THE LINEAR BOLTZMANN EQUATION WITH SPACE PERIODIC ELECTRIC FIELD

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This paper is dedicated to Nina Uraltceva for her 75th birthday.

ABSTRACT. We investigate the well posedness of the stationary linear Boltzmann equation with space periodic electric field. We discuss the different behaviors that occur depending if the average electric field vanishes or not. The existence follows by perturbation techniques and stability arguments under uniform a priori estimates. The uniqueness of the weak solution holds for space periodic electric fields with non vanishing average, under the constraint of given current. The main ingredients of the proof rely on the dissipation properties of the linear collision operator and the derivation of refined estimates.

INTRODUCTION

This paper is concerned with the free space linear Boltzmann equation

$$v(p)\partial_x f + F(x)\partial_p f = Q(f), \qquad (x,p) \in \mathbb{R}^2. \tag{0.1}$$

The unknown $f = f(x,p)$ represents the number density of a population of charged particles, with $x \in \mathbb{R}$ the space variable and $p \in \mathbb{R}$ the momentum variable. The velocity $p \mapsto v(p)$ is defined by

$$v(p) = \frac{p}{m}\left(1 + \frac{p^2}{m^2 c_0^2}\right)^{-1/2} \tag{0.2}$$

where m is the mass of the particles and c_0 is the light speed in vacuum. The kinetic energy associated to $v(p)$ is then given by

$$\mathcal{E}(p) = mc_0^2\left(\left(1 + \frac{p^2}{m^2 c_0^2}\right)^{1/2} - 1\right)$$

so that $\mathcal{E}'(p) = v(p), p \in \mathbb{R}$. The collision operator Q, which is an integral operator with respect to the variable p, is defined as follows:

$$Q(g)(p) = M_\theta(p)\langle g\rangle_s(p) - \sigma(p)g(p), \qquad p \in \mathbb{R}$$

with
(0.3)

$$\langle g\rangle_s(p) = \int_{\mathbb{R}} s(p,p')g(p')\,\mathrm{d}p' \qquad \text{and} \qquad \sigma(p) = \int_{\mathbb{R}} s(p,p')M_\theta(p')\,\mathrm{d}p' = \langle M_\theta\rangle_s(p)\ ;$$

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and

$$M_\theta(p) = \exp\left(-\frac{\mathcal{E}(p)}{\theta}\right)\left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(p')}{\theta}\right)\,\mathrm{d}p'\right)^{-1}. \tag{0.4}$$

The force field is given by $F = qE$ where q is the charge of the particles and E is the electric field the particles are subject to.

The definition (0.2) means that we are dealing with relativistic particles and it will be crucial for the analysis to observe that

$$-c_0 < v(p) < c_0. \tag{0.5}$$

We also remark that

$$mc_0^2 + \mathcal{E}(p) \geq c_0|p| \geq v(p)p \geq 0. \tag{0.6}$$

Throughout the paper the scattering cross-section $s(p, p')$ is required to satisfy

$$s(p,p') = s(p',p)\ , \qquad 0 < s_0 \leq s(p,p') \leq s_1 < +\infty, \qquad p,p' \in \mathbb{R}. \tag{0.7}$$

Consequently, for any integrable function g we have

$$0 < \langle g \rangle s_0 \leq \langle g \rangle_s(p) \leq \langle g \rangle s_1 < +\infty, \qquad p \in \mathbb{R},$$

and in particular the collision frequency $\sigma(\cdot)$ verifies

$$0 < s_0 \leq \sigma(p) \leq s_1 < +\infty, \qquad p \in \mathbb{R}.$$

The equation (0.1) models charge transport phenomena, with applications in semi-conductor theory or plasma physics. We refer to [4] for further details on the model as well as for the basis of its mathematical analysis. The boundary value problem for (0.1) has been studied in [6]. The analysis of [6] uses comparison principles based on the observation that

$$\mathcal{M}_{\theta,\phi}(x,p) = \exp\left(-\frac{\mathcal{E}(p) + q\phi(x)}{\theta}\right), \qquad \phi' = -E$$

is a particular solution for (0.1), since this function makes vanish both the transport operator $v(p)\partial_x + F(x)\partial_p$ and the collision operator Q. Hence, existence results for the boundary value problem associated to (0.1) can be established dealing with incoming data comparable with $\mathcal{M}_{\theta,\phi}$. The existence theory has been extended recently to general integrable data in [3].

The aim of this article is to analyze the free space problem (0.1). As said before, the function $\mathcal{M}_{\theta,\phi}$, and obviously all the multiple of $\mathcal{M}_{\theta,\phi}$, are solutions of (0.1). However we shall see that the equation admit other solutions. This can be already seen in the specific case where the electric field and the scattering function are constant: if $E(x) = E \neq 0$ and $s(p,p') = \frac{1}{\tau}$, τ being the relaxation time, we can find particle densities f depending only on the momentum by solving analytically the ordinary differential equation

$$\frac{1}{\tau} f(p) + qE\frac{df}{dp} = \frac{1}{\tau} M_\theta(p), \qquad \int_{\mathbb{R}} f(p)\,\mathrm{d}p = 1. \tag{0.8}$$

This example already appears in [5] in the non relativistic case (*i.e.* $v(p) = p/m$) and the properties of the solution can be used to investigate the associated boundary value and Milne problems, see [5], [2]. This remark raises the question of selecting the physically relevant solution of (0.1) by some appropriate criterion. Still in the specific case of constant non vanishing electric field it seems reasonable to select the solution given by (0.8) which remains bounded on $\mathbb{R}^2$, while the distribution $\mathcal{M}_{\theta,\phi}$

becomes unbounded (as $x \to -\infty$ if $qE > 0$ and as $x \to +\infty$ if $qE < 0$). In some sense, the lack of boundary conditions has to be compensated by imposing a uniform behaviour of the solution with respect to the space variable, at least when dealing with bounded electric fields. We shall call "permanent regimes" (with respect to the space variable) such bounded solutions. Observe also that since (0.1) is linear we only can expect uniqueness up to a multiplicative constant. Eventually this constant can be determined by imposing the current $j = q\int_{\mathbb{R}} v(p)f \, \mathrm{d}p$ since this quantity does not depend on the space variable: we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)f \, \mathrm{d}p = 0, \qquad x \in \mathbb{R},$$

due to the conservative character of the collision operator. Hence, a legitimate uniqueness result for permanent regimes is

Uniqueness. *Consider f, g two permanent solutions for (0.1) having the same current*

$$q\int_{\mathbb{R}} v(p)f \, \mathrm{d}p = q\int_{\mathbb{R}} v(p)g \, \mathrm{d}p.$$

Then the solutions f, g coincide.

Having defined relevant criteria for uniqueness, we are nevertheless still left with the task of establishing the existence of such permanent solutions, in the general case of space varying electric fields. In this work we restrict to the situation where the electric field is space periodic and we seek space periodic solutions. It does not answer the general question we address, but up to our knowledge it is the first work on this direction. Besides the physical relevance of this case, the study is interesting from the mathematical point of view. Moreover we expect that similar results can be established for more general frameworks, as the almost periodic one, by adapting the same techniques. Our main result is the following

We assume that $F(x) = qE(x)$ is a given L-periodic bounded force field. We consider the periodic domain $\mathbb{T} = \mathbb{R}/(L\mathbb{Z})$.

Theorem 0.1. *Assume that $E \in L^\infty(\mathbb{R})$ is a bounded L-periodic electric field.*

a) *If $\int_{\mathbb{T}} E(x) \, \mathrm{d}x = 0$ then all the periodic solutions of the linear Boltzmann equation (0.1) are of the form $k\mathcal{M}_{\theta,\phi}$ with $k \in \mathbb{R}$.*

b) *If $\int_{\mathbb{T}} E(x) \, \mathrm{d}x \neq 0$ then there exists a periodic weak solution f of the linear Boltzmann equation (0.1) such that $f(x,p) \geq 0$ and $\frac{1}{L}\int_{\mathbb{T}}\int_{\mathbb{R}} f(x,p) \, \mathrm{d}p \, \mathrm{d}x = 1$. Moreover, this solution satisfies*

$$\begin{cases} (1+\mathcal{E}(p))f \in L^1([0,L]\times\mathbb{R}) = L^1(\mathbb{T}\times\mathbb{R}), \\ f \in L^\infty([0,L]\times\mathbb{R}) = L^\infty(\mathbb{T}\times\mathbb{R}), \\ \langle f \rangle := \int_{\mathbb{R}} f \, \mathrm{d}p \in L^\infty(\mathbb{T}). \end{cases}$$

c) *The current associated to the solution exhibited in b) does not vanish and we have*

$$q\int_{\mathbb{R}} v(p)\, f(x,p) \, \mathrm{d}p \, \int_{\mathbb{T}} E(x) \, \mathrm{d}x > 0.$$

d) *Assume $\int_{\mathbb{T}} E(x) \, \mathrm{d}x \neq 0$. For any $j \in \mathbb{R}$ there is a unique periodic weak solution f of the linear Boltzmann equation (0.1) verifying $q\int_{\mathbb{R}} v(p)f \, \mathrm{d}p = j$. The solution has constant sign, given by* $\operatorname{sgn}(f) = \operatorname{sgn}\left(j/\int_{\mathbb{T}} E(x) \, \mathrm{d}x\right)$

Our paper is organized as follows. In Section 1 we set up a few definition that will be necessary for the existence-uniqueness theory. Section 2 is devoted to the uniqueness result. The proof is based on new sharp dissipative properties for the linear collision operator. In the last section we discuss the existence of periodic weak solution: we analyze a penalized periodic problem, we establish a priori estimates and we conclude by stability arguments.

1. Weak solutions

Let us introduce the corresponding notion of weak solution for (0.1).

Definition 1.1. Assume that F belongs to $L^\infty(\mathbb{T})$. We say that $f \in L^1(\mathbb{T}\times\mathbb{R})$ is a periodic weak solution for (0.1) iff

$$-\int_{\mathbb{T}}\int_{\mathbb{R}} f(x,p)(v(p)\partial_x\varphi + F(x)\partial_p\varphi)\,\mathrm{d}p\,\mathrm{d}x = \int_{\mathbb{T}}\int_{\mathbb{R}} Q(f)\varphi(x,p)\,\mathrm{d}p\,\mathrm{d}x \tag{1.1}$$

for any function $\varphi \in C^1_c(\mathbb{T}\times\mathbb{R})$.

It is easily seen that the formulation (1.1) holds true for any test function $\varphi \in C^1_b(\mathbb{T}\times\mathbb{R})$ (*i.e.,* the set of bounded C^1 functions with bounded partial derivatives). Since f belongs to $L^1(\mathbb{T}\times\mathbb{R})$ and the relativistic velocity is bounded, see (0.5), the function $v(p)f \in L^1(\mathbb{T}\times\mathbb{R})$ and therefore the current $j(x) = q\int_{\mathbb{R}} v(p)f\,\mathrm{d}p$ is well defined for a.a. $x \in \mathbb{T}$. In particular taking $\varphi = \varphi(x) \in C^1(\mathbb{T})$ in (1.1) yields

$$\int_{\mathbb{T}} \varphi'(x)j(x)\,\mathrm{d}x = 0$$

implying that the current is preserved along $x \in \mathbb{T}$.

Proposition 1.2. *Let f be a periodic weak solution of (0.1). Then the current $j = q\int_{\mathbb{R}} v(p)f\,\mathrm{d}p$ is constant.*

2. Uniqueness of the periodic weak solution

Consider $f, g \in L^1(\mathbb{T}\times\mathbb{R})$ two periodic weak solutions for (0.1). By linearity we have

$$v(p)\partial_x(f-g) + F(x)\partial_p(f-g) = Q(f-g), \qquad (x,p)\in\mathbb{T}\times\mathbb{R} \tag{2.1}$$

and by standard computations one gets in $\mathcal{D}'(\mathbb{T}\times\mathbb{R})$

$$v(p)\partial_x|f-g| + F(x)\partial_p|f-g| - \mathrm{sgn}(f-g)Q(f-g) = 0. \tag{2.2}$$

After integration with respect to momentum we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)|f-g|\,\mathrm{d}p - \int_{\mathbb{R}} \mathrm{sgn}(f-g)Q(f-g)\,\mathrm{d}p = 0, \qquad \text{in} \qquad \mathcal{D}'(\mathbb{T}). \tag{2.3}$$

Following the idea in [1] we can write

$$
\begin{aligned}
&-\int_{\mathbb{R}} \operatorname{sgn}(f-g) Q(f-g)\ \mathrm{d}p \\
&= \int_{\mathbb{R}} \operatorname{sgn}(f-g) \int_{\mathbb{R}} s(p,p')\Big\{ M_\theta(p')(f-g)(x,p) - M_\theta(p)(f-g)(x,p')\Big\}\ \mathrm{d}p'\ \mathrm{d}p \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} s(p,p')\Big\{ M_\theta(p')(f-g)(x,p) - M_\theta(p)(f-g)(x,p')\Big\} \\
&\qquad \times \Big\{\operatorname{sgn}(M_\theta(p')(f-g)(x,p)) - \operatorname{sgn}(M_\theta(p)(f-g)(x,p'))\Big\}\ \mathrm{d}p'\ \mathrm{d}p \\
&\geq 0
\end{aligned}
\tag{2.4}
$$

with equality iff $\operatorname{sgn}(f-g)$ is constant with respect to p. Integrating now (2.3) with respect to x and using the periodicity of f and g implies

$$
-\int_{\mathbb{T}}\int_{\mathbb{R}} \operatorname{sgn}(f-g) Q(f-g)\ \mathrm{d}p\ \mathrm{d}x = 0.
$$

Therefore for a.a. $x \in \mathbb{T}$ we have $-\int_{\mathbb{R}} \operatorname{sgn}(f-g)Q(f-g)\ \mathrm{d}p = 0$ and thus $\operatorname{sgn}(f-g)$ depends only on x. Eventually (2.1) can be written now

$$
v(p)\partial_x|f-g| + F(x)\partial_p|f-g| = Q(|f-g|), \qquad (x,p) \in \mathbb{T}\times\mathbb{R}
$$

implying that

$$
\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)|f-g|\ \mathrm{d}p = 0, \qquad x \in \mathbb{T}
$$

but this is not enough in order to guarantee the uniqueness of the periodic weak solution. Actually we will see that, in the particular case of electric fields satisfying $\int_{\mathbb{T}} E(x)\ \mathrm{d}x = 0$, the above arguments allow to determine all the periodic solutions. When the average of E does not vanish a sharper estimate will be necessary.

2.1. **Vanishing electric field average.** Let us assume that $\int_{\mathbb{T}} E\ \mathrm{d}x = 0$ holds. In such a case, the potential $\phi(x) = -\int_0^x E(y)\ \mathrm{d}y$ is also L-periodic. Since for any fixed $c \in \mathbb{R}$ the function $c\mathcal{M}_{\theta,\phi}(x,p)$ solves (0.1) we can replace (2.1) by

$$
v(p)\partial_x(f-g-c\mathcal{M}_{\theta,\phi}) + F(x)\partial_p(f-g-c\mathcal{M}_{\theta,\phi}) = Q(f-g-c\mathcal{M}_{\theta,\phi}).
\tag{2.5}
$$

Following the same steps as before we find for any $c \in \mathbb{R}$

$$
-\int_{\mathbb{R}} \operatorname{sgn}(f-g-c\mathcal{M}_{\theta,\phi}) Q(f-g-c\mathcal{M}_{\theta,\phi})\ \mathrm{d}p = 0, \qquad \text{a.e. } x \in \mathbb{T}.
\tag{2.6}
$$

Notice that the periodicity of the potential is crucial when writing

$$
\int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)|f-g-c\mathcal{M}_{\theta,\phi}|\ \mathrm{d}p\ \mathrm{d}x = 0.
$$

Hence (2.6) implies that, for a.a. $x \in \mathbb{T}$ and any $c \in \mathbb{R}$, the function $p \mapsto (f - g - c\mathcal{M}_{\theta,\phi})(x,p)$ has a constant sign. In particular it holds true for $c = c(x)$ given by $\int_{\mathbb{R}}(f-g)(x,p)\ \mathrm{d}p = c(x)\int_{\mathbb{R}} \mathcal{M}_{\theta,\phi}(x,p)\ \mathrm{d}p$. Since the function $p \mapsto (f-g-c\mathcal{M}_{\theta,\phi})(x,p)$ has a constant sign and zero integral with respect to $p \in \mathbb{R}$ we have

$$
f(x,p) - g(x,p) = c(x)\mathcal{M}_{\theta,\phi}(x,p) = \langle f-g\rangle(x) M_\theta(p), \qquad (x,p) \in \mathbb{T}\times\mathbb{R}.
\tag{2.7}
$$

Inserting now (2.7) in (2.1) we deduce that

$$
f(x,p) - g(x,p) = k\mathcal{M}_{\theta,\phi}(x,p), \qquad (x,p) \in \mathbb{T}\times\mathbb{R}
$$

for some real constant k. This conclusion holds for every two periodic solutions, and for $g = 0$ it tells us that all the periodic solutions of (0.1) when $\int_{\mathbb{T}} E \, dx = 0$ are $k\mathcal{M}_{\theta,\phi}$, $k \in \mathbb{R}$. Clearly, these solutions remain bounded. Observe also that the current of these solutions vanishes since $\int_{\mathbb{R}} v(p)\mathcal{M}_{\theta,\phi} \, dp = 0$. This already proves part a) of Theorem 0.1.

2.2. **Non vanishing electric field average.** Let us analyze the case of electric fields with non vanishing average: from now on we assume that

$$E \text{ is } L\text{-periodic with } \int_0^L E(x) \, dx \neq 0.$$

Now $\mathcal{M}_{\theta,\phi}$ is not periodic with respect to the space variable. Let us show that there is at most one periodic solution with a given current.

Proposition 2.1. *Assume that $E \in L^\infty(\mathbb{T})$ such that $\int_{\mathbb{T}} E \, dx \neq 0$ and let $f, g \in L^1(\mathbb{T} \times \mathbb{R})$ be two periodic weak solutions for (0.1) with the same current*

$$q \int_{\mathbb{R}} v(p) f \, dp = q \int_{\mathbb{R}} v(p) g \, dp.$$

Then we have $f = g$.

The proof exploits new dissipation properties of the linear collision operator Q. We have seen that the inequality (2.4) is not strong enough for our purposes. Actually a better minoration for the dissipation term $-\int_{\mathbb{R}} \operatorname{sgn}(f-g) Q(f-g) \, dp$ is available, at least in the relativistic case.

Lemma 2.2. *Let $h = h(p)$ be a function of $L^1(\mathbb{R})$ with vanishing current $\int_{\mathbb{R}} v(p)h(p) \, dp = 0$. Then we have the following inequality*

$$-\int_{\mathbb{R}} \operatorname{sgn}(h(p)) Q(h)(p) \, dp \geq \frac{s_0}{c_0} \left| \int_{\mathbb{R}} v(p)|h(p)| \, dp \right|. \tag{2.8}$$

Proof. We consider the sets

$$A_+ = \{p \in \mathbb{R} \, : \, h(p) \geq 0\}, \qquad A_- = \{p \in \mathbb{R} \, : \, h(p) < 0\}.$$

Since $\int_{\mathbb{R}} v(p)h(p) \, dp = 0$, we have

$$\int_{\mathbb{R}} v(p)|h(p)| \mathbf{1}_{A_+}(p) \, dp = \int_{\mathbb{R}} v(p)|h(p)| \mathbf{1}_{A_-}(p) \, dp = \frac{1}{2} \int_{\mathbb{R}} v(p)|h(p)| \, dp.$$

Observe that

$$\begin{aligned} -\int_{\mathbb{R}} \operatorname{sgn}(h(p)) Q(h)(p) \, dp &= \int_{\mathbb{R}} \operatorname{sgn}(h(p)) \int_{\mathbb{R}} s(p,p') \Big(M_\theta(p')h(p) - M_\theta(p)h(p') \Big) \, dp' \, dp \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} s(p,p') h(p) M_\theta(p') \Big(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p')) \Big) \, dp' \, dp. \end{aligned}$$

But for any $(p, p') \in \mathbb{R}^2$ we have

$$h(p) M_\theta(p')(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p'))) \geq 0, \qquad s(p,p') \geq s_0 > 0,$$

and thus, by taking into account (0.5), we can write

$$h(p) M_\theta(p')(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p'))) \geq \pm \frac{v(p)}{c_0} h(p) M_\theta(p')(\operatorname{sgn}(h(p)) - \operatorname{sgn}(h(p'))).$$

Combining these computations yields

$$
\begin{aligned}
-\int_{\mathbb{R}} \mathrm{sgn}(h)Q(h)\,\mathrm{d}p \quad &\geq \quad \frac{\pm s_0}{c_0}\int_{\mathbb{R}}\int_{\mathbb{R}} v(p)h(p)M_\theta(p')(\mathrm{sgn}(h(p)) - \mathrm{sgn}(h(p')))\,\mathrm{d}p\,\mathrm{d}p' \\
&\geq \quad \frac{\pm 2s_0}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\mathbf{1}_{A_+}(p)\,\mathrm{d}p \int_{\mathbb{R}} M_\theta(p')\mathbf{1}_{A_-}(p')\,\mathrm{d}p' \\
&\quad\; + \frac{\pm 2s_0}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\mathbf{1}_{A_-}(p)\,\mathrm{d}p \int_{\mathbb{R}} M_\theta(p')\mathbf{1}_{A_+}(p')\,\mathrm{d}p' \\
&\geq \quad \frac{\pm s_0}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\,\mathrm{d}p \int_{\mathbb{R}} M_\theta(p')(\mathbf{1}_{A_-} + \mathbf{1}_{A_+})(p')\,\mathrm{d}p' \\
&\geq \quad \frac{\pm s_0}{c_0}\int_{\mathbb{R}} v(p)|h(p)|\,\mathrm{d}p.
\end{aligned}
$$

□

Remark 2.3. When the light speed c_0 becomes large, the inequality (2.8) degenerates to (2.4). In particular in the non-relativistic case the conclusion of Lemma 2.2 reduces to the well-known inequality (2.4) which is not enough for the uniqueness of the periodic weak solution.

Proof of Proposition 2.1. Consider the function $h = f - g - c\mathcal{M}_{\theta,\phi}$ with $c \in \mathbb{R}$. This function belongs to $L^1([a,b]\times\mathbb{R})$ for any $a < b$. It has vanishing current $\int_{\mathbb{R}} v(p)h(x,p)\,\mathrm{d}p = 0, x \in \mathbb{R}$ and satisfies in $\mathcal{D}'(\mathbb{R}^2)$

$$v(p)\partial_x h + F(x)\partial_p h = Q(h).$$

We obtain

$$v(p)\partial_x|h| + F(x)\partial_p|h| - \mathrm{sgn}(h)Q(h) = 0. \tag{2.9}$$

Integrating with respect to $p \in \mathbb{R}$ and combining with Lemma 2.2 yield

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)|h|\,\mathrm{d}p + \frac{s_0}{c_0}\left|\int_{\mathbb{R}} v(p)|h|\,\mathrm{d}p\right| \\
&\qquad\leq \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)|h|\,\mathrm{d}p - \int_{\mathbb{R}} \mathrm{sgn}(h)Q(h)\,\mathrm{d}p = 0.
\end{aligned}
\tag{2.10}
$$

Let us set

$$u(x) = \int_{\mathbb{R}} v(p)|h(x,p)|\,\mathrm{d}p.$$

This function is not periodic but satisfies the bounds

$$\sup_{n\in\mathbb{Z}} |u(x+nL)| < +\infty, \qquad \text{a.e. } x \in \mathbb{R}. \tag{2.11}$$

Indeed we have for any $n \in \mathbb{Z}$

$$
\begin{aligned}
|u(x+nL)| &= \left| \int_{\mathbb{R}} v(p)\Big(|h(x+nL,p)| - |c\mathcal{M}_{\theta,\phi}(x+nL,p)| \Big) \,\mathrm{d}p \right| \\
&\le \int_{\mathbb{R}} |v(p)| \, \Big| \, |h(x+nL,p)| - |c\mathcal{M}_{\theta,\phi}(x+nL,p)| \, \Big| \,\mathrm{d}p \\
&\le c_0 \int_{\mathbb{R}} \Big| h(x+nL,p) + c\mathcal{M}_{\theta,\phi}(x+nL,p) \Big| \,\mathrm{d}p \\
&\le c_0 \int_{\mathbb{R}} \Big| f(x+nL,p) - g(x+nL,p) \Big| \,\mathrm{d}p \\
&\le c_0 \int_{\mathbb{R}} |f(x,p)| \,\mathrm{d}p + c_0 \int_{\mathbb{R}} |g(x,p)| \,\mathrm{d}p.
\end{aligned}
$$

We shall see now that $u(x)$ actually vanishes. By (2.10) we know that $u'(x) + \frac{\pm s_0}{c_0} u(x) \le 0$, $x \in \mathbb{R}$, which implies

$$
\frac{\mathrm{d}}{\mathrm{d}x} \left\{ u(x) \exp\left(\frac{\pm s_0 x}{c_0} \right) \right\} \le 0, \qquad x \in \mathbb{R}. \tag{2.12}
$$

Consider first the case with the sign $+$. Let us integrate (2.12) between $x - nL$ and x with $n \in \mathbb{N}$. We deduce that

$$
u(x) \le u(x-nL) \exp\left(\frac{-nLs_0}{c_0} \right).
$$

We let $n \to +\infty$: by using (2.11) we get $u(x) \le 0$ for a.a. $x \in \mathbb{R}$. Similarly, for the case with the sign $-$, we integrate over $[x, x+nL]$ with $n \in \mathbb{N}$ and and we let $n \to +\infty$. We get $u(x) \ge 0$ for a.a. $x \in \mathbb{R}$. Therefore we have $u = 0$ and coming back to (2.10) we deduce that

$$
\int_{\mathbb{R}} \operatorname{sgn}(h) Q(h) \,\mathrm{d}p = 0, \qquad \text{a.e. } x \in \mathbb{R}. \tag{2.13}
$$

At this stage let us point out that one can not obtain (2.13) as in the case of periodic potentials, by integrating (2.9) over $\mathbb{T} \times \mathbb{R}$. Indeed, in this case h is not periodic and thus

$$
\int_{\mathbb{T}} \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} v(p) |h(x,p)| \,\mathrm{d}p \,\mathrm{d}x \ne 0.
$$

Therefore Lemma 2.2 is crucial when establishing (2.13) for non periodic potentials.

¿From now on we follow the same steps as for periodic potentials. We deduce that there is a constant $k \in \mathbb{R}$ such that

$$
f(x,p) - g(x,p) = k\mathcal{M}_{\theta,\phi}(x,p), \qquad (x,p) \in \mathbb{R}^2.
$$

Since f and g are periodic and $\mathcal{M}_{\theta,\phi}$ is not periodic we must have $k = 0$ and therefore $f = g$. Indeed, the weak formulation (1.1) applied to $f - g$ implies

$$
k \int_{\mathbb{T}} \int_{\mathbb{R}} \mathcal{M}_{\theta,\phi}(x,p) (v(p)\partial_x \varphi + F(x) \partial_p \varphi) \,\mathrm{d}p \,\mathrm{d}x = 0, \qquad \varphi \in C_b^1(\mathbb{T} \times \mathbb{R})
$$

and after integration by parts we get

$$
k \int_{\mathbb{R}} v(p) \varphi(0,p) (\mathcal{M}_{\theta,\phi}(L,p) - \mathcal{M}_{\theta,\phi}(0,p)) \,\mathrm{d}p = 0.
$$

Since the potential is not periodic we obtain

$$k \int_{\mathbb{R}} v(p)\varphi(0,p)M_\theta(p)\ \mathrm{d}p = 0, \qquad \varphi \in C_b^1(\mathbb{T}\times\mathbb{R}).$$

In particular taking $\varphi(x,p) = v(p),\ (x,p) \in \mathbb{T}\times\mathbb{R}$ yields

$$k \int_{\mathbb{R}} |v(p)|^2 M_\theta(p)\ \mathrm{d}p = 0$$

saying that $k = 0$. □

3. Existence of periodic weak solution

In order to construct a periodic solution for the linear Boltzmann equation we appeal to perturbation techniques. We start by considering smooth electric fields, the regularity will be removed by a standard approximation argument. We consider the periodic equation with a damping term and a source:

$$\alpha f(x,p) + v(p)\partial_x f + F(x)\partial_p f = Q(f) + S(x,p), \qquad (x,p) \in \mathbb{T}\times\mathbb{R}. \tag{3.1}$$

where $\alpha > 0$. To this end, we adapt Definition 1.1 as follows.

Definition 3.1. Let $F \in L^\infty(\mathbb{T}), S \in L^1(\mathbb{T}\times\mathbb{R})$ and $\alpha > 0$. We say that $f \in L^1(\mathbb{T}\times\mathbb{R})$ is a periodic weak solution for (3.1) iff

$$\begin{aligned} -\int_{\mathbb{T}}\int_{\mathbb{R}} f(x,p)(-\alpha\varphi(x,p) + v(p)\partial_x\varphi + F(x)\partial_p\varphi)\ \mathrm{d}p\ \mathrm{d}x \\ = \int_{\mathbb{T}}\int_{\mathbb{R}} Q(f)\varphi(x,p)\ \mathrm{d}p\ \mathrm{d}x + \int_{\mathbb{T}}\int_{\mathbb{R}} S(x,p)\varphi(x,p)\ \mathrm{d}p\ \mathrm{d}x \end{aligned} \tag{3.2}$$

for any function $\varphi \in C_c^1(\mathbb{T}\times\mathbb{R})$.

We check that the above formulation holds true for any $\varphi \in C_b^1(\mathbb{T}\times\mathbb{R})$ as well. Furthermore the following identity holds by integrating (3.1) with respect to the momentum variable p

$$\alpha \int_{\mathbb{R}} f(x,p)\ \mathrm{d}p + \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p)f(x,p)\ \mathrm{d}p = \int_{\mathbb{R}} S(x,p)\ \mathrm{d}p, \qquad x \in \mathbb{T}\ . \tag{3.3}$$

The basis of the existence proof relies on the following claim, where the problem with source term in $L^1(\mathbb{T}\times\mathbb{R})$ is investigated. The solution is shown to satisfy estimates uniformly with respect to the penalization parameter $\alpha > 0$.

Proposition 3.2. *Assume that $S \in L^1(\mathbb{T}\times\mathbb{R}), E \in W^{1,\infty}(\mathbb{T})$ and $\alpha > 0$. Then there is a unique periodic solution of* (3.1) *satisfying*

$$\|f\|_{L^1(\mathbb{T}\times\mathbb{R})} \le \frac{1}{\alpha}\|S\|_{L^1(\mathbb{T}\times\mathbb{R})}. \tag{3.4}$$

Moreover the following properties hold

a) *if $S \ge 0$ then $f \ge 0$,*
b) *if $S \in L^\infty(\mathbb{T}; L^1(\mathbb{R}))$ then*

$$\left\|\int_{\mathbb{R}} v(p)|f(\cdot,p)|\ \mathrm{d}p\right\|_{L^\infty(\mathbb{T})} \le \frac{c_0}{\alpha}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))},$$

c) *if $S \ge 0,\ S \in L^\infty(\mathbb{T}\times\mathbb{R})$ and $\int_{\mathbb{R}} v(p)pf(\cdot,p)\ \mathrm{d}p \in L^\infty(\mathbb{T})$ then $f \in L^\infty(\mathbb{T}\times\mathbb{R}),\ \langle f\rangle \in L^\infty(\mathbb{T})$.*

This statement is obtained as a consequence of the following well-posedness result.

Lemma 3.3. *Let $E \in W^{1,\infty}(\mathbb{T}), S \in L^1(\mathbb{T}\times\mathbb{R})$ and $\alpha > 0$. Then there is a unique periodic weak solution $f \in L^1(\mathbb{T}\times\mathbb{R})$ of the problem*

$$(\alpha + \sigma(p))f(x,p) + v(p)\partial_x f + F(x)\partial_p f = S(x,p), \qquad (x,p) \in \mathbb{T}\times\mathbb{R} \tag{3.5}$$

satisfying

$$\|f\|_{L^1(\mathbb{T}\times\mathbb{R})} \le \frac{1}{\alpha+s_0}\|S\|_{L^1(\mathbb{T}\times\mathbb{R})}. \tag{3.6}$$

If $S \ge 0$ then $f \ge 0$, and more generally if $S \in L^\infty(\mathbb{T}\times\mathbb{R})$ then

$$-\frac{1}{\alpha+s_0}\|S_-\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le f(x,p) \le \frac{1}{\alpha+s_0}\|S_+\|_{L^\infty(\mathbb{T}\times\mathbb{R})}, \qquad a.e.\ (x,p) \in \mathbb{T}\times\mathbb{R} \tag{3.7}$$

where $S_\pm = \max(0, \pm S)$. Finally, if $S \in L^\infty(\mathbb{T}; L^1(\mathbb{R}))$ then

$$\left\| \int_{\mathbb{R}} v(p)|f(\cdot,p)|\,\mathrm{d}p \right\|_{L^\infty(\mathbb{T})} \le \frac{c_0}{\alpha+s_0}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))} \tag{3.8}$$

Proof. We start by proving the uniqueness: for any two solutions f, g we have

$$(\alpha+\sigma(p))|f-g| + v(p)\partial_x|f-g| + F(x)\partial_p|f-g| = 0$$

and therefore

$$(\alpha+s_0)\int_{\mathbb{T}}\int_{\mathbb{R}}|f-g|\,\mathrm{d}p\,\mathrm{d}x \le \int_{\mathbb{T}}\int_{\mathbb{R}}(\alpha+\sigma(p))|f-g|\,\mathrm{d}p\,\mathrm{d}x = 0$$

implying that $f = g$.

Owing to the regularity of the electric field, let us consider the characteristics (X, P) defined by

$$\frac{\mathrm{d}X}{\mathrm{d}s} = v(P(s;x,p)), \qquad \frac{\mathrm{d}P}{ds} = qE(X(s;x,p)),$$

with the conditions

$$X(0;x,p) = x, \qquad P(0;x,p) = p.$$

Integrating equation (3.1) along the characteristics yields

$$f(x,p) = \int_{-\infty}^{0} e^{\int_0^s \{\alpha+\sigma(P(\tau;x,p))\}\,\mathrm{d}\tau} S(X(s;x,p),P(s;x,p))\,\mathrm{d}s \tag{3.9}$$

is a weak solution for (3.5). Since E is periodic, we have

$$X(s;x+L,p) = X(s;x,p) + L, \qquad P(s;x+L,p) = P(s;x,p)$$

and therefore f is also L-periodic.

The L^1 bound (3.6) follows by using $\sigma(p) \ge s_0$ and integrating over $\mathbb{T}\times\mathbb{R}$ the inequality

$$(\alpha+\sigma(p))|f(x,p)| + v(p)\partial_x|f| + F(x)\partial_p|f| \le |S(x,p)|.$$

The L^∞ bounds (3.7) follow immediately from the explicit formula (3.9).

It remains to justify the estimate on the current (3.8). Assume that $S \in L^\infty(\mathbb{T}; L^1(\mathbb{R}))$ and observe that

$$(\alpha+s_0)\int_{\mathbb{R}}|f(x,p)|\,\mathrm{d}p + \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}}v(p)|f(x,p)|\,\mathrm{d}p \le \int_{\mathbb{R}}|S(x,p)|\,\mathrm{d}p.$$

We set $u(x) = \int_{\mathbb{R}} v(p)|f(x,p)|\ \mathrm{d}p$. Then, the relativistic bound (0.5), implies $c_0 \int_{\mathbb{R}} |f(x,p)|\ \mathrm{d}p \geq |u(x)|$, so one gets

$$\frac{\pm(\alpha+s_0)}{c_0} u(x) + u'(x) \leq \int_{\mathbb{R}} |S(x,p)|\ \mathrm{d}p, \qquad x \in \mathbb{R}.$$

With the + sign, we integrate between $x - nL$ and x, with $n \in \mathbb{N}$, and we deduce that

$$\begin{aligned} u(x) \quad &\leq \quad u(x-nL) \exp\left(-\frac{(\alpha+s_0)nL}{c_0}\right) \\ &\quad + \|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))} \int_{x-nL}^{x} \exp\left(-\frac{(\alpha+s_0)(x-y)}{c_0}\right)\ \mathrm{d}y. \end{aligned}$$

Letting $n \to +\infty$ we obtain $u(x) \leq \frac{c_0}{\alpha+s_0}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))}$ (and clearly we can dominate it by $\frac{c_0}{s_0}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))}$, which provides an estimate uniform with respect to α). Proceeding similarly with the $-$ sign, we deduce that $u(x) \geq -\frac{c_0}{\alpha+s_0}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))} \geq -\frac{c_0}{s_0}\|S\|_{L^\infty(\mathbb{T};L^1(\mathbb{R}))}$. □

Proof of Proposition 3.2. We consider the sequence of periodic weak solutions $(f_\pm^{(n)})_{n\in\mathbb{N}}$ defined by

$$\sigma(p) f_\pm^{(0)}(x,p) + \alpha f_\pm^{(0)}(x,p) + v(p)\partial_x f_\pm^{(0)} + F(x)\partial_p f_\pm^{(0)} = S_\pm(x,p), \qquad (x,p) \in \mathbb{T}\times\mathbb{R}$$

and for any $n \in \mathbb{N}$

(3.10)

$$\sigma(p) f_\pm^{(n+1)} + \alpha f_\pm^{(n+1)} + v(p)\partial_x f_\pm^{(n+1)} + F(x)\partial_p f_\pm^{(n+1)} = M_\theta\langle f_\pm^{(n)}\rangle_s + S_\pm, \qquad (x,p) \in \mathbb{T}\times\mathbb{R}$$

where $S_\pm$ are the positive/negative parts of S. Thanks to Lemma 3.3 the sequence $(f_\pm^{(n)})_{n\in\mathbb{N}}$ is well defined. We have $f_\pm^{(0)} \geq 0$ and we check recursively that $0 \leq f_\pm^{(n)} \leq f_\pm^{(n+1)}$ for any $n \in \mathbb{N}$. Integrating over $\mathbb{T}\times\mathbb{R}$ we get

$$\begin{aligned} \int_{\mathbb{T}}\int_{\mathbb{R}} (\alpha + \sigma(p))\, f_\pm^{(n+1)}(x,p)\ \mathrm{d}p\ \mathrm{d}x \quad &= \quad \int_{\mathbb{T}}\int_{\mathbb{R}} M_\theta(p)\langle f_\pm^{(n)}\rangle_s\ \mathrm{d}p\ \mathrm{d}x + \int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm\ \mathrm{d}p\ \mathrm{d}x \\ &= \quad \int_{\mathbb{T}}\int_{\mathbb{R}} \sigma(p) f_\pm^{(n)}\ \mathrm{d}p\ \mathrm{d}x + \int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm\ \mathrm{d}p\ \mathrm{d}x \\ &\leq \quad \int_{\mathbb{T}}\int_{\mathbb{R}} \sigma(p) f_\pm^{(n+1)}\ \mathrm{d}p\ \mathrm{d}x + \int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm\ \mathrm{d}p\ \mathrm{d}x \end{aligned}$$

implying that $\sup_{n\in\mathbb{N}} \int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm^{(n)}\ \mathrm{d}p\ \mathrm{d}x \leq \alpha^{-1}\int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm\ \mathrm{d}p\ \mathrm{d}x$. By the monotone convergence theorem we deduce that $(f_\pm^{(n)})_n$ converges in $L^1(\mathbb{T}\times\mathbb{R})$. Let $f_\pm = \lim_{n\to\infty} f_\pm^{(n)} \geq 0$. Passing to the limit for $n \to +\infty$ we deduce that $f_\pm \geq 0$ are periodic weak solutions of

$$\alpha f_\pm + v(p)\partial_x f_\pm + F(x)\partial_p f_\pm = Q(f_\pm) + S_\pm, \qquad (x,p) \in \mathbb{T}\times\mathbb{R}$$

satisfying

$$\int_{\mathbb{T}}\int_{\mathbb{R}} f_\pm\ \mathrm{d}p\ \mathrm{d}x = \alpha^{-1}\int_{\mathbb{T}}\int_{\mathbb{R}} S_\pm\ \mathrm{d}p\ \mathrm{d}x.$$

Therefore $f = f_+ - f_-$ is a periodic weak solution of (3.1) satisfying (3.4).

Assume now that S belongs to $L^\infty(\mathbb{T}; L^1(\mathbb{R}))$. The estimate in b) follows exactly as in the proof of Lemma 3.3 since we have

$$\alpha \int_{\mathbb{R}} |f| \,\mathrm{d}p + \frac{\mathrm{d}}{\mathrm{d}x} \int_{\mathbb{R}} v(p)|f| \,\mathrm{d}p = \int_{\mathbb{R}} Q(f)\mathrm{sgn}(f) \,\mathrm{d}p + \int_{\mathbb{R}} S \,\mathrm{sgn}(f) \,\mathrm{d}p \le \int_{\mathbb{R}} |S| \,\mathrm{d}p.$$

The final step consists in proving the L^∞ estimate. Let $S \in L^\infty(\mathbb{T}\times\mathbb{R}), S \ge 0$ and assume that there is $\mathcal{K} > 0$ such that

$$\int_{\mathbb{R}} v(p)pf_+^{(n)} \,\mathrm{d}p \le \int_{\mathbb{R}} v(p)pf_+ \,\mathrm{d}p = \int_{\mathbb{R}} v(p)pf \,\mathrm{d}p \le \mathcal{K}, \qquad \text{a.e. } x \in \mathbb{T}.$$

Let $\tilde{\mathcal{K}} > 1$ which will be defined later on, such that $\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le s_0\tilde{\mathcal{K}}/2$.

By Lemma 3.3 we know that

$$\|f_+^{(0)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le \frac{1}{s_0+\alpha}\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le \frac{\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})}}{s_0} \le \tilde{\mathcal{K}}.$$

Then we wish to find some $\tilde{\mathcal{K}}$ such that the property extends to the whole sequence.

Assume that $f_+^{(n)} \in L^\infty(\mathbb{T}\times\mathbb{R})$ with $\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le \tilde{\mathcal{K}}$. Then, on one hand we can estimate the integral of $f_+^{(n)}$ with respect to momentum p by splitting the domain of integration as

$$\begin{aligned}
\langle f_+^{(n)} \rangle &= \int_{\mathbb{R}} f_+^{(n)} \mathbf{1}_{\{|p|<R\}} \,\mathrm{d}p + \int_{\mathbb{R}} f_+^{(n)} \mathbf{1}_{\{|p|\ge R\}} \,\mathrm{d}p \\
&\le 2R\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \frac{1}{Rv(R)} \int_{\mathbb{R}} v(p)pf_+^{(n)} \mathbf{1}_{\{|p|\ge R\}} \,\mathrm{d}p \\
&\le 2R\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \frac{1}{Rv(R)}\mathcal{K} \\
&\le 2R\tilde{\mathcal{K}} + \frac{1}{Rv(R)}\mathcal{K}.
\end{aligned}$$

Hence, we define $R = R(\tilde{\mathcal{K}})$ such that $2R\tilde{\mathcal{K}} = \frac{\mathcal{K}}{Rv(R)}$ to obtain the inequality

$$\|\langle f_+^{(n)} \rangle\|_{L^\infty(\mathbb{T})} \le 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}.$$

Since $v(R)/R \to 1/m$ as $R \to 0$, we observe that the relation $R(\tilde{\mathcal{K}})^2 v(R(\tilde{\mathcal{K}})) = \mathcal{K}/2\tilde{\mathcal{K}}$ implies that $R(\tilde{\mathcal{K}})$ tends to 0 as $\tilde{\mathcal{K}} \to \infty$ (it behaves like $\tilde{\mathcal{K}}^{-1/3}$).

On the other hand, a direct application of Lemma 3.3, see (3.7), leads to

$$\begin{aligned}
\|f_+^{(n+1)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} &\le \frac{s_1}{s_0}\left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(p')}{\theta}\right) \mathrm{d}p'\right)^{-1} \|\langle f_+^{(n)} \rangle\|_{L^\infty(\mathbb{T})} + \frac{1}{s_0}\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \\
&\le \frac{s_1}{s_0} 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}} \left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(p')}{\theta}\right) \mathrm{d}p'\right)^{-1} + \frac{1}{s_0}\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})}.
\end{aligned}$$

Therefore, the uniform estimates relies on the possibility to find $\tilde{\mathcal{K}}$ such that $\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le s_0\tilde{\mathcal{K}}/2$ and

$$\frac{s_1}{s_0} 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}} \left(\int_{\mathbb{R}} \exp\left(-\frac{\mathcal{E}(p')}{\theta}\right) \mathrm{d}p'\right)^{-1} + \frac{1}{s_0}\|S\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le \tilde{\mathcal{K}}. \tag{3.11}$$

This is obviously possible for $\tilde{\mathcal{K}}$ large enough since $\lim_{\tilde{\mathcal{K}}\to+\infty} R(\tilde{\mathcal{K}}) = 0$.

Accordingly, we have obtained the estimates

$$\|f_+^{(n)}\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le \tilde{\mathcal{K}}, \qquad \|\langle f_+^{(n)} \rangle\|_{L^\infty(\mathbb{T})} \le 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}, \qquad n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} f_+^{(n)}(x,p) = f_+(x,p) = f(x,p)$ for a.a. $(x,p) \in \mathbb{T}\times\mathbb{R}$ we deduce that $\|f\|_{L^\infty(\mathbb{T}\times\mathbb{R})} \le \tilde{\mathcal{K}}$. The monotone convergence theorem also proves that $\lim_{n\to\infty}\langle f_+^{(n)}\rangle(x) = \langle f\rangle(x)$ for a.a. $x\in\mathbb{T}$ and thus

$$\|\langle f\rangle\|_{L^\infty(\mathbb{T})} \le 4R(\tilde{\mathcal{K}})\tilde{\mathcal{K}}.$$

□

By Proposition 3.2 we know that for any $\alpha > 0$ there is a unique periodic weak solution f_α for the problem

$$\alpha f_\alpha(x,p) + v(p)\partial_x f_\alpha + F(x)\partial_p f_\alpha = Q(f_\alpha) + \alpha M_\theta(p), \qquad (x,p)\in\mathbb{T}\times\mathbb{R}. \tag{3.12}$$

These solutions are non negative and satisfy for any $\alpha > 0$

$$\begin{aligned} &\int_{\mathbb{T}}\int_{\mathbb{R}} f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x = \int_{\mathbb{T}}\int_{\mathbb{R}} M_\theta(p)\,\mathrm{d}p\,\mathrm{d}x = L,\\ &\left\|\int_{\mathbb{R}} v(p) f_\alpha(\cdot,p)\,\mathrm{d}p\right\|_{L^\infty(\mathbb{T})} \le c_0. \end{aligned} \tag{3.13}$$

We split the end of the proof of Theorem 0.1 into several steps.

Step 1: Existence of a non trivial solution

By applying the weak formulation of (3.12) to the test function $\mathcal{E}(p) + q\phi(x)$ one gets

$$\begin{aligned} \alpha\int_{\mathbb{T}}\int_{\mathbb{R}} f_\alpha(\mathcal{E}(p)+q\phi(x))\,\mathrm{d}p\,\mathrm{d}x \quad &+\quad q\phi(L)\int_{\mathbb{R}} v(p) f_\alpha(L,p)\,\mathrm{d}p - q\phi(0)\int_{\mathbb{R}} v(p) f_\alpha(0,p)\,\mathrm{d}p\\ &+\quad \int_{\mathbb{T}}\int_{\mathbb{R}} (\sigma(p) f_\alpha - M_\theta(p)\langle f_\alpha\rangle_s)(\mathcal{E}(p)+q\phi(x))\,\mathrm{d}p\,\mathrm{d}x\\ &=\quad \alpha\int_{\mathbb{T}}\int_{\mathbb{R}} M_\theta(p)(\mathcal{E}(p)+q\phi(x))\,\mathrm{d}p\,\mathrm{d}x. \end{aligned}$$

Therefore by taking into account (3.13) and

$$\int_{\mathbb{T}}\int_{\mathbb{R}} M_\theta(p)\langle f_\alpha\rangle_s \mathcal{E}(p)\,\mathrm{d}p\,\mathrm{d}x \le s_1\int_{\mathbb{T}}\langle f_\alpha\rangle\,\mathrm{d}x\int_{\mathbb{R}} M_\theta(p)\mathcal{E}(p)\,\mathrm{d}p = s_1 L\int_{\mathbb{R}} M_\theta(p)\mathcal{E}(p)\,\mathrm{d}p$$

we deduce that

$$\sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} f_\alpha(x,p)\mathcal{E}(p)\,\mathrm{d}p\,\mathrm{d}x < +\infty.$$

Multiplying now (3.12) by p one gets

$$\int_{\mathbb{R}} (\alpha+\sigma(p)) p f_\alpha\,\mathrm{d}p + \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p = F(x)\langle f_\alpha\rangle + \int_{\mathbb{R}} p M_\theta(p)\langle f_\alpha\rangle_s\,\mathrm{d}p. \tag{3.14}$$

This relation proves that $x\mapsto \int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p$ is continuous since $\int_{\mathbb{R}}(\alpha + \sigma(p)) p f_\alpha\,\mathrm{d}p$ and $F(x)\langle f_\alpha\rangle + \int_{\mathbb{R}} p M_\theta(p)\langle f_\alpha\rangle_s\,\mathrm{d}p$ are integrable on $\mathbb{T}$, uniformly with respect to α. Then (0.6) leads to

$$\sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x \le \sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} (mc_0^2 + \mathcal{E}(p)) f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x < +\infty.$$

Therefore there exists $x_\alpha\in\mathbb{T}$ such that

$$\sup_{0<\alpha<1}\int_{\mathbb{R}} v(p) p f_\alpha(x_\alpha,p)\,\mathrm{d}p \le \frac{1}{L}\sup_{0<\alpha<1}\int_{\mathbb{T}}\int_{\mathbb{R}} v(p) p f_\alpha(x,p)\,\mathrm{d}p\,\mathrm{d}x < +\infty.$$

We integrate now (3.14) between x_α and x for any $x \in [x_\alpha, x_\alpha + L]$. Using (0.6) again we deduce that there is a constant C such that

$$\sup_{0<\alpha<1} \left\| \int_{\mathbb{R}} v(p) p f_\alpha(\cdot, p) \, \mathrm{d}p \right\|_{L^\infty(\mathbb{T})} \leq C$$
$$\sup_{0<\alpha<1} \int_{\mathbb{T}} \int_{\mathbb{R}} (mc_0^2 + \mathcal{E}(p)) f_\alpha(x,p) \, \mathrm{d}p \, \mathrm{d}x < +\infty$$

and thus applying Proposition 3.2-c) we obtain uniform L^∞ bounds for f_α and $\langle f_\alpha \rangle$. Therefore there is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ converging towards zero such that

$$\lim_{n\to\infty} f_{\alpha_n} = f \qquad \text{weakly-}\star \text{ in } L^\infty(\mathbb{T}\times\mathbb{R}),$$
$$\lim_{n\to\infty} \langle f_{\alpha_n} \rangle = \rho \qquad \text{weakly-}\star \text{ in } L^\infty(\mathbb{T}),$$
$$\lim_{n\to\infty} \int_{\mathbb{R}} v(p) f_{\alpha_n} \, \mathrm{d}p = j \qquad \text{weakly-}\star \text{ in } L^\infty(\mathbb{T}).$$

It is easily seen that

$$(1 + \mathcal{E}(p)) f \in L^1(\mathbb{T}\times\mathbb{R}),$$
$$f \in L^\infty(\mathbb{T}\times\mathbb{R}), \qquad f \geq 0, \qquad \int_{\mathbb{T}} \int_{\mathbb{R}} f \, \mathrm{d}p \, \mathrm{d}x = L,$$
$$\rho \in L^1(\mathbb{T}) \cap L^\infty(\mathbb{T})$$

and we check immediately that $\rho = \langle f \rangle$ and $j = \int_{\mathbb{R}} v(p) f \, \mathrm{d}p \in [-c_0, c_0]$. Now passing to the limit for $n \to +\infty$ in the weak formulation of (3.1) we deduce that f is a periodic weak solution of (0.1). This ends the proof of b) in Theorem 0.1 at least for smooth electric fields.

Step 2: Extension to rough electric fields
Notice that all the bounds of the above solution depend on $m, c_0, q, s_0, s_1, \theta$ and $\|E\|_{L^\infty(\mathbb{T})}$ but not on $\|E'\|_{L^\infty(\mathbb{T})}$. Therefore, we can onsider $(E_n)_{n\in\mathbb{N}}$ a sequence of smooth fields —- for each n, $E_n \in W^{1,\infty}(\mathbb{T})$ —- which converges a.e. towards $E \in L^\infty(\mathbb{T})$, with $\|E_n\|_{L^\infty(\mathbb{T})} \leq \|E\|_{L^\infty(\mathbb{T})}$. Let f_n be the associated sequence of periodic weak solution, as given by Step 1. We have the uniform bounds

$$\sup_{n\in\mathbb{N}} \|f_n\|_{L^\infty(\mathbb{T}\times\mathbb{R})} + \sup_{n\in\mathbb{N}} \int_{\mathbb{T}} \int_{\mathbb{R}} (1 + v(p)p) f_n(x,p) \, \mathrm{d}p \, \mathrm{d}x < +\infty$$

$$\sup_{n\in\mathbb{N}} \|\langle f_n \rangle\|_{L^\infty(\mathbb{T})} < +\infty, \qquad \int_{\mathbb{T}} \int_{\mathbb{R}} f_n(x,p) \, \mathrm{d}p \, \mathrm{d}x = L, \qquad \left| \int_{\mathbb{R}} v(p) f_n \, \mathrm{d}p \right| \leq c_0.$$

Therefore, we can extract a weakly-$\star$ convergent subsequence $(f_{n_k})_{k\in\mathbb{N}}$ in $L^\infty(\mathbb{T}\times\mathbb{R})$, and letting k go to ∞ proves part b) of Theorem 0.1 for any bounded electric field.

Step 3: Properties of the current
Let us discuss the sign of the current of the solution obtained in b), that we denote by f_* from now on. To this end we use another entropy dissipation relation. We combine the equalities

$$(v(p)\partial_x + F(x)\partial_p)(f_* \ln f_*) = Q(f_*)(1 + \ln f_*)$$

$$\frac{1}{\theta}(v(p)\partial_x + F(x)\partial_p)(f_*(\mathcal{E}(p) + q\phi(x))) = \frac{1}{\theta} Q(f_*)(\mathcal{E}(p) + q\phi(x))$$

and we deduce that

$$(v(p)\partial_x+F\partial_p)(f_*(\ln f_*+\theta^{-1}(\mathcal{E}(p)+q\phi(x)))) = Q(f_*)(1+\ln f_*+\theta^{-1}(\mathcal{E}(p)+q\phi(x))).$$

Taking into account the periodicity of f_* and the fact that $\int_{\mathbb{R}} Q(f_*)\ \mathrm{d}p = 0$, integration with respect to $(x,p) \in \mathbb{T}\times\mathbb{R}$ yields

$$\frac{q}{\theta}(\phi(L)-\phi(0))\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p - \int_{\mathbb{T}}\int_{\mathbb{R}} Q(f_*)\ln\frac{f_*}{M_\theta(p)}\ \mathrm{d}p\ \mathrm{d}x = 0. \tag{3.15}$$

Using now the monotonicity of the ln function we have

$$\begin{aligned}
-\int_{\mathbb{R}} Q(f_*)\ln\frac{f_*}{M_\theta}\ \mathrm{d}p &= -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} s(p,p')\{M_\theta(p)f_*(x,p') - M_\theta(p')f_*(x,p)\} \\
&\quad \times\left\{\ln\frac{f_*(x,p)}{M_\theta(p)} - \ln\frac{f_*(x,p')}{M_\theta(p')}\right\}\ \mathrm{d}p'\ \mathrm{d}p \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} s(p,p')M_\theta(p)M_\theta(p')\left\{\frac{f_*(x,p')}{M_\theta(p')} - \frac{f_*(x,p)}{M_\theta(p)}\right\} \\
&\quad \times\left\{\ln\frac{f_*(x,p')}{M_\theta(p')} - \ln\frac{f_*(x,p)}{M_\theta(p)}\right\}\ \mathrm{d}p'\ \mathrm{d}p \\
&\geq 0
\end{aligned} \tag{3.16}$$

and therefore $q\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p\int_{\mathbb{T}} E\ \mathrm{d}x \geq 0$. It is worth pointing out that, in the specific case of a constant scattering function, (3.15) leads to the following remarkable identity

$$\frac{q}{\theta}(\phi(L)-\phi(0))\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p + \frac{1}{\tau}\int_{\mathbb{T}}\int_{\mathbb{R}}(f_* - \langle f_*\rangle M_\theta(p))\ln\frac{f_*}{\langle f_*\rangle M_\theta(p)}\ \mathrm{d}p\ \mathrm{d}x = 0.$$

Let us now show that

$$\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p \neq 0.$$

We argue by contradiction. If the current of f_* vanishes then by (3.15), (3.16) we know that f_*/M_θ do not depend on p, implying that $f_* = \langle f_*\rangle M_\theta$ and therefore

$$(v(p)\partial_x + F(x)\partial_p)(\langle f_*\rangle M_\theta) = Q(\langle f_*\rangle M_\theta) = 0.$$

Hence, we have $f_* = \langle f_*\rangle M_\theta = k\mathcal{M}_{\theta,\phi}$ which is periodic only if $k=0$ (since $\int_{\mathbb{T}} E\ \mathrm{d}x \neq 0$). But it would contradict $\int_{\mathbb{T}}\int_{\mathbb{R}} f_*\ \mathrm{d}p\ \mathrm{d}x = L$.

Step 4: Solution with a given current
It remains to prove the existence of periodic solution for any given current $j\in\mathbb{R}$, when the electric field has a non vanishing average.

Given $j \in \mathbb{R}$, we can consider

$$f = \frac{j}{q\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p}\ f_*,$$

which defines a periodic weak solution with current equal to j. Observe that $\mathrm{sgn} f = \mathrm{sgn}\big(\frac{j}{q\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p}\big)$. Since $q\int_{\mathbb{R}} v(p)f_*\ \mathrm{d}p\int_{\mathbb{T}} E\ \mathrm{d}x > 0$, we conclude that $\mathrm{sgn} f = \mathrm{sgn}\big(\frac{j}{\int_{\mathbb{T}} E\ \mathrm{d}x}\big)$. The proof of Theorem 0.1 is now completed. □

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