{\rho} - \overline{\rho}|^2 + |\widetilde{J}|^2 \right),$$

where the constant C only depends on $\overline{\rho}$, and the norms $\|\widetilde{\rho} - \overline{\rho}\|_{L^\infty([0,t]\times\mathbb{R})}$, $\|\widetilde{J}\|_{L^\infty([0,t]\times\mathbb{R})}$. To conclude, we use the definition (35), and we thus derive the bounds

$$\frac{1}{C} |W|^2 \le \eta(\widetilde{\rho}, \widetilde{J}) \le C \, |W|^2,$$

where the constant C only depends on $\overline{\rho}$, and the norm $\|W\|_{L^\infty([0,t]\times\mathbb{R}^d)}$, but is independent of ε. Eventually, we obtain:

$$\|W(t)\|_0^2 + \frac{1}{\varepsilon} \int_0^t \|W_2(\tau)\|_0^2 \, d\tau \le C(\|W_2\|_{L^\infty([0,t]\times\mathbb{R})}) \, N_\varepsilon(0)^2. \tag{40}$$

2.3.2 The $L^\infty(H^1)$ estimate of W

One first differentiates (36) with respect to the space variable x, then takes the scalar product with $\partial_x W$, and integrates over the strip $[0,t] \times \mathbb{R}$. Defining the matrix

$$A(W_2) := A_0(W_2)^{-1} A_1(W_2) = \begin{pmatrix} 0 & \psi \circ \mathbb{G}(W_2) - \mathbb{G}(W_2)\, \psi' \circ \mathbb{G}(W_2) \\ 1 & \psi' \circ \mathbb{G}(W_2) \end{pmatrix}, \tag{41}$$

and performing some simplifications (see equations (52)-(53)-(54)-(55) in [16]), we obtain the relation

$$\frac{1}{2} \int_{\mathbb{R}} A_0(W_2) \partial_x W \cdot \partial_x W \, dx \Big|_0^t + \frac{1}{\varepsilon} \int_0^t \!\! \int_{\mathbb{R}} \mathbb{G}'(W_2) (\partial_x W_2)^2 \, dx \, ds = \int_0^t \!\! \int_{\mathbb{R}} \left[\frac{1}{2} (T_1 + T_2 - T_3) + T_4 \right] dx \, ds, \tag{42}$$

where we have set:

$$
\begin{aligned}
T_1 &:= (\partial_t W_2) A_0'(W_2) \partial_x W \cdot \partial_x W\,, \\
T_2 &:= (\partial_x W_2) A_0'(W_2) A(W_2) \partial_x W \cdot \partial_x W\,, \\
T_3 &:= (\partial_x W_2) A_0(W_2) A'(W_2) \partial_x W \cdot \partial_x W\,, \\
T_4 &:= \frac{1}{\varepsilon}\,(\partial_x W_2)\, A_0'(W_2) A_0^{-1}(W_2) \begin{pmatrix} 0 \\ \mathbb{G}(W_2) \end{pmatrix} \cdot \partial_x W\,.
\end{aligned}
$$

Using Cauchy-Schwarz' inequality, and the Sobolev imbedding $H^2(\mathbb{R}) \subset W^{1,\infty}(\mathbb{R})$, we obtain

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathbb{R}} |T_2| + |T_3|\, dx\, ds &\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\, \|\partial_x W\|_{L^\infty([0,t]\times\mathbb{R})} \int_0^t\!\!\int_{\mathbb{R}} |\partial_x W_2| |\partial_x W|\, dx\, ds \\
&\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\, N_\varepsilon(t)^3\,.
\end{aligned}
$$

Using (36)-(37)-(38), we compute the scalar equation

$$
\partial_t W_2 + \partial_x W_1 + \psi' \circ \mathbb{G}(W_2)\, \partial_x W_2 = -\frac{1}{\varepsilon}\, \frac{\mathbb{G}(W_2)}{\mathbb{G}'(W_2)}\,. \tag{43}
$$

Thanks to (43), we also obtain

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathbb{R}} |T_1|\, dx\, ds &\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} (|\partial_x W| + \frac{1}{\varepsilon}\,|W_2|)\, |\partial_x W_2|\, |\partial_x W|\, dx\, ds \\
&\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\, \|\partial_x W\|_{L^\infty([0,t]\times\mathbb{R})} \int_0^t\!\!\int_{\mathbb{R}} |\partial_x W|\, |\partial_x W_2| + \frac{1}{\varepsilon}\,|W_2|\, |\partial_x W_2|\, dx\, ds \\
&\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\, N_\varepsilon(t)^3\,.
\end{aligned}
$$

Eventually, we have

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathbb{R}} |T_4|\, dx\, ds &\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} \frac{1}{\varepsilon}\,|W_2|\, |\partial_x W_2|\, |\partial_x W|\, dx\, ds \\
&\leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\, N_\varepsilon(t)^3\,.
\end{aligned}
$$

The left hand side of (42) is easily estimated from below, and summing up, we obtain the $L^\infty(H^1)$ uniform estimate:

$$
\|\partial_x W(t)\|_0^2 + \frac{1}{\varepsilon} \int_0^t \|\partial_x W_2(\tau)\|_0^2\, d\tau \leq C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\, \left(N_\varepsilon(0)^2 + N_\varepsilon(t)^3\right)\,. \tag{44}
$$

2.3.3 The $L^\infty(H^2)$ estimate of W

The beginning is the same as in the former paragraph. One differentiates twice (36) with respect to the space variable x, then takes the scalar product with $\partial_{xx} W$, and integrates over the strip $[0,t] \times \mathbb{R}$. Using the relations (58)-(59) of [16], we are led to the relation

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}} A_0(W_2) \partial_{xx} W \cdot \partial_{xx} W\, dx \Big|_0^t + \frac{1}{\varepsilon} \int_0^t\!\!\int_{\mathbb{R}} \mathbb{G}'(W_2) (\partial_{xx} W_2)^2\, dx\, ds \\
&\qquad = \int_0^t\!\!\int_{\mathbb{R}} \Big[\frac{1}{2}(S_1 + S_2) + 2(S_3 - S_4) + S_5 + S_6 - S_7 - S_8 - S_9\Big]\, dx\, ds\,,
\end{aligned} \tag{45}
$$

where we have set:

$$
\begin{aligned}
S_1 &:= (\partial_t W_2)\, A_0'(W_2)\partial_{xx}W\cdot\partial_{xx}W\,,\\
S_2 &:= (\partial_x W_2)\, A_1'(W_2)\partial_{xx}W\cdot\partial_{xx}W\,,\\
S_3 &:= \frac{\partial_x W_2}{\varepsilon}\, A_0'(W_2)\partial_x\left[A_0^{-1}(W_2)\begin{pmatrix}0\\ \mathbb{G}(W_2)\end{pmatrix}\right]\cdot\partial_{xx}W\,,\\
S_4 &:= (\partial_x W_2)\, A_0(W_2)A'(W_2)\partial_{xx}W\cdot\partial_{xx}W\,,\\
S_5 &:= \frac{\partial_{xx} W_2}{\varepsilon}\, A_0'(W_2)A_0^{-1}(W_2)\begin{pmatrix}0\\ \mathbb{G}(W_2)\end{pmatrix}\cdot\partial_{xx}W\,,\\
S_6 &:= \frac{(\partial_x W_2)^2}{\varepsilon}\, A_0''(W_2)A_0^{-1}(W_2)\begin{pmatrix}0\\ \mathbb{G}(W_2)\end{pmatrix}\cdot\partial_{xx}W\,,\\
S_7 &:= (\partial_{xx} W_2)\, A_0(W_2)A'(W_2)\partial_{x}W\cdot\partial_{xx}W\,,\\
S_8 &:= (\partial_x W_2)^2\, A_0(W_2)A''(W_2)\partial_{x}W\cdot\partial_{xx}W\,,\\
S_9 &:= \frac{1}{\varepsilon}\,\mathbb{G}''(W_2)\,(\partial_x W_2)^2\,\partial_{xx}W_2\,.
\end{aligned}
$$

Let us start with the easy terms. Since $\mathbb{G}''$ is an odd function, we have:

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathbb{R}} |S_9|\,dx\,ds &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}\frac{|W_2|}{\varepsilon}\,(\partial_x W_2)^2\,|\partial_{xx}W_2|\,dx\,ds\\
&\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,\|\partial_x W_2\|^2_{L^\infty([0,t]\times\mathbb{R})}\int_0^t\!\!\int_{\mathbb{R}}\frac{|W_2|\,|\partial_{xx}W_2|}{\varepsilon}\,dx\,ds\\
&\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^4\,.
\end{aligned}
$$

In a similar way, we have

$$
\int_0^t\!\!\int_{\mathbb{R}} |S_8|\,dx\,ds \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^4\,,
$$

and

$$
\int_0^t\!\!\int_{\mathbb{R}} |S_7|\,dx\,ds \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3\,.
$$

Observe that the first column of $A'(W_2)$ is zero, see (41), hence we have

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathbb{R}} |S_4|\,dx\,ds &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}|\partial_x W_2|\,|\partial_{xx}W_2|\,|\partial_{xx}W|\,dx\,ds\\
&\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,\|\partial_x W\|_{L^\infty([0,t]\times\mathbb{R})}\int_0^t\!\!\int_{\mathbb{R}}|\partial_{xx}W_2||\partial_{xx}W|\,dx\,ds\\
&\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3\,.
\end{aligned}
$$

The five remaining terms in the right-hand side of (45) are estimated in a slightly different way. We shall use the following version of Hölder's inequality:

$$
\begin{aligned}
\int_0^t\!\!\int_{\mathbb{R}} |fgh(s,x)|\,dx\,ds &\le \int_0^t \|f(s,\cdot)\|_{L^\infty(\mathbb{R})}\,\|g(s,\cdot)\|_{L^2(\mathbb{R})}\,\|h(s,\cdot)\|_{L^2(\mathbb{R})}\,ds\\
&\le \|g\|_{L^\infty(0,t;L^2(\mathbb{R}))}\,\|f\|_{L^2(0,t;L^\infty(\mathbb{R}))}\,\|h\|_{L^2((0,t)\times\mathbb{R})}\,.
\end{aligned}
$$

Combining this inequality to the Sobolev embedding leads to

$$
\int_0^t\!\!\int_{\mathbb{R}} |fgh(s,x)|\,dx\,ds \le \|g\|_{L^\infty(0,t;L^2(\mathbb{R}))}\,\|f\|_{L^2(0,t;H^1(\mathbb{R}))}\,\|h\|_{L^2((0,t)\times\mathbb{R})}\,.
$$

This is where our analysis of the source terms differs from what was done in [16].

Applying the strategy described just above, we get

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |S_6|\,dx\,ds &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} \frac{(\partial_x W_2)^2}{\varepsilon}\,|W_2|\,|\partial_{xx}W|\,dx\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,\|\partial_x W\|_{L^\infty([0,t]\times\mathbb{R})} \int_0^t\!\!\int_{\mathbb{R}} \frac{|\partial_x W_2|\,|W_2|}{\varepsilon}\,|\partial_{xx}W|\,dx\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)\int_0^t \|\partial_{xx}W(s)\|_{L^2(\mathbb{R})}\,\frac{\|\partial_x W_2(s)\|_{L^2(\mathbb{R})}}{\sqrt{\varepsilon}}\,\frac{\|W_2(s)\|_{L^\infty(\mathbb{R})}}{\sqrt{\varepsilon}}\,ds\,,\end{aligned}$$

and now, we use the obvious inequalities (see (39)):

$$\|\partial_{xx}W(s)\|_{L^2(\mathbb{R})} \le N_\varepsilon(s) \le N_\varepsilon(t)\,, \quad \|W_2(s)\|_{L^\infty(\mathbb{R})} \le C\,\|W_2(s)\|_{H^1(\mathbb{R})}\,,$$

and we obtain

$$\int_0^t\!\!\int_{\mathbb{R}} |S_6|\,dx\,ds \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^4\,.$$

In a similar way, we have

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |S_5|\,dx\,ds &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} \frac{|\partial_{xx}W_2|\,|W_2|}{\varepsilon}\,|\partial_{xx}W|\,dx\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t \|\partial_{xx}W(s)\|_{L^2(\mathbb{R})}\,\frac{\|\partial_{xx} W_2(s)\|_{L^2(\mathbb{R})}}{\sqrt{\varepsilon}}\,\frac{\|W_2(s)\|_{L^\infty(\mathbb{R})}}{\sqrt{\varepsilon}}\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3\,.\end{aligned}$$

To estimate S_3, we remark that the vector

$$\partial_x \left[A_0^{-1}(W_2) \begin{pmatrix} 0 \\ \mathbb{G}(W_2) \end{pmatrix} \right]$$

can be written under the form $(\partial_x W_2)\vartheta(W_2)$, for some appropriate vector $\vartheta(W_2)$. Therefore, we have

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |S_3|\,dx\,ds &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} \frac{(\partial_x W_2)^2}{\varepsilon}\,|\partial_{xx}W|\,dx\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t \|\partial_{xx}W(s)\|_{L^2(\mathbb{R})}\,\frac{\|\partial_x W_2(s)\|_{L^2(\mathbb{R})}}{\sqrt{\varepsilon}}\,\frac{\|\partial_x W_2(s)\|_{L^\infty(\mathbb{R})}}{\sqrt{\varepsilon}}\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3\,,\end{aligned}$$

where, for the last inequality, we have used Sobolev's imbedding:

$$\|\partial_x W_2(s)\|_{L^\infty(\mathbb{R})} \le C\,\|W_2(s)\|_{H^2(\mathbb{R})}\,.$$

Eventually, we have

$$\begin{aligned}\int_0^t\!\!\int_{\mathbb{R}} |S_2|\,dx\,ds &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} |\partial_x W_2|\,|\partial_{xx}W|^2\,dx\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t \|\partial_x W_2(s)\|_{L^\infty(\mathbb{R})}\,\|\partial_{xx}W(s)\|^2_{L^2(\mathbb{R})}\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)\int_0^t \|W_2(s)\|_{H^2(\mathbb{R})}\,\|\partial_x W(s)\|_{H^1(\mathbb{R})}\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3\,.\end{aligned}$$

Using (43), we also have

$$\int_0^t\!\!\int_{\mathbb{R}} |S_1|\,dx\,ds \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t\!\!\int_{\mathbb{R}} (|\partial_x W| + \frac{|W_2|}{\varepsilon})\,|\partial_{xx} W_2|\,|\partial_{xx} W|\,dx\,ds$$
$$\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,\|\partial_x W\|_{L^\infty([0,t]\times\mathbb{R})} \int_0^t\!\!\int_{\mathbb{R}} |\partial_{xx} W_2|\,|\partial_{xx} W|\,dx\,ds$$
$$+ C(\|W\|_{L^\infty([0,t]\times\mathbb{R})}) \int_0^t \frac{\|W_2(s)\|_{L^\infty(\mathbb{R})}}{\sqrt{\varepsilon}}\,\frac{\|\partial_{xx} W_2(s)\|_{L^2(\mathbb{R})}}{\sqrt{\varepsilon}}\,\|\partial_{xx} W(s)\|_{L^2(\mathbb{R})}\,ds$$
$$\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3\,.$$

Going back to (45), the sum of all the estimates above yields the expected $L^\infty(H^2)$ bound:

$$\|\partial_{xx} W(t)\|_0^2 + \frac{1}{\varepsilon}\int_0^t \|\partial_{xx} W_2(\tau)\|_0^2\,d\tau \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,\big(N_\varepsilon(0)^2 + N_\varepsilon(t)^3 + N_\varepsilon(t)^4\big)\,. \quad (46)$$

2.3.4 The $L^2(H^1)$ estimate of $\partial_x W_1$

In this paragraph, we follow the method developed in [16]. Recall that the diffusion coefficient d, that is given by Assumption 3, satisfies $d = \mathbb{G}'(0) = \psi(0)$.

We begin with the following elementary result:

Lemma 5. *Let K denote the matrix*

$$K := \begin{pmatrix} 0 & 1/d \\ -1 & 0 \end{pmatrix}.$$

Then $K\,A_0(0)$ is skew-symmetric, and

$$K\,A_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & -d \end{pmatrix}.$$

Following [16], we rewrite (36) as

$$A_0(0)\,\partial_t W + A_1(0)\,\partial_x W = H_\varepsilon(W_2, \partial_x W)\,, \quad (47)$$

with

$$H_\varepsilon(W_2,\partial_x W) := \big[A_1(0) - A_1(W_2) + (A_0(0) - A_0(W_2))A(W_2)\big]\,\partial_x W$$
$$- \frac{1}{\varepsilon}\big[(A_0(0) - A_0(W_2))A_0^{-1}(W_2) + I\big]\begin{pmatrix} 0 \\ \mathbb{G}(W_2) \end{pmatrix}. \quad (48)$$

We multiply (47) by εK, (K is the matrix defined in Lemma 5), then take the scalar product with $\partial_x W$, and integrate over the strip $[0,t]\times\mathbb{R}$:

$$\int_0^t\!\!\int_{\mathbb{R}} \varepsilon\,(KA_0(0)\partial_t W + KA_1(0)\partial_x W)\cdot\partial_x W\,dx\,ds = \int_0^t\!\!\int_{\mathbb{R}} \varepsilon\,[K\,H_\varepsilon(W_2,\partial_x W)]\cdot\partial_x W\,dx\,ds\,. \quad (49)$$

Using relation (70) in [16], we have

$$\int_0^t\!\!\int_{\mathbb{R}} \varepsilon\,KA_0(0)\partial_t W\cdot\partial_x W\,dx\,ds = -\frac{\varepsilon}{2}\int_{\mathbb{R}} KA_0(0)\partial_x W\cdot W\,dx\Big|_0^t \ge -C\,\varepsilon\,(\|W(t)\|_1^2 + \|W(0)\|_1^2)\,,$$

and Lemma 5 gives

$$\int_0^t\!\!\int_{\mathbb{R}} \varepsilon\,KA_1(0)\partial_x W\cdot\partial_x W\,dx\,ds = \varepsilon\int_0^t\!\!\int_{\mathbb{R}} (\partial_x W_1)^2\,dx\,ds - d\,\varepsilon\int_0^t\!\!\int_{\mathbb{R}} (\partial_x W_2)^2\,dx\,ds\,.$$

Now, we observe that

$$\left|\left[A_1(0) - A_1(W_2) + (A_0(0) - A_0(W_2))A(W_2)\right]\partial_x W\right| \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,|W_2|\,|\partial_x W|\,,$$
$$\frac{1}{\varepsilon}\left|(A_0(0) - A_0(W_2))A_0^{-1}(W_2)\begin{pmatrix}0\\ \mathbb{G}(W_2)\end{pmatrix}\right| \le \frac{1}{\varepsilon}\,C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,|W_2|^2\,,$$

and we compute

$$-K\begin{pmatrix}0\\ \mathbb{G}(W_2)\end{pmatrix}\cdot\partial_x W = -\frac{1}{d}\,\mathbb{G}(W_2)\,\partial_x W_1\,.$$

Consequently, using (48), we obtain

$$\begin{aligned}&\int_0^t\!\!\int_{\mathbb{R}}\left|\varepsilon\left[K\,H_\varepsilon(W_2,\partial_x W)\right]\cdot\partial_x W\right|dx\,ds\\ &\qquad\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}\varepsilon\,|W_2||\partial_x W|^2 + |W_2|^2\,|\partial_x W| + |W_2|\,|\partial_x W_1|\,dx\,ds\\ &\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3 + \frac{\varepsilon}{2}\int_0^t\!\!\int_{\mathbb{R}}(\partial_x W_1)^2\,dx\,ds + \frac{C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})}{\varepsilon}\int_0^t\!\!\int_{\mathbb{R}}(W_2)^2\,dx\,ds\,.\end{aligned}$$

Using these estimates in (49) gives

$$\varepsilon\int_0^t\|\partial_x W_1(\tau)\|_0^2\,d\tau \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\left(N_\varepsilon(0)^2 + N_\varepsilon(t)^3 + \|W(t)\|_1^2 + \frac{1}{\varepsilon}\int_0^t\|W_2(\tau)\|_1^2\,d\tau\right)\,.$$

We now use (40) and (44) to obtain

$$\varepsilon\int_0^t\|\partial_x W_1(\tau)\|_0^2\,d\tau \le C(\|W_2\|_{L^\infty([0,t]\times\mathbb{R})})\,(N_\varepsilon(0)^2 + N_\varepsilon(t)^3 + N_\varepsilon(t)^4)\,. \tag{50}$$

For the second derivative of W_1,we proceed in the same way. We differentiate (47) with respect to x, multiply by $\varepsilon\,K$, take the scalar product with $\partial_{xx}W$, and integrate over the strip $[0,t]\times\mathbb{R}$:

$$\int_0^t\!\!\int_{\mathbb{R}}\varepsilon\,(KA_0(0)\partial_{tx}W + KA_1(0)\partial_{xx}W)\cdot\partial_{xx}W\,dx\,ds = \int_0^t\!\!\int_{\mathbb{R}}\varepsilon\left[K\left(\partial_x H_\varepsilon(W_2,\partial_x W)\right)\right]\cdot\partial_x W\,dx\,ds\,. \tag{51}$$

Following what was done earlier, see [16] for the details, we can first derive the lower bound

$$\begin{aligned}&\int_0^t\!\!\int_{\mathbb{R}}\varepsilon\,(KA_0(0)\partial_{tx}W + KA_1(0)\partial_{xx}W)\cdot\partial_{xx}W\,dx\,ds\\ &\qquad\ge -C\,\varepsilon\,(\|W(t)\|_2^2 + \|W(0)\|_2^2) + \varepsilon\int_0^t\|\partial_{xx}W_1(\tau)\|_0^2\,d\tau - d\,\varepsilon\int_0^t\|\partial_{xx}W_2(\tau)\|_0^2\,d\tau\,.\end{aligned}$$

Starting from (48), we can write $\partial_x[H_\varepsilon(W_2,\partial_x W_2)]$ under the form

$$\begin{aligned}\partial_x[H_\varepsilon(W_2,\partial_x W)] &= \left[A_1(0) - A_1(W_2) + (A_0(0) - A_0(W_2))A(W_2)\right]\partial_{xx}W + (\partial_x W_2)\,\mathcal{A}_\flat(W_2)\,\partial_x W\\ &\quad - \frac{1}{\varepsilon}\left[(A_0(0) - A_0(W_2))A_0^{-1}(W_2) + I\right]\begin{pmatrix}0\\ \mathbb{G}'(W_2)\,\partial_x W_2\end{pmatrix} + \frac{\partial_x W_2}{\varepsilon}\,\mathcal{A}_\sharp(W_2)\begin{pmatrix}0\\ \mathbb{G}(W_2)\end{pmatrix}\,,\end{aligned}$$

where $\mathcal{A}_\flat(W_2)$ and $\mathcal{A}_\sharp(W_2)$ are matrices that depend only on W_2, and whose exact expression is useless. We thus obtain

$$\begin{aligned}
&\int_0^t\!\!\int_{\mathbb{R}} \left|\varepsilon\left[K\left(\partial_x H_\varepsilon(W_2,\partial_x W)\right)\right]\cdot\partial_{xx}W\right| dx\,ds \\
&\qquad\qquad \le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\int_0^t\!\!\int_{\mathbb{R}}\Big\{\varepsilon\,|W_2||\partial_{xx}W|^2+\varepsilon\,|\partial_x W_2|\,|\partial_x W|\,|\partial_{xx}W| \\
&\qquad\qquad\qquad\qquad + |W_2|\,|\partial_x W_2|\,|\partial_{xx}W| + |\partial_x W_2|\,|\partial_{xx}W_1|\Big\}\,dx\,ds \\
&\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,N_\varepsilon(t)^3+\frac{\varepsilon}{2}\int_0^t\|\partial_{xx}W_1(\tau)\|_0^2\,d\tau+\frac{C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})}{\varepsilon}\int_0^t\|W_2(\tau)\|_2^2\,d\tau\,.
\end{aligned}$$

In the same way as we derived (50), we obtain here:

$$\varepsilon\int_0^t\|\partial_{xx}W_1(\tau)\|_0^2\,d\tau\le C(\|W\|_{L^\infty([0,t]\times\mathbb{R})})\,(N_\varepsilon(0)^2+N_\varepsilon(t)^3+N_\varepsilon(t)^4)\,. \tag{52}$$

The sum of (40), (44), (46), (50), and (52) gives the result of Proposition 2.

2.4 End of the proof of Theorem 2

To conclude the proof, we follow [31]. Using Proposition 2, we first deduce that there exists a numerical constant $C_0\ge 1$ such that, if $W\in\mathcal{C}([0,T];H^2(\mathbb{R}))\cap\mathcal{C}^1([0,T];H^1(\mathbb{R}))$ is a solution to (36) that satisfies $N_\varepsilon(T)\le 1$, then W also satisfies

$$N_\varepsilon(T)^2\le C_0\,\left(N_\varepsilon(0)^2+N_\varepsilon(T)^3\right)\,.$$

The constant C_0 is, of course, independent of ε. Consequently, if W is a smooth solution on a time interval $[0,T]$ that satisfies $N_\varepsilon(T)\le 1/(2C_0)$, then W also satisfies

$$N_\varepsilon(T)\le\sqrt{2C_0}\,N_\varepsilon(0)\,. \tag{53}$$

Before going on, we observe that $N_\varepsilon(0)$ is independent of ε, see (39), since $N_\varepsilon(0)$ is just the H^2 norm of the initial data.

Consider an initial condition $W(0)\in H^2(\mathbb{R})$ such that $\|W(0)\|_2\le 1/[2(2C_0)^{3/2}]$. Assume that the corresponding smooth solution W to (36) is not global, and thus blows up in finite time, say at time $T_*>0$. This means that for some positive time T_0, one has

$$N_\varepsilon(T_0)=\frac{1}{4C_0}>N_\varepsilon(0)\,,\qquad\text{and}\qquad\forall\,t\in\,]T_0,T_*[\,,\quad N_\varepsilon(t)>\frac{1}{4C_0}\,.$$

Since $N_\varepsilon(T_0)<1/(2C_0)$, there exists a time $T_1\in\,]T_0,T_*[$ such that $N_\varepsilon(T_1)\le 1/(2C_0)$, and, applying (53), we obtain

$$N_\varepsilon(T_1)\le\sqrt{2C_0}\,N_\varepsilon(0)\le\frac{\sqrt{2C_0}}{2(2C_0)^{3/2}}\le\frac{1}{4C_0}\,.$$

We are led to a contradiction. The smooth solution is thus global in time for small enough initial data. The key point is that the smallness of the initial data is independent of ε. Moreover, when $\|W(0)\|_2\le 1/[2(2C_0)^{3/2}]$, one has the (global in time) uniform estimate

$$\forall\,t\ge 0\,,\quad N_\varepsilon(t)\le\min\left(\frac{1}{2C_0},\sqrt{2C_0}\,N_\varepsilon(0)\right)\,.$$

It remains to convert the result for the system (36) into a result for the system (34), and then into a result for the original system (32). Using (35), we first compute

$$\widetilde{\rho} = \overline{\rho}\; \mathbb{F}(W_2)\, \exp(W_1)\,, \quad \widetilde{u} = \mathbb{G}(W_2)\,.$$

Consequently, there exists a number $\delta_1 > 0$ that is independent of ε such that, if

$$\|\widetilde{\rho}_0 - \overline{\rho}\|_2 + \|\widetilde{u}_0\|_2 \le \delta_1\,,$$

then (34) has a global smooth solution $(\widetilde{\rho}, \widetilde{u})$, with initial data $(\widetilde{\rho}_0, \widetilde{u}_0)$, and that satisfies the global uniform estimate

$$\sup_{t\ge 0} (\|\widetilde{\rho}(t) - \overline{\rho}\|_2^2 + \|\widetilde{u}(t)\|_2^2) + \frac{1}{\varepsilon}\int_0^{+\infty} \|\widetilde{u}(t)\|_2^2\, dt \le C_1\, (\|\widetilde{\rho}_0 - \overline{\rho}\|_2^2 + \|\widetilde{u}_0\|_2^2)\,.$$

As far as system (32) is concerned, we deduce that there exists a number $\delta_2 > 0$, that is independent of ε, such that, if

$$\|\rho_0 - \overline{\rho}\|_2 + \|u_0\|_2 \le \delta_2\,,$$

then (32) has a global smooth solution (ρ, J), with initial data $(\rho_0, \rho_0 u_0/\varepsilon)$, and that satisfies

$$\sup_{t\ge 0} (\|\rho(t) - \overline{\rho}\|_2^2 + \|u(t)\|_2^2) + \frac{1}{\varepsilon^2}\int_0^{+\infty} \|u(t)\|_2^2\, dt \le C\, (\|\rho_0 - \overline{\rho}\|_2^2 + \|u_0\|_2^2)\,.$$

Recall that $J = \rho u/\varepsilon$. The proof of Theorem 2 is thus complete.

2.5 Proof of Theorem 3

Finally, we estimate how the kinetic density (24) approaches the solution f_ε to (1). We turn back to the notations of the introduction, and set $\widehat{u_\varepsilon} = \varepsilon\, \widehat{J_\varepsilon}/\widehat{\rho_\varepsilon}$. We know that this quantity is $\mathcal{O}(\varepsilon)$ in $L^2(\mathbb{R}^+; H^2(\mathbb{R}))$. We also know that $\widehat{u_\varepsilon}$ takes values in a compact interval $[-\eta, \eta]$, for some $0 < \eta < 1$ independent of ε.

Using (24), we compute

$$\begin{aligned}\int_0^\infty\!\!\int_{\mathbb{R}}\int_V \left|\widehat{f_\varepsilon} - \widehat{\rho_\varepsilon}\right|^2 d\mu(v)\, dx\, dt &= \int_0^\infty\!\!\int_{\mathbb{R}}\int_V (\widehat{\rho_\varepsilon})^2 \left|\frac{\exp\left(v\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})\right)}{\mathbb{F}\circ\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})} - 1\right|^2 d\mu(v)\, dx\, dt \\ &= \int_0^\infty\!\!\int_{\mathbb{R}} (\widehat{\rho_\varepsilon})^2 \left(\int_V \left[\frac{\exp\left(2v\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})\right)}{[\mathbb{F}\circ\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})]^2} - 2\frac{\exp\left(v\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})\right)}{\mathbb{F}\circ\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})} + 1\right] d\mu(v)\right) dx\, dt \\ &= \int_0^\infty\!\!\int_{\mathbb{R}} (\widehat{\rho_\varepsilon})^2\, \frac{\mathbb{F}(2\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})) - [\mathbb{F}\circ\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})]^2}{[\mathbb{F}\circ\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})]^2}\, dx\, dt.\end{aligned}$$

Taylor's formula gives the estimate

$$0 \le \mathbb{F}(2\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})) - [\mathbb{F}\circ\mathbb{G}^{(-1)}(\widehat{u_\varepsilon})]^2 \le C(\|\widehat{u_\varepsilon}\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})})\, |\widehat{u_\varepsilon}|^2\,,$$

and we obtain

$$\int_0^\infty\!\!\int_{\mathbb{R}}\int_V \left|\widehat{f_\varepsilon}(t,x,v) - \widehat{\rho_\varepsilon}(t,x)\right|^2 d\mu(v)\; dx\, dt \le C\int_0^\infty\!\!\int_{\mathbb{R}} (\widehat{\rho_\varepsilon}\, \widehat{u_\varepsilon})^2\, dx\, dt\,.$$

Using Theorem 2 and the Sobolev embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$, the density $\widehat{\rho_\varepsilon}$ lies in a bounded set of $L^\infty(\mathbb{R}^+ \times \mathbb{R})$, and we get

$$\|\widehat{f_\varepsilon} - \widehat{\rho_\varepsilon}\|_{L^2(\mathbb{R}^+\times\mathbb{R}\times V)} \le C\,\varepsilon\,.$$

Fix a time $T > 0$. The triangle inequality yields

$$\|\widehat{f}_\varepsilon - f_\varepsilon\|_{L^2((0,T)\times\mathbb{R}\times V)} \leq \|\widehat{f}_\varepsilon - \widehat{\rho}_\varepsilon\|_{L^2((0,T)\times\mathbb{R}\times V)} + \|\widehat{\rho}_\varepsilon - r\|_{L^2((0,T)\times\mathbb{R}\times V)} + \|r - f_\varepsilon\|_{L^2((0,T)\times\mathbb{R}\times V)} .$$

We have shown that the first term is $\mathcal{O}(\varepsilon)$, while the last term tends to 0 when $\varepsilon \to 0$ (see Theorem 1). It remains to show that $\|\widehat{\rho}_\varepsilon - r\|_{L^2((0,T)\times\mathbb{R}\times V)}$ tends to 0 (as a matter of fact, this term is $\mathcal{O}(\varepsilon)$). This last estimate can be obtained by following the arguments developed in [17] for the isothermal Euler system, and we postpone it to Appendix B.

3 Proof of Theorem 4

As in the preceeding section, we shall consider the equation (26) with $\gamma = 1$. (This amounts to changing the time and space variables by a constant factor). Before proving Theorem 4, we first introduce some smoothing operators, whose detailed construction is described in [1, page 97].

Lemma 6. *There exists a family*

$$S_\theta : \bigcup_{s\geq -1} H^s(\mathbb{R}) \longrightarrow \bigcap_{s\geq -1} H^s(\mathbb{R}) ,$$

that is defined for $\theta \geq 1$, and that satisfies the following properties

i) $\|S_\theta u\|_{H^s(\mathbb{R})} \leq C\,\theta^{(s-s')_+}\,\|u\|_{H^{s'}(\mathbb{R})}$, *for all $s, s' \geq -1$, with $x_+ = \max(x, 0)$,*

ii) $\|u - S_\theta u\|_{H^s(\mathbb{R})} \leq C\,\theta^{s-s'}\,\|u\|_{H^{s'}(\mathbb{R})}$, *for all $s \in [-1, s']$,*

iii) $\|\frac{d}{d\theta} S_\theta u\|_{H^s(\mathbb{R})} \leq C\,\theta^{s-s'-1}\,\|u\|_{H^{s'}(\mathbb{R})}$, *for all $s, s' \geq -1$,*

iv) *S_θ is selfadjoint on $L^2(\mathbb{R})$,*

v) *S_θ commutes with the operator ∂_x.*

The constants C above are uniform when s, s' belong to a bounded interval $[-1, M]$.

Set $r = \rho - \overline{\rho}$ and recall that $\mathbb{G}(\beta) = \beta\,\mathbb{D}(\beta)$, with $\mathbb{G}$ an increasing bounded function, see Lemma 4. Following [37, page 327], we are going to introduce a sort of Galerkin method in order to solve the nonlinear equation (26). We first rewrite (26) as

$$\partial_t r - \mathbb{G}'\left(\frac{\varepsilon\,\partial_x r}{\overline{\rho} + r}\right)\partial_{xx} r + \varepsilon\,\mathbb{D}'\left(\frac{\varepsilon\,\partial_x r}{\overline{\rho} + r}\right)\frac{(\partial_x r)^3}{(\overline{\rho} + r)^2} = 0 , \quad t \in]0, T[, \quad x \in \mathbb{R} ,$$

with initial data

$$r(t = 0) = \overline{r} = \rho_0 - \overline{\rho} \in H^4(\mathbb{R}).$$

For simplicity, we introduce a short notation for the lower order term:

$$\forall\,(u, v) \in]-\overline{\rho}, \overline{\rho}[\times \mathbb{R} , \quad B_\varepsilon(u, v) := \mathbb{D}'\left(\frac{\varepsilon\,v}{\overline{\rho} + u}\right)\frac{v^3}{(\overline{\rho} + u)^2} . \tag{54}$$

Observe that B_ε is a $\mathcal{C}^\infty$ function that vanishes at the origin.

For all $\theta \geq 1$, we introduce the following *regularized* problem

$$\begin{cases} \partial_t r = S_\theta\left\{\mathbb{G}'\left(\dfrac{\varepsilon\,S_\theta\partial_x r}{\overline{\rho} + S_\theta r}\right)\partial_{xx} S_\theta r\right\} - \varepsilon\,S_\theta\big[B_\varepsilon(S_\theta r, \partial_x S_\theta r)\big] , & t \in]0, T[, \quad x \in \mathbb{R} , \\ r_{|t=0} = S_\theta\,\overline{r} . \end{cases} \tag{55}$$

Assuming that $\overline{r}$ is not too large in $H^4(\mathbb{R})$, so that, for instance, $\|S_\theta\,\overline{r}\|_{L^\infty(\mathbb{R})} \le \overline{\rho}/2$, the existence of a unique solution $r_{\varepsilon,\theta} \in \mathcal{C}^1([0,T_{\varepsilon,\theta}];H^4(\mathbb{R}))$ to (55), for some $T_{\varepsilon,\theta} > 0$, follows from Cauchy-Lipschitz' Theorem. Moreover, using the equation (55), we have $r_{\varepsilon,\theta} \in \mathcal{C}^1([0,T_{\varepsilon,\theta}];H^k(\mathbb{R}))$ for all $k \in \mathbb{N}$. Our aim is to show that $r_{\varepsilon,\theta}$ exists on some time interval $[0,T_*]$, where $T_* > 0$ can be chosen independent of $\varepsilon \in]0,1]$ and $\theta \ge 1$, and that the family $\{r_{\varepsilon,\theta},\, \varepsilon \in]0,1],\, \theta \ge 1\}$ satisfies a uniform bound in $\mathcal{C}([0,T_*];H^4(\mathbb{R}))$. In the end, we shall show that for a suitable sequence $\big(\theta_n\big)_{n\in\mathbb{N}}$ that tends to infinity, $\big(r_{\varepsilon,\theta_n}\big)_{n\in\mathbb{N}}$ converges in $\mathcal{C}([0,T_{**}];H^1(\mathbb{R}))$, where $T_{**} > 0$, and $T_{**} \le T_*$ is independent of ε. The analysis is performed in the next paragraphs by using some classical nonlinear estimates.

3.1 Uniform bound in the high norm

We are going to show the following intermediate result:

Proposition 3. *There exists $\alpha > 0$, and there exists an increasing function $C : [0,\overline{\rho}[\to \mathbb{R}^+$, such that for all $\theta \ge 1$ and for all $\varepsilon \in]0,1]$, if $\overline{r} \in H^4(\mathbb{R})$ satisfies $\|\overline{r}\|_{H^4(\mathbb{R})} \le \alpha$, and if $r_{\varepsilon,\theta} \in \mathcal{C}^1([0,T_{\varepsilon,\theta}];H^4(\mathbb{R}))$ is a solution to (55), then one has:*

$$\forall\, t \in [0,T_{\varepsilon,\theta}]\,, \quad \frac{d}{dt}\|r_{\varepsilon,\theta}(t)\|^2_{H^4(\mathbb{R})} \le \varepsilon\, C(\|S_\theta r_{\varepsilon,\theta}(t)\|_{W^{2,\infty}(\mathbb{R})})\, \|r_{\varepsilon,\theta}(t)\|^2_{H^4(\mathbb{R})}\,.$$

Using Lemma 6 and Sobolev's imbeddings, we know that there exists a constant $\alpha_1 \ge 1$ such that for all integer $k \in \{-1,\dots,4\}$, the inequality

$$\forall\, u \in H^k(\mathbb{R})\,, \quad \forall\, \theta \ge 1\,, \quad \|S_\theta u\|_{H^k(\mathbb{R})} \le \alpha_1\, \|u\|_{H^k(\mathbb{R})}\,,$$

holds as well as for all integer $k \in \{0,\dots,3\}$,

$$\forall\, u \in H^{1+k}(\mathbb{R})\,, \quad \forall\, \theta \ge 1\,, \quad \|S_\theta u\|_{W^{k,\infty}(\mathbb{R})} \le \alpha_1\, \|u\|_{H^{1+k}(\mathbb{R})}\,.$$

We now define $\alpha := \overline{\rho}/(4\alpha_1^2)$, and we are going to show that for this positive number α, the result of Proposition 3 holds. In this proof, we shall use the following adaptation of a classical nonlinear estimate (see e.g. [1, page 101]):

Lemma 7. *Let $g :]-R,R[^q \to \mathbb{R}$ be a $\mathcal{C}^\infty$ function that vanishes at the origin. Then for all $s \ge 0$, there exists a nonnegative, nondecreasing, function $C_s : [0,R[\to \mathbb{R}^+$, such that for all $u_1,\dots,u_q \in L^\infty(\mathbb{R}) \cap H^s(\mathbb{R})$ satisfying $\max_j \|u_j\|_{L^\infty(\mathbb{R})} < R$, for all $\varepsilon \in]0,1]$, one has $g(\varepsilon\, u_1,\dots,u_q) \in L^\infty(\mathbb{R}) \cap H^s(\mathbb{R})$ and*

$$\|g(\varepsilon\, u_1,\dots,u_q)\|^2_{H^s(\mathbb{R})} \le C_s(\max_j \|u_j\|_{L^\infty(\mathbb{R})}) \sum_j \|u_j\|^2_{H^s(\mathbb{R})}\,.$$

The function C_s only depends on g.

Proof. Assume that $\overline{r} \in H^4(\mathbb{R})$ satisfies $\|\overline{r}\|_{H^4(\mathbb{R})} \le \alpha$, and that $r_{\varepsilon,\theta} \in \mathcal{C}^1([0,T_{\varepsilon,\theta}];H^5(\mathbb{R}))$ is a solution to (55). As said before, such solutions exist thanks to Cauchy-Lipschitz' Theorem, and thanks to the estimate $\|S_\theta\overline{r}\|_{W^{1,\infty}(\mathbb{R})} \le \overline{\rho}/4$.

To simplify the calculations, r denotes the solution $r_{\varepsilon,\theta}$, where ε and θ are kept fixed in all the proof of Proposition 3.

Let $m \in \{0,\dots,4\}$. Integrating by parts, we compute

$$\begin{aligned}\frac{d}{dt}\|\partial_x^m r(t)\|^2_{L^2(\mathbb{R})} =& -2\big\langle \partial_x^{m+1} S_\theta r(t); \mathbb{G}'(\dots)\partial_x^{m+1} S_\theta r(t)\big\rangle_{L^2(\mathbb{R})}\\ &- 2\big\langle \partial_x^{m} S_\theta r(t); \partial_x(\mathbb{G}'(\dots))\partial_x^{m+1} S_\theta r(t)\big\rangle_{L^2(\mathbb{R})}\\ &+ 2\big\langle \partial_x^{m} S_\theta r(t); [\partial_x^m, \mathbb{G}'(\dots)]\partial_{xx} S_\theta r(t)\big\rangle_{L^2(\mathbb{R})}\\ &- 2\varepsilon\, \big\langle \partial_x^{m} S_\theta r(t); \partial_x^m B_\varepsilon(S_\theta r(t), \partial_x S_\theta r(t))\big\rangle_{L^2(\mathbb{R})}\,, \end{aligned} \qquad (56)$$

where $\mathbb{G}'(...)$ is here a short notation for

$$\mathbb{G}'\left(\frac{\varepsilon\,\partial_x S_\theta r(t)}{\overline{\rho}+S_\theta r(t)}\right).$$

We first remark that

$$-2\big\langle \partial_x^{m+1} S_\theta r(t); \mathbb{G}'(...)\partial_x^{m+1} S_\theta r(t)\big\rangle_{L^2(\mathbb{R})} \le -2\,c(\|S_\theta r(t)\|_{W^{1,\infty}(\mathbb{R})})\,\|\partial_x^{m+1} S_\theta r(t)\|^2_{L^2(\mathbb{R})},$$

for some positive decreasing function $c : [0,\overline{\rho}[\to]0,+\infty[$, that is independent of ε. Using Lemma 7 and the definition (54), we also obtain

$$\begin{aligned} &-2\varepsilon\,\big\langle \partial_x^m S_\theta r(t); \partial_x^m B_\varepsilon(S_\theta r(t), \partial_x S_\theta r(t))\big\rangle_{L^2(\mathbb{R})} \\ &\qquad\le \varepsilon\;C(\|S_\theta r(t)\|_{W^{1,\infty}(\mathbb{R})})\,\|S_\theta r(t)\|_{H^4(\mathbb{R})}\,(\|S_\theta r(t)\|_{H^4(\mathbb{R})}+\|\partial_x S_\theta r(t)\|_{H^4(\mathbb{R})}), \end{aligned}$$

for some positive increasing function $C : [0,\overline{\rho}[\to\mathbb{R}$, that is independent of ε. Next, we compute

$$\partial_x(\mathbb{G}'(...)) = \varepsilon\,\mathbb{G}''(...)\left[\frac{\partial_{xx} S_\theta r(t)}{\overline{\rho}+S_\theta r(t)} - \frac{(\partial_x S_\theta r(t))^2}{(\overline{\rho}+S_\theta r(t))^2}\right],$$

and we thus obtain

$$\begin{aligned} &-2\big\langle \partial_x^m S_\theta r(t); \partial_x(\mathbb{G}'(...))\partial_x^{m+1} S_\theta r(t)\big\rangle_{L^2(\mathbb{R})} \\ &\qquad\le \varepsilon\,C(\|S_\theta r(t)\|_{W^{2,\infty}(\mathbb{R})})\,\|S_\theta r(t)\|_{H^4(\mathbb{R})}\,\|\partial_x S_\theta r(t)\|_{H^4(\mathbb{R})}, \end{aligned}$$

where, again, $C : [0,\overline{\rho}[\to\mathbb{R}$ is a nonnegative, nondecreasing function, which is independent of ε. We deduce that the inequality

$$\begin{aligned} \frac{d}{dt}\|\partial_x^m r(t)\|^2_{L^2(\mathbb{R})} &\le -2c(\|S_\theta r(t)\|_{W^{1,\infty}(\mathbb{R})})\,\|\partial_x^{m+1} S_\theta r(t)\|^2_{L^2(\mathbb{R})} \\ &+\varepsilon\,C(\|S_\theta r(t)\|_{W^{2,\infty}(\mathbb{R})})\,\|S_\theta r(t)\|^2_{H^4(\mathbb{R})} + \varepsilon\,C(\|S_\theta r(t)\|_{W^{2,\infty}(\mathbb{R})})\,\|S_\theta r(t)\|_{H^4(\mathbb{R})}\,\|\partial_x S_\theta r(t)\|_{H^4(\mathbb{R})} \\ &+2\|S_\theta r(t)\|_{H^4(\mathbb{R})}\,\|[\partial_x^m, \mathbb{G}'(...)]\partial_{xx} S_\theta r(t)\|_{L^2(\mathbb{R})} \end{aligned} \tag{57}$$

holds. To estimate the commutator $[\partial_x^m, \mathbb{G}'(...)]\partial_{xx} S_\theta r(t)$, we use the classical Moser type inequality (see e.g. [1, page 100]):

$$\begin{aligned} &\|[\partial_x^m, \mathbb{G}'(...)]\partial_{xx} S_\theta r(t)\|_{L^2(\mathbb{R})} \\ &\quad\le C\left(\|\partial_x(\mathbb{G}'(...))\|_{L^\infty(\mathbb{R})}\|\partial_x^{m+1} S_\theta r(t)\|_{L^2(\mathbb{R})} + \|\partial_{xx} S_\theta r(t)\|_{L^\infty(\mathbb{R})}\|\partial_x(\mathbb{G}'(...))\|_{H^{m-1}(\mathbb{R})}\right). \end{aligned}$$

We have already given the expression of $\partial_x(\mathbb{G}'(...))$, and it is easy to derive an upper bound for $\|\partial_x(\mathbb{G}'(...))\|_{L^\infty(\mathbb{R})}$. Moreover, applying Lemma 7, we obtain

$$\|\partial_x(\mathbb{G}'(...))\|_{H^{m-1}(\mathbb{R})} \le \varepsilon\,C(\|S_\theta r(t)\|_{W^{2,\infty}(\mathbb{R})})\,(\|S_\theta r(t)\|_{H^4(\mathbb{R})}+\|\partial_x S_\theta r(t)\|_{H^4(\mathbb{R})}).$$

Going back to (57), we end up with

$$\begin{aligned} \frac{d}{dt}\|\partial_x^m r(t)\|^2_{L^2(\mathbb{R})} &\le -2c(\|S_\theta r(t)\|_{W^{1,\infty}(\mathbb{R})})\,\|\partial_x^{m+1} S_\theta r(t)\|^2_{L^2(\mathbb{R})} \\ &\quad+\varepsilon\,C(\|S_\theta r(t)\|_{W^{2,\infty}(\mathbb{R})})\,\|S_\theta r(t)\|_{H^4(\mathbb{R})}(\|S_\theta r(t)\|_{H^4(\mathbb{R})}+\|\partial_x S_\theta r(t)\|_{H^4(\mathbb{R})}). \end{aligned}$$

Summing over $m = 0,\ldots,4$, then using Young's inequality as well as Lemma 6, we finish the proof. □

When the initial data $\overline{r}$ satisfies $\|\overline{r}\|_{H^4(\mathbb{R})} \leq \alpha$, one has

$$\|S_\theta \overline{r}\|_{H^4(\mathbb{R})} \leq \frac{\overline{\rho}}{4\alpha_1}\,, \quad \text{and} \quad \|S_\theta \overline{r}\|_{W^{2,\infty}(\mathbb{R})} \leq \frac{\overline{\rho}}{4}\,.$$

Using Proposition 3, we deduce in a classical way that there exists a time $T_* > 0$ such that, for all $\varepsilon \in]0,1]$, and for all $\theta \geq 1$, the solution $r_{\varepsilon,\theta}$ to (55) exists on $[0,T_*]$, and sastisfies

$$\forall\, t \in [0,T_*]\,, \quad \|r_{\varepsilon,\theta}(t)\|_{H^4(\mathbb{R})} \leq \|S_\theta \overline{r}\|_{H^4(\mathbb{R})} \exp(\varepsilon\, C\, t)\,, \quad \|S_\theta r_{\varepsilon,\theta}(t)\|_{W^{2,\infty}(\mathbb{R})} \leq \frac{\overline{\rho}}{2}\,, \tag{58}$$

for a suitable numerical constant C (that is independent of ε and θ).

In the next paragraph, we always consider initial data $\overline{r}$ that satisfy $\|\overline{r}\|_{H^4(\mathbb{R})} \leq \alpha$, so that (58) holds, with T_* independent of ε and θ. Defining $\theta_n = 2^n$, we are going to prove that the sequence $\big(r_{\varepsilon,\theta_n}\big)_{n\in\mathbb{N}}$ converges in $\mathcal{C}([0,T_{**}];H^1(\mathbb{R}))$, with T_{**} small enough (independent of ε and n).

3.2 Convergence in the low norm

In this section, we let $\theta_n = 2^n$, and define

$$\mathbb{G}'_n := \mathbb{G}'\left(\frac{\varepsilon\,\partial_x S_{\theta_n} r_{\varepsilon,\theta_n}(t)}{\overline{\rho} + S_{\theta_n} r_{\varepsilon,\theta_n}(t)}\right)\,.$$

Recall that the family $\big\{r_{\varepsilon,\theta_n},\, \varepsilon \in]0,1],\, n \in \mathbb{N}\big\}$ is bounded in $\mathcal{C}([0,T_*];H^4(\mathbb{R}))$, and therefore it is also bounded in $\mathcal{C}([0,T_*];W^{3,\infty}(\mathbb{R}))$. Consequently, there exist two positive constants c and C such that

$$\forall\,(t,x) \in [0,T_*] \times \mathbb{R}\,, \forall n \in \mathbb{N}\,, \quad \mathbb{G}'_n(t,x) \geq c\,, \quad \|\partial_x(\mathbb{G}'_n(t,\cdot))\|_{W^{1,\infty}(\mathbb{R})} \leq C\,\varepsilon\,.$$

With such estimates on the coefficients, we obtain the following Lemma:

Lemma 8. *Let $n \in \mathbb{N}$, $T > 0$, and let $u \in \mathcal{C}^1([0,T];H^4(\mathbb{R}))$ be a solution to*

$$\begin{cases} \partial_t u - S_{\theta_n}[\mathbb{G}'_n \partial_{xx} S_{\theta_n} u] = g\,, & t \in]0,T[\,, x \in \mathbb{R}\,, \\ u_{|t=0} = u_0\,, & x \in \mathbb{R}\,. \end{cases}$$

Then u satisfies the estimate

$$\forall\, t \in [0,T]\,, \quad \|u(t)\|_{H^1(\mathbb{R})} \leq C(T)\left(\|u_0\|_{H^1(\mathbb{R})} + \|g\|_{L^2([0,T]\times\mathbb{R})} + \frac{1}{\theta_n}\,\|u\|_{L^2([0,T];H^3(\mathbb{R}))}\right)\,.$$

The proof is standard. The equation yields a L^2 bound by integrating by parts. Note that the parabolic term only gives control of $\partial_x S_{\theta_n} u$ in $L^2_{t,x}$, and one thus uses a decomposition

$$\big\langle u(t); g(t)\big\rangle_{L^2(\mathbb{R})} = \big\langle (u - S_{\theta_n} u)(t); g(t)\big\rangle_{H^1(\mathbb{R}),H^{-1}(\mathbb{R})} + \big\langle S_{\theta_n} u(t); g(t)\big\rangle_{H^1(\mathbb{R}),H^{-1}(\mathbb{R})}\,.$$

The norm $\|(u - S_{\theta_n} u)(t)\|_{H^1(\mathbb{R})}$ is estimated by $C\,(\theta_n)^{-1}\,\|u(t)\|_{H^2(\mathbb{R})}$, thanks to Lemma 6. For the H^1 estimate, one commutes the equation with ∂_x and uses the L^2 estimate. To achieve this part, one needs a $L^\infty_t(W^{2,\infty}_x)$ control of the coefficient $\mathbb{G}'_n$.

We now apply Lemma 8 with $T \leq T_*$, and $u = r_{\varepsilon,\theta_{n+1}} - r_{\varepsilon,\theta_n}$. We compute

$$\begin{cases} \partial_t (r_{\varepsilon,\theta_{n+1}} - r_{\varepsilon,\theta_n}) - S_{\theta_n}[\mathbb{G}'_n \partial_{xx} S_{\theta_n}(r_{\varepsilon,\theta_{n+1}} - r_{\varepsilon,\theta_n})] = g_n\,, & t \in]0,T[\,, x \in \mathbb{R}\,, \\ (r_{\varepsilon,\theta_{n+1}} - r_{\varepsilon,\theta_n})_{|t=0} = (S_{\theta_{n+1}} - S_{\theta_n})\overline{r}\,, & x \in \mathbb{R}\,, \end{cases}$$

with

$$
\begin{aligned}
g_n := &- \varepsilon\, S_{\theta_n} \left[B_\varepsilon(S_{\theta_{n+1}} r_{\varepsilon,\theta_{n+1}}, \partial_x S_{\theta_{n+1}} r_{\varepsilon,\theta_{n+1}}) - B_\varepsilon(S_{\theta_n} r_{\varepsilon,\theta_n}, \partial_x S_{\theta_n} r_{\varepsilon,\theta_n}) \right] \\
&- \varepsilon\, (S_{\theta_{n+1}} - S_{\theta_n}) B_\varepsilon(S_{\theta_{n+1}} r_{\varepsilon,\theta_{n+1}}, \partial_x S_{\theta_{n+1}} r_{\varepsilon,\theta_{n+1}}) \\
&+ (S_{\theta_{n+1}} - S_{\theta_n}) \left[\mathbb{G}_n'(\partial_{xx} S_{\theta_n} r_{\varepsilon,\theta_{n+1}}) \right] \\
&+ S_{\theta_{n+1}} \left[(\mathbb{G}_{n+1}' - \mathbb{G}_n')(\partial_{xx} S_{\theta_n} r_{\varepsilon,\theta_{n+1}}) \right] \\
&+ S_{\theta_{n+1}} \left[\mathbb{G}_{n+1}'(\partial_{xx} (S_{\theta_{n+1}} - S_{\theta_n}) r_{\varepsilon,\theta_{n+1}}) \right] .
\end{aligned}
$$

Recall that for all integer k, the quantities $S_{\theta_k} r_{\varepsilon,\theta_k}(t,x)$ and $\partial_x S_{\theta_k} r_{\varepsilon,\theta_k}(t,x)$ belong to the closed interval $[-\overline{\rho}/2, \overline{\rho}/2]$. Therefore, using Lemma 8, as well as the estimate

$$
\|(S_{\theta_{n+1}} - S_{\theta_n}) v\|_{H^s(\mathbb{R})} \leq \frac{C}{\theta_n} \|v\|_{H^{s+1}(\mathbb{R})} ,
$$

and estimating each term of g_n in $L^\infty([0,T]; L^2(\mathbb{R}))$, we end up with

$$
\sup_{t\in[0,T]} \|(r_{\varepsilon,\theta_{n+1}} - r_{\varepsilon,\theta_n})(t)\|_{H^1(\mathbb{R})} \leq C(T) \left(\frac{1}{\theta_n} + \sqrt{T} \sup_{t\in[0,T]} \|(r_{\varepsilon,\theta_{n+1}} - r_{\varepsilon,\theta_n})(t)\|_{H^1(\mathbb{R})} \right) .
$$

Of course, $C(T)$ is independent of ε and n. Choosing $T = T_{**}$ small enough, independent of ε, so that $C(T_{**})\sqrt{T_{**}} \leq 1/2$, we obtain that $\left(r_{\varepsilon,\theta_n}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([0,T_{**}]; H^1(\mathbb{R}))$, and therefore it converges towards some $r_\varepsilon \in \mathcal{C}([0,T_{**}]; H^1(\mathbb{R}))$. Since, by construction, $r_{\varepsilon,\theta_n}(t=0)$ converges in $H^1(\mathbb{R})$ towards $\overline{r}$, we have $r_\varepsilon(t=0) = \overline{r}$. It remains to show that r_ε solves the nonlinear equation

$$
\partial_t r - \mathbb{G}'\left(\frac{\varepsilon\, \partial_x r}{\overline{\rho} + r}\right) \partial_{xx} r + \varepsilon\, \mathbb{D}'\left(\frac{\varepsilon\, \partial_x r}{\overline{\rho} + r}\right) \frac{(\partial_x r)^3}{(\overline{\rho} + r)^2} = 0 , \quad t \in]0, T_{**}[, \quad x \in \mathbb{R} ,
$$

with initial data $r(t=0) = \overline{r} \in H^4(\mathbb{R})$.

3.3 End of the proof of Theorem 4

The sequence $\left(r_{\varepsilon,\theta_n}\right)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}([0,T_{**}]; H^4(\mathbb{R}))$, and the sequence $\left(\partial_t r_{\varepsilon,\theta_n}\right)_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}([0,T_{**}]; H^2(\mathbb{R}))$. Moreover, we have seen that the sequence $\left(r_{\varepsilon,\theta_n}\right)_{n\in\mathbb{N}}$ converges in $\mathcal{C}([0,T_{**}]; H^1(\mathbb{R}))$ towards r_ε as $n \to \infty$. Consequently, up to a subsequence, we have

$$
\begin{aligned}
r_{\varepsilon,\theta_n} &\rightharpoonup r_\varepsilon \quad L^\infty(0,T_{**}; H^4(\mathbb{R})) \quad \text{weak-}\star , \\
\partial_t r_{\varepsilon,\theta_n} &\rightharpoonup \partial_t r_\varepsilon \quad L^\infty(0,T_{**}; H^2(\mathbb{R})) \quad \text{weak-}\star .
\end{aligned}
$$

In particular, we have $r_\varepsilon \in Lip(0,T_{**}; H^2(\mathbb{R}))$. Moreover, thanks to the convexity properties of the norms in H^s, we have $r_{\varepsilon,\theta_n} \to r_\varepsilon$ in all the spaces $\mathcal{C}([0,T_{**}]; H^{4-\delta}(\mathbb{R}))$, $\delta > 0$, and in particular $r_{\varepsilon,\theta_n} \to r_\varepsilon$ in the space $\mathcal{C}([0,T_{**}]; \mathcal{C}^3(\mathbb{R}))$. With such strong convergences, it is easy to show that $r_\varepsilon \in L^\infty(0,T_{**}; H^4(\mathbb{R})) \cap Lip(0,T_{**}; H^2(\mathbb{R}))$ is a solution to the nonlinear equation

$$
\partial_t r_\varepsilon - \partial_x \left(\mathbb{D}\Big(\frac{\varepsilon\, \partial_x r_\varepsilon}{\overline{\rho} + r_\varepsilon}\Big) \partial_x r_\varepsilon \right) = 0 ,
$$

with initial data $\overline{r}$. Following [37] and using the uniform estimates of Proposition 3, we claim that $r_\varepsilon \in \mathcal{C}([0,T_{**}]; H^4(\mathbb{R})) \cap \mathcal{C}^1([0,T_{**}]; H^2(\mathbb{R}))$. Passing to the limit in (58), we also obtain the $\mathcal{C}_t(H_x^4)$ bound for r_ε.

To end the proof of Theorem 4, it only remains to show the convergence of $\overline{\rho} + r_\varepsilon$ towards the solution to the heat equation. This can be performed by the standard energy estimates in L^2 for the heat equation, and we do not give the details. The convergence result for the microscopic densities $\widehat{f}_\varepsilon$, and f_ε is obtained, as in the preceeding section, by a suitable triangle inequality.

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A Proof of Lemma 4

We write $\mathbb{F}(\beta) = \int_V \cosh(\beta v)\ d\mu(v) + \int_V \sinh(\beta v)\ d\mu(v) = \int_V \cosh(\beta v)\ d\mu(v)$, thanks to Assumption 2; hence $\mathbb{F}$ is even. The Laplace transform $\mathbb{F}$ is clearly $\mathcal{C}^\infty$ thanks to Lebesgue's differentiability Theorem, and so is $\mathbb{G}$. We compute:

$$\mathbb{G}'(\beta) = \frac{\mathbb{F}(\beta)\,\mathbb{F}''(\beta) - \mathbb{F}'(\beta)^2}{\mathbb{F}(\beta)^2}\,,$$

and the Cauchy-Schwarz' inequality yields

$$\mathbb{F}'(\beta)^2 = \left(\int_V v \exp(\beta v)\, d\mu(v)\right)^2 \le \left(\int_V \exp(\beta v)\, d\mu(v)\right)\left(\int_V v^2 \exp(\beta v)\, d\mu(v)\right) = \mathbb{F}(\beta)\,\mathbb{F}''(\beta)\,.$$

Thus $\mathbb{G}$ is nondecreasing. Now, assume that $\mathbb{G}'(\beta_0) = 0$ for some $\beta_0 \in \mathbb{R}$. This means that we are in the Cauchy-Schwarz' equality case. A simple analysis shows that this is possible if, and only if, $v = 0$ μ-almost everywhere, and this is excluded by Assumption 3. Thus $\mathbb{G}$ is increasing, and it remains to compute the limits of $\mathbb{G}$ at $\pm\infty$. Recall that

$$1 - \mathbb{G}(\beta) = \frac{\int_V (1-v)\ \exp(\beta v)\, d\mu(v)}{\int_V \exp(\beta v)\, d\mu(v)} \ge 0\,,$$

and that the maximum of the support of μ is $+1$. Let $\alpha > 0$. Thanks to the assumption on the support of μ, we know that $\mu([1-\alpha/2, 1]) > 0$. First, we observe that

$$\frac{\int_{[1-\alpha,1]} (1-v)\ \exp(\beta v)\, d\mu(v)}{\int_V \exp(\beta v)\, d\mu(v)} \le \alpha\,.$$

Then, for $\beta \ge 0$, we use the inequalities

$$\int_{[-1,1-\alpha[} (1-v)\ \exp(\beta v)\, d\mu(v) \le 2\ \exp[\beta(1-\alpha)]\,,$$

$$\int_V \exp(\beta v)\, d\mu(v) \ge \int_{[1-\alpha/2,1]} \exp(\beta v)\, d\mu(v) \ge \exp[\beta\,(1-\alpha/2)]\,\mu([1-\alpha/2,1])\,,$$

and we deduce that

$$\frac{\int_{[-1,1-\alpha[} (1-v)\ \exp(\beta v)\, d\mu(v)}{\int_V \exp(\beta v)\, d\mu(v)} \le \frac{2}{\mu([1-\alpha/2,1])}\ \exp(-\beta\,\alpha/2) \le \alpha$$

holds for β greater than some $\beta_0(\alpha) > 0$. This shows that $\mathbb{G}(\beta)$ tends to 1 as β tends to $+\infty$. The other limit is computed in the same way, and the proof is complete.

B Convergence to the heat equation

The solutions of (23) that are given by Theorem 2 satisfy the following estimate

$$\begin{cases} \sup\limits_{\varepsilon>0,\ t\ge 0} \|\widehat{\rho}_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \le C\,, \\ \sup\limits_{\varepsilon>0,\ t\ge 0} \varepsilon^2 \displaystyle\int_{\mathbb{R}} |\widehat{J}_\varepsilon(t,x)|^2\ dx \le C\,, \\ \sup\limits_{\varepsilon>0} \displaystyle\int_0^{+\infty}\int_{\mathbb{R}} |\widehat{J}_\varepsilon(t,x)|^2\ dx\ dt \le C\,. \end{cases} \tag{59}$$

Furthermore, $\varepsilon\widehat{J}_\varepsilon/\widehat{\rho}_\varepsilon$ takes values in some compact set $[-\eta,+\eta]$, independent of ε.

Let r be the solution to the heat equation (6) with initial data ρ_0. Following [17], we rewrite the equation

$$\partial_t(\widehat{\rho}_\varepsilon - r) + \partial_x(\widehat{J}_\varepsilon + d\partial_x r) = 0\,,$$

as the divergence free (with respect to t,x) condition satisified by a stream function

$$\begin{cases} \partial_x z_\varepsilon := \widehat{\rho}_\varepsilon - r\,, \\ \partial_t z_\varepsilon := -(\widehat{J}_\varepsilon + d\partial_x r)\,. \end{cases} \tag{60}$$

We normalize by choosing $z_\varepsilon(t=0)=0$. Then, we multiply the second equation in (23) by z_ε and we obtain

$$\varepsilon^2\int_0^T\int_{\mathbb{R}} \partial_t\widehat{J}_\varepsilon\; z_\varepsilon\;dx\;dt + \int_0^T\int_{\mathbb{R}} \partial_x\Big(\widehat{\rho}_\varepsilon\psi\big(\varepsilon\frac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}\big) - d\,r\Big)\; z_\varepsilon\;dx\;dt = -\int_0^T\int_{\mathbb{R}}(\widehat{J}_\varepsilon + d\partial_x r)\; z_\varepsilon\;dx\;dt\,. \tag{61}$$

Using (60), the right-hand side of (61) equals

$$\Big[\int_{\mathbb{R}}\frac{z_\varepsilon^2}{2}\;dx\Big]_0^T = \int_{\mathbb{R}}\frac{z_\varepsilon(T,x)^2}{2}\;dx\,.$$

Integrating by parts, the left-hand side of (61) equals

$$\varepsilon^2\Big[\int_{\mathbb{R}}\widehat{J}_\varepsilon\; z_\varepsilon\;dx\Big]_0^T + \varepsilon^2\int_0^T\int_{\mathbb{R}}\widehat{J}_\varepsilon\;(\widehat{J}_\varepsilon + d\partial_x r)\;dx\;dt - \int_0^T\int_{\mathbb{R}}\Big(\widehat{\rho}_\varepsilon\psi\big(\varepsilon\frac{\widehat{J}_\varepsilon}{\widehat{\rho}_\varepsilon}\big) - dr\Big)\;\Big(\widehat{\rho}_\varepsilon - r\Big)\;dx\;dt. \tag{62}$$

The last term in the sum can be rewritten as

$$-d\int_0^T\int_{\mathbb{R}}\big|\widehat{\rho}_\varepsilon - r\big|^2\;dx\;dt - \int_0^T\int_{\mathbb{R}}\widehat{\rho}_\varepsilon(\psi(\widehat{u}_\varepsilon) - \psi(0))(\widehat{\rho}_\varepsilon - r)\,dx\;dt\,.$$

Recalling that $\psi(0)=d$, $\psi'(0)=0$, and using (59), it can be dominated by

$$-d\int_0^T\int_{\mathbb{R}}\big|\widehat{\rho}_\varepsilon - r\big|^2\;dx\;dt + C\,\varepsilon^2\,.$$

Moreover, the Cauchy-Schwarz' inequality yields

$$\int_{\mathbb{R}}\varepsilon^2\;\widehat{J}_\varepsilon(T)\;z_\varepsilon(T)\;dx \le \varepsilon^4\int_{\mathbb{R}}\widehat{J}_\varepsilon(T)^2\;dx + \frac{1}{4}\int_{\mathbb{R}}z_\varepsilon(T)^2\;dx\,.$$

Therefore, we deduce from (61)-(62) the inequality

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}}z_\varepsilon(T)^2\;dx + d\int_0^T\int_{\mathbb{R}}\big|\widehat{\rho}_\varepsilon - r\big|^2\;dx\;dt \le C\,\varepsilon^2 \\ + \varepsilon^2\,\Big(\|\widehat{J}_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R})} + \|\widehat{J}_\varepsilon\|_{L^2([0,T]\times\mathbb{R})}\|d\partial_x r\|_{L^2([0,T]\times\mathbb{R})}\Big) + \varepsilon^4\int_{\mathbb{R}}\widehat{J}_\varepsilon(T,x)^2\;dx\,.\end{aligned} \tag{63}$$

Using the uniform estimates (59), we deduce that

$$d\int_0^T\int_{\mathbb{R}}\big|\widehat{\rho}_\varepsilon - r\big|^2\;dx\;dt \le C\,\varepsilon^2\,,$$

and the constant is uniform with respect to T. We can thus pass to the limit $T\to+\infty$ and prove the first estimate of Theorem 3.

References

[1] S. Alinhac, P. Gérard. *Opérateurs pseudo-différentiels et théorème de Nash-Moser.* InterEditions, 1991.

[2] C. Bardos, F. Golse, B. Perthame, and R. Sentis. The nonaccretive radiative transfer equations: existence of solutions and Rosseland approximation. *J. Funct. Anal.*, 77(2):434–460, 1988.

[3] C. Bardos, R. Santos, and R. Sentis. Diffusion approximation and computation of the critical size. *Trans. Amer. Math. Soc.*, 284(2):617–649, 1984.

[4] E. Bonneaux. *Résolution de problèmes de transport par fermeture non linéaire des équations de moments. Applications à la neutronique.* PhD thesis, University Bordeaux 1, 1997.

[5] S. Buet and C. Cordier. Asymptotic preserving scheme for radiative hydrodynamics models. *C. R. Acad. Sci. Paris Sér. I Math.*, to appear, 2004.

[6] J.-F. Collet. Extensive Lyapounov functionals for moment-preserving evolution equations. *C. R. Math. Acad. Sci. Paris*, 334(5):429–434, 2002.

[7] J.-F. Coulombel, F. Golse, and T. Goudon. Global weak solutions of entropy-based moment closure for kinetic equations. *Preprint*, 2005.

[8] J.-F. Coulombel and T. Goudon. The strong relaxation limit of the multidimensional isothermal euler equations. *Preprint*, 2004.

[9] W. Dreyer, M. Junk, and M. Kunik. On the approximation of the Fokker-Planck equation by moment systems. *Nonlinearity*, 14(4):881–906, 2001.

[10] B. Dubroca, J.-L. Feugeas. Étude théorique et numérique d'une hiérarchie de modèles aux moments pour le transfert radiatif. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(10):915–920, 1999.

[11] S. Goldstein. On diffusion by discontinuous movements, and on the telegraph equation. *Quart. J. Mech. Appl. Math.*, 4:129–156, 1951.

[12] F. Golse. From kinetic to macroscopic models. In *Kinetic equations and asymptotic theory,* L. Desvillettes, B. Perthame Eds., volume 4 of *Series in Appl. Math.*, pages 41–126. Gauthier-Villars, 2000.

[13] T. Goudon, A. Mellet. Homogenization and diffusion asymptotics of the linear Boltzmann equation. *ESAIM Control Optim. Calc. Var.*, 9:371–398, 2003.

[14] T. Goudon, F. Poupaud. Approximation by homogenization and diffusion of kinetic equations. *Comm. Partial Differential Equations*, 26(3-4):537–569, 2001.

[15] T. Goudon, F. Poupaud. Homogenization of transport equations; weak mean field approximation. *SIAM J. Math. Anal.*, 36(3):856–881, 2004.

[16] B. Hanouzet, R. Natalini. Global existence of smooth solutions for partially dissipative hyperbolic systems with a convex entropy. *Arch. Ration. Mech. Anal.*, 169(2):89–117, 2003.

[17] S. Junca, M. Rascle. Strong relaxation of the isothermal Euler system to the heat equation. *Z. Angew. Math. Phys.*, 53(2):239–264, 2002.

[18] M. Junk. Maximum entropy for reduced moment problems. *Math. Models Methods Appl. Sci.*, 10(7):1001–1025, 2000.

[19] M. Kac. A stochastic model related to the telegrapher's equation. *Rocky Mountain J. Math.*, 4:497–509, 1974. Reprinting of an article published in 1956, from Magnolia Forever Petroleum Company Colloquium Lectures in the Pure and Applied Sciences (Univ. Alberta, 1972).

[20] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58(3):181–205, 1975.

[21] C. D. Levermore. A Chapman-Enskog approach to flux limited diffusion theory. *Technical Report, Lawrence Livermore Laboratory*, UCID-18229, 1979.

[22] C. D. Levermore. Moment closure hierarchies for kinetic theories. *J. Statist. Phys.*, 83(5-6):1021–1065, 1996.

[23] C. D. Levermore. Entropy-based moment closures for kinetic equations. *Transport Theory Statist. Phys.*, 26:591–606, 1997.

[24] C. D. Levermore, G. C. Pomraning. A flux-limited diffusion theory. *Astrophys. J.*, 248:321–334, 1981.

[25] C. D. Levermore, W. J. Morokoff. The Gaussian moment closure for gas dynamics. *SIAM J. Appl. Math.*, 59(1):72–96, 1999.

[26] P.-L. Lions, G. Toscani. Diffusive limit for finite velocity Boltzmann kinetic models. *Rev. Mat. Iberoamericana*, 13(3):473–513, 1997.

[27] H. P. McKean. The central limit theorem for Carleman's equation. *Israel J. Math.*, 21(1):54–92, 1975.

[28] L. Mieussens. Discrete velocity model and implicit scheme for the BGK equation of rarefied gas dynamics. *Math. Models Methods Appl. Sci.*, 10(8):1121–1149, 2000.

[29] L. Mieussens. Discrete-velocity models and numerical schemes for the Boltzmann-BGK equation in plane and axisymmetric geometries. *J. Comput. Phys.*, 162(2):429–466, 2000.

[30] G. N. Minerbo. Maximum entropy Eddington factors. *J. Q. S. R. T.*, 20:541–545, 1978.

[31] T. Nishida. *Nonlinear hyperbolic equations and related topics in fluid dynamics.* Département de Mathématique, Université de Paris-Sud, Orsay, 1978.

[32] G. C. Pomraning. A comparison of various flux limiters and Eddington factors. *Technical Report, Lawrence Livermore Laboratory*, UCID-19220, 1981.

[33] C. Ringhofer, C. Schmeiser, and A. Zwirchmayr. Moment methods for the semiconductor Boltzmann equation on bounded position domains. *SIAM J. Numer. Anal.*, 39(3):1078–1095, 2001.

[34] C. Schmeiser, A. Zwirchmayr. Convergence of moment methods for linear kinetic equations. *SIAM J. Numer. Anal.*, 36(1):74–88, 1999.

[35] Y. Shizuta, S. Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14(2):249–275, 1985.

[36] G. I. Taylor. Diffusion by continuous movements. *Proc. London Math. Soc.*, 20:196–212, 1921/2.

[37] M. E. Taylor. *Partial differential equations. III*, volume 117 of *Applied Mathematical Sciences*. Springer-Verlag, 1997.

[38] R. Turpault. Construction d'un modèle M1-multigroupe pour les équations du transfert radiatif. *C. R. Math. Acad. Sci. Paris*, 334(4):331–336, 2002.

[39] W.-A. Yong. Entropy and global existence for hyperbolic balance laws. *Arch. Ration. Mech. Anal.*, 172(2):247–266, 2004.