Analysis of a model of self-propelled agents interacting through pheromone

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Abstract

We establish well-posedness for a model of self-propelled agents interacting through pheromone which they themselves produce. The model consists of an arbitrary number of agents modeled by a system of ordinary differential equations, for which the acceleration term includes the influence of a chemical signal, or pheromone, which induces a turning-like behaviour. The signal is produced by the agents themselves and obeys a diffusion equation. We prove that the resulting system, which is non-local in both time and space, enjoys well-posedness properties, using a fixed point method, and show some numerical results.

Keywords: Self-propelled agents; Self organisation; Collective behaviour; Ant navigation; Individual-based model; Pheromone landscape; Animal movement.

MSC classification: 92D50, 34C60, 34A12, 70F99

1 Introduction

Despite very limited communication abilities, certain populations of animals, or even robots, are able to self-organize. This organization is shaped by the reaction of the individuals to signals [2, 9, 12, 17, 19, 20], which can be either external, or produced by the individuals themselves. Ant societies provide examples of such remarkable pattern formations, quite easily accessible to observations and experiments. The emergence of collective behavior in ant societies is attributed to chemical communication by way of pheromones, chemical substances secreted by the individuals and deposited on the substrate to provide information to the others: ants use pheromone trails to communicate between themselves information about food sources – such as direction, distance, quality, or abundance –, danger, direction to the nest, and so on [3, 5, 7, 8, 13]. Mathematical modeling is intended to shed some light on the emergence of such collective behavior, based

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on limited exchanges of information and simple individual rules; as initiated in the seminal work [19, 20]. In the specific case of ants, it is worth mentioning the works of [5, 6, 10, 11, 14, 16] which offer a large variety of approaches by using individual-based models or more macroscopic PDEs systems.

Therefore, ants move in response to local concentrations of pheromones and, in turn, modify these same concentrations by the deposit of a certain amount of pheromone. A key point of the modelling relies on the description of the detection capabilities of the individuals. It turns out that ant antennae seem to be oriented towards the front of the ant, with a restricted detection angle [13, 14]. Hence, their behaviour is driven by pheromone concentrations on the trails ahead, with a turning rate based on the differences of pheromone concentration in the detection area, weighted by the total amount of pheromone detected; this is known as Weber's law. In [1], we introduced an individual-based model for ant navigation using this Weber's law, based on an individual description of each ant as a self-propelled agent whose velocity depends on a pheromone, or signal, landscape. In particular, [1] brings out the fundamental role of the angle of the detection area in the stability of trails, in response to a given pheromone concentration. Here, we extend the model of [1] by considering that the pheromones are emitted by the individuals. This induces a self-consistent field which shapes the individuals' displacement. We address the question of the well-posedness of the underlying system of differential equations, in order to provide a rigorous basis to the modeling.

The difficulties in achieving this result are related to the fact that, mathematically, the model consists of a system of ordinary differential equations which are coupled nonlinearly and nonlocally in both time and space, through the influence of the pheromone signal. For this reason, it is necessary to establish Lipschitz-like properties for the acceleration terms. This will bring into play detailed properties of the signal distribution: on one hand, the distribution is a solution to a diffusion equation with measure source, and on the other hand, it must have some regularity in order to allow for the agents' sensing.

The paper is organized as follows. In Section 2 we introduce the model: the concentration of pheromones obeys a diffusion equation, with a source term concentrated on individuals' trajectories. We consider two cases for the detection capabilities: the case of a full sensing cone, where the detection domain is a full circular section, and the extreme situation where the detection is restricted to the endpoints of two antennae. Section 2.1 provides numerical illustrations showing the ability of the model in reproducing the spontaneous formation of trails. The main results are stated in Section 2.2, while Section 2.3 details a general result on parabolic equations with measure source which can be of independent interest. The analysis of the full sector sensing model is performed in Section 3, by means of a fixed point reasoning. The pointwise sensing model requires refined estimates and it is studied in Section 4.

2 Coupling the self-propelled agents' trajectories with pheromone signal production

Although for the purposes of [1] the agents' trajectories evolved in the space $\mathbb{R}^2$, it is natural and convenient to consider as the physical domain the two-dimensional torus $\mathbb{T}^2$, which is equivalent to setting periodic boundary conditions on the square $[0,1]\times[0,1]$. This has the advantage of providing a bounded domain – so the agents cannot run off to infinity – while eliminating any artefacts produced by boundary conditions. Indeed, it is known [14, 10] that ants may tend to aggregate on the edges of experimental domains for reasons that are not quite well understood. Also, it is not clear what boundary conditions the agents should verify at an individual level. Setting the problem in $\mathbb{T}^2$ obviates the need for boundary conditions, while only slightly complicating the analysis. Furthermore, as shown in the numerical simulations below, a square with periodic boundary conditions is a very natural setting to perform computations.

We consider for $t \geq 0$ a population of N self-propelled agents, or "ants", living in the two-dimensional torus $\mathbb{T}^2$. Each ant has a position/velocity pair $(\mathbf{x}_i(t), \mathbf{v}_i(t)) \in \mathbb{T}^2 \times \mathbb{R}^2$. We recall here the model from [1] where each ant deposits pheromone continuously in the substrate and so the ants interact indirectly through the pheromone distribution. For $t > 0$, we have the system of $4N$ ODEs

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = -\dfrac{1}{\tau}\big(\mathbf{v}_i - F(\mathbf{x}_i, \mathbf{v}_i, \mathcal{P})\big), \end{cases} \tag{2.1}$$

for $i = 1, \dots, N$, where $\mathcal{P} : (t, \mathbf{x}) \mapsto \mathcal{P}(t, \mathbf{x})$ is the pheromone concentration at the point $\mathbf{x} \in \mathbb{T}^2$ at time t, and the desired velocity function F is defined in (2.6)-(2.9) below. The system is supplemented with initial conditions $(\mathbf{x}_i^0, \mathbf{v}_i^0)$.

We suppose that $\mathcal{P}$ satisfies $\mathcal{P}(t = 0, \mathbf{x}) = \mathcal{P}^0(\mathbf{x})$ and obeys a diffusion equation of the form

$$\partial_t \mathcal{P} - D\Delta\mathcal{P} + \gamma\mathcal{P} = \sum_{j=1}^{N} \delta_{\mathbf{x}=\mathbf{x}_j(t)}, \tag{2.2}$$

so that $\mathcal{P}(t, \mathbf{x})$ couples the equations for the different ants in (2.1). The solution of (2.2) is given by the explicit formula

$$\begin{aligned} \mathcal{P}(t, \mathbf{x}) = &\sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\gamma t - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}} \mathcal{P}^0(\mathbf{y})\, d\mathbf{y} \\ &+ \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_0^t \sum_{j=1}^{N} \frac{1}{4\pi(t-s)} e^{-\gamma(t-s) - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{x}_j(s)|^2}{4(t-s)}}\, ds. \end{aligned} \tag{2.3}$$

In [1], the pheromone distribution was supposed to be a given function, describing the case where the self-propelled agents are navigating in a pre-existing signal landscape. Here, in contrast, the signal $\mathcal{P}$ is generated by the agents, and in turn influences their trajectories through the function F.

The model relies on the notions of *desired velocity* and *effective signal* that we describe now. Having a signal distribution $\mathcal{P}(t, \mathbf{x})$, we consider the *effective signal* given by $\mathcal{P}_e(t, \mathbf{x}) = \Phi(\mathcal{P}(t, \mathbf{x}))$, where $\Phi : \mathbb{R} \to \mathbb{R}$ is a non decreasing function such that for all $z \in \mathbb{R}$

$$\Phi(z) \in [c_*, c^*], \quad \Phi(z) = c^* \text{ if } z \geq c^*, \tag{2.4}$$

Roughly speaking, we think of Φ as the function

$$\Phi(z) = \min\big(c^*, \max(c_*, z)\big), \tag{2.5}$$

but other relevant functions, more regular, can be readily constructed. The effective signal arises naturally from the fact that signal concentrations above a maximum threshold have no additional effect, and that very low signal concentrations are treated as if the concentration level were at the lower threshold, which provides a continuity property of the agent's behaviour when the signal level is very small. We refer the reader to [1] for a more in-depth discussion.

To complete the model (2.1), we must define the *desired velocity* $F(\mathbf{x}, \mathbf{v}, \mathcal{P})$. Let us recall from [1] some definitions. The *sensing area* $B(\mathbf{v}, \ell, \beta)$ is given by

$$B(\mathbf{v}, \ell, \beta) = \{\mathbf{y} \in \mathbb{R}^2 : \angle(\mathbf{v}, \mathbf{y}) \in (-\beta, \beta),\, \|\mathbf{y}\| \leq \ell\}, \tag{2.6}$$

where $\ell > 0$ is its radius, $\angle(\mathbf{v}, \mathbf{v}') \in [-\pi, \pi]$ the angle from the vector $\mathbf{v}$ to the vector $\mathbf{v}'$, and β is half the angle of the sensing area. Hence, $B(\mathbf{v}, \ell, \beta)$ is a circular sector of radius ℓ and angle 2β, centered at zero and aligned with the direction of the velocity vector $\mathbf{v}$. From (2.6), we see that

$$|B(\mathbf{v}, \ell, \beta)| = \beta\ell^2. \tag{2.7}$$

Thus, we will consider the two models discussed in [1], namely the full sector sensing model

$$F_s(\mathbf{x}_i, \mathbf{v}_i, \mathcal{P}) = \int_{B(\mathbf{v}_i)} \mathbf{y} \frac{\mathcal{P}_e(\mathbf{x}_i + \mathbf{y})}{\int_{B(\mathbf{v}_i)} \mathcal{P}_e(\mathbf{x}_i + \mathbf{y}')\, d\mathbf{y}'}\, d\mathbf{y}, \tag{2.8}$$

and the two point sensing model

$$F_d(\mathbf{x}_i, \mathbf{v}_i, \mathcal{P}) = \frac{\mathbf{y}_{iL}\mathcal{P}_e(\mathbf{x}_i + \mathbf{y}_{iL}) + \mathbf{y}_{iR}\mathcal{P}_e(\mathbf{x}_i + \mathbf{y}_{iR})}{\mathcal{P}_e(\mathbf{x}_i + \mathbf{y}_{iL}) + \mathcal{P}_e(\mathbf{x}_i + \mathbf{y}_{iR})}, \tag{2.9}$$

with $\mathbf{v}_i = \|\mathbf{v}_i\|(\cos\Theta_i, \sin\Theta_i)$ and (recalling that β is the sensing half-angle)

$$\mathbf{y}_{iL} = \ell \begin{pmatrix} \cos(\Theta_i + \beta) \\ \sin(\Theta_i + \beta) \end{pmatrix}, \qquad \mathbf{y}_R = \ell \begin{pmatrix} \cos(\Theta_i - \beta) \\ \sin(\Theta_i - \beta) \end{pmatrix}. \tag{2.10}$$

Note that the model (2.9) corresponds exactly to taking the Dirac measure $\delta_{\mathbf{y}=\mathbf{y}_L} + \delta_{\mathbf{y}=\mathbf{y}_R}$ in place of the Lebesgue measure in (2.8); thus the agent senses only the signal concentration at the two points $\mathbf{y}_L$ and $\mathbf{y}_R$ which, in the case of the application to ant movement, represent the points where the tips of the antennae touch the substrate. In contrast, in the model (2.8), all points in the sector B contribute equally to the response.

Let us note here that, as mentioned earlier, the two models presented here are based upon Weber's law; indeed, both (2.8) and (2.9) can be seen as a weighted average of signal concentration around the agent, normalised by the total signal present. We again refer to [1] for more details.

2.1 Numerical examples

In this section, we briefly present some numerical experiments to illustrate the emergence of collective organisation from the system analysed in this work. These results are not meant to be an in-depth investigation of the formation of collective patterns such as trails, but serve only to illustrate the fact that, numerically at least, agents evolving according to the system (2.1),(2.2) exhibit spontaneous trail formation.

We implemented an algorithm in `C++` to simulate (2.1),(2.2) with the two point sensing (2.9). The ODEs are advanced in time according to the Euler scheme (which is sufficient for our illustrative purposes) on a square with periodic boundary conditions. To approximate the pheromone field, the following approach is used: every k iterations (in the simulations below, $k = 20$ and the time step is 0.2), the agents drop a pheromone "droplet" at their current position. This droplet then evolves according to the explicit solution of the diffusion equation without source term. In this way, the signal trail left by each agent, given by (2.2),(2.3), is approximated by a sum of successive droplets along the trajectory. The advantage is that this requires no mesh in order to simulate the time evolution of the diffusion equation. It is easy to see that this procedure amounts to nothing more than a quadrature rule for the time integral in (2.3), computed using a discrete approximation of the agent's trajectory. The signal initial data is zero.

We show two experiments, the first with 10 agents (Fig.1), and the second with 100 (Fig.2). In both cases, the initial positions and directions of the agents are taken randomly, and this is the only source of randomness in the procedure. One can see the self-organising process taking place, with the spontaneous formation of trails.

2.2 Main results

The analysis of the system (2.1),(2.2) with a desired velocity given either by (2.8) or (2.9) uses the fact that the agents' speed is bounded from above and below, as observed in [1, Propositions 3.1 & 3.3] in the case of a given pheromone distribution. However, the proof for the model with an arbitrary number of agents interacting through a pheromone field carries over in exactly the same way.

Lemma 2.1 ([1]). *Suppose that the sensing half-angle β satisfies $0 < \beta < \pi/2$, and $\mathbf{v}(0) \neq 0$. Let $0 < c_* < c^*$ be the pheromone detection thresholds in* (2.4),

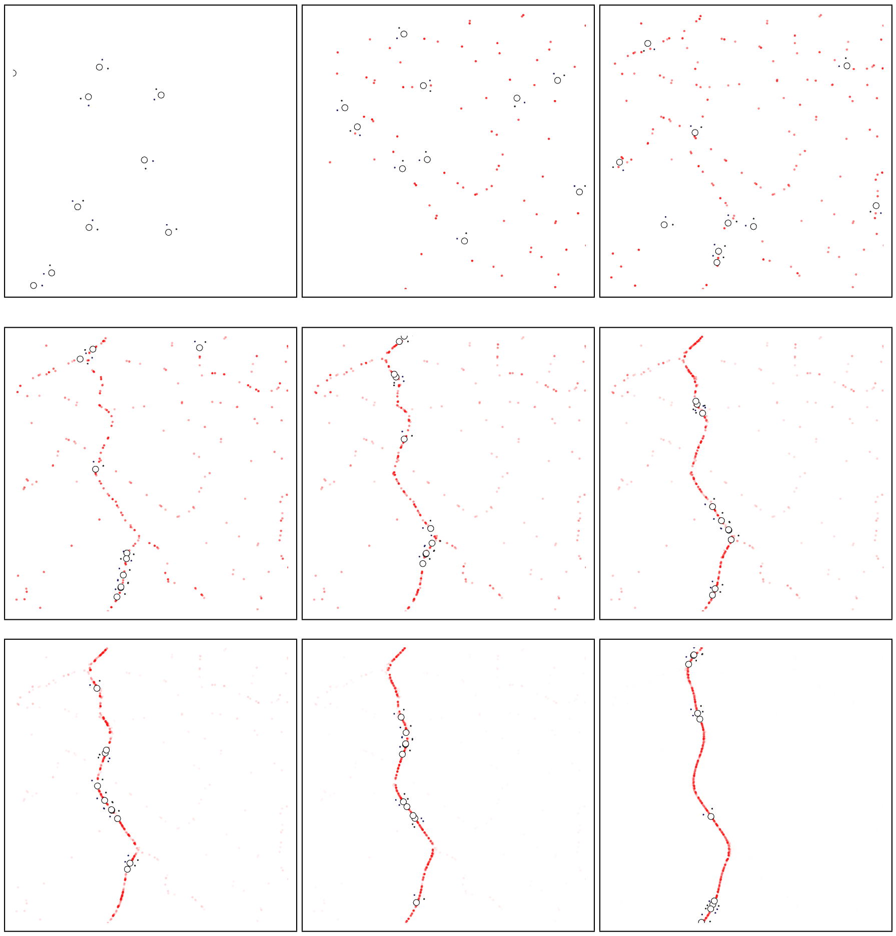

Figure 1: Snapshots of the simulation of (2.1),(2.2),(2.9) with 10 agents. Pheromone droplets are coloured red, and the opacity of the droplets indicates their age.

and let $\mathcal{P}_e$ *be any effective pheromone concentration. Then, there exist constants* C_1*,* $C_2 > 0$*, depending explicitly on* ℓ, c_*, c^*, β*, and the initial data, but not on* t*, such that for all* $t \geq 0$*,*

$$\min\big(|\mathbf{v}(0)|, C_1\big) \leq |\mathbf{v}(t)| \leq \max\big(|\mathbf{v}(0)|, C_2\big), \tag{2.11}$$

where $t \mapsto (\mathbf{x}(t), \mathbf{v}(t))$ *is a solution to* (2.1),(2.2)*, with the desired velocity given either by* (2.8) *or* (2.9)*.*

Our main results establish the existence-uniqueness of solutions for both the full sector sensing model, and the pointwise sensing model. For the latter, we

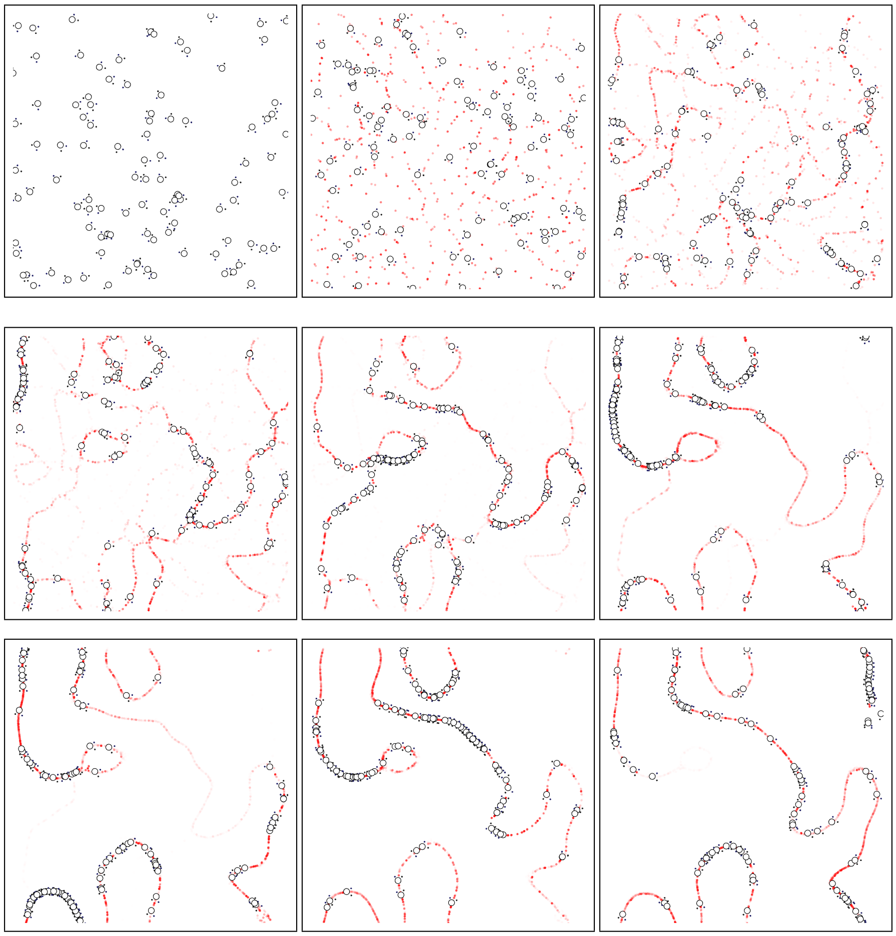

Figure 2: Snapshots of the simulation of (2.1),(2.2),(2.9) with 100 agents. Pheromone droplets are coloured red, and the opacity of the droplets indicates their age.

need to assume that the initial positions of the agents are separated, in a way that at $t = 0$ no sensing point lies on another agent. Since we are working within the periodic framework, we make use of the following distance defined on $\mathbb{T}^2$:

$$d_{\mathbb{T}^2}(\mathbf{x},\mathbf{y}) = \inf\left\{|\mathbf{x}-\mathbf{y}+\mathbf{z}|,\ \mathbf{z}\in\mathbb{Z}^2\right\} = \inf\left\{|\mathbf{x}-\mathbf{y}+\mathbf{z}|,\ \mathbf{z}\in\mathbb{Z}^2,\ |\mathbf{z}|_\infty \le 1\right\} \in \left[0, \frac{\sqrt{2}}{2}\right].$$

Theorem 2.2 (Well-posedness for the full sector sensing model)**.** *For $i \in \{1,\dots,N\}$, let $\mathbf{x}_i(0)\in\mathbb{T}^2$ be the initial positions, and $\mathbf{v}_i(0)\in\mathbb{R}^2\setminus\{(0,0)\}$ the initial velocities of the agents. Suppose that the initial signal $\mathcal{P}(t=0,\mathbf{x}) = \mathcal{P}^0(\mathbf{x})$*

is such that $\mathcal{P}^0 = \mathcal{P}_e^0$ *and*

$$\|\nabla \mathcal{P}_e^0\|_{L^1(\mathbb{T}^2)} < \infty. \tag{2.12}$$

Then, for any $t > 0$*, there exists a unique solution*

$$(\mathbf{x}_i(t), \mathbf{v}_i(t)) \in C^1([0,t];\mathbb{T}^2) \times C([0,t];\mathbb{R}^2)$$

of the system (2.1),(2.2) *with desired velocity given by* (2.8).

Theorem 2.3 (Well-posedness for the pointwise sensing model)**.** *For* $i \in \{1,\dots,N\}$*, let* $\mathbf{x}_i(0) \in \mathbb{T}^2$ *be the initial positions, and* $\mathbf{v}_i(0) \in \mathbb{R}^2 \setminus \{(0,0)\}$ *the initial velocities of the agents. Suppose that the initial signal* $\mathcal{P}(t=0,\mathbf{x}) = \mathcal{P}^0(\mathbf{x})$ *is such that* $\mathcal{P}^0 = \mathcal{P}_e^0$ *and*

$$\|\nabla \mathcal{P}_e^0\|_{L^\infty(\mathbb{T}^2)} < \infty, \tag{2.13}$$

and that the initial positions satisfy

$$d(\mathbf{x}_i^0, \mathbf{x}_j^0) > \ell, \qquad \forall i,j \in \{1,\dots,N\},\ i \neq j. \tag{2.14}$$

Then, for any $T > 0$*, there exists a unique solution*

$$(\mathbf{x}_i(t), \mathbf{v}_i(t)) \in C^2([0,T];\mathbb{T}^2) \times C^1([0,T];\mathbb{R}^2)$$

of the system (2.1),(2.2) *with desired velocity given by* (2.9).

Remark 2.4. In Theorem 2.2 above, the number of agents N is arbitrary. In Theorem 2.3, though, the condition (2.14) on the initial positions places an upper bound on the number of agents, which is roughly $N \lesssim 1/\ell^2$. However, we believe this limitation is only technical. Indeed, as will be explained in the analysis below, the separation condition (2.14) is only necessary for very short times, and in no way implies that the trajectories must remain separated for larger times. It is related to the fact that arbitrary initial data will, in general, not be "compatible" with the type of singularities originated by the equation. If, however, the initial data *already has* infinite singularities at the agents locations, of the same type which are generated by the equation, then there would be no need for the separation condition (2.14) and the number of agents would be arbitrary even in Theorem 2.3.

2.3 A particular case of parabolic equation with measure data

The next lemma focuses on the properties of the pheromone distribution $\mathcal{P}$ produced by one agent. At first sight, $\mathcal{P}$ is solution of a parabolic equation with measure data; the general theory of such problems, after the seminal ideas in [18] for instance, provides solutions with a gradient in L^q, for some $q < 2$, see e. g. [4, 15]. Such a regularity is not enough for the pointwise sensing model to make sense. Hence, we need strengthened regularity, which, in the present context, can be derived by direct inspection of the solution, expressed by means of convolution with the heat kernel.

Proposition 2.5. *Let $\mathcal{P}^0 \in W^{1,1}(\mathbb{T}^2)$, let $t > 0$, and let $\zeta : [0,t] \to \mathbb{T}^2$ be a C^1 curve. Then, the function*

$$\begin{aligned}\mathcal{P}(t,\mathbf{x}) = \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\gamma t - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}} \mathcal{P}^0(\mathbf{y})\, d\mathbf{y} \\ + \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_0^t \frac{1}{4\pi(t-s)} e^{-\gamma(t-s) - \frac{|\mathbf{x}+\mathbf{z}-\zeta(s)|^2}{4(t-s)}}\, ds,\end{aligned} \tag{2.15}$$

which is a solution to the Cauchy problem

$$\partial_t \mathcal{P} - \Delta \mathcal{P} + \gamma \mathcal{P} = \delta_{\mathbf{x}=\zeta(t)}, \quad \mathcal{P}(0,\mathbf{x}) = \mathcal{P}^0(\mathbf{x}), \quad t \geq 0,\, \mathbf{x} \in \mathbb{T}^2, \tag{2.16}$$

verifies

$$\mathcal{P} \in C\big([0,t]; L^1(\mathbb{T}^2)\big) \tag{2.17}$$

and

$$\nabla\mathcal{P} \in C\big([0,t]; L^1(\mathbb{T}^2)\big). \tag{2.18}$$

Moreover, the following assertions hold:

i) for any $\eta, \epsilon > 0$, $\mathcal{P}$ is C^∞ over the set $\mathscr{U}_{\eta,\epsilon} = \{(t,\mathbf{x}) \in [0,\infty) \times \mathbb{T}^2,\ t > \eta,\ d_{\mathbb{T}^2}(\mathbf{x},\zeta(t)) > \epsilon\}$;

ii) for any $\tau > 0$, there exists $\epsilon > 0$ such that $\mathcal{P}(t,\mathbf{x}) > c^*$ if $t \geq \tau$ and $d_{\mathbb{T}^2}(\mathbf{x},\zeta(t)) \leq \epsilon$;

iii) for any $0 < \tau < t < \infty$,

$$\|\nabla \mathcal{P}_e\|_{L^\infty([\tau,t]\times\mathbb{T}^2)} < L, \tag{2.19}$$

where L may be unbounded as $\tau \searrow 0$, and the truncation $\mathcal{P}_e$ is defined in (2.4).

Finally, suppose that $\mathcal{P}^0 \in W^{1,\infty}(\mathbb{T}^2)$ and for $t^* > 0$ and $\delta > 0$ let

$$\mathcal{T}_{t^*,\delta} = \big\{(t,\mathbf{x}) \in [0,t^*] \times \mathbb{T}^2 : d_{\mathbb{T}^2}(\mathbf{x},\zeta(s)) \geq \delta, \quad \textit{for any } 0 \leq s \leq t\big\};$$

then, there exists a constant $L > 0$ such that

$$\sup_{(t,\mathbf{x})\in\mathcal{T}_{t^*,\delta}} |\nabla\mathcal{P}(t,\mathbf{x})| \leq L.$$

Essentially, Proposition 2.5 states that for each $t > 0$ the function $\mathcal{P}$ has an integrable infinite singularity at the point $\mathbf{x} = \zeta(t)$, but is C^∞ elsewhere. This singularity appears instantaneously for $t > 0$ but, for small values of t, its influence can be mostly localised to a vanishingly small neighbourhood of the signal deposition point.

Proof: First of all, we set $\gamma = 0$ in the proof without loss of generality. Note that if f is integrable on $\mathbb{R}^2$, then

$$\sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} f(\mathbf{x}+\mathbf{z})\, d\mathbf{x} = \int_{\mathbb{R}^2} f(\mathbf{x})\, d\mathbf{x}. \tag{2.20}$$

To prove (2.17), consider first the term involving the initial data $\mathcal{P}^0$. We have

$$\begin{aligned}
\int_{\mathbb{T}^2} \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}} \mathcal{P}^0(\mathbf{y})\, d\mathbf{y}\, d\mathbf{x} \\
&= \int_{\mathbb{T}^2} \mathcal{P}^0(\mathbf{y}) \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}}\, d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbb{T}^2} \mathcal{P}^0(\mathbf{y}) \int_{\mathbb{R}^2} \frac{1}{4\pi t} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}}\, d\mathbf{x}\, d\mathbf{y} \\
&= \int_{\mathbb{T}^2} \mathcal{P}^0(\mathbf{y})\, d\mathbf{y},
\end{aligned}$$

and so the initial data gives a finite contribution. Next we compute with (2.20), Fubini's Theorem, and the change of variable $y = \frac{\mathbf{x}}{\sqrt{4(t-s)}}$ (but still writing the integral with $\mathbf{x}$),

$$\begin{aligned}
\int_{\mathbb{T}^2} \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}+\mathbf{z}-\zeta(s)|^2}{4(t-s)}}\, ds\, d\mathbf{x} &= \int_{\mathbb{R}^2} \int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}-\zeta(s)|^2}{4(t-s)}}\, ds\, d\mathbf{x} \\
&= \int_0^t \int_{\mathbb{R}^2} \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}-\zeta(s)|^2}{4(t-s)}}\, d\mathbf{x}\, ds \\
&= \int_0^t \int_{\mathbb{R}^2} e^{-|\mathbf{x}-\frac{\zeta(s)}{\sqrt{4(t-s)}}|^2}\, d\mathbf{x}\, ds \\
&= \int_0^t \int_{\mathbb{R}^2} \frac{1}{\pi} e^{-|\mathbf{x}|^2}\, d\mathbf{x}\, ds = t
\end{aligned}$$

(note that this mass growth law is in agreement with the equation satisfied by $\mathcal{P}$). Therefore, accounting for the influence of γ, we find $\|\mathcal{P}\|_{L^\infty(0,t;L^1(\mathbb{T}^2))} < \infty$. Additionally, we see that the contribution to the $L^1(\mathbb{T}^2)$ norm of the second term of (2.15) tends to zero with t. This, in conjunction with the classical properties of the heat kernel, gives

$$\lim_{t\to 0} \|\mathcal{P}(t,\cdot) - \mathcal{P}^0(\cdot)\|_{L^1(\mathbb{T}^2)} = 0.$$

When we prove (2.18), we will know that for each $t > 0$, the solution $\mathcal{P}(t,\mathbf{x}) \in W^{1,1}(\mathbb{T}^2)$. Therefore, due to the semigroup property, we can prove the continuity result above at any $t > 0$. This proves (2.17).

To prove (2.18), note that using (2.20) as above, the first part of (2.15) can be written as

$$\int_{\mathbb{R}^2} \frac{1}{4\pi t} e^{-\frac{|\mathbf{x}-y|^2}{4t}} \overline{\mathcal{P}^0}(\mathbf{y})\, d\mathbf{y} = (K * \overline{\mathcal{P}^0})(t,\mathbf{x}),$$

where $\overline{\mathcal{P}^0} : \mathbb{R}^2 \to \mathbb{R}$ is the periodic extension of $\mathcal{P}^0$ and $K(t, \mathbf{x})$ is the fundamental solution of the heat equation. Therefore, by the well-known properties of the convolution and of the fundamental solution,

$$\begin{aligned}\int_{\mathbb{T}^2} |\nabla(K * \overline{\mathcal{P}^0})(t, \mathbf{x}) - \nabla\overline{\mathcal{P}^0}(t, \mathbf{x})| \, d\mathbf{x} \\ = \int_{\mathbb{T}^2} |(K * \nabla\overline{\mathcal{P}^0})(t, \mathbf{x}) - \nabla\overline{\mathcal{P}^0}(t, \mathbf{x})| \, d\mathbf{x} \underset{t\to 0}{\longrightarrow} 0.\end{aligned} \tag{2.21}$$

For the second term in (2.15), we find with $\sigma = \frac{1}{t-s}$

$$\int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}+\mathbf{z}-\zeta(s)|^2}{4(t-s)}} \, ds \quad = \frac{1}{4\pi} \int_{1/t}^{+\infty} \frac{1}{\sigma} e^{-\frac{\sigma}{4}|\mathbf{x}+\mathbf{z}-\zeta(t-1/\sigma)|^2} \, d\sigma.$$

Taking the gradient and integrating over $\mathbb{T}^2$, we find for the L^1 norm of the gradient of the second term in (2.15), using (2.20) and then $\mathbf{y} = \sqrt{\sigma}\mathbf{x}$,

$$\begin{aligned}&\frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{1/t}^{+\infty} |\mathbf{x} - \zeta(t-1/\sigma)| e^{-\frac{\sigma}{4}|\mathbf{x}-\zeta(t-1/\sigma)|^2} \, d\sigma \, d\mathbf{x} \\ &\quad = \frac{1}{8\pi} \int_{1/t}^{+\infty} \int_{\mathbb{R}^2} \frac{|\mathbf{y} - \sqrt{\sigma}\zeta(t-1/\sigma)|}{\sigma^{3/2}} e^{-\frac{1}{4}|\mathbf{y}-\sqrt{\sigma}\zeta(t-1/\sigma)|^2} \, d\mathbf{y} \, d\sigma \\ &\quad = \frac{1}{8\pi} \int_{1/t}^{+\infty} \frac{1}{\sigma^{3/2}} \int_{\mathbb{R}^2} |\mathbf{y}| e^{-\frac{1}{4}|\mathbf{y}|^2} \, d\mathbf{y} \, d\sigma \\ &\quad = C\sqrt{t}.\end{aligned}$$

These observations together prove the continuity in (2.18) at $t = 0$. To prove (2.18), it suffices to apply the same reasoning for any $t > 0$.

Let us now prove i-iii). Let $\eta, \epsilon > 0$ be small but arbitrary. The set $\mathscr{U}_{\eta.\epsilon}$ is an open set in $[0, \infty) \times \mathbb{T}^2$. Let T denote the distribution $\delta_{\mathbf{x}=\zeta(t)} \in \mathscr{D}'([0, \infty) \times \mathbb{T}^2)$; we have $\mathscr{U}_{\eta,\epsilon} \cap \operatorname{supp}(T) = \emptyset$ so that T is C^∞ over $\mathscr{U}_{\eta,\epsilon}$. Since the differential operator $\partial_t - \Delta + \gamma$ is hypoelliptic, the solution $\mathcal{P}$ is C^∞ over $\mathscr{U}_{\eta,\epsilon}$ too.

In order to justify (2.19), we need some more information about how $\mathcal{P}$ blows up near $\mathbf{x} = \zeta(t)$. We are going to show that for every $t, M > 0$ there exists $\epsilon > 0$ such that $d_{\mathbb{T}^2}(\mathbf{x}, \zeta(t)) \leq \epsilon$ implies $\mathcal{P}(t, \mathbf{x}) > M$. Let $\mathbf{z}_0(s, \mathbf{x}) \in \mathbb{Z}^2$ be such that $d_{\mathbb{T}^2}(\mathbf{x}, \zeta(s)) = |\mathbf{x} - \zeta(s) + \mathbf{z}_0(s, \mathbf{x})|$. The second part of (2.15) is evaluated as follows

$$\begin{aligned}
&\sum_{\mathbf{z}\in\mathbb{Z}^2}\int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}+\mathbf{z}-\zeta(s)|^2}{4(t-s)}}\,ds \\
&\quad = \sum_{\mathbf{z}\in\mathbb{Z}^2}\int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}-\zeta(s)+\mathbf{z}_0(s,\mathbf{x})+(\mathbf{z}-\mathbf{z}_0(s,\mathbf{x}))|^2}{4(t-s)}}\,ds \\
&\quad > \int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{|\mathbf{x}-\zeta(s)+\mathbf{z}_0(s,\mathbf{x})|^2}{4(t-s)}}\,ds \\
&\qquad\quad \text{since all terms of the sum are positive} \\
&\quad > \int_0^t \frac{1}{4\pi(t-s)} e^{-\frac{d_{\mathbb{T}^2}(\mathbf{x},\zeta(s))^2}{4(t-s)}}\,ds \\
&\quad > \frac{1}{4\pi}\int_{1/t}^{+\infty} \frac{1}{\sigma} e^{-\frac{\sigma}{4} d_{\mathbb{T}^2}(\mathbf{x},\zeta(t-\frac{1}{\sigma}))^2}\,d\sigma \\
&\qquad\quad \text{with the change of variable } \sigma = \tfrac{1}{t-s}.
\end{aligned} \tag{2.22}$$

Let

$$V = \max\{|\zeta'(s)|,\ 0 \le s \le t\} > 0$$

denote the maximum velocity of ζ on $[0,t]$, and let $\epsilon \le tV$. Then, $1/t \le V/\epsilon$ and so

$$(2.22) > \frac{1}{4\pi}\int_{V/\epsilon}^{+\infty} \frac{1}{\sigma} e^{-\frac{\sigma}{4} d_{\mathbb{T}^2}(\mathbf{x},\zeta(t-\frac{1}{\sigma}))^2}\,d\sigma.$$

Also, if $d_{\mathbb{T}^2}(\mathbf{x},\zeta(t)) \le \epsilon$, we find for $\sigma \ge V/\epsilon$

$$\begin{aligned}
d_{\mathbb{T}^2}(\mathbf{x},\zeta(t-1/\sigma)) &\le d_{\mathbb{T}^2}(\mathbf{x},\zeta(t)) + d_{\mathbb{T}^2}(\zeta(s),\zeta(t-1/\sigma)) \le \epsilon + V/\sigma \\
&\le 2\epsilon.
\end{aligned}$$

Therefore, when $d_{\mathbb{T}^2}(\mathbf{x},\zeta(t)) \le \epsilon$, we get

$$(2.22) > \frac{1}{4\pi}\int_{V/\epsilon}^{+\infty} \frac{1}{\sigma} e^{-\sigma\epsilon^2}\,d\sigma = \frac{1}{4\pi}\int_{V\epsilon}^{+\infty} \frac{e^{-\rho}}{\rho}\,d\rho,$$

with the change of variable $\rho = \epsilon^2\sigma$, and this quantity bounds (2.15) from below. Since $\rho \mapsto \frac{e^{-\rho}}{\rho}$ is not integrable on $[0,1]$, we have

$$\lim_{\epsilon\to 0}\int_{V\epsilon}^{+\infty} \frac{e^{-\rho}}{\rho}\,d\rho = \infty.$$

More exactly, observe that, for $\epsilon < 1$,

$$\int_{V\epsilon}^{+\infty} \frac{e^{-\rho}}{\rho}\,d\rho > \int_{V\epsilon}^{V} \frac{e^{-\rho}}{\rho}\,d\rho \ge e^{-V}|\log\epsilon|,$$

and so it suffices to take $\epsilon \le e^{-4\pi M e^V}$ to ensure that this integral is larger than $M > 0$ (and this choice may not be optimal). To summarise, we have proved that for any $t, M > 0$, if $d_{\mathbb{T}^2}(\mathbf{x}, \zeta(t)) \le \epsilon$, with $\epsilon \le \min(tV, e^{-4\pi M e^V}, 1)$, then $\mathcal{P}(t, \mathbf{x}) > M$. In particular, we note that ϵ depends on t and $\epsilon \to 0$ when $t \to 0$.

Accordingly, for any $\tau > 0$, there exists $\epsilon > 0$ such that for any $t > \tau > 0$, $\mathcal{P}(t, \mathbf{x}) > c^*$, $\forall \mathbf{x} \in B_{\zeta(t),\epsilon}$. To be specific, rewording what precedes,

$$\text{in ii), we can take } \epsilon = \frac{1}{2}\min(\tau V, e^{-4\pi c^* e^V}, 1). \tag{2.23}$$

Conversely, if $\mathbf{x}$ lies in $\{\mathbf{x} \in \mathbb{T}^2 : \mathcal{P}(t, \mathbf{x}) \le c^*\}$, then $d_{\mathbb{T}^2}(\mathbf{x}, \zeta(t)) > \epsilon$, and $(t, \mathbf{x}) \in \mathscr{U}_{\tau,\epsilon}$. We deduce that $\mathbf{x} \mapsto \mathcal{P}(t, \mathbf{x})$ as well as its derivatives are smooth and uniformly continuous on $\{\mathbf{x} \in \mathbb{T}^2 : \mathcal{P}(t, \mathbf{x}) \le c^*\}$, which is therefore a compact set, and so the truncation $\mathcal{P}_e$ verifies (2.19).

We go back to (2.15) and evaluate the gradient on short time and far from the singularity. The term coming from the initial data reads

$$\sum_{\mathbf{z} \in \mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\gamma t - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}} \nabla \mathcal{P}^0(y)\, dy$$

and it is dominated by $\|\nabla \mathcal{P}^0\|_{L^\infty(\mathbb{T}^2)}$. For the contribution of the singular source term, we rewrite the sum by introducing $\mathbf{z}_0(s, \mathbf{x}) \in \mathbb{Z}^2$ such that

$$\delta \le d_{\mathbb{T}^2}(\mathbf{x}, \zeta(s)) = |\mathbf{x} + \mathbf{z}_0(s, \mathbf{x}) - \zeta(s)| \le 1;$$

we thus consider

$$\sum_{\mathbf{z} \in \mathbb{Z}^2} \int_0^t \frac{\mathbf{x} - \zeta(s) + \mathbf{z}_0(s, \mathbf{x}) + \mathbf{z}}{8\pi(t-s)^2} e^{-\frac{|\mathbf{x} - \zeta(s) + \mathbf{z}_0(s,\mathbf{x}) + \mathbf{z}|^2}{4(t-s)}}\, ds$$

We split the sum depending whether $|\mathbf{z}| \le 1$ or $|\mathbf{z}| > 1$. The former is dominated by

$$\int_0^t \frac{1}{4\pi(t-s)^2} e^{-\frac{\delta}{4(t-s)}}\, ds = \int_{1/t}^\infty \frac{1}{4\pi} e^{-\delta\sigma/4}\, d\sigma = \frac{e^{-\delta/(4t)}}{\pi\delta}$$

since $\delta \le |\mathbf{x} - \zeta(s) + \mathbf{z}_0(s, \mathbf{x}) + \mathbf{z}| \le 2$. The latter is evaluated by using $|\mathbf{x} - \zeta(s) + \mathbf{z}_0(s, \mathbf{x}) + \mathbf{z}| \ge |\mathbf{z}| - |\mathbf{x} - \zeta(s) + \mathbf{z}_0(s, \mathbf{x})| \ge |\mathbf{z}| - 1$; it can be dominated by

$$\begin{aligned}
\sum_{|\mathbf{z}|>1} \int_0^t \frac{1 + |\mathbf{z}|}{8\pi(t-s)^2} e^{-\frac{|\mathbf{z}|-1}{4(t-s)}}\, ds \quad &= \quad \sum_{|\mathbf{z}|>1} \int_{1/t}^\infty \frac{1 + |\mathbf{z}|}{8\pi} e^{-\sigma(|\mathbf{z}|-1)/4}\, d\sigma \\
&= \quad \sum_{|\mathbf{z}|>1} \frac{1 + |\mathbf{z}|}{2\pi} e^{-\sigma(|\mathbf{z}|-1)/(4t)}.
\end{aligned}$$

Therefore, we can dominate $\nabla\mathcal{P}$ uniformly over $\mathcal{T}_{t^*,\delta}$ by a constant L which depends on t^*, δ and $\|\nabla \mathcal{P}^0\|_{L^\infty(\mathbb{T}^2)}$. (Note that this claim might becomes useless as time becomes large since the trajectory can fill the entire domain $\mathbb{T}^2$ so that $\mathcal{T}_{t^*,\delta}$ becomes the empty set.) This finishes the proof of Proposition 2.5.

3 Well-posedness for the full sector sensing model

3.1 Continuity property of the desired velocity

The proof of Theorem 2.2 is based on a fixed-point argument, which relies on a Lipschitz continuity property of the desired velocity F. For the full sector sensing model (2.8), it states as follows.

Lemma 3.1. *Let*

$$\mathcal{V} = \{\mathbf{v} \in \mathbb{R}^2 : |\mathbf{v}| \geq C_1 > 0, \text{ for some constant } C_1\}.$$

Let $\mathcal{E}$ be the set of of functions $\mathbf{x} \mapsto \mathcal{Q}(\mathbf{x}) \in L^1(\mathbb{T}^2)$ such that there exists a constant $L > 0$ verifying

$$\|\nabla \mathcal{Q}_e\|_{L^1(\mathbb{T}^2)} < M_e, \tag{3.1}$$

where $\mathcal{Q}_e$ is the truncation defined in (2.4)*. Then, the desired velocity function $F(\mathbf{x}, \mathbf{v}, \mathcal{Q})$ in* (2.8) *satisfies a Lipschitz property on the set $\mathbb{T}^2 \times \mathcal{V} \times \mathcal{E}$. More exactly, there exists a constant $C_F > 0$ (depending on $C_1, M_e, \ell, \beta, c_*, c^*$) such that for every $\mathbf{x}_i, \mathbf{v}_i, \mathcal{Q}_i \in \mathbb{T}^2 \times \mathcal{V} \times \mathcal{E}$, $i \in \{1, 2\}$, it holds*

$$\begin{aligned}\big|F(\mathbf{x}_1, \mathbf{v}_1, \mathcal{Q}_1) - &F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_2)\big| \\ &\leq C_F \Big(d_{\mathbb{T}^2}(\mathbf{x}_1, \mathbf{x}_2) + |\mathbf{v}_1 - \mathbf{v}_2| + \|\mathcal{Q}_{1e} - \mathcal{Q}_{2e}\|_{L^1(\mathbb{T}^2)}\Big).\end{aligned} \tag{3.2}$$

Proof. Let $(\mathbf{x}_i, \mathbf{v}_i) \in \mathbb{T}^2 \times \mathcal{V}$, and $\mathcal{Q}_i \in \mathcal{E}$, with $i = 1, 2$. The difference to be evaluated splits into three pieces

$$\begin{aligned}\big|F(\mathbf{x}_1, \mathbf{v}_1, \mathcal{Q}_1) - F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_2)\big| &\leq \big|F(\mathbf{x}_1, \mathbf{v}_1, \mathcal{Q}_1) - F(\mathbf{x}_2, \mathbf{v}_1, \mathcal{Q}_1)\big| \\ + \big|F(\mathbf{x}_2, \mathbf{v}_1, \mathcal{Q}_1) - F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_1)\big| &+ \big|F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_1) - F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_2)\big|.\end{aligned} \tag{3.3}$$

Now, omitting the subscripts in $\mathbf{v}_1$ and $\mathcal{Q}_1$, we obtain

$$\begin{aligned}&\big|F(\mathbf{x}_1, \mathbf{v}, \mathcal{Q}) - F(\mathbf{x}_2, \mathbf{v}, \mathcal{Q})\big| \\ &\quad\leq \int_{B(\mathbf{v})} \Big|\mathbf{y} \frac{\mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y})}{\int_{B(\mathbf{v})} \mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y}')\, d\mathbf{y}'} - \mathbf{y} \frac{\mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y})}{\int_{B(\mathbf{v})} \mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y}')\, d\mathbf{y}'}\Big|\, d\mathbf{y} \\ &\quad\leq \ell \int_{B(\mathbf{v})} \Big|\frac{\mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y})}{\int_{B(\mathbf{v})} \mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y}')\, d\mathbf{y}'} - \frac{\mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y})}{\int_{B(\mathbf{v})} \mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y}')\, d\mathbf{y}'}\Big|\, d\mathbf{y} \\ &\quad\leq \ell \int_B \frac{\Big|\mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y}) \int_B \mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y}')\, d\mathbf{y}' - \mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y}) \int_B \mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y}')\, d\mathbf{y}'\Big|}{\int_B \mathcal{Q}_e(\mathbf{x}_1 + \mathbf{y}')\, d\mathbf{y}' \int_B \mathcal{Q}_e(\mathbf{x}_2 + \mathbf{y}')\, d\mathbf{y}'}\, d\mathbf{y}.\end{aligned}$$

As $\mathcal{Q}_e \geq c_*$, the last term is bounded by

$$\begin{aligned}
&\frac{\ell}{|B|^2 c_*^2} \int_B \Big| \mathcal{Q}_e(\mathbf{x}_1+\mathbf{y}) \int_B \mathcal{Q}_e(\mathbf{x}_2+\mathbf{y}')\,d\mathbf{y}' - \mathcal{Q}_e(\mathbf{x}_2+\mathbf{y}) \int_B \mathcal{Q}_e(\mathbf{x}_1+\mathbf{y}')\,d\mathbf{y}' \Big|\,d\mathbf{y} \\
&\quad \leq \frac{\ell}{|B|^2 c_*^2} \int_B \Big| \mathcal{Q}_e(\mathbf{x}_1+\mathbf{y}') - \mathcal{Q}_e(\mathbf{x}_2+\mathbf{y}') \Big|\,d\mathbf{y}' \int_B \mathcal{Q}_e(\mathbf{x}_1+\mathbf{y})\,d\mathbf{y} \\
&\qquad + \frac{\ell}{|B|^2 c_*^2} \int_B \Big| \mathcal{Q}_e(\mathbf{x}_1+\mathbf{y}) - \mathcal{Q}_e(\mathbf{x}_2+\mathbf{y}) \Big|\,d\mathbf{y} \int_B \mathcal{Q}_e(\mathbf{x}_2+\mathbf{y}')\,d\mathbf{y}' \\
&\quad \leq \frac{2\ell}{|B|^2 c_*^2} |B| c^* \int_B \Big| \mathcal{Q}_e(\mathbf{x}_1+\mathbf{y}) - \mathcal{Q}_e(\mathbf{x}_2+\mathbf{y}) \Big|\,d\mathbf{y} \\
&\quad \leq \frac{2\ell c^*}{|B| c_*^2} d_{\mathbb{T}^2}(\mathbf{x}_1,\mathbf{x}_2) \int_B \int_0^1 |\nabla \mathcal{Q}_e(s(\mathbf{x}_1+\mathbf{y}) + (1-s)(\mathbf{x}_2+\mathbf{y}))|\,ds\,d\mathbf{y} \\
&\quad \leq \frac{2c^*}{\beta \ell c_*^2} d_{\mathbb{T}^2}(\mathbf{x}_1,\mathbf{x}_2) \int_{\mathbb{T}^2} |\nabla \mathcal{Q}_e(\mathbf{y})|\,d\mathbf{y},
\end{aligned}$$

where we have made the L^1 norm of the gradient of $\mathcal{Q}_e$ appear. Owing to (3.1), this is bounded by

$$\frac{2M_e c^*}{\beta \ell c_*^2} d_{\mathbb{T}^2}(\mathbf{x}_1,\mathbf{x}_2).$$

Therefore, we end up with

$$\Big| F(\mathbf{x}_1,\mathbf{v},\mathcal{Q}) - F(\mathbf{x}_2,\mathbf{v},\mathcal{Q}) \Big| \leq \frac{2M_e c^*}{\beta \ell c_*^2} d_{\mathbb{T}^2}(\mathbf{x}_1,\mathbf{x}_2). \tag{3.4}$$

Returning to (3.3), we find (omitting the subscripts in $\mathbf{x}, \mathcal{Q}$)

$$\begin{aligned}
&\Big| F(\mathbf{x},\mathbf{v}_1,\mathcal{Q}) - F(\mathbf{x},\mathbf{v}_2,\mathcal{Q}) \Big| \\
&\quad \leq \Big| \int_{B(\mathbf{v}_1)} \mathbf{y} \frac{\mathcal{Q}_e(\mathbf{x}+\mathbf{y})}{\int_{B(\mathbf{v}_1)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}'}\,d\mathbf{y} - \int_{B(\mathbf{v}_2)} \mathbf{y} \frac{\mathcal{Q}_e(\mathbf{x}+\mathbf{y})}{\int_{B(\mathbf{v}_2)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}'}\,d\mathbf{y} \Big| \\
&\quad = \frac{\Big| \int_{B(\mathbf{v}_1)} \mathbf{y}\mathcal{Q}_e(\mathbf{x}+\mathbf{y})\,d\mathbf{y} \int_{B(\mathbf{v}_2)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}' - \int_{B(\mathbf{v}_2)} \mathbf{y}\mathcal{Q}_e(\mathbf{x}+\mathbf{y})\,d\mathbf{y} \int_{B(\mathbf{v}_1)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}' \Big|}{\int_{B(\mathbf{v}_1)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}' \int_{B(\mathbf{v}_2)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}'} \\
&\quad \leq \frac{1}{|B|^2 c_*^2} \Bigg| \int_{B(\mathbf{v}_1)} \mathbf{y}\mathcal{Q}_e(\mathbf{x}+\mathbf{y})\,d\mathbf{y} \int_{B(\mathbf{v}_2)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}' \\
&\qquad\qquad - \int_{B(\mathbf{v}_2)} \mathbf{y}\mathcal{Q}_e(\mathbf{x}+\mathbf{y})\,d\mathbf{y} \int_{B(\mathbf{v}_1)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\,d\mathbf{y}' \Bigg|.
\end{aligned}$$

Observe that, formally,

$$\begin{aligned}
&\left|\int_{B_1} A_1 \int_{B_2} A_2 - \int_{B_2} A_1 \int_{B_1} A_2\right| \\
&\qquad\le \left|\int_{B_1} A_1 - \int_{B_2} A_1\right| \int_{B_2} |A_2| + \int_{B_2} |A_1| \left|\int_{B_2} A_2 - \int_{B_1} A_2\right| \\
&\qquad\le \int_{B_1 \triangle B_2} |A_1| \int_{B_2} |A_2| + \int_{B_2} |A_1| \int_{B_1 \triangle B_2} |A_2|,
\end{aligned}$$

where $A \triangle B$ denotes the symmetric difference $(A \cup B) \setminus (A \cap B)$. Applying this calculation to the previous estimate gives (omitting the constant $\frac{1}{|B|^2 c_*^2}$),

$$\begin{aligned}
&\int_{B(\mathbf{v}_1)\triangle B(\mathbf{v}_2)} |\mathbf{y}| \mathcal{Q}_e(\mathbf{x}+\mathbf{y})\, d\mathbf{y} \int_{B(\mathbf{v}_2)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\, d\mathbf{y}' \\
&\qquad + \int_{B(\mathbf{v}_2)} |\mathbf{y}| \mathcal{Q}_e(\mathbf{x}+\mathbf{y})\, d\mathbf{y} \int_{B(\mathbf{v}_1)\triangle B(\mathbf{v}_2)} \mathcal{Q}_e(\mathbf{x}+\mathbf{y}')\, d\mathbf{y}'.
\end{aligned} \tag{3.5}$$

Setting $2\theta = \angle(\mathbf{v}_1, \mathbf{v}_2)$, we find from (2.7)

$$|B(\mathbf{v}_1) \triangle B(\mathbf{v}_2)| = \ell^2 \theta. \tag{3.6}$$

Now note that some elementary trigonometry gives the relation

$$\cos\theta = 1 - \frac{1}{2}\Big|\frac{\mathbf{v}_1}{|\mathbf{v}_1|} - \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\Big|^2$$

and so

$$\theta = \arccos\Big(1 - \frac{1}{2}\Big|\frac{\mathbf{v}_1}{|\mathbf{v}_1|} - \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\Big|^2\Big).$$

Now consider the function $f(z) = \arccos(1 - \frac{1}{2}z^2)$ defined for $z \in [0, 2]$. It is easy to show that $f''(z) > 0$, $f(0) = 0$, $f(2) = \pi$. Therefore, being convex, $f(z)$ is below the line $y = \frac{\pi}{2}z$, which gives $f(z) \le \frac{\pi}{2}z$. We conclude that

$$\theta \le \frac{\pi}{2}\Big|\frac{\mathbf{v}_1}{|\mathbf{v}_1|} - \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\Big|.$$

Next, we wish an estimate by $|\mathbf{v}_1 - \mathbf{v}_2|$. To this end, we use the fact that $|\mathbf{v}_1|, |\mathbf{v}_2|$ are bounded from below by C_1: we have

$$\begin{aligned}
\Big|\frac{\mathbf{v}_1}{|\mathbf{v}_1|} - \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\Big| &= \Big|\frac{\mathbf{v}_1 - \mathbf{v}_2}{|\mathbf{v}_2|} + \mathbf{v}_1 \frac{|\mathbf{v}_2| - |\mathbf{v}_1|}{|\mathbf{v}_1|\,|\mathbf{v}_2|}\Big| \\
&\le \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{|\mathbf{v}_2|} + \frac{\big||\mathbf{v}_1| - |\mathbf{v}_2|\big|}{|\mathbf{v}_2|}.
\end{aligned}$$

Next, we remark that

$$\big||\mathbf{v}_1| - |\mathbf{v}_2|\big| \le \big|\mathbf{v}_1 - \mathbf{v}_2\big|,$$

as can be seen by squaring both sides and using the Cauchy-Schwartz inequality. Therefore, we get

$$\Big|\frac{\mathbf{v}_1}{|\mathbf{v}_1|} - \frac{\mathbf{v}_2}{|\mathbf{v}_2|}\Big| \leq |\mathbf{v}_1 - \mathbf{v}_2|\frac{2}{|\mathbf{v}_2|} \leq \frac{2}{C_1}|\mathbf{v}_1 - \mathbf{v}_2|. \tag{3.7}$$

Returning to (3.6) we deduce that

$$|B(\mathbf{v}_1) \triangle B(\mathbf{v}_2)| = \ell^2 \frac{\pi}{C_1}|\mathbf{v}_1 - \mathbf{v}_2|.$$

Coming back to (3.5), we conclude that

$$\big|F(\mathbf{x}, \mathbf{v}_1, \mathcal{Q}) - F(\mathbf{x}, \mathbf{v}_2, \mathcal{Q})\big| \leq C|\mathbf{v}_1 - \mathbf{v}_2|,$$

for $C = \frac{1}{|B|^2 c_*^2} 2|B(\mathbf{v}_1) \triangle B(\mathbf{v}_2)||B|\ell(c^*)^2 = \frac{1}{|B|c_*^2} 2\ell^2\ell(c^*)^2 \frac{\pi}{C_1} = \big(\frac{c^*}{c_*}\big)^2 \frac{2\ell\pi}{\beta C_1}$.

It remains to bound the term

$$\big|F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_1) - F(\mathbf{x}_2, \mathbf{v}_2, \mathcal{Q}_2)\big|.$$

With computations similar to the ones before, we find (omitting the subscripts in $\mathbf{x}_2, \mathbf{v}_2$)

$$\begin{aligned}
&\big|F(\mathbf{x}, \mathbf{v}, \mathcal{Q}_1) - F(\mathbf{x}, \mathbf{v}, \mathcal{Q}_2)\big| \\
&\leq \frac{1}{|B|^2 c_*^2}\Big| \int_{B(\mathbf{v})} \mathbf{y}\mathcal{Q}_{1e}(\mathbf{x}+\mathbf{y})\, d\mathbf{y} \int_{B(\mathbf{v})} \mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y}')\, d\mathbf{y}' \\
&\qquad\qquad - \int_{B(\mathbf{v})} \mathbf{y}\mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y})\, d\mathbf{y} \int_{B(\mathbf{v})} \mathcal{Q}_{1e}(\mathbf{x}+\mathbf{y}')\, d\mathbf{y}'\Big| \\
&\leq \frac{1}{|B|^2 c_*^2}\Big(\int_{B(\mathbf{v})} |\mathbf{y}|\big|\mathcal{Q}_{1e}(\mathbf{x}+\mathbf{y}) - \mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y})\big|\, d\mathbf{y} \int_{B(\mathbf{v})} \mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y}')\, d\mathbf{y}' \\
&\qquad + \int_{B(\mathbf{v})} |\mathbf{y}|\mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y})\, d\mathbf{y} \int_{B(\mathbf{v})} \big|\mathcal{Q}_{1e}(\mathbf{x}+\mathbf{y}') - \mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y}')\big|\, d\mathbf{y}'\Big) \\
&\leq \frac{2\ell c^*}{|B|c_*^2} \int_{B(\mathbf{v})} \big|\mathcal{Q}_{1e}(\mathbf{x}+\mathbf{y}) - \mathcal{Q}_{2e}(\mathbf{x}+\mathbf{y})\big|\, d\mathbf{y} \\
&\leq \frac{2c^*}{\ell\beta c_*^2}\|\mathcal{Q}_{1e} - \mathcal{Q}_{2e}\|_{L^1(\mathbb{T}^2)}.
\end{aligned}$$

This concludes the proof of Lemma 3.1. □

3.2 Proof of Theorem 2.2

We will use a fixed point argument. In the proof, we will use the notation $\mathbf{X}$ as a shorthand for $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$. We introduce the set

$$\mathscr{V} = \big\{\mathbf{V} \in \big(L^1(0,t;\mathbb{R}^2)\big)^N : 0 < C_1^0 \leq |\mathbf{v}_i(t)| \leq C_2^0,\ \forall i \in \{1, \ldots, N\}\big\},$$

with $C_1^0 = \min(|\mathbf{v}_i(0)|, C_1)$, $C_2^0 = \max(|\mathbf{v}_i(0)|, C_2)$, for the constants C_1, C_2 given in Lemma 2.1. We define the mapping $\Psi : \mathscr{V} \to \mathscr{V}$ in the following steps. Take $\mathbf{V} \in \mathscr{V}$;

1. Define $\mathbf{x}_i(t) = \mathbf{x}_i(0) + \int_0^t \mathbf{v}_i(s)\,ds \in C([0,t];\mathbb{T}^2)$, $i \in \{1,\dots,N\}$;
2. From $\mathbf{X}(t)$, let $\mathcal{P}(t,\mathbf{x}) = \sum_{i=1}^N \mathcal{P}_i(t,\mathbf{x})$, where $\mathcal{P}_i$ is the solution of the diffusion equation (2.2) with initial data $\mathcal{P}_0$. Therefore, $\mathcal{P}_i$ is given by
$$\begin{aligned}\mathcal{P}_i(t,\mathbf{x}) = &\sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\gamma t - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}} \mathcal{P}^0(\mathbf{y})\,d\mathbf{y} \\ &+ \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_0^t \frac{1}{4\pi(t-s)} e^{-\gamma(t-s) - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{x}_i(s)|^2}{4(t-s)}}\,ds;\end{aligned}$$
3. From $\mathcal{P}(t,\mathbf{x})$ obtain the truncated signal $\mathcal{P}_e(t,\mathbf{x})$ according to (2.4);
4. Build the desired velocity function $F(\mathbf{y},\mathbf{w},\mathcal{P})$ with $\mathbf{y}\in\mathbb{T}^2$, $\mathbf{w}\in\mathbb{R}^2$ with
$$F(\mathbf{y},\mathbf{w},\mathcal{P}) = \int_{B(\mathbf{w})} \mathbf{y}' \frac{\mathcal{P}_e(t,\mathbf{y}+\mathbf{y}')}{\int_{B(\mathbf{w})} \mathcal{P}_e(t,\mathbf{y}+\mathbf{y}'')\,d\mathbf{y}''}\,d\mathbf{y}';$$
5. Finally, take $\Psi(\mathbf{V}) = (\mathbf{w}_1,\dots,\mathbf{w}_N)$, with $t \mapsto (\mathbf{y}_i,\mathbf{w}_i)(t)$ as the solution (in the integral sense) of the differential system
$$\begin{array}{ll} \mathbf{y}_i' = \mathbf{w}_i, & \mathbf{w}_i' = -\frac{1}{\tau}\big(\mathbf{w}_i - F(\mathbf{y}_i,\mathbf{w}_i,\mathcal{P})\big), \\ \mathbf{y}_i(0) = \mathbf{x}_i(0), & \mathbf{w}_i(0) = \mathbf{v}_i(0), \end{array} \tag{3.8}$$
with $\mathcal{P}$ and $\mathcal{P}_e$ given in the previous steps.

The first task is to verify that the mapping Ψ is well defined. For this we must ensure that the system of $2N$ ODEs in step 5 is well-posed. Recall that here, $\mathcal{P}_e$ is a fixed function, so this is a standard ODE system. To apply the classical Cauchy–Lipschitz Theorem, we should check that the right-hand side (so, the function F) is a continuous function of t, where the time dependency comes through $\mathcal{P}_e$, and a Lipschitz function of $\mathbf{y}_i$ and $\mathbf{w}_i$. This is ensured by the Lipschitz property of Lemma 3.1 and the time continuity property (2.17) in Proposition 2.5. Note that using the Lipschitz property of Lemma 3.1 here requires that $\mathcal{P}_e$ has integrable gradient for $s \in [0,t]$, but this is proved to be the case in Proposition 2.5. Therefore, the system in step 5 above is well posed.

Next, we must show that the mapping Ψ takes $\mathscr{V}$ into $\mathscr{V}$. Let $\mathbf{V} \in \mathscr{V}$, and $\mathbf{W} = \Psi(\mathbf{V})$. Since $\mathbf{X}$ is a solution of an ODE similar to (2.1), then by Lemma 2.1, $\mathbf{W}$ is bounded by the same constants as $\mathbf{V}$, which only depend on the data of the problem.

It remains to see that Ψ is a contraction, at least for some short time $t > 0$. Let $\mathbf{V}^{1,2} \in \mathscr{V}$. By steps 2, 3, and 4, we obtain signal distributions $\mathcal{P}^{1,2}$ and $\mathcal{P}_e^{1,2}$ and desired velocities which we write as
$$F^{1,2} = F(\mathbf{y},\mathbf{w},\mathcal{P}^{1,2}).$$
Set $\mathbf{W}^{1,2} = \Psi(\mathbf{V}^{1,2})$. Let us denote for $j \in \{1,\dots,N\}$
$$\zeta_j(s) = \int_0^s |\mathbf{w}_j^1(r) - \mathbf{w}_j^2(r)|\,dr,$$

so that

$$d_{\mathbb{T}^2}(\mathbf{y}_j^1(s), \mathbf{y}_j^2(s)) \le \zeta_j(s).$$

By using (3.8), the Lipschitz property (3.2), and (2.17), we have, for $s \in (0,t)$,

$$\begin{aligned}
\zeta_j(s) &= \int_0^s \Big| \int_0^r \dot{\mathbf{w}}_j^1(\sigma) - \dot{\mathbf{w}}_j^2(\sigma)\, d\sigma \Big|\, dr \\
&\le \frac{1}{\tau} \int_0^s \int_0^r |\mathbf{w}_j^1(\sigma) - \mathbf{w}_j^2(\sigma)|\, d\sigma\, dr + \frac{1}{\tau} \int_0^s \int_0^r |F^1(\sigma) - F^2(\sigma)|\, d\sigma\, dr \\
&\le \frac{1}{\tau} \int_0^s \int_0^r |\mathbf{w}_j^1(\sigma) - \mathbf{w}_j^2(\sigma)|\, d\sigma\, dr + \frac{C_F}{\tau} t^2 \|\mathcal{P}_e^1 - \mathcal{P}_e^2\|_{L^\infty((0,t);L^1(\mathbb{T}^2))} \\
&\quad + \frac{C_F}{\tau} \int_0^s \int_0^r \big(d_{\mathbb{T}^2}(\mathbf{y}_j^1(\sigma), \mathbf{y}_j^2(\sigma)) + |\mathbf{w}_j^1(\sigma) - \mathbf{w}_j^2(\sigma)| \big)\, d\sigma\, dr \\
&\le \frac{1 + 2C_F}{\tau} t \int_0^s \zeta_j(r)\, dr + \frac{C_F}{\tau} t^2 \|\mathcal{P}_e^1 - \mathcal{P}_e^2\|_{L^\infty((0,t);L^1(\mathbb{T}^2))}.
\end{aligned}$$

Note that while the equation (3.8) holds in the integral sense, the above computation remains valid with the obvious modifications. Using Grönwall's Lemma, we obtain that, in fact,

$$\zeta_j(s) = \int_0^s |\mathbf{w}_j^1(r) - \mathbf{w}_j^2(r)|\, dr \le \frac{C_F}{\tau} t^2 e^{t^2(1+2C_F)/\tau} \|\mathcal{P}_e^1 - \mathcal{P}_e^2\|_{L^\infty((0,t);L^1(\mathbb{T}^2))}. \tag{3.9}$$

We proceed with an estimate of $\|\mathcal{P}_e^1 - \mathcal{P}_e^2\|_{L^\infty((0,t);L^1(\mathbb{T}^2))}$. Using Fubini's theorem and (2.20), we find

$$\begin{aligned}
&\int_{\mathbb{T}^2} \big| \mathcal{P}^1(\mathbf{x}) - \mathcal{P}^2(\mathbf{x}) \big|\, d\mathbf{x} \\
&\le \sum_{\mathbf{z} \in \mathbb{Z}^2} \sum_{j=1}^N \int_{\mathbb{T}^2} \int_0^t \frac{1}{4\pi(t-s)} e^{-\gamma(t-s)} \Big| e^{-\frac{|\mathbf{x}+\mathbf{z}-\mathbf{x}_j^1(s)|^2}{4(t-s)}} - e^{-\frac{|\mathbf{x}+\mathbf{z}-\mathbf{x}_j^2(s)|^2}{4(t-s)}} \Big|\, ds\, d\mathbf{x} \\
&\le \sum_{j=1}^N \int_0^t \int_{\mathbb{R}^2} \frac{1}{4\pi(t-s)} \Big| e^{-\frac{|\mathbf{x}+\mathbf{z}_0(s)-\mathbf{x}_j^1(s)|^2}{4(t-s)}} - e^{-\frac{|\mathbf{x}-\mathbf{x}_j^2(s)|^2}{4(t-s)}} \Big|\, d\mathbf{x}\, ds,
\end{aligned}$$

where $\mathbf{z}_0(s) \in \mathbb{Z}^2$, $|\mathbf{z}_0(s)|_\infty \le 1$ is chosen such that

$$|\mathbf{x}_j^1(s) - \mathbf{x}_j^2(s) + \mathbf{z}_0(s)| = d_{\mathbb{T}^2}(\mathbf{x}_j^1(s), \mathbf{x}_j^2(s)).$$

Now with the change of variable $\mathbf{y} = \frac{\mathbf{x}}{\sqrt{4(t-s)}}$ (but still writing the integral with $\mathbf{x}$), we get for this last expression

$$\sum_{j=1}^N \int_0^t \int_{\mathbb{R}^2} \frac{1}{\pi} \Big| e^{-\left|\mathbf{x} - \frac{\mathbf{x}_j^1(s)+\mathbf{z}_0(s)}{\sqrt{4(t-s)}}\right|^2} - e^{-\left|\mathbf{x} - \frac{\mathbf{x}_j^2(s)}{\sqrt{4(t-s)}}\right|^2} \Big|\, d\mathbf{x}\, ds.$$

Now, we have for $\theta \in [0,1]$ and $\overline{\theta} = \theta \frac{\mathbf{x}_j^1(s)+\mathbf{z}_0(s)}{\sqrt{4(t-s)}} + (1-\theta)\frac{\mathbf{x}_j^2(s)}{\sqrt{4(t-s)}}$,

$$
\begin{aligned}
&\int_{\mathbb{R}^2} \Big| e^{-|\mathbf{x} - \frac{\mathbf{x}_j^1(s)+\mathbf{z}_0(s)}{\sqrt{4(t-s)}}|^2} - e^{-|\mathbf{x} - \frac{\mathbf{x}_j^2(s)}{\sqrt{4(t-s)}}|^2} \Big| \, d\mathbf{x} \\
&\quad = \int_{\mathbb{R}^2} \int_0^1 2\big|\mathbf{x} - \overline{\theta}\big| e^{-|\mathbf{x}-\overline{\theta}|^2} \frac{|\mathbf{x}_j^1(s) - \mathbf{x}_j^2(s) + \mathbf{z}_0(s)|}{\sqrt{4(t-s)}} \, d\theta \, d\mathbf{x} \\
&\quad = \frac{d_{\mathbb{T}^2}(\mathbf{x}_j^1(s), \mathbf{x}_j^2(s))}{\sqrt{4(t-s)}} \int_0^1 \int_{\mathbb{R}^2} 2\big|\mathbf{x} - \overline{\theta}\big| e^{-|\mathbf{x}-\overline{\theta}|^2} \, d\mathbf{x} \, d\theta \\
&\quad = \frac{d_{\mathbb{T}^2}(\mathbf{x}_j^1(s), \mathbf{x}_j^2(s))}{\sqrt{4(t-s)}} \int_0^1 \int_{\mathbb{R}^2} 2\big|\mathbf{x}\big| e^{-|\mathbf{x}|^2} \, d\mathbf{x} \, d\theta \\
&\quad = M \frac{d_{\mathbb{T}^2}(\mathbf{x}_j^1(s), \mathbf{x}_j^2(s))}{\sqrt{4(t-s)}},
\end{aligned}
$$

with $M = 2\int |\mathbf{x}| e^{-|\mathbf{x}|^2} \, d\mathbf{x}$. Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^2} \big|\mathcal{P}^1(\mathbf{x}) - \mathcal{P}^2(\mathbf{x})\big| \, d\mathbf{x} &\leq \frac{M}{\pi} \sum_{j=1}^{N} \int_0^t \frac{d_{\mathbb{T}^2}(\mathbf{x}_j^1(s), \mathbf{x}_j^2(s))}{\sqrt{4(t-s)}} \, ds \\
&\leq \frac{M}{\pi} \sum_{j=1}^{N} \sqrt{t} \sup_{s \in (0,t)} d_{\mathbb{T}^2}(\mathbf{x}_j^1(s), \mathbf{x}_j^2(s)).
\end{aligned} \tag{3.10}
$$

Going back to (3.9), using (3.10) and the fact that the truncation is itself a (non-strict) contraction, we see that

$$
\begin{aligned}
\int_0^t |\mathbf{w}_j^1(r) - \mathbf{w}_j^2(r)| \, dr &\leq \frac{MC_F}{\tau\pi} t^{5/2} e^{t^2(1+2C_F)} \sum_{k=1}^{N} \sup_{(0,t)} d_{\mathbb{T}^2}(\mathbf{x}_k^1(s), \mathbf{x}_k^2(s)) \\
&\leq \frac{MC_F}{\tau\pi} t^{5/2} e^{t^2(1+2C_F)} \sum_{k=1}^{N} \int_0^t |\mathbf{v}_k^1(s) - \mathbf{v}_k^2(s)| \, ds.
\end{aligned}
$$

Finally, this gives

$$
\begin{aligned}
\|\mathbf{W}^1 - \mathbf{W}^2\|_{\mathscr{V}} &\equiv \sum_{k=1}^{N} \int_0^t |\mathbf{w}_k^1(r) - \mathbf{w}_k^2(r)| \, dr \\
&\leq \frac{NMC_F}{\tau\pi} t^{5/2} e^{t^2(1+2C_F)} \sum_{k=1}^{N} \int_0^t |\mathbf{v}_k^1(s) - \mathbf{v}_k^2(s)| \, ds \\
&= \frac{NMC_F}{\tau\pi} t^{5/2} e^{t^2(1+2C_F)} \|\mathbf{V}^1 - \mathbf{V}^2\|_{\mathscr{V}}
\end{aligned}
$$

which implies that Ψ is a contraction for t small enough. The fact that the solutions $\mathbf{W}$ satisfy the estimate in Lemma 2.1, gives existence of solution for all times.

4 The pointwise sensing model: proof of Theorem 2.3

In the proof, we use the notation $\mathbf{X}$ as a shorthand for $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Although we also use a fixed point argument, the strategy here is different than in the case of the full sector sensing model. We will consider the uniform topology on the space of C^1 curves and prove directly the existence of solution to the ODE system, bypassing the application of the Cauchy-Lipschitz Theorem inside the fixed point proof. This is more convenient in this case, since the sensing by Dirac deltas requires continuity properties of the signal function.

Fix initial positions $\mathbf{X}^0$ verifying the assumptions of the theorem, initial velocities $\mathbf{V}^0$, and define the set

$$\mathscr{X} = \Big\{ \mathbf{X} \in \big(C^1([0,t];\mathbb{T}^2)\big)^N \text{such that:} \quad 0 < \frac{C_1^0}{2} \le |\dot{\mathbf{X}}(s)| \le 2C_2^0 \text{ for any } s \in [0,t], \quad \mathbf{X}(0) = \mathbf{X}^0,\ \dot{\mathbf{X}}(0) = \mathbf{V}^0 \Big\}, \tag{4.1}$$

with $C_1^0 = \min(|\mathbf{v}_1(0)|, ..., |\mathbf{v}_N(0)|, C_1)$, $C_2^0 = \max(|\mathbf{v}_1(0)|, ..., |\mathbf{v}_N(0)|, C_2)$, the constants that appear in the velocity bounds (2.11), endowed with the standard uniform topology

$$\|\mathbf{X}\|_{\mathscr{X}} = \sum_{i=1}^N \|\mathbf{x}_i\|_{C^1([0,t])} = \sum_{i=1}^N \Big(\|\mathbf{x}_i\|_{L^\infty(0,t;\mathbb{T}^2)} + \|\dot{\mathbf{x}}_i\|_{L^\infty(0,t;\mathbb{R}^2)} \Big), \tag{4.2}$$

$\mathscr{X}$ is a closed and complete subset of $C^1([0,t])$. We define the mapping $\Gamma : \mathscr{X} \to \mathscr{X}$ in the following steps. Take $\mathbf{X} \in \mathscr{X}$;

1. From $\mathbf{X}$, let $\mathcal{P}(t,\mathbf{x}) = \sum_{i=1}^N \mathcal{P}_i(t,\mathbf{x})$, where $\mathcal{P}_i$ is the solution of the diffusion equation (2.2) with initial data $\frac{1}{N}\mathcal{P}_0$. Therefore, the total signal is
$$\mathcal{P}(t,\mathbf{x}) = \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_{\mathbb{T}^2} \frac{1}{4\pi t} e^{-\gamma t - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{y}|^2}{4t}} \mathcal{P}^0(\mathbf{y})\, d\mathbf{y} + \sum_{i=1}^N \sum_{\mathbf{z}\in\mathbb{Z}^2} \int_0^t \frac{1}{4\pi(t-s)} e^{-\gamma(t-s) - \frac{|\mathbf{x}+\mathbf{z}-\mathbf{x}_i(s)|^2}{4(t-s)}}\, ds;$$
2. From $\mathcal{P}(t,\mathbf{x})$ obtain the truncated signal $\mathcal{P}_e(t,\mathbf{x})$ according to (2.4); (When it will be necessary we shall use the notation $\mathcal{P}[\mathbf{X}](t,\mathbf{x})$, and $\mathcal{P}_e[\mathbf{X}](t,\mathbf{x})$ in order to keep track of the dependence on the trajectories $\mathbf{X}$.)
3. Build the desired velocity function $F(\mathbf{x},\mathbf{v},\mathcal{P})$ with $\mathbf{x}\in\mathbb{T}^2$, $\mathbf{v}\in\mathbb{R}^2$ with
$$F(\mathbf{x},\mathbf{v},\mathcal{P}) = \frac{\mathbf{y}_L \mathcal{P}_e(t,\mathbf{x}+\mathbf{y}_L) + \mathbf{y}_R \mathcal{P}_e(t,\mathbf{x}+\mathbf{y}_R)}{\mathcal{P}_e(t,\mathbf{x}+\mathbf{y}_R) + \mathcal{P}_e(t,\mathbf{x}+\mathbf{y}_R)},$$
according to (2.9),(2.10) (recall that $\mathbf{y}_{L,R}$ depends on $\mathbf{v}$);

4. Define the new velocities by

$$\Upsilon(\mathbf{v}_i(t)) = \mathbf{v}_i^0 - \frac{1}{\tau}\int_0^t \big(\mathbf{v}_i(s) - F(\mathbf{x}_i(s), \mathbf{v}_i(s), \mathcal{P})\big)\,ds; \qquad (4.3)$$

5. Finally, take $\Gamma(\mathbf{X})$ as

$$\Gamma(\mathbf{x}_i(t)) = \mathbf{x}_i^0 + \int_0^t \Upsilon(\mathbf{v}_i(s))\,ds. \qquad (4.4)$$

The first task is to verify that the mapping Γ takes $\mathscr{X}$ into $\mathscr{X}$, at least for t small enough. For this, notice first that

$$|\Upsilon(\mathbf{v}_i(s))| \leq |\mathbf{v}_i(0)| + t(2C_2^0 + \ell).$$

Indeed, the function F verifies $|F| \leq \ell$, no matter what are the arguments of the function $\mathcal{P}$, as is easily checked from (2.9). By definition, initially $|\mathbf{v}_i(0)| \leq C_2^0$. Therefore, taking

$$t = \frac{1}{2}\frac{C_2^0}{2C_2^0 + \ell} > 0$$

gives that the upper bound on the velocity in (4.1) is verified. For the lower bound, we have similarly that

$$|\Upsilon(\mathbf{v}_i(s))| \geq |\mathbf{v}_i(0)| - t(2C_2^0 + \ell),$$

with $|\mathbf{v}_i(0)| \geq C_1^0$, so that taking

$$t = \frac{1}{2}\min_{i\in\{1,\dots,N\}}\frac{C_1^0}{2C_2^0 + \ell} > 0$$

proves the lower bound on the on the velocity in (4.1). This proves that Γ maps $\mathscr{X}$ into $\mathscr{X}$, for sufficiently small t.

Now we must show that Γ is a contraction. Consider $\mathbf{X}^1, \mathbf{X}^2 \in \mathscr{X}$, along with the corresponding velocities $\mathbf{V}^1, \mathbf{V}^2$. Denote by $\mathcal{P}^{1,2}(s,\mathbf{x}) = \mathcal{P}[\mathbf{X}^{1,2}](s,\mathbf{x})$ the signal functions generated, respectively, by the families of curves $\mathbf{X}^1$ and $\mathbf{X}^2$, according to step 1 above. That Γ is a contraction on X, on a small enough time interval $[0,t]$, is a consequence of the following Lipschitz-type estimate:

$$\big|F(\mathbf{x}_j^1(s), \mathbf{v}_j^1(s), \mathcal{P}^1) - F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^2)\big| \leq \Lambda\big\|\mathbf{X}^1 - \mathbf{X}^2\big\|_{\mathscr{X}}, \qquad (4.5)$$

which holds for $s \in [0,t]$ and $j \in \{1,\dots,N\}$, with a uniform constant $\Lambda > 0$, which only depends on the parameters of the problem. With (4.5), we can show that Γ admits a unique fixed point in $\mathscr{X}$, provided t is small enough. It defines a solution of the nonlinear differential system (2.1), (2.2), with (2.9); since Lemma 2.1 applies, this solution satisfies the strengthened estimate (2.11), which allows us to eventually extend the solution to any time interval.

The proof of (4.5) turns out to be quite involved. The strategy is to distinguish:

- what happens for very short times: the trajectories do not intersect and the individuals do not feel each other. Hence the argument relies on the regularity of $\mathcal{P}$ far from the deposition points, inherited from the initial data.

- what happens for positive times, by using the regularizing estimates of Proposition 2.5.

The following considerations are essential in the proof; they allow us to use the properties of the initial state to derive the necessary estimates on the earliest stages of the time evolution. Let us pick some $0 < \delta \ll \ell$ such that for any $j \neq k$, $d_{\mathbb{T}^2}(\mathbf{x}_j^0, \mathbf{x}_k^0) > \ell + 4\delta$. The trajectories in $\mathscr{X}$ have maximal speed C_2^0. Therefore, for any $\mathbf{X} \in \mathscr{X}$, $0 \leq t \leq t^*$ and $j \in \{1, ..., N\}$

- the ball $B(\mathbf{x}_j(t), \delta)$ remains included in $B(\mathbf{x}_j^0, \delta + C_2^0 t^*)$;

- any point located at a distance ℓ from $\mathbf{x}_j(t)$ is at least at a distance $\ell - C_2^0 t \geq \ell - C_2^0 t^*$ far from $\mathbf{x}_j^0$.

Hence, we can find $t^* > 0$ small enough so that

- $\inf\big\{d_{\mathbb{T}^2}(\mathbf{x}_j^1(t), \mathbf{x}_k^2(s)),\ 0 \leq t, s \leq t^*,\ j, k \in \{1, ..., N\},\ j \neq k,\ \mathbf{X}^1, \mathbf{X}^2 \in \mathscr{X}\big\} \geq \ell + \delta$,

- $\inf\big\{d_{\mathbb{T}^2}(S(\mathbf{x}_j^1(t), \ell), B(\mathbf{x}_k^2(s), \delta)),\ 0 \leq t, s \leq t^*,\ j, k \in \{1, ..., N\},\ \mathbf{X}^1, \mathbf{X}^2 \in \mathscr{X}\big\} \geq \delta$,

where $S(\mathbf{x}, \ell)$ stands for the sphere of center $\mathbf{x}$ and radius $\ell > 0$. The consequence from this claim is that on the time interval $[0, t^*]$, the sensing points (antennae), which lie on the spheres $S(\mathbf{x}_j^1(s), \ell)$, stay at a positive distance away from all signal deposition points $\mathbf{x}_k^2(r)$, $r \leq s$, of curves in $\mathscr{X}$. With notation similar to that of Proposition 2.5, we introduce the compact set

$$\mathcal{T}_{t^*,\delta} = \big\{(t, \mathbf{x}) \in [0, t^*] \times \mathbb{T}^2,\ d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_j^1(s)) \geq \delta,\ j \in \{1, ..., N\},\ \mathbf{X}^1 \in \mathscr{X},\ 0 \leq s \leq t\big\}.$$

We turn to the Lipschitz estimate. We have

$$\begin{aligned}
\big|F(\mathbf{x}_j^1(s), \mathbf{v}_j^1(s), \mathcal{P}^1) - F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^2)\big| \\
&\leq \big|F(\mathbf{x}_j^1(s), \mathbf{v}_j^1(s), \mathcal{P}^1) - F(\mathbf{x}_j^2(s), \mathbf{v}_j^1(s), \mathcal{P}^1)\big| \\
&\quad + \big|F(\mathbf{x}_j^2(s), \mathbf{v}_j^1(s), \mathcal{P}^1) - F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^1)\big| \\
&\quad + \big|F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^1) - F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^2)\big| \\
&\quad = A + B + C.
\end{aligned} \tag{4.6}$$

We are going to estimate the three terms one after the other, and for all three we obtain the estimate by considering first what happens on a short enough time interval $[0,t^*]$, and second dealing with positive times in (t^*,t). We begin by estimating A:

$$A = \left| \frac{\mathbf{y}^1_{jL}(s)\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)) + \mathbf{y}^1_{jR}(s)\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jr}(s))}{\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)) + \mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jR}(s))} \right.$$
$$\left. - \frac{\mathbf{y}^1_{jL}(s)\mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s)) + \mathbf{y}^1_{jR}(s)\mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jr}(s))}{\mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s)) + \mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jR}(s))} \right|,$$

which after a somewhat lengthy computation using only $c_* \le \mathcal{P}_e \le c^*$ and $|\mathbf{y}_{L,R}| = \ell$, gives

$$A \le \frac{c^*\ell}{2c_*^2}\Big(\big|\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)) - \mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s))\big| \\ + \big|\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jR}(s)) - \mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jR}(s))\big|\Big). \tag{4.7}$$

According to Proposition 2.5, we can find $L>0$ such that $\|\nabla\mathcal{P}^1\|_{L^\infty(\mathcal{T}_{t^*,\delta})} \le L$, and thus $\|\nabla\mathcal{P}^1_e\|_{L^\infty(\mathcal{T}_{t^*,\delta})} \le L$ too. For any $0\le s\le t^*$, the domain $\mathcal{T}_s = \complement\big(\bigcup_{j=1}^N B(\mathbf{x}^1_j(s),\delta)\big)$ is path connected, and it contains both $\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)$ and $\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s)$. So there exists a curve $\zeta:[0,1]\to\mathcal{T}_s$ connecting $\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)$ and $\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s)$ such that $\int_0^1|\zeta'(r)|\,dr \le D d_{\mathbb{T}^2}(\mathbf{x}^1_j(s),\mathbf{x}^2_j(s))$ for some $0<D\le\pi\delta$, and we find

$$\begin{aligned}\big|\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)) - \mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s))\big| &= \left|\int_0^1 \frac{d}{dr}\mathcal{P}^1_e(s,\zeta(r))\,dr\right| \\ &\le \int_0^1 |\nabla\mathcal{P}^1_e(s,\zeta(r))|\,|\zeta'(r)|\,dr \\ &\le DL\, d_{\mathbb{T}^2}(\mathbf{x}^1_j(s),\mathbf{x}^2_j(s)).\end{aligned}$$

We proceed similarly for $\mathbf{x}^1_j(s)+\mathbf{y}^1_{jR}(s)$ and $\mathbf{x}^2_j(s)+\mathbf{y}^1_{jR}(s)$, and coming back to (4.7), we conclude that

$$A \le \frac{DLc^*\ell}{2c_*^2}\, d_{\mathbb{T}^2}(\mathbf{x}^1_j(s),\mathbf{x}^2_j(s))$$

holds at least for $s\in[0,t^*]$.

For $s\in(t^*,t]$, property (2.19) holds, and the previous precautions are not necessary: we have simply

$$\big|\mathcal{P}^1_e(s,\mathbf{x}^1_j(s)+\mathbf{y}^1_{jL}(s)) - \mathcal{P}^1_e(s,\mathbf{x}^2_j(s)+\mathbf{y}^1_{jL}(s))\big| \le L(t^*)d_{\mathbb{T}^2}(\mathbf{x}^1_j(s),\mathbf{x}^2_j(s)),$$

with $L(t^*)$ from (2.19). In any case, for all $0\le s\le t$, we have

$$A \le \max\left(\frac{DLc^*\ell}{2c_*^2}, L(t^*)\right) d_{\mathbb{T}^2}(\mathbf{x}^1_j(s),\mathbf{x}^2_j(s)) \le \Lambda\big\|\mathbf{X}^1-\mathbf{X}^2\big\|_{\mathscr{X}}, \tag{4.8}$$

where the constant Λ depends on $\ell, \delta, t^*, c_*, c^*$.

Now consider the term B in (4.6). To keep the presentation less cluttered, we now omit some sub- and superscripts that do not change. So, writing $\mathbf{x} = \mathbf{x}_j^2(s), \mathbf{v}^{1,2} = \mathbf{v}_j^{1,2}(s)$, $\mathbf{y}_{L,R}^{1,2} = \mathbf{y}_{jL,R}^{1,2}(s)$ and $\mathcal{P}_e(\cdot) = \mathcal{P}_e^1(s,\cdot)$, we find

$$
\begin{aligned}
B &= \left|F(\mathbf{x},\mathbf{v}^1,\mathcal{P}) - F(\mathbf{x},\mathbf{v}^2,\mathcal{P})\right| \\
&= \left|\frac{\mathbf{y}_L^1\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) + \mathbf{y}_R^1\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)} - \frac{\mathbf{y}_L^2\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2) + \mathbf{y}_R^2\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2)}\right| \\
&\le \left|\frac{\mathbf{y}_L^1\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)} - \frac{\mathbf{y}_L^2\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2)}\right| \\
&\quad + \left|\frac{\mathbf{y}_R^1\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)} - \frac{\mathbf{y}_R^2\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2)}\right| \\
&= B_1 + B_2.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
B_1 &\le d_{\mathbb{T}^2}(\mathbf{y}_L^1, \mathbf{y}_L^2)\frac{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)} \\
&\quad + \mathbf{y}_L^2\left|\frac{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)} - \frac{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2)}{\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2) + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2)}\right| \\
&\le \frac{c^*}{2c_*} d_{\mathbb{T}^2}(\mathbf{y}_L^1, \mathbf{y}_L^2) + \frac{1}{2c_*^2}\Big|(\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) - \mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2))\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2) \\
&\quad + \mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2)(\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2) - \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1))\Big| \\
&\le \frac{c^*}{2c_*} d_{\mathbb{T}^2}(\mathbf{y}_L^1, \mathbf{y}_L^2) \\
&\quad + \frac{c^*}{2c_*^2}\Big(\left|\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) - \mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2)\right| + \left|\mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^2) - \mathcal{P}_e(\mathbf{x}+\mathbf{y}_R^1)\right|\Big).
\end{aligned}
$$

The points $\mathbf{x}_j^2(s)+\mathbf{y}_{jL}^1(s)$ and $\mathbf{x}_j^2(s)+\mathbf{y}_{jL}^2(s)$ are both in $S(\mathbf{x}_j^2(s),\ell)$. Therefore, as we did above, for each pair of such points we can again find a curve $\zeta : [0,1] \to \mathcal{T}_s$ (with a slight modification of the definition of $\mathcal{T}_s$) connecting the given points such that $\int_0^1 |\zeta'(r)|\, dr \le D\, d_{\mathbb{T}^2}(\mathbf{y}_L^1, \mathbf{y}_L^2)$. In this way,

$$
\begin{aligned}
\left|\mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^1) - \mathcal{P}_e(\mathbf{x}+\mathbf{y}_L^2)\right| &= \Big|\int_0^1 \frac{d}{dr}\mathcal{P}_e^1(s,\zeta(r))\, dr\Big| \\
&\le DL\, d_{\mathbb{T}^2}(\mathbf{y}_L^1(s), \mathbf{y}_L^2(s)),
\end{aligned}
$$

using Proposition 2.5. This gives

$$
B_1 \le \frac{c^*}{2c_*}\, d_{\mathbb{T}^2}(\mathbf{y}_{jL}^1(s), \mathbf{y}_{jL}^2(s)) + \frac{LDc^*}{c_*^2}\Big(d_{\mathbb{T}^2}(\mathbf{y}_{jL}^1(s), \mathbf{y}_{jL}^2(s)) + d_{\mathbb{T}^2}(\mathbf{y}_{jR}^1(s), \mathbf{y}_{jR}^2(s))\Big).
$$

The term B_2 satisfies a similar estimate, by exchanging $\mathbf{y}_L$ and $\mathbf{y}_R$. Now observe that, in fact, $d_{\mathbb{T}^2}(\mathbf{y}_{jL}^1(s), \mathbf{y}_{jL}^2(s)) = d_{\mathbb{T}^2}(\mathbf{y}_{jR}^1(s), \mathbf{y}_{jR}^2(s)) = \big|\frac{\mathbf{v}_j^1(s)}{|\mathbf{v}_j^1(s)} - \frac{\mathbf{v}_j^2(s)}{|\mathbf{v}_j^2(s)|}\big|$. By using (3.7) – which relies on the fact that the speed of the elements of $\mathscr{X}$ is bounded from below – this is dominated by $\frac{4}{C_1^0}|\mathbf{v}_1 - \mathbf{v}_2|$. Therefore, we find

$$B \le \Big(\frac{c^*}{2c_*} + \frac{8LDc^*}{C_1^0 c_*^2}\Big)\big|\mathbf{v}_j^1(s) - \mathbf{v}_j^2(s)\big|.$$

This is valid for $s \in [0, t^*]$. For $s > t^*$, property (2.19) holds and so, we finally obtain, for $0 \le s \le t$,

$$B \le \Lambda\big|\mathbf{v}_j^1(s) - \mathbf{v}_j^2(s)\big| \le \Lambda\big\|\mathbf{X}^1 - \mathbf{X}^2\big\|_{\mathscr{X}}. \tag{4.9}$$

Eventually, writing $\mathbf{x} = \mathbf{x}_j^2(s)$, $\mathbf{y}_{L,R} = \mathbf{y}_{jL,R}^2(s)$ and $\mathcal{P}_e^{1,2}(\cdot) = \mathcal{P}_e^{1,2}(s, \cdot)$, we consider from (4.6)

$$\begin{aligned} C &= \big|F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^1) - F(\mathbf{x}_j^2(s), \mathbf{v}_j^2(s), \mathcal{P}^2)\big| \\ &= \left|\frac{\mathbf{y}_L\mathcal{P}_e^1(\mathbf{x}+\mathbf{y}_L) + \mathbf{y}_R\mathcal{P}_e^1(\mathbf{x}+\mathbf{y}_R)}{\mathcal{P}_e^1(\mathbf{x}+\mathbf{y}_L) + \mathcal{P}_e^1(\mathbf{x}+\mathbf{y}_R)} - \frac{\mathbf{y}_L\mathcal{P}_e^2(\mathbf{x}+\mathbf{y}_L) + \mathbf{y}_R\mathcal{P}_e^2(\mathbf{x}+\mathbf{y}_R)}{\mathcal{P}_e^2(\mathbf{x}+\mathbf{y}_L) + \mathcal{P}_e^2(\mathbf{x}+\mathbf{y}_R)}\right|. \end{aligned}$$

After some very similar manipulations as before, we can bound this term with differences of the form

$$\frac{\ell c^*}{2c_*^2}\big|\mathcal{P}_e^1(s, \mathbf{x}_j^2(s) + \mathbf{y}_{jL}^2(s)) - \mathcal{P}_e^2(s, \mathbf{x}_j^2(s) + \mathbf{y}_{jL}^2(s))\big|, \tag{4.10}$$

or with $\mathbf{y}_R$ instead of $\mathbf{y}_L$.

To start with, notice that it is enough to estimate this difference in the particular case where the families of curves $(\mathbf{x}_1^1, \dots, \mathbf{x}_N^1)$ and $(\mathbf{x}_1^2, \dots, \mathbf{x}_N^2)$ differ only by one of their components, say, $\mathbf{x}_1^1 \ne \mathbf{x}_1^2$ but $\mathbf{x}_k^1 = \mathbf{x}_k^2$, $k \in \{2, \dots, N\}$. Indeed, in the general case we have

$$\begin{aligned} \big|\mathcal{P}_e[\mathbf{X}^1] - \mathcal{P}_e[\mathbf{X}^2]\big| \le\ & \big|\mathcal{P}_e[(\mathbf{x}_1^1, \mathbf{x}_2^1, \dots, \mathbf{x}_N^1)] - \mathcal{P}_e[(\mathbf{x}_1^2, \mathbf{x}_2^1, \dots, \mathbf{x}_N^1)]\big| \\ & + \big|\mathcal{P}_e[(\mathbf{x}_1^2, \mathbf{x}_2^1, \dots, \mathbf{x}_N^1)] - \mathcal{P}_e[(\mathbf{x}_1^2, \mathbf{x}_2^2, \dots, \mathbf{x}_N^1)]\big| \\ & + \cdots + \big|\mathcal{P}_e[(\mathbf{x}_1^2, \dots, \mathbf{x}_N^1)] - \mathcal{P}_e[(\mathbf{x}_1^2, \dots, \mathbf{x}_N^2)]\big|, \end{aligned}$$

where in each difference, only one curve in each family is different. Taking this into account, we now denote $\mathcal{P}^1(s, \mathbf{x}) = \mathcal{P}[\mathbf{X}^1](s, \mathbf{x})$ and $\mathcal{P}^2(s, \mathbf{x}) = \mathcal{P}_e[(\mathbf{x}_1^2, \mathbf{x}_2^1, \dots, \mathbf{x}_N^1)](s, \mathbf{x})$ and $\mathbf{w}_j(s) = \mathbf{x}_j^2(s) + \mathbf{y}_{jL}^2(s)$.

We have (note that we are not considering the truncation now, but the difference between the truncated functions is bounded by the difference of the non-truncated functions)

$$\begin{aligned}&\big|\mathcal{P}^1(s,\mathbf{w}_j(s))-\mathcal{P}^2(s,\mathbf{w}_j(s))\big|\\&\leq\sum_{\mathbf{z}\in\mathbb{Z}^2}\sum_{k=1}^N\int_0^s\frac{1}{4\pi(s-r)}\Big|e^{-\frac{|\mathbf{w}_j(s)+\mathbf{z}-\mathbf{x}_k^1(r)|^2}{4(s-r)}}-e^{-\frac{|\mathbf{w}_j(s)+\mathbf{z}-\mathbf{x}_k^2(r)|^2}{4(s-r)}}\Big|\,dr\\&\leq\sum_{\mathbf{z}\in\mathbb{Z}^2}\frac{1}{4\pi}\int_{1/s}^{+\infty}\frac{1}{\sigma}\Big|e^{-\frac{\sigma}{4}|\mathbf{w}_j(s)+\mathbf{z}-\mathbf{x}_1^1(s-\frac{1}{\sigma})|^2}-e^{-\frac{\sigma}{4}|\mathbf{w}_j(s)+\mathbf{z}-\mathbf{x}_1^2(s-\frac{1}{\sigma})|^2}\Big|\,d\sigma.\end{aligned}$$

We are going to adapt some reasonings used when justifying Proposition 2.5. Let $\mathbf{z}_1\in\mathbb{Z}^2$, with $|\mathbf{z}_1|_\infty\leq 1$ (which thus depends on s and σ) be such that

$$\begin{aligned}\min&\big\{|\mathbf{w}_j(s)+\mathbf{z}-\mathbf{x}_1^1(s-1/\sigma)|,|\mathbf{w}_j(s)+\mathbf{z}-\mathbf{x}_1^2(s-1/\sigma)|,\ \mathbf{z}\in\mathbb{Z}^2\big\}\\&=\min\big\{|\mathbf{w}_j(s)+\mathbf{z}_1-\mathbf{x}_1^1(s-1/\sigma)|,|\mathbf{w}_j(s)+\mathbf{z}_1-\mathbf{x}_1^2(s-1/\sigma)|\big\}.\end{aligned}\tag{4.11}$$

Then, we can rewrite the previous estimate as follows

$$\big|\mathcal{P}^1(s,\mathbf{w}_j(s))-\mathcal{P}^2(s,\mathbf{w}_j(s))\big|\quad\leq\sum_{\mathbf{z}\in\mathbb{Z}^2}\frac{1}{4\pi}\int_{1/s}^{+\infty}\frac{1}{\sigma}\Big|e^{-\frac{\sigma}{4}\mathcal{A}(\mathbf{z})^2}-e^{-\frac{\sigma}{4}\mathcal{B}(\mathbf{z})^2}\Big|\,d\sigma,$$

where we have set

$$\begin{aligned}\mathcal{A}(\mathbf{z})&=|\mathbf{w}_j(s)+\mathbf{z}_1-\mathbf{x}_1^1(s-1/\sigma)+\mathbf{z}|,\\\mathcal{B}(\mathbf{z})&=|\mathbf{w}_j(s)+\mathbf{z}_1-\mathbf{x}_1^2(s-1/\sigma)+\mathbf{z}|.\end{aligned}$$

Notice that for $\sigma,a,b\geq 0$, using the mean value theorem,

$$|e^{-\frac{\sigma}{4}a^2}-e^{-\frac{\sigma}{4}b^2}|=\frac{\sigma}{2}ce^{-\frac{\sigma}{4}c^2}|b-a|\leq\frac{\sigma}{2}\max(a,b)e^{-\frac{\sigma}{4}\min(a,b)^2}|b-a|.$$

We find

$$\begin{aligned}&\big|\mathcal{P}^1(s,\mathbf{w}_j(s))-\mathcal{P}^2(s,\mathbf{w}_j(s))\big|\\&\leq\sum_{\mathbf{z}\in\mathbb{Z}^2}\frac{1}{8\pi}\int_{1/s}^{+\infty}\max(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))e^{-\frac{\sigma}{4}\min(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))^2}\big|\mathbf{x}_1^1(s-1/\sigma)-\mathbf{x}_1^2(s-1/\sigma)\big|\,d\sigma\\&\leq\big\|\mathbf{X}^1-\mathbf{X}^2\big\|_{\mathscr{X}}\sum_{\mathbf{z}\in\mathbb{Z}^2}\frac{1}{8\pi}\int_{1/s}^{+\infty}\max(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))e^{-\frac{\sigma}{4}\min(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))^2}\,d\sigma.\end{aligned}\tag{4.12}$$

We observe that $\max(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))\leq 2+\ell+|\mathbf{z}|$. We treat separately the sums over $|\mathbf{z}|\leq 1$ and $|\mathbf{z}|>1$. For the former, we use that $d_{\mathbb{T}^2}(\mathbf{w}_j(s),\mathbf{x}_1^k(s-1/\sigma))\geq\delta$ holds for any $0\leq s\leq t^*$ and $\sigma\geq 1/s$, so that $\min(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))\geq\delta$, and for the latter we remark that $\mathcal{A}(\mathbf{z})$ and $\mathcal{B}(\mathbf{z})$ are bounded from below by $|\mathbf{z}|-1$, when

$|\mathbf{z}| > 1$. It follows that, for some $K > 0$

$$\begin{aligned}
&\left|\mathcal{P}^1(s, \mathbf{w}_j(s)) - \mathcal{P}^2(s, \mathbf{w}_j(s))\right| \\
&\quad \leq \left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}} K \left(\int_{1/s}^{+\infty} e^{-\delta^2 \frac{\sigma}{4}} \, d\sigma + \sum_{|\mathbf{z}|>1} \int_{1/s}^{+\infty} (1 + |\mathbf{z}|) e^{-(|\mathbf{z}|-1)\frac{\sigma}{4}} \, d\sigma \right) \\
&\quad \leq \left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}} 4K \left(\frac{1}{\delta^2} e^{-\delta^2/(4t^*)} + \sum_{|\mathbf{z}>1} (1 + \mathbf{z}|) e^{-(|\mathbf{z}|-1)/(4t^*)} \right).
\end{aligned}$$

holds when $0 \leq s \leq t^*$.

We now discuss the case where $s \in (t^*, t]$. It relies on the properties stated in Proposition 2.5. The idea is that, once some time has passed from $s = 0$, Proposition 2.5 ensures that there is a small ball around the deposition point where $\mathcal{P}_e \equiv c^*$, and so, even if now the sensing points overlap with another agent's position, the truncation eliminates the undefined behavior of a Dirac delta evaluating a singular function. Consider then any $\mathbf{x} \in \mathbb{T}^2$. We are going to estimate $\left|\mathcal{P}_e^1(s, \mathbf{x}) - \mathcal{P}_e^2(s, \mathbf{x})\right|$, still assuming that $\mathbf{X}^1$ and $\mathbf{X}^2$ only differ by one of their components: $\mathbf{x}_1^1 \neq \mathbf{x}_1^2$ but $\mathbf{x}_k^1 = \mathbf{x}_k^2$, $k \in \{2, \ldots, N\}$.

We know that:

$$\text{for each } s > 0 \text{ there exists } \epsilon \in (0,1) \text{ such that } \mathcal{P}_e^{1,2}(s, \mathbf{x}) \equiv c^* \text{ if } \mathbf{x} \in \bigcup_{k=1}^{N} B(\mathbf{x}_k^1(s), \epsilon).$$

We consider three cases.

Case 1: $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^1(s)) < \epsilon$ and $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^2(s)) < \epsilon$. In this case, there is nothing to prove, as Proposition 2.5 ensures that $|\mathcal{P}_e^1(s, \mathbf{x}) - \mathcal{P}_e^2(s, \mathbf{x})| = |c^* - c^*| = 0$.

Case 2: $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^1(s)) > \epsilon$ and $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^2(s)) > \epsilon$. First, note that in case we also have $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_k^1(s)) < \epsilon$ for some $k \in \{2, \ldots, N\}$, then actually $\mathcal{P}_e^1(s, \mathbf{x}) = c^* = \mathcal{P}_e^2(s, \mathbf{x})$. Otherwise, as $\mathbf{x}$ is far away from the singular points $\mathbf{x}_1^{1,2}(s)$, we may use the fact that $|\mathcal{P}_e^1 - \mathcal{P}_e^2| \leq |\mathcal{P}^1 - \mathcal{P}^2|$.

To be more precise, by continuity, we can find $r^* < s$ such that for $r^* \leq r \leq s$ we have $d_{\mathbb{T}^2}(\mathbf{x}_1^1(r), \mathbf{x}_1^1(s)), d_{\mathbb{T}^2}(\mathbf{x}_1^2(r), \mathbf{x}_1^2(s)) \leq \epsilon$. We can now make the same calculation that leads to (4.12) but with $\mathbf{x}$ in place of $\mathbf{w}_j^2(s)$ to get, with obvious notation,

$$\begin{aligned}
&\left|\mathcal{P}^1(s, \mathbf{x}) - \mathcal{P}^2(s, \mathbf{x})\right| \\
&\quad \leq K \left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}} \sum_{\mathbf{z} \in \mathbb{Z}^2} \int_{1/s}^{+\infty} \max(\mathcal{A}(\mathbf{z}), \mathcal{B}(\mathbf{z})) e^{-\frac{\sigma}{4} \min(\mathcal{A}(\mathbf{z}), \mathcal{B}(\mathbf{z}))^2} \, d\sigma
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}(\mathbf{z}) &= |\mathbf{x} + \mathbf{z}_1 - \mathbf{x}_1^1(s - 1/\sigma) + \mathbf{z}|, \\
\mathcal{B}(\mathbf{z}) &= |\mathbf{x} + \mathbf{z}_1 - \mathbf{x}_1^2(s - 1/\sigma) + \mathbf{z}|.
\end{aligned}$$

So, if σ^* is such that $s - 1/\sigma^* = r^*$ (which amounts to $\sigma^* = (s-r^*)^{-1}$),

$$\begin{aligned}\sum_{\mathbf{z}\in\mathbb{Z}^2}\int_{1/s}^{+\infty} &\max(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))e^{-\frac{\sigma}{4}\min(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))^2}\,d\sigma\\ &= \sum_{\mathbf{z}\in\mathbb{Z}^2}\left(\int_{1/s}^{\sigma^*}+\int_{\sigma^*}^{+\infty}\right)\max(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))e^{-\frac{\sigma}{4}\min(\mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}))^2}\,d\sigma \\ &= I + J.\end{aligned} \tag{4.13}$$

Let us first take care of the term I. We have

$$\big(|\mathbf{z}|_\infty - 1\big)^+ \le \mathcal{A}(\mathbf{z}),\mathcal{B}(\mathbf{z}) \le 1+|\mathbf{z}|, \tag{4.14}$$

where $(a)^+ = \max(0,a)$. It follows that

$$\begin{aligned} I &\le \sum_{\mathbf{z}\in\mathbb{Z}^2}\int_{1/s}^{\sigma^*}\big(1+|\mathbf{z}|\big)e^{-\frac{\sigma}{4}(|\mathbf{z}|_\infty-1)^2}\,d\sigma\\ &\le \sum_{|\mathbf{z}|_\infty\le 1}\int_{1/s}^{\sigma^*}(1+|\mathbf{z}|)\,d\sigma + \sum_{|\mathbf{z}|_\infty>1}\int_{1/s}^{\sigma^*}\big(1+|\mathbf{z}|\big)e^{-\frac{\sigma}{4}(|\mathbf{z}|_\infty-1)^2}\,d\sigma\\ &\le 9(1+\sqrt{2})(\sigma^*-1/s) + \sum_{|\mathbf{z}|_\infty>1}\frac{1+|\mathbf{z}|}{(|\mathbf{z}|_\infty-1)^2}\left[e^{-\frac{\sigma}{4}(|\mathbf{z}|_\infty-1)^2}\right]_{1/s}^{\sigma^*}\\ &\le 9(1+\sqrt{2})\sigma^* + \sum_{|\mathbf{z}|_\infty>1}(1+|\mathbf{z}|)e^{-\frac{\sigma^*}{4}(|\mathbf{z}|_\infty-1)^2}.\end{aligned}$$

Let us now detail how this estimate depends on s through the parameter σ^*. Since all the velocities are bounded by the constant $2C_2^0$, we have necessarily $s - r^* = 1/\sigma^* \ge \epsilon/(2C_2^0)$: the trajectories entered the ball $B(\mathbf{x}_1^{1,2}(s),\epsilon)$ earlier than at time $s-\epsilon/(2C_2^0)$. For a similar reason, we have also $s-r^* = 1/\sigma^* \le 2\epsilon/C_1^0$. Therefore, we get

$$I \le 9(1+\sqrt{2})\frac{2C_2^0}{\epsilon} + \sum_{|\mathbf{z}|_\infty>1}(1+|\mathbf{z}|)e^{-\frac{C_1^0}{8\epsilon}(|\mathbf{z}|_\infty-1)^2},$$

where still ϵ depends on s. However, we are focusing on the situation where $s \ge t^* > 0$ so that, going back to (2.23), we can find two constants $K_1, K_2 > 0$ such that $K_1/t^* \le \epsilon^{-1} \le K_2/t^*$ holds for any $s \ge t^*$. We are thus led to the uniform estimate $I \le \Lambda$, with a constant $\Lambda > 0$ depending only on the data of the problem through t^* (which, we recall, is a small but fixed time depending on the separation of the initial positions).

Now we turn to J in (4.13). For $\sigma \in (\sigma^*, +\infty)$ we have

$$d_{\mathbb{T}^2}(\mathbf{x}_1^1(s-1/\sigma),\mathbf{x}_1^1(s)), d_{\mathbb{T}^2}(\mathbf{x}_1^2(s-1/\sigma),\mathbf{x}_1^2(s)) \le \epsilon,$$

and so, since we are in Case 2, $\mathbf{x}$ does not lie on the trajectories $\mathbf{x}_1^1(s-1/\sigma)$ or $\mathbf{x}_1^2(s-1/\sigma)$. We can still use (4.14) and the estimate on σ^* discussed before, and we arrive at the uniform estimate $J \le \Lambda$. Gathering the estimates on I and J, we conclude that

$$\left|\mathcal{P}^1(s,\mathbf{x}) - \mathcal{P}^2(s,\mathbf{x})\right| \le \Lambda \left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}}, \tag{4.15}$$

as desired.

Case 3: $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^1(s)) < \epsilon$ (say) but $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^2(s)) > \epsilon$. Note that in Proposition 2.5-ii), any $\epsilon' < \epsilon$ will also fulfil the conclusion. Therefore, (eventually having to take a smaller ϵ), we can suppose that in addition to (4), it also holds that $\mathcal{P}_e^{1,2}(s,\mathbf{x}) \equiv c^*$ if $\mathbf{x} \in \cup_{k=1}^N B(\mathbf{x}_k^1(s), 2\epsilon)$, and this changes nothing in the proofs of Cases 1 and 2. Now, Case 3 follows by the following reasoning. If $d_{\mathbb{T}^2}(\mathbf{x}_1^1(s), \mathbf{x}_1^2(s)) \le \epsilon$, then necessarily $d_{\mathbb{T}^2}(\mathbf{x}, \mathbf{x}_1^2(s)) \le 2\epsilon$, and so $\mathcal{P}_e^2(s,\mathbf{x}) = c^*$, and $|\mathcal{P}_e^1(s,\mathbf{x}) - \mathcal{P}_e^2(s,\mathbf{x})| = 0$. If $d_{\mathbb{T}^2}(\mathbf{x}_1^1(s), \mathbf{x}_1^2(s)) > \epsilon$, then $\left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}} > \epsilon$ and then we can estimate brutally using (2.23)

$$\begin{aligned}|\mathcal{P}_e^1(s,\mathbf{x}) - \mathcal{P}_e^2(s,\mathbf{x})| &\le \frac{2c^*}{\epsilon}\left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}} \\ &\le \Lambda \left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}}.\end{aligned}$$

We can now go back to (4.10), which can be dominated by $\Lambda\left\|\mathbf{X}^1 - \mathbf{X}^2\right\|_{\mathscr{X}}$, and so the term C in (4.6) satisfies the same bound for all $s \in [0,t]$. Together with the estimates (4.8),(4.9), this gives (4.5).

Note that for extending the solution from $[0,t]$ to an arbitrary time interval, we make use of the estimates of Lemma 2.1 satisfied by the solution on $[0,t]$, and we repeat the same reasonings which have permitted to handle the interval $[t^*, t]$. This concludes the proof of Theorem 2.3.

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