Ann Univ Ferrara manuscript No.
(will be inserted by the editor)

Mean field limit for particles interacting with a vibrating medium

Thierry Goudon · Arthur Vavasseur

Received: date / Accepted: date

Abstract We are interested in models describing the motion of particles that exchange momentum and energy with their environment, represented as a vibrating field. As the number of particles goes to ∞, we derive a Vlasov–like equation for the particle distribution function, when adopting a mean–field rescaling of the particle system. We also investigate this question when, additionally, the particles are subjected to friction and Brownian motion.

Keywords Vlasov–like equations · Interacting particles · Inelastic Lorentz gas · Mean–field regime.

Mathematics Subject Classification (2000) 82C70, · 70F45, · 37K05, · 74A25.

1 Introduction

In [6], L. Bruneau and S. De Bièvre introduced a mathematical model describing the motion of a classical particle through a homogeneous dissipative medium. The particle, which can also be subjected to the effect of an external potential V, exchanges momentum and energy with the medium, which is thought of as a vibrating field. Denoting by m the mass of the particle and by $t \mapsto q(t)$ the position of the particle at time t, the equations of motion read

$$\begin{cases} m\ddot{q}(t) = -\nabla V(q(t)) - \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \nabla_x \sigma_1(q(t)-z)\ \sigma_2(y)\ \Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z, \\ \partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\sigma_1(x-q(t)), \qquad x\in\mathbb{R}^d,\ y\in\mathbb{R}^n. \end{cases} \tag{1}$$

Here, Ψ represents the state of the environment of the particle. It creates the potential

$$\Phi(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(x-z)\ \sigma_2(y)\ \Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z \tag{2}$$

which, in turn, influences the trajectory of the particle. The coupling is embodied in the form factor functions σ_1 and σ_2, which are both non negative, infinitely smooth and compactly supported functions. In this approach the environment can be thought of as a continuous set of membranes that vibrate with

T. Goudon
Université Côte d'Azur, Inria, CNRS, LJAD
Parc Valrose, F-06108 Nice, France
E-mail: thierry.goudon@inria.fr

A. Vavasseur
Université Côte d'Azur, Inria, CNRS, LJAD
Parc Valrose, F-06108 Nice, France
E-mail: arthur.vavasseur@unice.fr

wave speed $c > 0$ in directions ($y \in \mathbb{R}^n$) perpendicular to the particles motion ($q(t) \in \mathbb{R}^d$). The model (1) has a Hamiltonian structure and the following energy conservation holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\frac{m}{2}\Big|\frac{\mathrm{d}}{\mathrm{d}t}q(t)\Big|^2 + V(q(t)) + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\Psi(t,x,y)|^2 + c^2|\nabla_y\Psi(t,x,y)|^2\Big)\,\mathrm{d}y\,\mathrm{d}x \\ + \iint_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_2(y)\sigma_1(q(t)-x))\Psi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\Bigg\} = 0.\end{aligned}$$

In [6], the existence–uniqueness of solutions of (1) is established, together with a deep discussion on the asymptotic behavior of the system: roughly speaking, the interaction with the vibrating field acts as a friction force on the particle. We refer the reader to [1,9–11,22,17] for thorough investigation of the asymptotic properties of the model, based either on analytical treatments or on numerical evidence. The question is reminiscent of the analysis of the Lorentz gas [3,7,13,15,19]; here, hard scatterers are replaced by the soft interacting potential created by the vibrating environment.

The modeling can be readily adapted in order to consider a set of N particles, all of them interacting with the vibrating medium. We are thus led to the following system of differential equations, for $j \in \{1, ..., N\}$

$$\begin{cases} m\ddot{q}_j(t) = -\nabla V(q_j(t)) - \nabla\Phi(t, q_j(t)), \\ \partial^2_{tt}\Psi(t,x,y) - c^2\Delta_y\Psi(t,x,y) = -\sigma_2(y)\displaystyle\sum_{k=1}^{N}\sigma_1(x - q_k(t)), \end{cases} \tag{3}$$

where the self–consistent potential Φ is still defined by (2). The system is completed by the initial data

$$(q_j(0), \dot{q}_j(0)) = (q_{0,j}, p_{0,j}) \quad \text{for } j \in \{1, ..., N\}, \qquad (\Psi, \partial_t\Psi)\Big|_{t=0} = (\Psi_0, \Psi_1). \tag{4}$$

Note that the large time behavior for a system of $N > 1$ particles is likely to be much more intricate than for a single particle as analyzed in [6]. For further purposes, it is worth observing that energy conservation still holds for the N-particles system; it takes the following form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\frac{m}{2}\sum_{j=1}^{N}\Big|\frac{\mathrm{d}}{\mathrm{d}t}q_j(t)\Big|^2 + \sum_{j=1}^{N}V(q_j(t)) \\ + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^n}\Big(|\partial_t\Psi(t,x,y)|^2 + c^2|\nabla_y\Psi(t,x,y)|^2\Big)\,\mathrm{d}y\,\mathrm{d}x \\ + \sum_{j=1}^{N}\iint_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_2(y)\sigma_1(q_j(t)-x))\Psi(t,x,y)\,\mathrm{d}y\,\mathrm{d}x\Bigg\} = 0.\end{aligned} \tag{5}$$

In [8], we have revisited the model in the framework of kinetic equations. Instead of considering a particle or a set of particles described by the position–velocity pair $t \mapsto \big(q(t), \dot{q}(t)\big)$, we work with the particle distribution function in phase space $f(t,x,v) \geq 0$, with $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, the position and velocity variables respectively. We are thus led to the following PDE system

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f - \nabla_x(V + \Phi)\cdot\nabla_v f = 0, \\ &\big(\partial^2_{tt}\Psi - c^2\Delta_y\Psi\big)(t,x,y) = -\sigma_2(y)\int_{\mathbb{R}^d}\sigma_1(x-z)\rho(t,z)\,\mathrm{d}z, \\ &\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \end{aligned} \tag{6}$$

with (2) defining the interaction potential again. The system (6) is completed by initial data

$$f\Big|_{t=0} = f_0, \tag{7}$$

and

$$(\Psi, \partial_t \Psi)\Big|_{t=0} = (\Psi_0, \Psi_1). \tag{8}$$

We refer to this system as the *Vlasov–Wave system*, and we warn the reader that the wave equation for Ψ holds in a direction *transverse* to the space variable (in contrast to models inspired from the Vlasov–Maxwell system, as in [5]). Finally, it can be relevant to incorporate some dissipation effects in the Vlasov equation, as in [2], namely the kinetic equation for the particle distribution function becomes

$$\partial_t f + v \cdot \nabla_x f - \nabla_x (V + \Phi) \cdot \nabla_v f = \gamma \nabla_v \cdot (vf + \nabla_v f), \qquad \gamma > 0, \tag{9}$$

which involves the Fokker–Planck operator $\nabla_v \cdot (vf + \nabla_v f)$. We refer to this model as the *Vlasov–Wave–Fokker–Planck system.* The Fokker–Planck operator induces relaxation effects: equilibrium states can be identified (which, by the way, are also stationary solutions of the Vlasov–Wave system, see [2, Appendix A]) and the asymptotic trend to equilibrium can be established.

In this paper we wish to provide a rigorous derivation of the Vlasov–Wave system (6) from the equations of motion of N particles, as in (3), when the number of particles becomes large. We also propose a similar discussion to obtain (9): to this end, we need to modify the deterministic model (3) by introducing a drag force and particles' Brownian motion, which gives a stochastic nature to the particle model. In order to deal with the asymptotic regime of a large number of particles, the self–consistent potential has to be appropriately rescaled ("weak coupling scaling"). In Section 2 we briefly discuss the scaling issues and we present the interpretation of the asymptotic problem. In particular, it is relevant to consider both the empirical measure associated to the particle system and the joint probability measure of the system of N particles in the $N-$body phase space [4,14]. In Section 3, we set up a few technical tools that will be necessary for the asymptotic analysis. In Section 4 we investigate the mean field limit $N \to \infty$ and the derivation of the Vlasov–Wave system (6). In Section 5 we obtain the Vlasov–Wave–Fokker–Planck model (9). In both cases the analysis relies on fine estimates on the particle trajectories. For the former case, it allows us to establish directly the convergence of the empirical measure associated to the $N-$particle system to a solution of the kinetic equation, in the spirit of Dobrushin's work [12]. For the latter case, the analysis is more intricate due to the randomness induced by the Brownian motion. Thus, we study the behavior of the marginal of the $N-$particle distribution, following the arguments introduced by Sznitman [23]. The difficulty common to both situation is the result of the fact that the definition of the self–consistent potential (2) does not involve a smooth convolution with respect to the space variable only, but it is also non–local with respect to time.

2 Mean Field Regime and Weak Coupling Scaling

The derivation of the Vlasov–Wave system (6) from the model (3) for a set of $N \gg 1$ particles requires a certain rescaling of the interaction potential. In order to clarify the motivation of the rescaling, we start by rewriting the equations in dimensionless form. To this end, let us denote by $\mathcal{U}$ the typical value of the external potential. The dimension of $\mathcal{U}$ is mass $\times \left(\frac{\text{length}}{\text{time}}\right)^2$. With L and T the length and time units respectively, we thus set

$$V(q) = \mathcal{U}\ V'(q/\mathrm{L}),$$

where V' is a dimensionless quantity. Accordingly, we also set

$$q_j(t) = \mathrm{L} q_j'(t/\mathrm{T}), \qquad p_j(t) = \frac{\mathrm{L}}{\mathrm{T}} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} q_j'\right)(t/\mathrm{T}).$$

We shall use the notation $p_j'(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} q_j'(\tau)$, so that $\ddot{q}_j(t) = \frac{\mathrm{L}}{\mathrm{T}^2} \left(\frac{\mathrm{d}}{\mathrm{d}\tau} p_j'\right)(t/\mathrm{T})$. The self–consistent potential Φ scales like $\lambda\mathcal{U}$: $\mathcal{U}$ defines the units, while the strength of the coupling is embodied into the dimensionless parameter $\lambda > 0$. To be more specific, we denote by ℓ the unit of the variable $y \in \mathbb{R}^n$ (which is not necessarily a length) and we set

$$\Psi(t, x, y) = \bar{\psi}\ \tilde{\Psi}(t/\mathrm{T}, x/\mathrm{L}, y/\ell).$$

The coupling is defined by the product

$$\sigma_1(x)\sigma_2(y) = \bar{\sigma}\ \sigma_1'(x/\mathrm{L})\sigma_2'(y/\ell)$$

where we encapsulate the unit in the single parameter $\bar\sigma > 0$. Hence, taking into account the integration over $\mathbb{R}^d \times \mathbb{R}^n$ that defines Φ, we have

$$\lambda\mathcal{U} = \bar\sigma\bar\psi L^d\ell^n.$$

We find the dimension of $\bar\psi$ by comparing the terms in the energy balance: $m\frac{\dot q_j^2}{2}$ scales like $m\frac{\mathrm{L}^2}{\mathrm{T}^2}$, $V(q_j)$ scales like $\mathcal{U}$, and the quantities that involve the vibrating field, $|\partial_t\Psi|^2\,\mathrm{d}x\,\mathrm{d}y$ and $c^2|\nabla_y\Psi|^2\,\mathrm{d}x\,\mathrm{d}y$, should have the same dimension. Hence, we can set

$$\kappa\mathcal{U} = \bar\psi^2\frac{\mathrm{L}^d\ell^n}{\mathrm{T}^2},$$

with $\kappa > 0$ dimensionless.

In order to investigate the mean field regime, we assume that *the total mass is fixed:*

$$\sum_{j=1}^N m = Nm = \bar m \in (0,\infty)$$

does not depend on N. In other words, we have

$$m = \bar m/N.$$

Now, we compare the weights of the contributions to the energy balance (5):

$$\begin{aligned}
&\text{particles' kinetic energy} && mN\frac{\mathrm{L}^2}{\mathrm{T}^2} = \bar m\frac{\mathrm{L}^2}{\mathrm{T}^2},\\
&\text{particles' potential energy} && N\mathcal{U},\\
&\text{self consistent energy} && \lambda N\mathcal{U},\\
&\text{vibrational energy} && \frac{1}{2}\kappa\mathcal{U}\Big(1+\frac{c^2\mathrm{T}^2}{\ell^2}\Big).
\end{aligned}$$

Imposing all terms to be of order 1 with respect to $\mathcal{E} = \bar m\frac{\mathrm{L}^2}{\mathrm{T}^2}$ leads to the following relations

$$\mathcal{U} = \frac{1}{N}\times\mathcal{E}, \qquad \lambda = 1, \qquad \kappa = N, \qquad \frac{c^2\mathrm{T}^2}{\ell^2} = 1.$$

Having disposed of these observations, we can rewrite (3) in the following dimensionless form, where, for the sake of clarity, we ditch the prime symbol,

$$\begin{aligned}
&\dot q_j(t) = p_j(t),\\
&\dot p_j = -\nabla V(q_j(t)) - \nabla\Phi(t,q_j(t))\\
&\Phi(t,x) = \frac{1}{N}\int_{\mathbb{R}^d\times\mathbb{R}^n}\sigma_1(x-z)\sigma_2(y)\tilde\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}x,\\
&\partial_{tt}^2\tilde\Psi(t,x,y) - \Delta_y\tilde\Psi(t,x,y) = -\Lambda\sigma_2(y)\sum_{k=1}^N\sigma_1(x-q_k(t)),
\end{aligned}$$

where the coefficient $\Lambda > 0$ is given by

$$\Lambda = N\times\frac{\bar\sigma}{\bar\psi}\mathrm{T}^2 = N\times\bar\sigma\bar\psi L^d\ell^n\times\frac{\mathrm{T}^2}{\bar\psi^2\mathrm{L}^d\ell^n} = N\times\frac{\lambda\mathcal{U}}{\kappa\mathcal{U}} = 1.$$

In fact, we rescale the field (in dimensionless variables) as follows

$$\frac{1}{N}\,\tilde\Psi(t,x,y) = \Psi(t,x,y).$$

Beyond the notational convenience, this actually contains the assumption that the rescaled initial data (Ψ_0, Ψ_1) for the field are of order $\mathscr{O}(1)$. With this notation we arrive at the following system

$$\begin{aligned}
&\dot{q}_j(t) = p_j(t),\\
&\dot{p}_j = -\nabla V(q_j(t)) - \nabla \Phi(t, q_j(t))\\
&\Phi(t,x) = \int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(x-z)\sigma_2(y)\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}x,\\
&\partial^2_{tt}\Psi(t,x,y) - \Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \frac{1}{N}\sum_{k=1}^N \sigma_1(x-q_k(t)).
\end{aligned} \tag{10}$$

We warn the reader not to be confused by the fact that we are actually using the same notation for both (3) and the rescaled problem (10), bearing un mind that the asymptotic limit $N \to \infty$ will be considered for (10).

For obtaining the model with the Fokker–Planck operator, we add to the model (10) a friction force, namely a force proportional to the particles velocity, and a Brownian motion. We will thus deal with the following analog to (10), for $j \in \{1, ..., N\}$

$$\begin{cases}
\mathrm{d}q_j = p_j\,\mathrm{d}t,\\
\mathrm{d}p_j(t) = -\nabla V(q_j(t))\,\mathrm{d}t - \ \mathrm{d}t\displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^n} \sigma_1(q_j(t)-z)\ \sigma_2(y)\ \nabla_x\Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z\\
\qquad\qquad -\gamma p_j(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B_j(t),\\
\partial^2_{tt}\Psi(t,x,y) - \Delta_y\Psi(t,x,y) = -\sigma_2(y)\ \displaystyle\frac{1}{N}\sum_{k=1}^N \sigma_1(x-q_k(t)),
\end{cases} \tag{11}$$

with B_j a Brownian motion on $\mathbb{R}^d$.

We wish to investigate the regime $N \to \infty$ from (10) and (11), completed by the initial condition

$$(q_j(0), p_j(0)) = (q_{0,j}, p_{0,j}) \tag{12}$$

and

$$(\Psi, \partial_t\Psi)\Big|_{t=0} = (\Psi_0, \Psi_1). \tag{13}$$

We shall establish this way a connection with the kinetic models (6) and (9), respectively. From the rescaled systems (10) and (11), we can define two relevant quantities.

- The empirical measure of the $N-$particle system is simply defined by

$$\widehat{\mu}^N(t) = \frac{1}{N}\sum_{k=1}^N \delta_{(q_k(t), p_k(t))}. \tag{14}$$

  A direct computation shows that $f = \widehat{\mu}^N(t)$ actually satisfies (6), see Lemma 4. Assuming the convergence of the initial state $\widehat{\mu}^N_0 \to f_0(x,v)$ in some suitable sense, the question we address is thus nothing but a stability property of the system (6) for measure valued solutions.
- Considering the initial data $\big(q_1(0), p_1(0), ..., q_N(0), p_N(0)\big)$ as independent random variables distributed according to the same probability measure f_0 on $\mathbb{R}^d \times \mathbb{R}^d$, we can also deal with the joint probability measure μ^N, which is a probability measure on the $N-$body phase space $(\mathbb{R}^d\times\mathbb{R}^d)^N$.

For investigating the connection between the N-particle system (10) and the Vlasov–Wave system, it is enough to deal with the empirical measure. However, for (11) the trajectory of a particle is by nature a random variable, due to the Brownian motion. Thus, even if the initial data is purely deterministic, the (q_j, p_j)'s are random variables and the empirical measure $\widehat{\mu}^N$ becomes a random variable too. For this problem, the analysis is performed by dealing with the $N-$particle measure μ^N instead, or more precisely with its first marginal. Further comments and statements on these notions can be found in the surveys [4, 14].

3 Technical preliminaries

3.1 Main assumptions

Let us collect here the assumptions on the parameters of the model (coupling form functions, external potential), and on the initial data. Throughout the paper, we impose

$$\begin{cases} \sigma_1 \in C_c^\infty(\mathbb{R}^d, \mathbb{R}), \ \sigma_2 \in C_c^\infty(\mathbb{R}^n, \mathbb{R}), \\ \sigma_1(x) \geq 0, \ \sigma_2(y) \geq 0 \text{ for any } x \in \mathbb{R}^d, \ y \in \mathbb{R}^n, \\ \sigma_1, \sigma_2 \text{ are radially symmetric.} \end{cases} \tag{\textbf{H1}}$$

$$\Psi_0, \Psi_1 \in L^2(\mathbb{R}^d \times \mathbb{R}^n). \tag{\textbf{H2}}$$

$$f_0 \geq 0, \qquad f_0 \in \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d). \tag{\textbf{H3}}$$

$$\begin{cases} V \in W^{2,\infty}_{\text{loc}}(\mathbb{R}^d), \\ \text{and there exists } C \geq 0 \text{ such that } V(x) \geq -C(1+|x|^2) \text{ for any } x \in \mathbb{R}^d. \end{cases} \tag{\textbf{H4}}$$

We shall need another technical assumption on the external potential, that will be detailed in Section 3.4. In [8], the existence–uniqueness of solutions to (6)–(8) is established under this set of assumptions, with (**H3**) strengthened into $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. The extension to the framework of measure–valued solutions that we present here unifies the N-particles viewpoint and the PDE viewpoint.

3.2 An overview on the Kantorowich–Rubinstein distance

The use of the dual space $\big(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\big)'$ appeared naturally in the analysis of [8]; this functional framework turns out to be well–adapted to establish a well–posedness theory for the model (6). This is strongly related to the Kantorowich–Rubinstein distance, which can be used to make the space of the probability measures a metric space. We refer the reader to [12,24] for a detailed introduction to this notion.

Definition 1 (Kantorowich–Rubinstein distance) Let (S, d) be a separable metric space. Let μ, ν two probability measures on S. The Kantorowich–Rubinstein distance between μ and ν is defined by

$$W_1(\mu, \nu) = \inf_\pi \Big\{ \int_{S^2} d(x,y) \, \mathrm{d}\pi(x,y) \Big\} = \inf_{X,Y} \mathbb{E}\left[d(X,Y)\right]$$

where the infimum is taken in the first equality over measures π having μ and ν as marginals and in the second equality over all the random variables X and Y having the probability μ and ν, respectively.

The definition of W_1 is meaningful on the whole space of probability measures on S, when the distance d is bounded on $S \times S$. When $S = \mathbb{R}^d \times \mathbb{R}^d$, we can take $d(x,y) = |x-y| \wedge 1$, where $a \wedge b = \min(a,b)$. It is well known (see [12] or [24, Chapter 6]) that, when d is bounded on $S \times S$, W_1 metrizes the tight convergence in $\mathcal{M}^1(S)$ (the weak convergence of measures seen as linear forms on the space of the continuous and bounded real valued functions on S). This result will be used to prove the compactness of the sequence of the empirical measures $(\widehat{\mu}^N)_{N \in \mathbb{N}}$ in the forthcoming Sections.

Another interpretation of W_1 is given by the following Kantorowich–Rubinstein duality formula, which makes the connection with the dual of $W^{1,\infty}$ appear [24, Theorem 5.10, Chapter 5 and, even more precisely, Remark 6.5, Chapter 6]

$$W_1(\mu, \nu) = \sup_{\|f\|_{\text{Lip}} \leq 1} \left| \int_S f \, \mathrm{d}(\mu - \nu) \right|, \qquad \|f\|_{\text{Lip}} = \sup_{x,y \in S} \frac{|f(x) - f(y)|}{d(x,y)}.$$

In the specific case $S = \mathbb{R}^d \times \mathbb{R}^d$ and $d(x,y) = |x-y| \wedge 1$, the Kantorowich–Rubinstein formula becomes

$$\inf_\pi \Big\{ \int_{(\mathbb{R}^d \times \mathbb{R}^d)^2} (|x-y| \wedge 1) \, \mathrm{d}\pi(x,y) \Big\} = \sup_{2\|f\|_\infty, \|\nabla f\|_\infty \leq 1} \left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f \, \mathrm{d}(\mu - \nu) \right| \tag{15}$$

(the infimum in the left hand side is taken over measures π having μ and ν as marginals). As a matter of fact, the distances associated to W_1 and to the $\big(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\big)'$ norm are equivalent

$$\frac{1}{2}\|\mu-\nu\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}\leq W_1(\mu,\nu)\leq 2\|\mu-\nu\|_{\left(W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)\right)'}.$$

We also point out that the Kantorowich–Rubinstein distance can be extended into a norm on the space of bounded mesures, as a consequence of the duality formula; we refer the reader to [24, Bibliographical Notes to Chapter 6] for further information on this point.

As mentioned above, the distance W_1 and relation (15) will play a crucial role in the analysis; in order to simplify the computations, from now on, we slightly modify the definition of the norm on $W^{1,\infty}(\mathbb{R}^d\times\mathbb{R}^d)$ and we set

$$\|f\|_{\mathrm{Lip}}=2\|f\|_\infty\wedge\|\nabla f\|_\infty.$$

In what follows we will deal with measures parametrized by the time variable $\mu : t\in[0,T]\longmapsto\mu_t\in\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)$. We will say that μ lies in $C([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)-\text{tight})$, when, for any continuous and bounded function $\chi:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$, we have

$$\left(t\mapsto\int\chi(x,v)\,\mathrm{d}\mu_t(x,v)\right)\in C([0,T]).$$

The natural distance induced by W_1 between two measures valued functions $\mu,\nu\in C([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)-\text{tight})$ is then given by

$$\|W_1(\mu,\nu)\|_{L^\infty(0,T)}=\sup_{0\leq t\leq T}W_1(\mu_t,\nu_t).$$

This distance makes $C([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)-\text{tight})$ a Banach space and we shall use the shorthand notation $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. For instance, saying that a sequence $\big(\mu_N\big)_{N\in\mathbb{N}}$ converges to μ in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$ means

$$\lim_{N\to\infty}\Big(\sup_{0\leq t\leq T}W_1(\mu_N(t),\mu(t))\Big)=0,$$

and, equivalently, for any $\chi\in L^\infty\cap C(\mathbb{R}^d\times\mathbb{R}^d)$

$$\lim_{N\to\infty}\int\chi(x,v)\,\mathrm{d}\mu_N(t,x,v)=\int\chi(x,v)\,\mathrm{d}\mu(t,x,v)$$

holds uniformly over $t\in[0,T]$.

3.3 Expression of the self–consistent potential

As already remarked in [8], it is convenient to rewrite the interaction potential Φ as an integral operator acting on the macroscopic density $\rho(t,x)=\int_{\mathbb{R}^d}f(t,x,v)\,\mathrm{d}v$. To this end, let us set

$$t\longmapsto p(t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{\sin(c|\xi|t)}{c|\xi|}\left|\widehat{\sigma_2}(\xi)\right|^2\,\mathrm{d}\xi,$$

where $\widehat{\cdot}$ stands for the Fourier transform with respect to the variable $y\in\mathbb{R}^n$. We also set

$$\Phi_0(t,x)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^d}\sigma_1(x-z)\left(\widehat{\Psi_0}(z,\xi)\cos(c|\xi|t)+\widehat{\Psi_1}(z,\xi)\frac{\sin(c|\xi|t)}{c|\xi|}\right)\widehat{\sigma_2}(\xi)\,\mathrm{d}z\,\mathrm{d}\xi$$

which is clearly associated to the solution of the homogeneous wave equation with initial conditions (Ψ_0, Ψ_1). Finally, we define the operator $\mathcal{L}$ which associates to a measure valued function $f : (0, \infty) \to \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$ the quantity

$$\begin{aligned} \mathcal{L}(f)(t,x) &= \int_0^t p(t-s) \left(\int_{\mathbb{R}^d} \Sigma(x-z)\rho(s,z)\,\mathrm{d}z \right) \mathrm{d}s, \\ \rho(t,x) &= \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \qquad \Sigma = \sigma_1 \underset{x}{*} \sigma_1. \end{aligned} \tag{16}$$

Note that the regularity of the form functions σ_1, σ_2 imply that $\mathcal{L}(f)$ is a well defined smooth function, while $f(t,\cdot)$ is only measure valued. We refer the reader to [8, Section 2] for detailed proofs of the following statements. In fact [8] assumes that $f(t,\cdot)$ is an integrable function, but the results clearly apply to the measure framework as well.

Lemma 1 *Assume* (**H1**)–(**H2**). *Let f in $C_{W_1}\big(\mathbb{R}_+; \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)\big)$. Then, the self consistent potential Φ defined by* (2) *with Ψ solution of the wave equation*

$$\partial^2_{tt}\Psi - c^2 \Delta_y \Psi = -\sigma_2(y) \int_{\mathbb{R}^d \times \mathbb{R}^d} \sigma_1(x-z) f(t,z,v)\,\mathrm{d}v\,\mathrm{d}z$$

can be recast as $\Phi = \Phi_0 - \mathcal{L}(f)$.

Lemma 2 (Estimates on the self-consistent potential) *Let $0 < T < \infty$. The following properties hold:*

i) $\mathcal{L}$ belongs to the space $\mathcal{A}_T$ of the continuous operators on $C\big([0,T]; \big(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\big)'\big)$ with values in $C\big([0,T]; W^{2,\infty}(\mathbb{R}^d)\big)$. Its norm is evaluated as follows

$$|||\mathcal{L}|||_{\mathcal{A}_T} \le \|\sigma_1\|^2_{W^{3,2}(\mathbb{R}^d)} \|\sigma_2\|^2_{L^2(\mathbb{R}^n)} \, \frac{T^2}{2};$$

ii) $\mathcal{L}$ belongs to the space $\mathcal{B}_T$ of the continuous operators on $C\big([0,T]; \big(W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)\big)'\big)$ with values in $C^1\big([0,T]; L^\infty(\mathbb{R}^d)\big)$. Its norm is evaluated as follows

$$|||\mathcal{L}|||_{\mathcal{B}_T} \le \|\sigma_1\|^2_{W^{1,2}(\mathbb{R}^d)} \|\sigma_2\|^2_{L^2(\mathbb{R}^n)} \, \Big(T + \frac{T^2}{2}\Big);$$

iii) Φ_0 satisfies

$$\|\Phi_0(t,\cdot)\|_{W^{2,\infty}(\mathbb{R}^d)} \le \|\sigma_1\|_{W^{2,2}(\mathbb{R}^d)} \|\sigma_2\|_{L^2(\mathbb{R}^n)} \left(\|\Psi_0\|_{L^2(\mathbb{R}^n)} + t\|\Psi_1\|_{L^2(\mathbb{R}^n)} \right),$$

for any $0 \le t \le T$, and, moreover

$$\|\Phi_0\|_{C^1([0,T];L^\infty(\mathbb{R}^d))} \le \|\sigma_1\|_{L^2(\mathbb{R}^d)} \|\sigma_2\|_{W^{1,2}(\mathbb{R}^n)} \left(2\|\Psi_0\|_{L^2(\mathbb{R}^n)} + (1+T)\|\Psi_1\|_{L^2(\mathbb{R}^n)} \right).$$

3.4 Estimates on the characteristic curves

For an external potential V that satisfies (**H4**) and for a given function Φ which lies in $C^0([0,\infty); W^{2,\infty}(\mathbb{R}^d)) \cap C^1([0,\infty); L^\infty(\mathbb{R}^d))$, we can define the characteristic curves, solutions of the ODE system

$$\begin{cases} \dot X(t) = \xi(t), & \dot\xi(t) = -\nabla V(X(t)) - \nabla\Phi(t, X(t)), \\ X(\alpha) = x_0, & \xi(\alpha) = v_0. \end{cases} \tag{17}$$

We denote by $\varphi_\alpha^{\Phi,\beta}(x_0, v_0)$ the solution $t \mapsto (X(t), \xi(t))$ of (17) at time β; it can be interpreted as the position–velocity pair at time β of a particle subjected to the force field$-\nabla(V + \Phi)$, with state (x_0, v_0) at time α. The analysis relies crucially on the properties of the solutions of the differential system (17).

Lemma 3 *Let V satisfy* (**H4**) *and let $\Phi \in C^0([0,\infty); W^{2,\infty}(\mathbb{R}^d)) \cap C^1([0,\infty); L^\infty(\mathbb{R}^d))$.*

a) There exists a function

$$R : (\mathscr{N}, t, x, v) \in [0, \infty) \times [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \longmapsto R(\mathscr{N}, t, x, v) \in [0, \infty),$$

non decreasing with respect to the first two variables, such that the solution $t \mapsto (X(t), \xi(t))$ *of* (17) *satisfies the following estimate, for any* $t \in \mathbb{R}$,

$$(X(t), \xi(t)) \in B_t(x_0, v_0),$$
$$B_t(x_0, v_0) = B\left(0, R\left(\|\Phi\|_{C^1([0,t];L^\infty(\mathbb{R}^d))}, |t|, x_0, v_0\right)\right) \subset \mathbb{R}^d \times \mathbb{R}^d.$$

b) Moreover, we have

$$|\nabla_{x,v}\varphi_0^{\Phi,t}(x, v)| \le \exp\left(\int_0^t (1 + \|\nabla^2(V + \Phi(s))\|_{L^\infty(B_s(x,v))}\, \mathrm{d}s\right).$$

c) Taking two additional potential Φ_1 *and* Φ_2*, the following estimate holds for any* $t > 0$

$$|(\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t})(x_0, v_0)|$$
$$\le \int_0^t \|(\Phi_1 - \Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)} \exp\left(\int_s^t \|\nabla^2(\Phi_1(\tau) + V)\|_{L^\infty(\tilde{B}_\tau(x_0,v_0))}\, \mathrm{d}\tau\right) \mathrm{d}s,$$

where we have set

$$\tilde{B}_\tau(x, v) = B\left(0, R\left(\max_{i=1,2} \|\Phi_i\|_{C^1([0,\tau];L^\infty(\mathbb{R}^d))}, \tau, x, v\right)\right).$$

Proof of Lemma 3. We refer the reader to [8, Section 3] for items a) and c); we prove b) here. Let (X_1, ξ_1) and (X_2, ξ_2) stand for the solutions (17) with initial data (x_1, v_1) and (x_2, v_2), respectively, at $\alpha = 0$. We have

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(X_1 - X_2)(t) = (\xi_1 - \xi_2)(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\xi_1 - \xi_2)(t) = -\nabla V(X_1(t)) + \nabla V(X_2(t)) - \nabla\Phi(t, X_1(t)) + \nabla\Phi(t, X_2(t)). \end{cases}$$

Let us set

$$K_t = \bigcup_{i=1,2} B\left(0, R\left(\|\Phi\|_{C^1([0,t];L^\infty(\mathbb{R}^d))}, |t|, x_i, v_i\right)\right)$$

Using a), we obtain (at least in the sense of distributions)

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}|X_1 - X_2| \le |\xi_1 - \xi_2|, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}|\xi_1 - \xi_2| \le \|\nabla^2(V + \Phi(t))\|_{L^\infty(K_t)}|X_1(t) - X_2(t)|. \end{cases}$$

The Grönwall Lemma yields

$$|(X_1, \xi_1) - (X_2, \xi_2)| \le |(x_1, v_1) - (x_2, v_2)| \exp\left(\int_0^t \|\nabla^2(V + \Phi(t))\|_{L^\infty(K_s)}\, \mathrm{d}s\right).$$

By definition, we have $(X_i, \xi_i) = \varphi_0^{\Phi,t}(x_i, v_i)$ for $i = 1, 2$. Letting (x_2, v_2) converge to (x_1, v_1), we get

$$|\nabla\varphi_0^{\Phi_1,t}(x_1, v_1)| \le \exp\left(\int_0^t \left(1 + \|\nabla^2(V + \Phi(s))\|_{L^\infty(B_s(x_1,v_1))}\right) \mathrm{d}s\right),$$

by using the continuity of R. ■

We can now introduce an additional technical requirement on the external potential. Given Ψ_0 and Ψ_1 satisfying (**H2**), we set

$$r(t, x, v) = R(\|\Phi_0\|_{C^1([0,t];L^\infty(\mathbb{R}^d))} + \|\mathcal{L}\|_{\mathcal{B}_t}, t, x, v), \tag{18}$$

where R is the function defined in Lemma 3-a) and the quantity $\|\Phi_0\|_{C^1([0,t];L^\infty(\mathbb{R}^d))}$ is well defined by virtue of Lemma 2. Then, we assume that

$$\mathscr{K}_T(\mu_0) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \exp\left(\int_0^T \|\nabla^2 V\|_{L^\infty(B(0,r(t,x,v)))}\,\mathrm{d}t\right)\mathrm{d}\mu_0 < \infty. \tag{\textbf{H5}}$$

We refer the reader to [8] for further comments on this assumption. In particular, due to the regularity of V in (**H4**), we notice that any finite and compactly supported measure satisfies this assumption.

Assumptions (**H1**)–(**H5**) are supposed to be fulfilled throughout the paper.

4 Mean field Limit for the Vlasov–Wave system

4.1 Particle viewpoint vs. kinetic viewpoint

According to Lemma 1, it is equivalent to consider a solution (f,Ψ) to the system (6)-(8) and a solution f of

$$\begin{cases} \partial_t f + v\cdot\nabla_x f = \nabla_x\left(V+\Phi_0-\mathcal{L}(f)\right)\cdot\nabla_v f, \\ f(0,x,v)=f_0(x,v), \end{cases} \tag{19}$$

with $\mathcal{L}$ defined by (16). It allows us to establish that the empirical measure $\widehat{\mu}^N$ associated to (10) satisfies (6), with $c=1$ (since (10) is set in dimensionless form).

Lemma 4 *The following properties are satisfied:*

i) If Ψ and $(q_j,p_j)_{j\in\{1,\dots,N\}}$ is solution of (10) *with* (12)–(13) *then $\widehat{\mu}^N$ is solution of* (19) *with initial data $f_0=\frac{1}{N}\sum_{j=1}^N \delta_{(q_{0,j},p_{0,j})}$.*
ii) Moreover, if μ is a solution of (19) *with initial data $f_0=\frac{1}{N}\sum_{j=1}^N\delta_{(q_{0,j},p_{0,j})}$, then for all $t\geq 0$, we can find Ψ and $(q_j,p_j)_{j\in\{1,\dots,N\}}$ solution of* (10) *with* (12)–(13)*, such that μ is given by* (14).

Proof. We split the proof into two steps.
Step 1: Proof of i)
Let Ψ and $(q_j,p_j)_{j\in\{1,\dots,N\}}$ be a solution of (10) with (12)–(13). We associate to this solution the empirical measure $\widehat{\mu}^N$ given by (14). Let $\chi\in C\cap L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. We have

$$\left\langle \widehat{\mu}_t^N|\chi\right\rangle = \frac{1}{N}\sum_{j=1}^N \chi(q_j(t),p_j(t)).$$

As a matter of fact, we observe that, on the one hand,

$$\left\langle \widehat{\mu}_t^N|\chi\right\rangle\Big|_{t=0} = \frac{1}{N}\sum_{j=1}^N \chi(q_{0,j},p_{0j}),$$

and, on the other hand, the self–consistent potential can be cast as

$$\begin{aligned} &\Phi(t,x) = \int_{\mathbb{R}^d}\sigma_1(x-z)\sigma_2(y)\ \Psi(t,z,y)\,\mathrm{d}y\,\mathrm{d}z, \\ &\partial_{tt}^2\Psi - c^2\Delta_y\Psi = -\sigma_2(y)\ \left\langle\widehat{\mu}_t^N|\sigma_1(x)\otimes\mathbf{1}(v)\right\rangle. \end{aligned}$$

Now, let $\psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d)$ and compute the distribution bracket

$$\begin{aligned}
&\left\langle\!\left\langle \big(\partial_t + v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v\big)\widehat{\mu}_t^N \,\middle|\, \psi \right\rangle\!\right\rangle \\
&= -\left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} - \int_0^\infty \left\langle \widehat{\mu}_t^N \middle| \big(\partial_t + v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v\big)\psi(t,\cdot)\right\rangle \mathrm{d}t \\
&= -\left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} - \frac{1}{N}\sum_{j=1}^N \int_0^\infty \Big((\partial_t\psi)(t,q_j(t),p_j(t)) \\
&\qquad + p_j(t)\cdot\nabla_x\psi(t,q_j(t),p_j(t)) - \nabla(V+\Phi)(t,q_j(t))\cdot\nabla_v\psi(t,q_j(t),p_j(t))\Big)\,\mathrm{d}t \\
&= -\left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} - \frac{1}{N}\sum_{j=1}^N \int_0^\infty \Big((\partial_t\psi)(t,q_j(t),p_j(t)) \\
&\qquad + \dot{q}_j(t)\cdot\nabla_x\psi(t,q_j(t),p_j(t)) + \dot{p}_j(t)\cdot\nabla_v\psi(t,q_j(t),p_j(t))\Big)\,\mathrm{d}t \\
&= -\left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} - \frac{1}{N}\sum_{j=1}^N \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big[\psi\big(t,q_j(t),p_j(t)\big)\Big]\,\mathrm{d}t \\
&= -\left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} + \frac{1}{N}\sum_{j=1}^N \psi(0,q_{0,j},p_{0,j}) \\
&= -\left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} + \left\langle \widehat{\mu}_t^N \middle| \psi(t,\cdot)\right\rangle\Big|_{t=0} = 0.
\end{aligned}$$

It follows that $\widehat{\mu}^N$ is a weak solution of (6).

Step 2: Proof of ii)

Let Ψ and μ be a solution of (6) (with $c=1$) with initial data (Ψ_0,Ψ_1) and $\mu_0 = \frac{1}{N}\sum_{j=1}^N \delta_{(q_{0,j},p_{0,j})}$, respectively. Equivalently, μ satisfies (19). Then, given χ_0 in $C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, and $T\geq 0$, we define $\chi:[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ as to be the solution of the following Liouville equation

$$\begin{cases} \partial_t\chi + v\cdot\nabla_x\chi - \nabla_x(V+\Phi)\cdot\nabla_v\chi = 0, \\ \chi(T,x,v) = \chi_0. \end{cases}$$

Here, the potential Φ is given by $\Phi = \Phi_0 - \mathcal{L}(\mu)$. By virtue of Lemma 2, it is a smooth function and the solution χ can be obtained by integrating along characteristics, see (17). Namely, for any $(x,v)\in\mathbb{R}^d\times\mathbb{R}^d$, $t\mapsto \chi(t,\varphi_0^{\Phi,t}(x,v))$ does not depend on the time variable $t\in[0,T]$ and we have

$$\chi(t,x,v) = \chi_0\circ\varphi_t^{\Phi,T}(x,v).$$

Next, we observe that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu(t)|\chi(t)\rangle \\
&\quad = +\langle \mu(t)|(v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v)\chi(t)\rangle - \langle \mu(t)|(v\cdot\nabla_x - \nabla_x(V+\Phi)\cdot\nabla_v)\chi(t)\rangle \\
&\quad = 0.
\end{aligned}$$

Integrating this relation over $[0,T]$, we obtain

$$\langle \mu(T),\chi_0\rangle = \langle \mu_0,\chi(0,\cdot)\rangle = \frac{1}{N}\sum_{j=1}^N \chi_0(\varphi_0^{\Phi,T}(q_{0,j},p_{0,j})).$$

Therefore, since the final time $T\geq 0$ and the trial function χ_0 are arbitrary, we conclude that $\mu(t)$ is given by

$$\mu(t) = \frac{1}{N}\sum_{j=1}^N \delta_{(q_j(t),p_j(t))}, \qquad \text{with } \big(q_j(t),p_j(t)\big) = \varphi_0^{\Phi,t}(q_{0,j},p_{0,j}).$$

By definition of φ^Φ, and since $\Phi = \Phi_0 - \mathcal{L}(\mu)$, we check that $(q_j,p_j)_{j\in\{1,\dots,N\}}$ satisfy (10). ■

It is therefore equivalent to prove the existence–uniqueness of a solution of (6)-(8), and to prove the existence–uniqueness of a solution of (19) with the initial data $\widehat{\mu}_0^N = \frac{1}{N}\sum_{j=1}^N \delta_{(q_{0,j},p_{0,j})}$. We shall adopt the PDE viewpoint, so that we can conclude by adapting the reasoning in [8].

4.2 Existence theory for the Vlasov–Wave system

This Section is devoted to the proof of the following statement, which extends to the framework of measure–valued solutions the analysis of [8]. In particular it justifies the existence of solutions for (10).

Theorem 1 *Assume* (**H1**)-(**H5**). *Let* $0 < T < \infty$. *Then, there exists a unique* $\mu \in C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$ *solution of* (19) *on* $[0,T]$ *such that* $\mu(0) = f_0$.

Given $(q_{0,1}, p_{0,1}, ..., q_{0,N}, p_{0,N}) \in (\mathbb{R}^d \times \mathbb{R}^d)^N$ and $(\Psi_0, \Psi_1) \in L^2(\mathbb{R}^d \times \mathbb{R}^n)$, condition (**H5**) is fulfilled by $\widehat{\mu}_0^N = \frac{1}{N}\sum_{j=1}^N \delta_{(q_{0,j},p_{0,j})}$ since, as far as N is fixed, we can say that $(q_{0,1}, p_{0,1}, ..., q_{0,N}, p_{0,N})$ belongs to a compact set of $(\mathbb{R}^d \times \mathbb{R}^d)^N$. Some caution will be necessary when we will let N tend to ∞, see (**H6**) below. As a consequence of Lemma 1 and Lemma 4, we also obtain the following claim.

Corollary 1 *For all* $N \in \mathbb{N} \setminus \{0\}$, *for all* $(q_{0,j}, p_{0,j})_{j\in\{1,...N\}}$ *in* $(\mathbb{R}^d \times \mathbb{R}^d)^N$ *and for all* Ψ_0, Ψ_1 *in* $L^2(\mathbb{R}^d \times \mathbb{R}^n)$, *there exists a unique solution* $(\Psi, q_1, p_1, ...q_N, p_N)$ *of* (10)–(13).

The proof of Theorem 1 relies on a fixed point strategy. To this end, we introduce the following mapping. For a non negative finite measure μ_0, we denote by Λ (or Λ_{μ_0} if necessary) the mapping which associates to Φ in $C([0,\infty); W^{2,\infty}(\mathbb{R}^d)) \cap C^1([0,\infty); L^\infty(\mathbb{R}^d))$ the unique weak solution μ of the Liouville equation

$$\partial_t \mu + v \cdot \nabla_x \mu - \nabla_x (V + \Phi) \cdot \nabla_v \mu = 0,$$

with initial data μ_0. Owing to the regularity of V and Φ, the characteristic curves $\varphi_\alpha^{\Phi,\beta}(x_0, v_0)$ solution $t \mapsto (X(t), \xi(t))$ of (17), are well–defined. Thus, the solution $\mu = \Lambda_{\mu_0}(\Phi) \in C([0,\infty); \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d) - \text{weak-}\star)$ is defined as the pushforward of μ_0 by the flow; namely, it is given by the following duality formula: for any $\chi \in C \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\langle \Lambda_{\mu_0}(\Phi)(t)|\chi\rangle = \left\langle \mu_0|\chi \circ \varphi_0^{\Phi,t} \right\rangle = \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi(\varphi_0^{\Phi,t}(x,v))\,\mathrm{d}\mu_0(x,v).$$

Lemma 5 *The mapping* $(\mu_0, \Phi) \mapsto \Lambda_{\mu_0}(\Phi)$ *is continuous from* $\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d) \times C([0,\infty); W^{2,\infty}(\mathbb{R}^d)) \cap C^1([0,\infty); L^\infty(\mathbb{R}^d))$ *to* $C_{W_1}([0,\infty); \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$.

Proof. Let $0 < T < \infty$. We consider two pairs $(\mu_{0,1}, \Phi_1)$ and $(\mu_{0,2}, \Phi_2)$ and we wish to estimate the Kantorovich-Rubinstein distance between $\Lambda_{\mu_{0,1}}(\Phi_1)(t)$ and $\Lambda_{\mu_{0,2}}(\Phi_2)(t)$ for all $0 \le t \le T$. Owing to (15) we have

$$\begin{aligned} W_1\left(\Lambda_{\mu_{0,1}}(\Phi_1)(t), \Lambda_{\mu_{0,2}}(\Phi_2)(t)\right) &= \sup_{\|\chi\|_{\mathrm{Lip}}\le 1} \Big| \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\chi\circ\varphi_0^{\Phi_1,t}\,\mathrm{d}\mu_{0,1} - \chi\circ\varphi_0^{\Phi_2,t}\,\mathrm{d}\mu_{0,2}\right)\Big| \\ &\le \sup_{\|\chi\|_{\mathrm{Lip}}\le 1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\chi\circ\varphi_0^{\Phi_1,t} - \chi\circ\varphi_0^{\Phi_2,t}\right|\mathrm{d}\mu_{0,1} \\ &\quad + \sup_{\|\chi\|_{\mathrm{Lip}}\le 1} \Big|\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\circ\varphi_0^{\Phi_2,t}\,\mathrm{d}(\mu_{0,1}-\mu_{0,2})\Big|. \end{aligned} \tag{20}$$

In order to bound those two terms, we introduce the cut off function

$$\begin{array}{ll} \theta_R(z) = \theta(z/R), & \theta \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d), \\ |\nabla\theta(z)| \le 1 \text{ for any } z \in \mathbb{R}^d\times\mathbb{R}^d, & \theta(z) = 0 \text{ for } |z| \ge 2, \\ 0 \le \theta(z) \le 1 \text{ for any } z \in \mathbb{R}^d\times\mathbb{R}^d, & \theta(z) = 1 \text{ for } |z| \le 1. \end{array} \tag{21}$$

Let $\epsilon > 0$. We can find $R > 0$ depending on $\mu_{0,1}$ and ϵ, such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} (1-\theta_R)\,\mathrm{d}\mu_{0,1} \le \frac{\epsilon}{4} \tag{22}$$

For χ in $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, we split the first term arising in the right hand side of (20) into two parts

$$\begin{aligned} \int_{\mathbb{R}^d\times\mathbb{R}^d} |\chi\circ\varphi_0^{\Phi_1,t} - \chi\circ\varphi_0^{\Phi_2,t}|\,\mathrm{d}\mu_{0,1} \le \int_{\mathbb{R}^d\times\mathbb{R}^d} & \theta_R(z)\|\nabla\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right|(z)\,\mathrm{d}\mu_{0,1}(z) \\ & + \frac{\epsilon}{2}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}. \end{aligned}$$

Lemma 3 allows us to control $|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}|$. We set

$$A(\Phi_1, \mu_{0,1}, \epsilon) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \theta_R(z) \exp\left(\int_0^T \|\nabla^2(\Phi_1(\tau) + V)\|_{L^\infty(B_\tau(z))}\,\mathrm{d}\tau\right)\mathrm{d}\mu_{0,1}(z),$$

and we get

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} &|\chi\circ\varphi_0^{\Phi_1,t} - \chi\circ\varphi_0^{\Phi_2,t}|\,\mathrm{d}\mu_{0,1} \\ &\le A(\Phi_1,\mu_{0,1},\epsilon)\|\nabla\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\int_0^t \|(\Phi_1-\Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)}\,\mathrm{d}s + \frac{\epsilon}{2}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}.\end{aligned} \tag{23}$$

We turn to estimate the second term of the right hand side of (20). Owing to (22) and (15), we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}(1-\theta_R)\,\mathrm{d}\mu_{0,2} \le \frac{\epsilon}{4} + W_1(\mu_{0,1},\mu_{0,2}).$$

It allows us to split the integral as we did above, and we obtain

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi\circ\varphi_0^{\Phi_2,t}\,\mathrm{d}(\mu_{0,1}-\mu_{0,2}) \le &\int_{\mathbb{R}^d\times\mathbb{R}^d}\theta_R(\chi\circ\varphi_0^{\Phi_2,t})\,\mathrm{d}(\mu_{0,1}-\mu_{0,2}) \\ &+\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\left(\frac{\epsilon}{2} + W_1(\mu_{0,1},\mu_{0,2})\right)\end{aligned} \tag{24}$$

By using (15) we get

$$\begin{aligned}&\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\theta_R(\chi\circ\varphi_0^{\Phi_2,t})\,\mathrm{d}(\mu_{0,1}-\mu_{0,2})\right| \\ &\qquad\le \left(2\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\wedge\|\nabla(\theta_M(\chi\circ\varphi_0^{\Phi_2,t})\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\right)W_1(\mu_{0,1},\mu_{0,2}) \\ &\qquad\le\left(1 + B(\mu_{0,1},\epsilon)e^{\int_0^t\|\nabla^2\Phi_2(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s}\right)W_1(\mu_{0,1},\mu_{0,2})\|\chi\|_{\mathrm{Lip}}\end{aligned} \tag{25}$$

where we have used Lemma 3, again, to estimate $\nabla\varphi_0^{\Phi_2,t}$ and we have set

$$B(\mu_{0,1},\epsilon) = \sup_{|z|\le 2R}\exp\left(\int_0^T(1+\|\nabla^2 V\|_{L^\infty(B_s(z))})\,\mathrm{d}s\right).$$

Coming back to (20), and combining the intermediate estimates (23), (24) and (25), we conclude that, for all $\epsilon>0$, we can find $A(\Phi_1,\mu_1,\epsilon)$ and $B(\mu_1,\epsilon)$ such that, for all $0<t<T$, we have

$$\begin{aligned}W_1\left(\Lambda_{\mu_{0,1}}(\Phi_1)(t),\Lambda_{\mu_{0,2}}(\Phi_2)(t)\right) \le &\left(2 + B(\mu_{0,1},\epsilon)e^{\int_0^t\|\nabla^2\Phi_2(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s}\right)W_1(\mu_{0,1},\mu_{0,2}) \\ &+A(\Phi_1,\mu_{0,1},\epsilon)\int_0^t\|(\Phi_1-\Phi_2)(s)\|_{W^{1,\infty}(\mathbb{R}^d)}\,\mathrm{d}s + \frac{\epsilon}{2}.\end{aligned}$$

(Note that $A(\Phi_1,\mu_1,\epsilon)$ and $B(\mu_1,\epsilon)$ depend on $0<T<\infty$.) Letting both $W_1(\mu_{0,2},\mu_{0,1})$ and $\|(\Phi_1-\Phi_2)(s)\|_{L^\infty(0,T;W^{1,\infty}(\mathbb{R}^d))}$ go to 0, we conclude that $W_1\left(\Lambda_{\mu_{0,1}}(\Phi_1)(t),\Lambda_{\mu_{0,2}}(\Phi_2)(t)\right)$ tends to 0, uniformly for $t\in[0,T]$. (Note that we should assume that Φ_1 and Φ_2 remain bounded in $C([0,T];W^{2,\infty}(\mathbb{R}^d))$ and in $C^1([0,T];W^{1,\infty}(\mathbb{R}^d))$.) ■

We are now ready to justify the existence and uniqueness of solution of (6)–(8), or equivalently of (19).

Proof of Theorem 1. We turn to the fixed point reasoning. For μ given in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$, we set

$$\mu\longmapsto\mathcal{T}_{\mu_0}(\mu) = \Lambda_{\mu_0}(\Phi_0 - \mathcal{L}(\mu)).$$

It is clear that a fixed point of $\mathcal{T}_{\mu_0}$ is a solution to (19). Note also that, as a consequence of Lemma 2 and Lemma 5, $\mathcal{T}_{\mu_0}(\mu)(t)\in\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)$. More precisely, we know that $\mu\mapsto\mathcal{T}_{\mu_0}(\mu)$ is continuous with values in the space $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. We shall prove that $\mathcal{T}_{\mu_0}$ admits an iteration which is a contraction on $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. Let μ_1 and μ_2 be two elements of this set. We denote $\varphi_\alpha^{\Phi_i,t}$

the flow of (17) with $\Phi_i = \Phi_0 - \mathcal{L}(\mu_i)$. By using (20), we get

$$W_1\left(\mathcal{T}_{\mu_0}(\mu_1)(t), \mathcal{T}_{\mu_0}(\mu_2)(t)\right) = \sup_{\|\chi\|_{\text{Lip}}\leq 1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\chi \circ \varphi_0^{\Phi_1,t} - \chi \circ \varphi_0^{\Phi_2,t}\right) \mathrm{d}\mu_0 \qquad (26)$$
$$\leq \sup_{\|\chi\|_{\text{Lip}}\leq 1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \|\nabla\chi\|_\infty \left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right| (x,v)\, \mathrm{d}\mu_0$$
$$\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right| (x,v)\, \mathrm{d}\mu_0.$$

By using Lemma 3-b), we obtain

$$\left|\varphi_0^{\Phi_1,t} - \varphi_0^{\Phi_2,t}\right| (x,v)$$
$$\leq \bar{m}_T \int_0^t \|\mathcal{L}(\mu_1 - \mu_2)\|_{L^\infty(0,s;W^{2,\infty}(\mathbb{R}^d))}$$
$$\times \exp\left(\int_s^t \|\nabla^2 V\|_{L^\infty B(0,R(\|\Phi_0+\mathcal{L}(\mu_i)\|_{C^1([0,u];L^\infty(\mathbb{R}^d))},u,x_0,v_0))}\, \mathrm{d}u\right) \mathrm{d}s,$$

where we have also used

$$\exp\left(\int_0^T \|\nabla^2(\Phi_0(u) - \mathcal{L}(\mu_1)(u)\|_{L^\infty(\mathbb{R}^d)}\, \mathrm{d}u\right)$$
$$\leq \exp\left(\int_0^T \left(\|\nabla^2\Phi_0(u)\|_{L^\infty(\mathbb{R}^d)} + \|\mathcal{L}\|_{\mathcal{A}_u}\right) \mathrm{d}u\right) = \bar{m}_T.$$

Going back to (26) yields

$$W_1\left(\mathcal{T}_{\mu_0}(\mu_1)(t), \mathcal{T}_{\mu_0}(\mu_2)(t)\right) \leq \bar{m}_T \int_{\mathbb{R}^d\times\mathbb{R}^d} \int_0^t \|\mathcal{L}(\mu_1 - \mu_2)\|_{L^\infty(0,s;W^{2,\infty}(\mathbb{R}^d))}$$
$$\times \exp\left(\int_s^t \|\nabla^2 V\|_{L^\infty(B(0,r(u,x,v)))}\, \mathrm{d}u\right) \mathrm{d}s\, \mathrm{d}\mu_0(x,v).$$

Using Lemma 2 and (15), it recasts as

$$W_1\left(\mathcal{T}_{\mu_0}(\mu_1)(t), \mathcal{T}_{\mu_0}(\mu_2)(t)\right) \leq \bar{m}'_T \mathscr{K}_T \int_0^t \left(\sup_{0\leq\tau\leq s} W_1(\mu_{1,\tau}, \mu_{2,\tau})\right) \mathrm{d}s \qquad (27)$$

with

$$\bar{m}'_T = \bar{m}_T \times \sup_{0\leq s\leq T} \|\mathcal{L}\|_{\mathcal{A}_s}.$$

By induction, we deduce that

$$W_1\left(\mathcal{T}^\ell_{\mu_0}(\mu_1)(t), \mathcal{T}^\ell_{\mu_0}(\mu_2)(t)\right) \leq \frac{(t\bar{m}'_T\mathscr{K}_T)^\ell}{\ell!} \sup_{0\leq t\leq T} W_1(\mu_{1,t}, \mu_{2,t})$$

holds for any $\ell \in \mathbb{N}$ and $0 \leq t \leq T$. Finally, we are led to

$$\sup_{0\leq t\leq T} W_1\left(\mathcal{T}^\ell_{\mu_0}(\mu_1(t)), \mathcal{T}^\ell_{\mu_0}(\mu_2(t))\right) \leq \frac{(T\bar{m}'_T\mathscr{K}_T)^\ell}{\ell!} \|W_1(\mu_1(t), \mu_2(t))\|_{L^\infty(0,T)}.$$

It shows that an iteration of $\mathcal{T}_{\mu_0}$ is a contraction in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. Therefore, there exists a unique fixed point μ in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. Furthermore, the solution is continuous with respect to the parameters of the system. ■

4.3 Asymptotic analysis

We now wish to investigate the limit $N \to \infty$ in (10)–(13) and to justify that it allows us to derive (6)–(8). Since any finite measure f_0 can be obtained as the tight–limit of sums of Dirac masses, see [18, Chap. 2, Th. 6.9], by using Lemma 4, Theorem 1 and Corollary 1, we reduce the question we address to a stability issue. We suppose that Ψ_0 and Ψ_1 do not depend on N and we consider a sequence of initial data $\left(\mu_0^N\right)_{N\in\mathbb{N}}$. We associate to these data the corresponding solutions μ^N of (6)–(8). We are going to distinguish two situations. Either we suppose that

$$\begin{aligned}&(\mu_0^N)_{N\in\mathbb{N}} \text{ converges tightly to a finite measure } \mu_0 \text{ and}\\&\mathscr{K}_T = \sup_{N\in\mathbb{N}} \left\{ \int_{\mathbb{R}^d\times\mathbb{R}^d} \exp\left(\int_0^T \|\nabla^2 V\|_{L^\infty(B(0,r(t,x,v)))}\,\mathrm{d}t \right) \mathrm{d}\mu_0^N \right\} < \infty.\end{aligned} \tag{\textbf{H6}}$$

where r is defined by (18) (which does not depend upon N), or

$$\text{the sequence } (\mu_0^N)_{N\in\mathbb{N}} \text{ is tight.} \tag{\textbf{H6b}}$$

Clearly (**H6**) is stronger than (**H6b**), and it allows us to obtain sharper results. The analysis of the situation with (**H6b**) only relies on a compactness analysis.

Theorem 2 (Stability for the Vlasov–Wave system)

a) Assume (**H6**)*. Then there exists a measure–valued function μ solution of* (19) *with initial data μ_0 such that μ^N converges to μ in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$.*

b) Assume **(H6b)***. Then, we can extract a subsequence $\left(\mu^{N_\ell}\right)_{\ell\in\mathbb{N}}$ such that μ^{N_ℓ} converges to a measure–valued function μ, solution of* (19)*, in $C_{W_1}([0,\infty);\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$.*

Proof.

Step 1: Proof of a). We remind the reader that the solutions of (19) in Theorem 1 have been obtained as fixed points of the application $\mathcal{T}$; namely, we have

$$\mu^N = \mathcal{T}_{\mu_0^N}(\mu^N) = \Lambda_{\mu_0^N}(\Phi_0 - \mathcal{L}(\mu^N)), \qquad \mu = \mathcal{T}_{\mu_0}(\mu) = \Lambda_{\mu_0}(\Phi_0 - \mathcal{L}(\mu)).$$

Based on this, we write

$$W_1(\mu_t^N, \mu_t) \le W_1\left(\mathcal{T}_{\mu_0^N}(\mu^N)(t), \mathcal{T}_{\mu_0^N}(\mu)(t)\right) + W_1\left(\mathcal{T}_{\mu_0^N}(\mu)(t), \mathcal{T}_{\mu_0}(\mu)(t)\right). \tag{28}$$

Bearing in mind (27), (**H6**) leads to the following estimate

$$W_1\left(\mathcal{T}_{\mu_0^N}(\mu^N)(t), \mathcal{T}_{\mu_0^N}(\mu)(t)\right) \le \bar{m}'_T \mathscr{K}_T \int_0^t \left(\sup_{0\le\tau\le s} W_1(\mu_\tau^N, \mu_\tau) \right) \mathrm{d}s.$$

Let us set

$$\begin{aligned}\alpha^N(t) &= \sup_{0\le\tau\le t} W_1(\mu_\tau^N, \mu_\tau),\\ \beta^N(t) &= \sup_{0\le\tau\le t} W_1\left(\mathcal{T}_{\mu_0^N}(\mu)(\tau), \mathcal{T}_{\mu_0}(\mu)(\tau)\right).\end{aligned}$$

The previous inequality implies

$$\alpha^N(t) \le \beta^N(t) + \bar{m}'_T\mathscr{K}_T \int_0^t \alpha^N(s)\,\mathrm{d}s. \tag{29}$$

It can be cast as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\bar{m}'_T\mathscr{K}_T t} \int_0^t \alpha^N(\tau)\,\mathrm{d}\tau \right) \le \beta^N(t) e^{-\bar{m}'_T\mathscr{K}_T t}.$$

Integration over $[0,s]$ yields

$$\int_0^s \alpha^N(\tau)\,\mathrm{d}\tau \le \int_0^s \beta^N(\tau) e^{\bar{m}'_T\mathscr{K}_T(s-\tau)}\,\mathrm{d}\tau \le \frac{\beta^N(s)}{\bar{m}'_T\mathscr{K}_T}\left(e^{\bar{m}'_T\mathscr{K}_T s} - 1\right).$$

Going back to (29) leads to

$$\alpha^N(t) \le \beta^N(t) e^{\bar{m}'_T\mathscr{K}_T t}.$$

With $\Phi = \Phi_0 - \mathcal{L}(\mu)$, by definition of β^N and $\mathcal{T}$, this can be rewritten

$$\alpha^N(t) \le e^{T\bar{m}'_T\mathscr{K}_T} \sup_{0\le s\le T} W_1\left(\Lambda_{\mu_0^N}(\Phi), \Lambda_{\mu_0}(\Phi)\right).$$

We conclude by coming back to Lemma 5.

Step 2: Proof of b). We start by showing that the sequence $\big(\mu^N\big)_{N\in\mathbb{N}}$ is compact in $C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))$. By hypothesis, we note that

$$\bar{m}=\sup_{N\in\mathbb{N}}\|\mu_0^N\|_{\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)}<\infty.$$

Pick $\chi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^d)$. For any $0\le t\le T$, we have, on the one hand,

$$\begin{aligned}\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v)\right|&\le\|\mu_t^N\|_{\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\&\le\|\mu_0^N\|_{\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}\\&\le\bar{m}\|\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)},\end{aligned}\tag{30}$$

by mass conservation, and, on the other hand,

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v)\right|\\&=\left|\int_{\mathbb{R}^d\times\mathbb{R}^d}\big(v\cdot\nabla_x\chi-\nabla_x(V+\Phi_0-\mathcal{L}(\mu^N)(t))\cdot\nabla_v\chi\big)(x,v)\,\mathrm{d}\mu_t^N(x,v)\right|\\&\le\bar{m}\Big(\|v\cdot\nabla\chi-\nabla V\cdot\nabla_v\chi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d))}\\&\qquad+\Big(|\!|\!|\mathcal{L}|\!|\!|_{\mathcal{A}_T}\bar{m}+\|\Phi_0\|_{L^\infty([0,T];W^{1,\infty}(\mathbb{R}^d))}\Big)\|\chi\|_{L^\infty}\Big).\end{aligned}$$

Lemma 2 then ensures that the set

$$\Big\{t\mapsto\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v),\ N\in\mathbb{N}\Big\}$$

is equibounded and equicontinuous; hence, by virtue of Arzela–Ascoli's theorem it is relatively compact in $C([0,T])$. Going back to (30), a simple approximation argument allows us to extend the conclusion to any trial function χ in $C_0(\mathbb{R}^d\times\mathbb{R}^d)$, the space of continuous functions that vanish at infinity. This space is separable; consequently, by a diagonal argument, we can extract a subsequence, still labelled by $N\in\mathbb{N}$, and find a measure valued function $\mu\in C\big([0,\infty);\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)-\text{weak-}\star\big)$ such that

$$\lim_{N\to\infty}\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t^N(x,v)=\int_{\mathbb{R}^d\times\mathbb{R}^d}\chi(x,v)\,\mathrm{d}\mu_t(x,v)$$

holds uniformly on $[0,T]$, for any $\chi\in C_0(\mathbb{R}^d\times\mathbb{R}^d)$ and $0<T<\infty$. As a matter of fact, we note that, for any $0\le t\le T$,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{d}\mu_t(x,v)\le\bar{m}.$$

Next, we establish the tightness of the sequence of solutions. Let $\epsilon>0$ be fixed once for all. Owing to (**H6b**), we can find $M_\epsilon>0$ such that for all $N\ge0$,

$$\int_{x^2+v^2\ge M_\epsilon^2}\mathrm{d}\mu_0^N(x,v)\le\epsilon.$$

Let us set

$$A_\epsilon=\sup\{r(T,x,v),\ (x,v)\in B(0,M_\epsilon)\}$$

where we remind the reader that $r(T,x,v)$ has been defined in (18): $0<A_\epsilon<\infty$ is well defined by Lemma 2. Let $\varphi_\alpha^{N,t}$ stand for the flow associated to the characteristics of the equation satisfied by μ^N. For any $0\le t\le T$, we have $\varphi_0^{N,t}(B(0,M_\epsilon))\subset B(0,A_\epsilon)$ so that $\complement\left(\varphi_t^{k,0}(B(0,A_\epsilon))\right)=\varphi_t^{N,0}\big(\complement B(0,A_\epsilon)\big)\subset\complement B(0,M_\epsilon)$. It follows that

$$\begin{aligned}\int_{\complement B(0,A_\epsilon)}\mathrm{d}\mu_t^N(x,v)&=\int_{\complement\varphi_t^{k,0}(B(0,A_\epsilon))}\mathrm{d}\mu_0^N(x,v)\\&\le\int_{\complement B(0,M_\epsilon)}\mathrm{d}\mu_0^N(x,v)\le\epsilon.\end{aligned}$$

By a standard approximation, we check that the same estimate is satisfied by the limit μ. Finally, since the tight convergence is equivalent to the convergence with respect to the Kantorovich-Rubinstein distance W_1, we conclude that

$$\lim_{N\to\infty}\Big(\sup_{0\le t\le T} W_1(\mu_t^N,\mu_t)\Big)=0$$

According to Lemma 2 and Lemma 5, the following mapping

$$\begin{array}{ll}\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)\times C_{W_1}\big([0,T],\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)\big) & \longrightarrow C_{W_1}\big([0,T],\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d)\big)\\ (\mu_0,\mu) & \longmapsto \mathcal{T}_{\mu_0}(\mu)\end{array}$$

is continuous. Then, we get

$$\mu^N=\mathcal{T}_{\mu_0^N}(\mu^N)\xrightarrow[N\to\infty]{C_{W_1}([0,T];\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d))}\mathcal{T}_{\mu_0}(\mu).$$

It implies that $\mu=\mathcal{T}_{\mu_0}(\mu)$ and $\mu\in C_{W_1}\big([0,T],\mathcal{M}^1(\mathbb{R}^d\times\mathbb{R}^d\big)$ satisfies (19), which ends the proof. ■

5 Mean Field Limit for the Vlasov–Wave–Fokker-Planck system

5.1 Preliminary observations

In this section, we consider the case where the Fokker-Planck operator is added in the kinetic equation: namely the equation for the particle distribution in (6) is replaced by (9). We shall establish that this system can be obtained as the limit $N\to\infty$ from the system of stochastic differential equations (11). We remark that the right hand side of the wave equation in (11) is nothing but

$$-\sigma_2(y)\sigma_1\star\widehat{\rho}_t^N(x)$$

with $\widehat{\rho}_t^N(x)=\int_{\mathbb{R}^d}\mathrm{d}\widehat{\mu}_t^N(x,v)$ and $\widehat{\mu}_t^N=\frac{1}{N}\sum_{j=1}^N\delta_{(q_j(t),p_j(t))}$ is the empirical measure associated to (11). We then use Lemma 1 again to recast (11) as follows

$$\left\{\begin{array}{l}\mathrm{d}q_i^N(t)=p_i^N(t)\,\mathrm{d}t\\ \mathrm{d}p_i^N(t)=-\nabla_x(V+\Phi_0-\mathcal{L}(\widehat{\mu}^N))(t,q_i^N(t))\,\mathrm{d}t-\gamma p_i^N(t)\,\mathrm{d}t+\sqrt{2\gamma}\,\mathrm{d}B_i(t),\end{array}\right.\tag{31}$$

for any $i\in\{1,...,N\}$, where from now on we emphasize the dependence with respect to N. We also remind the reader that the $(B_j(t))_{t\ge0}$'s are independent Brownian motions. In this context the family of positions and velocities $t\mapsto(q_j^N(t),p_j^N(t))_{j\in\{1,...,N\}}$ is made of random variables depending on the Brownian motions and on the initial data which can be random (independently distributed) too. Indeed, the initial positions and velocities in (12) are supposed to be distributed according to

$$\mathbb{P}[(q_{0,j}^N,p_{0,j}^N)\in A]=\int_A\mathrm{d}\mu_0\quad\text{for any } j\in\{1,...,N\}.\tag{32}$$

In contrast to what happened for (3), here the empirical distribution $\widehat{\mu}^N$ is no longer a solution of (9). There are several arguments to convince ourselves of this fact [4]:

- due to the diffusion operator (with respect to velocity) in (9), we cannot expect that the solution of the kinetic equation remains a sum of Dirac masses for positive times,
- by nature $\widehat{\mu}^N$ is a random variable (due to the Brownian motions) while the solution of (9) is a deterministic quantity (at least when we work with determinstic initial data).

Actually, it is possible to compute the equation satisfied by $\widehat{\mu}^N$. Let us use the shorthand notation $z_j^N(t)=(q_j^N(t),p_j^N(t))$ to specify the solution of (11). Applying Itô's formula to integrate (11), we get

$$\begin{aligned}\varphi(z_j^N(t))-\varphi(z_j^N(0))=&\int_0^t\nabla_x\varphi(z_j^N(s))\cdot p_j^N(s)\,\mathrm{d}s\\ &-\int_0^t\nabla_v\varphi(z_j^N(s))\cdot\big(\nabla_x(V+\Phi_0(s)-\mathcal{L}(\widehat{\mu}^N)(s))(q_j^N(s))+\gamma p_j^N(s)\big)\,\mathrm{d}s\\ &+\gamma\int_0^t\Delta_v\varphi(z(s))\,\mathrm{d}s+\sqrt{2\gamma}\int_0^t\nabla_v\varphi(z_j^N(s))\cdot\,\mathrm{d}B_j(s)\end{aligned}$$

for any trial function $\varphi \in C(\mathbb{R}^d \times \mathbb{R}^d)$ and any $j \in \{1, ..., N\}$. Let us average over $j \in \{1, ..., N\}$. We obtain the following weak relation satisfied by $\widehat{\mu}^N$

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\widehat{\mu}_t^N(z) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\widehat{\mu}_0^N(z) = \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\varphi(z)\cdot v\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s \\ - \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot\big(\nabla_x(V + \Phi_0(s) - \mathcal{L}(\widehat{\mu}^N)(s))(x) + \gamma v\big)\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s \\ +\gamma\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \Delta_v\varphi(z)\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s + \mathcal{I}_N,\end{aligned} \tag{33}$$

where $\mathcal{I}_N$ is defined by

$$\mathcal{I}_N = \frac{\sqrt{2\gamma}}{N}\sum_{j=1}^N \int_0^t \nabla_v\varphi(z_j^N(s))\cdot\,\mathrm{d}B_j(s). \tag{34}$$

In general $\mathcal{I}_N \neq 0$ cannot be expressed by means of $\widehat{\mu}^N$, and the equation is not closed. However, we shall see that $\mathbb{E}[|\mathcal{I}_N|]$ is of order $\mathscr{O}(\frac{1}{\sqrt{N}})$; accordingly $\widehat{\mu}^N$ tends to be a solution of (9). Moreover, the martingale theory ensures that

$$\mathbb{E}[\mathcal{I}_N] = 0.$$

Indeed, since the Brownian motion is a martingale, $\mathcal{I}_N$ is a martingale too (see [16, Definition 2.9 Chapter 3]), and its expectation value does not depend on t and we conclude immediately since at $t = 0$ we have $\mathcal{I}_N = 0$. This observation motivates to introduce the measure $\mu^{(1,N)}$ defined by the following identity

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\mu_t^{(1,N)}(z) = \mathbb{E}\left[\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\widehat{\mu}_t^N\right]. \tag{35}$$

This measure can be related to the particle distribution in the $N-$body phase space: $\mu_t^{(N)}$ lies in $\mathcal{M}^1((\mathbb{R}^d\times\mathbb{R}^d)^N)$ and it is defined by

$$\mu_t^{(N)}(\mathscr{A}) = \mathbb{P}[(z_1^N(t), ..., z_N^N(t)) \in \mathscr{A}]$$

for any $\mathscr{A} \subset (\mathbb{R}^d \times \mathbb{R}^d)^N$. Since (11) and (32) do not change if we permute the particles $\{z_1^N, .., z_N^N\}$, all the random variables $z_j(t)$ share the same probability $\tilde{\mu}_t^{(1,N)}$, which is nothing but the first marginal of $\mu_t^{(N)}$: for any $A \subset \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\tilde{\mu}_t^{(1,N)}(A) = \int_{A\times(\mathbb{R}^d\times\mathbb{R}^d)^{N-1}} \mathrm{d}\mu^{(N)}(z) = \mathbb{P}[z_1^N(t) \in A].$$

We go back to (35) by the following simple computation: for any $A \subset \mathbb{R}^d \times \mathbb{R}^d$, we have

$$\mu_t^{(1,N)}(A) = \mathbb{E}[\widehat{\mu}_t^N(A)] = \frac{1}{N}\sum_{i=1}^N \mathbb{P}[z_i^N(t) \in A] = \frac{1}{N}\sum_{i=1}^N \mathbb{P}[z_1^N(t) \in A] = \tilde{\mu}_t^{(1,N)}(A).$$

We can thus identify the two measures $\mu_t^{(1,N)} = \tilde{\mu}_t^{(1,N)} \in \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$.

In (33), bearing in mind (16), we can write

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot\nabla_x\mathcal{L}(\widehat{\mu}^N)(s)(x)\,\mathrm{d}\widehat{\mu}_s^N(z)\,\mathrm{d}s \\ &\qquad = \int_0^t \int_0^s p(s-\tau)\sum_{1\leq i,j\leq N} \nabla_v\varphi(z_i^N(s))\cdot\nabla_x\Sigma(q_i^N(s) - q_j^N(\tau))\,\mathrm{d}\tau\,\mathrm{d}s.\end{aligned}$$

We take the expectation value in (33). We are led to the following weak equation satisfied by $\mu^{(1,N)}$

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z)\,\mathrm{d}\mu_t^{(1,N)}(z) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi(z) d\mu_0^{(1,N)}(z) = \int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_x\varphi(z)\cdot v\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s \qquad (36)\\
&\quad -\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z)\cdot\big(\nabla_x(V+\Phi_0(s))(x)+\gamma v\big)\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s\\
&\quad +\int_0^t\int_0^s\int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} p(s-\tau)\\
&\qquad\qquad \times\nabla\Sigma(x_1-x_2)\cdot\nabla_v\varphi(z_1)\,\mathrm{d}\left(\frac{N-1}{N}\mu_{s,\tau}^{(2,N)}+\frac{1}{N}\nu_{s,\tau}^{(2,N)}\right)(z_1,z_2)\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad +\gamma\int_0^t\int_{\mathbb{R}^d\times\mathbb{R}^d} \Delta_v\varphi(z)\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s+0,
\end{aligned}$$

where

– $\mu_{s,\tau}^{(2,N)}$ is the joint probability measure of two different particles at different times s and τ: for $B\subset(\mathbb{R}^d\times\mathbb{R}^d)^2$, we have
$$\mu_{s,\tau}^{(2,N)}(B)=\mathbb{P}\big((z_1^N(s),z_2^N(\tau))\in B\big),$$

– $\nu_{s,\tau}^{(2,N)}$ is the joint probability measure of one particle at two different times s and τ: for $B\subset(\mathbb{R}^d\times\mathbb{R}^d)^2$, we have
$$\nu_{s,\tau}^{(2,N)}(B)=\mathbb{P}\big((z_1^N(s),z_1^N(\tau))\in B\big).$$

Equation (36) is still not closed, due to the joint probability measures $\mu_{s,\tau}^{(2,N)}$ and $\nu_{s,\tau}^{(2,N)}$. Besides, we can write similarly the equations satisfied by $\mu^{(2,N)}$ or $\nu^{(2,N)}$, which imply the third order probability measures and so on... This BBGKY hierarchy is non standard because of the coupling with the wave equation which induces a half-convolution in time. Here, we focus on the behavior of (36) as $N\to\infty$. Since $\nu^{(2,N)}$ is a probability measure, it is clear that the corresponding term in (36) is of order $\mathscr{O}(1/N)$, and thus it goes to 0 as $N\to\infty$. The difficulties relies on the terms with $\mu^{(2,N)}$. Initially, the N particles are independent; this property is not conserved for positive times, but we expect that particles tend to be less and less correlated as N becomes large, which amounts to say that $\mu_{s,\tau}^{(2,N)}$ looks like the product $\mu_s^{(1,N)}\otimes\mu_\tau^{(1,N)}$. We shall make this intuition rigorous, which eventually justifies that $\mu^{(1,N)}$ tends to a solution of (9). As said above, new difficulties are related to the unusual half–convolution with respect to the time variable in the interaction operator. To handle this, we shall introduce a suitable notion of "multi-times propagation of chaos", which is inspired from the following Definition [23], see also [14,20, 21]

Definition 2 a) Let E be a separable metric space and let $(\mu^N)_{N\in\mathbb{N}\setminus\{0\}}$ be a sequence of symmetric probability measures on E^N. Let μ be a probability measure on E. We say that $(\mu^N)_{N\in\mathbb{N}\setminus\{0\}}$ is *μ-chaotic* if for any $k\in\mathbb{N}\setminus\{0\}$ and any $\varphi_1,..,\varphi_k$ in $C\cap L^\infty(E)$ the following identity holds

$$\lim_{N\to\infty}\int_{E^N}\varphi_1(z_1)...\varphi_k(z_k)\,\mathrm{d}\mu^N(z_1,..,z_N)=\prod_{i=1}^k\int_E\varphi_i(z)\,\mathrm{d}\mu(z). \qquad (37)$$

b) We say that a Markov process leading the evolution of a family $\big\{\mu^N: t\in[0,T]\mapsto\mu_t^N\in\mathcal{M}^1(E^N),\ N\in\mathbb{N}\setminus\{0\}\big\}$ of probability measures on E^N propagates the chaos if, given a sequence $(\mu_0^N)_{N\in\mathbb{N}\setminus\{0\}}$ of μ-chaotic initial data, the sequence $(\mu_t^N)_{N\in\mathbb{N}}$ is also $\mu-$chaotic for all $t>0$.

Applying Definition 2-a) with $k=1$ in (37), we see that μ is the weak limit in $\mathcal{M}^1(E)$ of the first marginal $\mu^{(1,N)}$ of μ^N as $N\to\infty$. In (32), we made a strong, but natural, assumption on the initial data, namely it factorizes: $\mu_0^N=\mu_0^{\otimes N}$. However, due to the interactions with the medium, this property has no reason to be preserved for positive times. The assumption for the initial data based on (37) is far weaker, and it is well–adapted to our purposes since it is preserved by the dynamics, as we shall see below in Corollary 2-ii) . Finally, to any sequence $(\mu^N)_{N\in\mathbb{N}\setminus\{0\}}$ of probability measures on E^N, we associate the family $(\widehat{\mu}^N)_{N\in\mathbb{N}\setminus\{0\}}$ of random measures on E defined by

$$\widehat{\mu}^N=\frac{1}{N}\sum_{i=1}^N\delta_{z_i}$$

where the random variable $(z_1, .., z_N)$ is distributed in E^N according to μ^N. The following result due to [23] makes the connection between the empirical distribution and the first marginal.

Proposition 1 *A sequence* $(\mu^N)_{N\in\mathbb{N}}$ *is* μ*-chaotic iff* (37) *is satisfied for* $k = 2$. *Equivalently,* $\widehat{\mu}^N$ *converges in law to* μ, *the weak limit in* $\mathcal{M}^1(E)$ *of the first marginal* $\mu^{(1,N)}$, *as* $N \to \infty$.

We explain now how we will proceed, following the arguments discussed in [23]. Assume that we have at hand a measure-valued solution $\mu \in C_{W_1}([0,\infty); \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$ of (9). We introduce the following system of stochastic differential equations

$$\begin{cases} \mathrm{d}\tilde{q}_i^N = \tilde{p}_i^N(t)\,\mathrm{d}t, \\ \mathrm{d}\tilde{p}_i^N = -\nabla_x(V + \Phi_0 - \mathcal{L}(\mu))(t, \tilde{q}_i^N(t))\,\mathrm{d}t - \gamma \tilde{p}_i^N(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B_i(t), \end{cases} \tag{38}$$

for $i \in \{1, ..., N\}$, where the Brownian motions $(B_i(t))_{t>0}$ are the same as in (11). The initial data for (38)

$$\tilde{q}_i^N(0) = q_{0,i}^N, \qquad \tilde{p}_i^N(0) = p_{0,i}^N \tag{39}$$

are also shared with (11). We suppose that (32) is fulfilled. The dynamics of these "fictitious" particles is driven by the measure μ. We are going to prove the following result, which shows that this dynamics is close to those of the original system.

Theorem 3 *Let* $(z_i^N)_{j\in\{1,...,N\}} = (q_i^N, p_i^N)_{j\in\{1,...,N\}}$ *be a solution of* (11), *with initial data given by* (32). *Let* $(\tilde{z}_i^N)_{j\in\{1,...,N\}} = (\tilde{q}_i^N, \tilde{p}_i^N)_{j\in\{1,...,N\}}$ *be a solution of* (38) *with the same initial data. Let* $0 < T < \infty$. *We can find a constant* C_T *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T} |z_i^N - \tilde{z}_i^N|(t)\right] \le \frac{C_T}{\sqrt{N}}.$$

We will deduce several consequences from this statement:

- it implies the convergence of the first marginal $\mu^{(1,N)}$ to μ for a certain Kantorowich–Rubinstein distance, see Corollary 2-i),
- the convergence in law of the empirical measure $\widehat{\mu}^N$ to the same limit then follows from Proposition 1,
- it allows us to establish the propagation of chaos for the solution of (11), see Corollary 2-ii),
- and, coming back to (33), (34) and (36), it allows us to prove that $\mathbb{E}[|\mathcal{I}_N|]$ goes to 0 as $N \to \infty$ while

$$\int_0^t \int_0^s \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} p(s-\tau)\nabla\sigma(x_1 - x_2) \cdot \nabla_v\varphi(z_1)\,\mathrm{d}\mu_{s,\tau}^{(2,N)}(z_1, z_2)\,\mathrm{d}\tau\,\mathrm{d}s$$

behaves like

$$\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z) \cdot \nabla_x\mathcal{L}(\mu^{(1,N)})(s,x)\,\mathrm{d}\mu_s^{(1,N)}(z)\,\mathrm{d}s$$

as expected since both converge to

$$\int_0^t \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\varphi(z) \cdot \nabla_x\mathcal{L}(\mu)(s,x)\,\mathrm{d}\mu_s(z)\,\mathrm{d}s.$$

These results justify (9) as the equation satisfied by the limit of the first marginal $\mu^{(1,N)}$ and the empirical distribution $\widehat{\mu}^N$ associated to (11) (which tend to coincide for a large number of particles) when N goes to infinity. This Section is organized as follows. Firstly, we prove the existence and uniqueness of the solution μ of (9) and of the random variables $(z_i^N)_{i\in\{1,...,N\}}$ and $(\tilde{z}_i^N)_{i\in\{1,...,N\}}$. Secondly, we establish Theorem 3 and its consequences.

5.2 Analysis of the stochastic equations and the PDE system

5.2.1 N–particles system

We first prove that the system (11) is well posed for data that verify (32).

Theorem 4 *The system* (11) *with* (32) *has a unique strong solution in the sense of the stochastic differential equation. It means that, fixing a family of Brownian motions* $(B_i)_{i\in\{1,\dots,N\}}$ *and a family of initial data* $(q^N_{0,i}, p^N_{0,i})_{i\in\{1,\dots,N\}}$ *there exists a unique continuous family* $(t\mapsto (q_i^N(t), p_i^N(t)))_{i\in\{1,\dots,N\}}$ *solution of* (11) *with* (32).

The proof is just an adaptation of the Cauchy–Lipschitz theorem. In order to simplify the forthcoming computations, let us set, for $\mu \in C_{W_1}([0,T];\mathbb{R}^d\times\mathbb{R}^d)$,

$$F(\mu)(t,q) = -\nabla_x(V+\Phi_0-\mathcal{L}(\mu))(t,q).$$

The following estimate on F will be useful for the analysis.

Lemma 6 *Let* μ_1,μ_2 *be to probability measures on* $C_{W_1}([0,T];\mathbb{R}^d\times\mathbb{R}^d)$. *Let* $z_1=(q_1,p_1)$ *and* $z_2=(q_2,p_2)$. *We set*

$$c_1(t) = \|\nabla^2 V\|_{L^\infty} + \|\nabla^2\sigma_1\|_{L^2}\|\sigma_2\|_{L^2}\left(\|\Psi_0\|_{L^2}+t\|\Psi_1\|_{L^2}\right) + \|\sigma_2\|^2_{L^2}\|\nabla\sigma_1\|^2_{L^2}\frac{t^2}{2}.$$

Then, we have:

$$|F(\mu_1)(t,q_1) - F(\mu_2)(t,q_2)| \le c_1(t)|z_1-z_2| + |\nabla\mathcal{L}(\mu_1-\mu_2)(t,q_1)|.$$

Proof. Lemma 2 allows us to obtain the following estimate

$$\begin{aligned} &|(-\nabla_x(V+\Phi_0-\mathcal{L}(\mu_1))(t,q_1)) - (-\nabla_x(V+\Phi_0-\mathcal{L}(\mu_2))(t,q_2)| \\ &\le (\|\nabla^2 V\|_{L^\infty(\mathbb{R}^d)} + \|\nabla^2\Phi_0(t)\|_{L^\infty(\mathbb{R}^d)})|q_1-q_2| + \\ &\quad + \|\nabla^2\mathcal{L}(\mu_1)(t)\|_{L^\infty(\mathbb{R}^d)}|q_1-q_2| + \|\nabla\mathcal{L}(\mu_1-\mu_2)(t)\|_{L^\infty(\mathbb{R}^d)} \\ &\le c_1(t)|q_1-q_2| + |\nabla\mathcal{L}(\mu_1-\mu_2)(t,q_1)|. \end{aligned}$$

■

Proof of Theorem 4. Let us introduce a few shorthand notations. We define $Z^N \in (\mathbb{R}^d\times\mathbb{R}^d)^N$ by

$$Z^N = (q_1,p_1,\dots,q_N,p_N).$$

Next we introduce the force field

$$F^N(\widehat{\mu}^N)(t,Z^N) = \begin{pmatrix} p_1 \\ F(\widehat{\mu}^N)(t,q_1)-\gamma p_1 \\ \vdots \\ p_N \\ F(\widehat{\mu}^N)(t,q_N)-\gamma p_N \end{pmatrix}.$$

We define the diffusion matrix Γ^N, which lies in $M_{2Nd}(\mathbb{R})$, by

$$\Gamma^N = \begin{pmatrix} 0 & & & & \\ & \sqrt{2\gamma}\mathrm{Id}_{\mathbb{R}^d} & & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & \sqrt{2\gamma}\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix}.$$

Finally, since the family of Brownian motions $(B_i(t))_{t\ge0}$ in $C([0,\infty);\mathbb{R}^d\times\mathbb{R}^d)$ can be described by the a single Brownian motion $(B^N(t))_{t\ge0}$ in $C([0,\infty);(\mathbb{R}^d\times\mathbb{R}^d)^N)$, the system (11) can be recast as

$$\begin{cases} \mathrm{d}Z^N = F^N(\widehat{\mu}^N_{Z^N})(t,Z^N)\,\mathrm{d}t + \Gamma^N\,\mathrm{d}B^N_t \\ Z^N(0) = Z^N_0 \end{cases} \tag{40}$$

with

$$\widehat{\mu}^N_{Z^N,t} = \frac{1}{N}\sum_{i=1}^N \delta_{(Z^N_{2i-1}(t), Z^N_{2i}(t))}.$$

We now fix the Brownian motion $(B^N(t))_{t\geq 0}$ and the initial data Z^N_0 verifying (32), and we are going to prove that (40) has a unique solution.

For any continuous function Z in $C([0,\infty);(\mathbb{R}^d\times\mathbb{R}^d)^N)$, we set

$$\mathcal{T}(Z)(t) = Z_0 + \int_0^t F^N(\widehat{\mu}^N_Z)(s, Z(s))\,\mathrm{d}s + \Gamma^N B^N_t.$$

Let Z^1 and Z^2 be in $C([0,\infty);(\mathbb{R}^d\times\mathbb{R}^d)^N)$. By using Lemma 2 we obtain

$$\begin{aligned}
|\nabla\mathcal{L}(\widehat{\mu}^N_{Z^1} - \widehat{\mu}^N_{Z^2})(t,x)| &\leq \int_0^t \frac{|p(t-s)|}{N}\sum_{i=1}^N |\nabla\Sigma(x - q^1_i(s)) - \nabla\Sigma(x - q^2_i(s))|\,\mathrm{d}s \\
&\leq \|\nabla^2\Sigma\|_{L^\infty(\mathbb{R}^d)}\frac{1}{N}\sum_{i=1}^N\int_0^t |p(t-s)||q^2_i(s) - q^1_i(s)|\,\mathrm{d}s \\
&\leq \|\nabla\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\int_0^t (t-s)|Z^1(s) - Z^2(s)|\,\mathrm{d}s \\
&\leq \|\nabla\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}\frac{t^2}{2}\|Z^1 - Z^2\|_{L^\infty(0,t)}.
\end{aligned} \tag{41}$$

Let us set $c_2 = \frac{1}{2}\|\nabla\sigma_1\|^2_{L^2(\mathbb{R}^d)}\|\sigma_2\|^2_{L^2(\mathbb{R}^n)}$. We now make use of Lemma 6 in order to obtain the estimate

$$\begin{aligned}
|\mathcal{T}(Z^1) - \mathcal{T}(Z^2)|(t) &= \left|\int_0^t \big(F(\widehat{\mu}^N_{Z^1})(s, Z^1(s)) - F(\widehat{\mu}^N_{Z^2})(s, Z^2(s))\big)\,\mathrm{d}s\right| \\
&\leq \int_0^t ((1+\gamma + c_1(s))|Z^1(s) - Z^2(s)| + \|\nabla\mathcal{L}(\widehat{\mu}^N_{Z^1} - \widehat{\mu}^N_{Z^2})(s)\|_{L^\infty})\,\mathrm{d}s \\
&\leq \int_0^t (1+\gamma + c_1(s) + c_2 s^2)\|Z^1 - Z^2\|_{L^\infty(0,s)}\,\mathrm{d}s.
\end{aligned}$$

Let $C_T = 1 + \gamma + c_1(T) + c_2T^2$. We get, for all $0 \leq t \leq T < \infty$,

$$\|\mathcal{T}(Z^1) - \mathcal{T}(Z^2)\|_{L^\infty(0,t)} \leq C_T\int_0^t \|Z^1 - Z^2\|_{L^\infty(0,s)}\,\mathrm{d}s.$$

By induction, we deduce that

$$\|\mathcal{T}^\ell(Z^1) - \mathcal{T}^\ell(Z^2)\|_{L^\infty(0,t)} \leq \frac{(tC_T)^\ell}{\ell!}\|Z^1 - Z^2\|_{L^\infty(0,T)}$$

holds for any $\ell\in\mathbb{N}$ and any $0 \leq t \leq T$. It shows that for ℓ such that $\frac{(tC_T)^\ell}{\ell!} < 1$, $\mathcal{T}^\ell$ is a contraction. Therefore, there exists a unique fixed point in $C([0,T];(\mathbb{R}^d\times\mathbb{R}^d)^N)$ for any $T > 0$, which ends the proof. ■

5.2.2 Vlasov–Wave–Fokker–Planck system

We now turn to prove the existence of solution for (9). To this end, we adopt a particle viewpoint by introducing the following system of stochastic differential equations

$$\begin{cases}
\mathrm{d}x = v(t)\,\mathrm{d}t, \\
\mathrm{d}v = -\nabla_x(V + \Phi_0(t) - \mathcal{L}(\mu)(t))(x)\,\mathrm{d}t - \gamma v(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B(t), \\
\mu_t(A) = \mathbb{P}[(x(t), v(t)) \in A], \\
\mu(0) = \mu_0,
\end{cases} \tag{42}$$

which is now non linear, in contrast to (38), since the trajectories depends on μ, their probability measure. In fact, this is nothing but a different viewpoint on (9). Indeed, if $z = (x, v)$ is a solution of (42), Itô's formula yields

$$\varphi(z(t)) - \varphi(z(0)) = \int_0^t (\nabla_x \varphi(z(s)) \cdot v(s) - \nabla_v \varphi(z(s)) \cdot F(\mu)(s, z(s)))\, \mathrm{d}s \\ + \sqrt{2\gamma} \int_0^t \nabla_v \varphi(z(s))\, \mathrm{d}B(s) + \gamma \int_0^t \Delta_v \varphi(z(s))\, \mathrm{d}s,$$

and by taking the expectation value, we get

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(z)\, \mathrm{d}\mu_t(z) - \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(z)\, \mathrm{d}\mu_0(z) \\ = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla_x \varphi(z) \cdot v - \nabla_v \varphi(z) \cdot F(\mu)(s, z))\, \mathrm{d}\mu_s(z)\, \mathrm{d}s \\ -\gamma \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (v \cdot \nabla_v \varphi(z) + \Delta \varphi)\, \mathrm{d}\mu_s(z)\, \mathrm{d}s.$$

It corresponds to the weak formulation of (9) (which is the analog of Lemma 4 in the stochastic framework). Based on this, we shall prove the following statement.

Theorem 5 *i) For any initial data μ_0 in $\mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$, there exists a unique solution μ of* (9) *in $C_{W_1}([0, +\infty); \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$,*
ii) the process (x, v) solution of (42) *is well defined.*

For analyzing (42), we need to go back to the definition of the Kantorowitch–Rubinstein distance in Section 3.2. It is convenient to change the framework and to work with measures defined on the space $S = C([0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ endowed with the distance $d(f, g) = \|f - g\|_{L^\infty(0,T)} \wedge 1$. We specify the corresponding Kantorowich–Rubinstein distance as D_T, namely

$$D_T(\mu, \nu) = \inf_\pi \Big\{ \int_{(C([0,T];\mathbb{R}^d \times \mathbb{R}^d))^2} (||f - g||_{L^\infty([0,T])} \wedge 1)\, \mathrm{d}\pi(f, g) \Big\}. \tag{43}$$

The interpretation in terms of tight convergence or strong convergence in the dual of Lipschitz functions remains true in that case but it is far more difficult to see concretely. Measurable sets of $S = C([0, T]; \mathbb{R}^d \times \mathbb{R}^d)$ are the elements of the $\sigma-$algebra generated by the sets of the form

$$B_{t,A} = \big\{ \phi(t) \in A \quad \text{for } \phi \in C([0, T]; \mathbb{R}^d \times \mathbb{R}^d) \big\},$$

where t spans $[0, T]$ and A spans the set of Borel-sets in $\mathbb{R}^d \times \mathbb{R}^d$. Given a measure μ on S, we can define a measure–valued function, that we still denote $\mu : t \in [0, T] \mapsto \mu_t \in \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d)$ by the relation

$$\mu_t(A) = \mu(B_{t,A}).$$

Looking at a process Z with probability μ in $\mathcal{M}^1(S)$, Z is almost surely continuous by definition of μ. Them, the dominated convergence theorem allows us to deduce

$$W_1(\mu_{t_1}, \mu_{t_2}) = \inf_{X,Y} \mathbb{E}[|X - Y| \wedge 1] \leq \inf_Z \mathbb{E}[|Z(t_1) - Z(t_2)| \wedge 1] \xrightarrow[t_1 \to t_2]{} 0$$

where the second infimum is taken over all the processes Z with probability μ. Accordingly, $\mathcal{M}^1(S)$ embeds in $C_{W_1}([0, T]; \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$. We can easily check that the distance D_T is stronger than the one we used previously on that set owing to the inequality

$$\sup_{0 \leq t \leq T} W_1(\mu_t, \nu_t) \leq D_T(\mu, \nu).$$

Reminding of (15), the following estimate

$$\left| \int_{\mathbb{R}^d \times \mathbb{R}^d} f\, \mathrm{d}(\mu_s - \nu_s) \right| \leq \|f\|_{\mathrm{Lip}} D_t(\mu, \nu), \tag{44}$$

holds for any $f \in W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$ and $0 \leq s \leq t$.

Proof of Theorem 5. The proof is based on a fixed point argument, again. Let $0 < T < \infty$ and set $S = C([0,T];\mathbb{R}^d \times \mathbb{R}^d)$. We shall use the Kantorowich–Rubinstein distance D_T defined by (43). Let μ be a finite measure on S. We consider the following system

$$\begin{cases} \mathrm{d}x = v(t)\,\mathrm{d}t, \\ \mathrm{d}v = -\nabla_x(V + \Phi_0 - \mathcal{L}(\mu)(t,x)\,\mathrm{d}t - \gamma v(t)\,\mathrm{d}t + \sqrt{2\gamma}\,\mathrm{d}B(t), \\ \mathbb{P}[(x_0,v_0) \in A] = \mu_0(A). \end{cases} \tag{45}$$

Since the field $(t,x,v) \mapsto (v, \nabla_x(V + \Phi_0 - \mathcal{L}(\mu))(t,x)$ is continuous with respect to time and Lipschitz with respect to the phase space variable (x,v), the solution of (45) is a well defined continuous process (see [16, Chapter 5, Theorem 2.9]) . We introduce the mapping

$$A \subset \mathbb{R}^d \times \mathbb{R}^d \longmapsto \mathbb{P}\big[(x(t),v(t)) \in A \quad \text{where } (x,v) \text{ satisfies (45) }\big],$$

which, in turn, defines a new probability measure on S that we denote $\mathscr{T}(\mu)$. Pick μ_1 and μ_2 in $\mathcal{M}^1(S)$. We denote $z_1 = (x_1,v_1)$ and $z_2 = (x_2,v_2)$ the processes associated to $\mathscr{T}(\mu_1)$ and $\mathscr{T}(\mu_2)$, respectively. We bear in mind that both the Brownian motion B and the initial data z_0 are fixed. Integrating (45) we get

$$\begin{cases} (x_1 - x_2)(t) = \displaystyle\int_0^t (v_1 - v_2)(s)\,\mathrm{d}s, \\ (v_1 - v_2)(t) = \displaystyle\int_0^t (F(\mu_1)(s,z_1(s)) - F(\mu_2,(s,z_2(s)) - \gamma(v_1 - v_2)(s))\,\mathrm{d}s. \end{cases}$$

We deduce that both $x_1 - x_2$ and $v_1 - v_2$ are derivable. Using Lemma 6, we get

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(z_1 - z_2)(t)\right| \le (1 + \gamma + c_1(t))|z_1 - z_2|(t) + \|\nabla_x \mathcal{L}(\mu_1 - \mu_2)(t)\|_{L^\infty(\mathbb{R}^d)}.$$

Applying the Grönwall lemma, we get

$$\begin{aligned} |z_1 - z_2|(t) &\le \int_0^t e^{\int_s^t (1+\gamma+c_1(s))\,\mathrm{d}s} \|\nabla_x \mathcal{L}(\mu_1 - \mu_2)(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s \\ &\le C_T \int_0^t \|\nabla_x \mathcal{L}(\mu_1 - \mu_2)(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s, \end{aligned}$$

for all $0 \le t \le T$, where

$$C_T := \exp\left(\int_0^T (1 + \gamma + c_1(s))\,\mathrm{d}s\right).$$

Notice that the right hand side is a deterministic monotically increasing function of t. Hence, we deduce that

$$D_t(\mathscr{T}(\mu_1),\mathscr{T}(\mu_2)) = \inf_{z_1,z_2} \mathbb{E}\left[||z_1 - z_2||_{L^\infty([0,t])} \wedge 1\right] \le C_T \int_0^t \|\nabla_x \mathcal{L}(\mu_1 - \mu_2)(s)\|_{L^\infty(\mathbb{R}^d)}\,\mathrm{d}s \tag{46}$$

holds. We now turn to dominate the right hand side by $D_t(\mu_1,\mu_2)$. Let $x \in \mathbb{R}^d$ and $0 \le s \le t$; owing to (44) and Lemma 2, we have

$$\begin{aligned} |\nabla_x \mathcal{L}\,(\mu_1 - \mu_2)(s,x)| &= \left|\int_0^s p(s-\tau) \int_{\mathbb{R}^d \times \mathbb{R}^d} \Sigma(x-y)\,\mathrm{d}(\mu_{1,\tau} - \mu_{2,\tau})(y,v)\,\mathrm{d}\tau\right| \mathrm{d}\tau \\ &\le \int_0^s |p(s-\tau)| \Big(2\|\Sigma(x-\cdot)\|_{L^\infty(\mathbb{R}^d)} + \|\nabla(\Sigma(x-\cdot)\|_{L^\infty(\mathbb{R}^d)}\Big) D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau \\ &\le \|\sigma_2\|_{L^2}^2 \|\sigma_1\|_{L^2}(2\|\sigma_1\|_{L^2} + \|\nabla\sigma_1\|_{L^2}) \int_0^s (s-\tau) D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau. \end{aligned}$$

Setting $c_3 = \|\sigma_2\|_{L^2}^2 \|\sigma_1\|_{L^2}(2\|\sigma_1\|_{L^2} + \|\nabla\sigma_1\|_{L^2})$, we obtain

$$\|\nabla_x \mathcal{L}(\mu_1 - \mu_2)(s)\|_{L^\infty(\mathbb{R}^d)} \le c_3 T \int_0^s D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau.$$

Finally, inserting this estimate into (46), we arrive at

$$D_t(\mathscr{T}(\mu_1),\mathscr{T}(\mu_2)) \le c_3 T C_T \int_0^t \int_0^s D_\tau(\mu_1,\mu_2)\,\mathrm{d}\tau\,\mathrm{d}s.$$

We deduce by induction that

$$D_t(\mathscr{T}^\ell(\mu_1), \mathscr{T}^\ell(\mu_2)) \le \frac{(c_3 T C_T t)^{2\ell}}{(2\ell)!} D_T(\mu_1, \mu_2)$$

holds for any $\ell \in \mathbb{N}$ and any $0 \le t \le T$. It shows that for ℓ such that $\frac{(c_3 C_T T^2)^{2\ell}}{(2\ell)!} < 1$, $\mathscr{T}^\ell$ is a contraction for the distance D_T. Therefore, $\mathscr{T}$ has a unique fixed point in $\mathcal{M}^1(S)$, and thus $\mu_t \in C_{W_1}([0,T]; \mathcal{M}^1(\mathbb{R}^d \times \mathbb{R}^d))$, for any $T > 0$. It ends the proof. ∎

5.2.3 Asymptotic analysis

This Section is devoted to the proof of Theorem 3. The analysis relies on many estimates that have been established above. We compare $(z_i^N)_{i\in\{1,\dots,N\}}$, solution of (11) and $(\tilde{z}_i^N)_{i\in\{1,\dots,N\}}$, solution of (38). The two equations start from the same initial data $(z_{i,0}^N)_{i\in\{1,\dots,N\}}$ and involve the same family of Brownian motions $(B_i(t))_{i\in\{1,\dots,N\}}$. In fact, for any fixed $i \in \{1, ..., N\}$, $t \mapsto (\tilde{q}_i^N(t), \tilde{p}_i^N(t)) = \tilde{z}_i^N(t)$ is the solution of (45) for the initial data $q_{0,i}^N, p_{0,i}^N$ distributed according to the common measure μ_0, see (32). In particular, we shall use the fact that

$$\mu_t(\,\mathrm{d}z) \text{ is the common law of the } (\tilde{q}_i^N(t), \tilde{p}_i^N(t))\text{'s.}$$

We estimate the difference

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(q_i^N - \tilde{q}_i^N)(t) = (p_i^N - \tilde{p}_i^N)(t) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(p_i^N - \tilde{p}_i^N)(t) = (-F(\widehat{\mu}^N)(t, q_i^N(t)) + F(\mu)(t, \tilde{q}_i^N(t))) - \gamma(p_i^N - \tilde{p}_i^N)(t). \end{cases}$$

Using Lemma 6, we get (at least in the sense of distributions)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|z_i^N - \tilde{z}_i^N\right|(t) \le (1 + \gamma + c_1(t))|z_i^N - \tilde{z}_i^N|(t) + |\nabla\mathcal{L}(\widehat{\mu}^N - \mu)(t, \tilde{q}_i^N(t))|.$$

Since the initial data coincide, we thus have

$$\left|z_i^N - \tilde{z}_i^N\right|(t) \le \int_0^t (1 + \gamma + c_1(s))|z_i^N - \tilde{z}_i^N|(s)\,\mathrm{d}s + \int_0^t |\nabla\mathcal{L}(\widehat{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|\,\mathrm{d}s.$$

In order to deal with the last term, we introduce the empirical density $\tilde{\mu}^N$ associated to the family $(\tilde{z}_i^N)_{i\in\{1,\dots,N\}}$ solution of (38)

$$\tilde{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{z}_i^N(t)}. \tag{47}$$

Then, using (41), we split

$$\begin{aligned} |\nabla\mathcal{L}(\widehat{\mu}^N - \mu)(s, \tilde{q}_i^N(s))| &\le |\nabla\mathcal{L}(\widehat{\mu}^N - \tilde{\mu}^N)(s, \tilde{q}_i^N(s))| + |\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))| \\ &\le c_2 \int_0^s \frac{s-\sigma}{N}\sum_{j=1}^N |z_j^N - \tilde{z}_j^N|(\sigma)\,\mathrm{d}\sigma + |\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|. \end{aligned}$$

We are thus led to

$$\begin{aligned} \sup_{0\le\tau\le t}\left|z_i^N - \tilde{z}_i^N\right|(\tau) \le &\int_0^t (1 + \gamma + c_1(s))|z_i^N - \tilde{z}_i^N|(s)\,\mathrm{d}s \\ &+ \int_0^t \int_0^s c_2 \frac{s-\sigma}{N}\sum_{j=1}^N |z_j^N - \tilde{z}_j^N|(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}s \\ &+ \int_0^t |\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|\,\mathrm{d}s. \end{aligned}$$

We shall take the expectation value in this inequality. As (11), with (32) and (38)-(39) do not change if we permute the indices $i \in \{1,..,N\}$ of the particles, all the random variables $|z_i^N - \tilde{z}_i^N|(t)$ share the same probability and the same expectation value. Therefore, we remark that, for any $i, j \in \{1,...,N\}$ and $s \geq 0$, we have

$$\mathbb{E} \sup_{0\leq\tau\leq s} \left|z_i^N - \tilde{z}_i^N\right|(\tau) = \mathbb{E} \sup_{0\leq\tau\leq s} \left|z_j^N - \tilde{z}_j^N\right|(\tau) = \frac{1}{N}\sum_{k=1}^{N} \mathbb{E} \sup_{0\leq\tau\leq s} \left|z_k^N - \tilde{z}_k^N\right|(\tau).$$

It allows us to obtain

$$\begin{aligned}\mathbb{E} \sup_{0\leq\tau\leq t} \left|(z_i^N - \tilde{z}_i^N)(\tau)\right| \leq \int_0^t \Big(1+\gamma+c_1(s)+c_2\frac{s^2}{2}\Big)\mathbb{E} \sup_{0\leq\sigma\leq s} |z_i^N - \tilde{z}_i^N|(s)\,\mathrm{d}s \\ + \int_0^t \mathbb{E}|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|\,\mathrm{d}s.\end{aligned}$$

Hence, by using the Grönwall lemma, we deduce that, for any $0 < T < \infty$ we can find $C_T > 0$ such that

$$\mathbb{E} \sup_{0\leq\tau\leq t} \left|z_i^N - \tilde{z}_i^N\right|(\tau) \leq C_T \int_0^t \mathbb{E}|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_i^N(s))|\,\mathrm{d}s \tag{48}$$

holds for any $0 \leq t \leq T$. We are going to estimate the right hand side as a consequence of the law of the large numbers. For any iid square integrable family of random variables $(Y_i)_{1\leq i\leq N}$ we indeed remind the reader that

$$\mathbb{E}\left[\Big|\frac{1}{N}\sum_{i=1}^{N} Y_i - \mathbb{E}[Y_1]\Big|\right] \leq \frac{(\mathrm{Var}(Y_1))^{1/2}}{\sqrt{N}} \tag{49}$$

holds. We now show that $\mathbb{E}\left[|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_1^N(s))|\right]$ can be written in a form similar to the left hand side of (49). By definition of $\tilde{\mu}^N$ as the empirical measure of the $\tilde{z}_i^N$, see (47), we have

$$\mathbb{E}\left[|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_1^N(s))|\right] = \mathbb{E}\left[\Big|\frac{1}{N}\sum_{i=1}^{N} \nabla\mathcal{L}(\delta_{\tilde{z}_i^N})(s, \tilde{q}_1^N(s)) - \nabla\mathcal{L}(\mu)(s, \tilde{q}_1^N(s))\Big|\right]. \tag{50}$$

In order to simplify the computations, we set $X_i^1(s) = \nabla\mathcal{L}(\delta_{\tilde{z}_i^N})(s, \tilde{q}_1^N(t))$ for $i \in \{1,...,N\}$. Assuming that we know $\tilde{z}_1$, the family $(X_2^1, X_3^1, ..., X_N^1)$ is made of iid random variables with the common expectation value

$$\begin{aligned}\mathbb{E}\left[\nabla\mathcal{L}(\delta_{\tilde{z}_2^N})(s, \tilde{q}_1^N(s))\Big|\tilde{z}_1\right] &= \mathbb{E}\left[\int_0^s p(s-\sigma)\nabla\mathit{\Sigma}(\tilde{q}_1^N(s) - \tilde{q}_2^N(\sigma))\,\mathrm{d}\sigma\Big|\,\tilde{z}_1\right] \\ &= \int_0^s p(s-\sigma)\mathbb{E}\left[\nabla\mathit{\Sigma}(\tilde{q}_1^N(s) - \tilde{q}_2^N(\sigma))|\tilde{z}_1\right]\,\mathrm{d}\sigma \\ &= \int_0^s p(s-\sigma)\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla\mathit{\Sigma}(\tilde{q}_1^N(s) - y)\,\mathrm{d}\mu_\sigma(y,v)\,\mathrm{d}\sigma \\ &= \nabla\mathcal{L}(\tilde{\mu})(s, \tilde{q}_1^N(s)).\end{aligned}$$

This observation allows us to split (50) as follows

$$\begin{aligned}\mathbb{E}\left[|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_1^N(s))|\right] \leq \frac{1}{N}\mathbb{E}\left[|X_1^1(s) - \mathbb{E}[X_2^1(s)|\tilde{z}_1]|\right] \\ + \frac{N-1}{N}\mathbb{E}\left[\left|\frac{1}{N-1}\sum_{i=2}^{N} X_i^1(s) - \mathbb{E}[X_2^1(s)|\tilde{z}_1]\right|\right].\end{aligned} \tag{51}$$

Since $(X_2^1, X_3^1, ..., X_N^1)$ are iid, we estimate the second term of the right hand side with (49)

$$\begin{aligned}\mathbb{E}\left[\left|\frac{1}{N-1}\sum_{i=2}^{N} X_i^1 - \mathbb{E}[X_2^1|\tilde{z}_1]\right|\right] &= \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{E}\left[\left|\frac{1}{N-1}\sum_{i=2}^{N} X_i^1 - \mathbb{E}[X_2^1|\tilde{z}_1]\right| \,\Big|\, \tilde{z}_1\right]\,\mathrm{d}\mu(\tilde{z}_1) \\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{1}{\sqrt{N-1}}(\mathrm{Var}(X_2^1|\tilde{z}_1))^{1/2}\,\mathrm{d}\mu(\tilde{z}_1).\end{aligned}$$

From the estimates on $\mathcal{L}$, we also have $|X_i^1(s)| \le \|\sigma_1\|_{L^2}\|\nabla\sigma_1\|_{L^2}\|\sigma_2\|_{L^2}^2 T =: c_4 T$ for any $0 \le s \le T < \infty$. Coming back to (51), we get

$$\mathbb{E}\left[|\nabla\mathcal{L}(\tilde{\mu}^N - \mu)(s, \tilde{q}_1^N(s))|\right] \le \frac{2c_4 T}{N} + \frac{c_4 T}{\sqrt{N}}$$

Eventually, we insert this in (48) and we conclude that

$$\mathbb{E}\left[\sup_{0\le t\le T} |z_i^N - \tilde{z}_i^N|(t)\right] \le \frac{3c_4 T C_T}{\sqrt{N}}$$

holds. It finishes the proof of Theorem 3. ■

Let us detail a few relevant consequences of Theorem 3.

Corollary 2 *Let μ be the solution of* (9) *We have*

i) With D_T the Kantorowich-Rubinstein distance on $\mathcal{M}^1(C([0,T];\mathbb{R}^d \times \mathbb{R}^d))$, defined by (43)*, we have*

$$D_T(\mu^{(1,N)}, \mu) \le \frac{C}{\sqrt{N}}.$$

ii) Multi-time propagation of chaos holds: it means that, taking $0 \le \tau \le s$ and φ_1, φ_2 in $C \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\lim_{N\to\infty} \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} \varphi_1(z_1)\varphi_2(z_2)\,\mathrm{d}\mu_{s,\tau}^{(2,N)}(z_1, z_2) = \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_1(z)\,\mathrm{d}\mu_s(z)\right)\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_2(z)\,\mathrm{d}\mu_\tau(z)\right).$$

In particular $\widehat{\mu}^N$ converges in law to μ.

This statement allows us to interpret (9) as the equation satisfied by the limit as $N \to \infty$ of both the first marginal $\mu^{(1,N)}$ and the empirical distribution $\widehat{\mu}^N$ for a system of particles $(q_i^N, p_i^N)_{i\in\{1,\dots,N\}}$ solution of (11) with (32).

Proof of Corollary 2. Item i) is a direct consequence of the definition of D_T and the estimate in Theorem 3. Let us discuss item ii). From the definition of W_1, we have

$$\begin{aligned} W_1(\mu_{s,\tau}^{(2,N)}, \mu_s \otimes \mu_\tau) &= \mathbb{E}\left[|(z_1^N(s), z_2^N(\tau)) - (\tilde{z}_1^N(s), \tilde{z}_2^N(\tau))| \wedge 1\right] \\ &\le \mathbb{E}\left[|z_1^N(s) - \tilde{z}_1^N(s)| \wedge 1\right] + \mathbb{E}\left[|z_2^N(s) - \tilde{z}_2^N(s)| \wedge 1\right] \\ &\le \frac{2C}{\sqrt{N}}. \end{aligned} \tag{52}$$

Since the Kantorovich-Rubinstein distance metrizes the tight convergence, we get for all φ in $C_b((\mathbb{R}^d \times \mathbb{R}^d)^2)$

$$\lim_{N\to\infty} \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} \varphi(z_1, z_2)\,\mathrm{d}\mu_{s,\tau}^{(2,N)}(z_1, z_2) = \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} \varphi(z_1, z_2)\,\mathrm{d}\mu_s(z_1)\mu_\tau(z_2).$$

With $\varphi = \varphi_1 \otimes \varphi_2$, we get the announced result. For $s = \tau$, we obtain (37) for $k = 2$. According to Proposition 1, $\widehat{\mu}^N$ converges in law to μ. ■

Actually, the convergence of $(\widehat{\mu}^N)_N$ can be more precise, with an explicit rate. Let χ be a bounded Lipschitz function on $\mathbb{R}^d \times \mathbb{R}^d$. We have

$$\begin{aligned} \mathbb{E}\left[\left|\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\widehat{\mu}_t^N - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\mu_t\right|\right] &= \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \chi(z_i(t) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\mu\right|\right] \\ &\le \mathbb{E}[\chi(z_1(t)) - \chi(\tilde{z}_1(t))] + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \chi(\tilde{z}_i(t) - \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi\,\mathrm{d}\mu\right|\right] \\ &\le \|\nabla\chi\|_{L^\infty}\frac{C}{\sqrt{N}} + \mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N \chi(\tilde{z}_i(t) - \mathbb{E}[\chi(\tilde{z}_1(t))]\right|\right] \\ &\le \|\nabla\chi\|_{L^\infty}\frac{C}{\sqrt{N}} + \frac{(\mathrm{Var}[\chi(\tilde{z_1}(t)])^{1/2}}{\sqrt{N}} \\ &\le \frac{C\|\nabla\chi\|_{L^\infty} + \|\chi\|_{L^\infty}}{\sqrt{N}}, \end{aligned}$$

where we have just applied Theorem 3 to estimate the first term and the law of large numbers to the family $(\chi(\tilde{z}_i(t))_{i\in\{1,\dots,N\}}$, which by construction is iid, to deal with the second term.

We now come back to the discussion of (33), and show that the remainder term $\mathcal{I}_N$ defined by (34) goes weakly to 0. We have

$$\begin{aligned}\mathbb{E}\left[\mathcal{I}_N^2\right] &= \mathbb{E}\left[\left|\frac{\sqrt{2\gamma}}{N}\sum_{i=1}^N\int_0^t \nabla_v\varphi(z_i^N(s))\,\mathrm{d}B_i(s)\right|^2\right]\\ &\le 4\gamma\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N(\nabla_v\varphi(z_i^N(s))-\nabla_v\varphi(\tilde{z}_i^N(s)))\,\mathrm{d}B_i(s)\right|^2\right]\\ &\qquad +4\gamma\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N\nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)\right|^2\right].\end{aligned}$$

We can get rid of the Brownian motion in the estimates of those two terms owing to

$$\mathbb{E}\left[\left|\int_0^t X(s)\,\mathrm{d}B_s\right|^2\right]=\int_0^t\mathbb{E}\left[|X(s)|^2\right]\,\mathrm{d}s,$$

which is a consequence of Itô's formula (see [16, Chapter 3.2.A]). On the one hand, we just have to use convexity inequality and apply Theorem 3 and we obtain

$$\begin{aligned}&\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N(\nabla_v\varphi(z_i^N(s))-\nabla_v\varphi(\tilde{z}_i^N(s)))\,\mathrm{d}B_i(s)\right|^2\right]\\ &\qquad=\int_0^t\mathbb{E}\left[\left|\frac{1}{N}\sum_{i=1}^N\nabla_v\varphi(z_i^N(s))-\nabla_v\varphi(\tilde{z}_i^N(s))\right|^2\right]\,\mathrm{d}s\\ &\qquad\le\frac{1}{N}\sum_{i=1}^N\int_0^t\mathbb{E}\left[\left|\nabla_v\varphi(z_i^N(s))-\nabla_v\varphi(\tilde{z}_i^N(s))\right|^2\right]\,\mathrm{d}s\\ &\qquad\le\int_0^t\left(\mathbb{E}\left[\left|\nabla_v\varphi(z_1^N(s))-\nabla_v\varphi(\tilde{z}_1^N(s))\right|\right]\right)^2\,\mathrm{d}s\\ &\qquad\le\int_0^t\|\nabla_v\varphi\|_{\mathrm{Lip}}^2(\mathbb{E}[|z_1(s)-\tilde{z}_1(s)|\wedge 1])^2\,\mathrm{d}s\\ &\qquad\le\frac{C^2\|\nabla_v\varphi\|_{\mathrm{Lip}}^2 t}{N}.\end{aligned}$$

On the other hand, the family of the N random variables $\int_0^t\nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)$ are iid and as such it is a martingale. We thus get

$$\mathbb{E}\left[\int_0^t\nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)\right]=0.$$

Applying the law of large numbers, we get

$$\begin{aligned}\mathbb{E}\left[\left|\int_0^t \frac{1}{N}\sum_{i=1}^N\nabla_v\varphi(\tilde{z}_i^N(s))\,\mathrm{d}B_i(s)\right|^2\right] &= \frac{1}{N}\mathbb{E}\left[\left|\int_0^t\nabla_v\varphi(\tilde{z}_1^N(s))\,\mathrm{d}B_1(s)\right|^2\right]\\ &=\frac{1}{N}\int_0^t\mathbb{E}\left[\left|\nabla_v\varphi(\tilde{z}_1^N(s))\right|^2\right]\,\mathrm{d}s\\ &\le\frac{t}{N}\|\nabla_v\varphi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^2.\end{aligned}$$

Finally, from Jensen inequality we get for all $0\le t\le T$,

$$\left(\mathbb{E}\left[|\mathcal{I}_N|\right]\right)^2\le\mathbb{E}\left[\mathcal{I}_N^2\right]\le\frac{4\gamma t}{N}\left(\|\nabla_v\varphi\|_{L^\infty(\mathbb{R}^d\times\mathbb{R}^d)}^2+C^2\|\nabla_v\varphi\|_{\mathrm{Lip}}^2\right).$$

Let us go back to (36), the equation satisfied by $\mu^{(1,N)}$. We use (52) and (15) with $\chi(z_1, z_2) = \nabla\Sigma(x_1 - x_2)\nabla_v\varphi(z_1)$. Hence, for all φ in $W^{1,\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, we obtain

$$\begin{aligned}\lim_{N\to\infty} \int_0^t \int_0^s \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} & p(s-\tau)\nabla\sigma(x_1-x_2).\nabla_v\varphi(z_1)\,\mathrm{d}\mu^{(2,N)}_{s,\tau}(z_1,z_2)\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \int_0^t \int_0^s \int_{(\mathbb{R}^d\times\mathbb{R}^d)^2} p(s-\tau)\nabla\sigma(x_1-x_2).\nabla_v\varphi(z_1)\,\mathrm{d}\mu_s(z_1)\,\mathrm{d}\mu_\tau(z_2)\,\mathrm{d}\tau\,\mathrm{d}s \\ &= \int_0^t \int_{(\mathbb{R}^d\times\mathbb{R}^d)} \nabla_v\varphi(z).\nabla_x\mathcal{L}(\mu)(s,x)\,\mathrm{d}\mu_s(z)\,\mathrm{d}s,\end{aligned}$$

with a rate of convergence at least $\mathscr{O}(\frac{1}{\sqrt{N}})$.

Acknowledgements We thank Stephan De Bièvre for introducing the subject to us and for many passionate and fruitful discussions.

References

1. Aguer, B., De Bièvre, S., Lafitte, P., Parris, P.E.: Classical motion in force fields with short range correlations. J. Stat. Phys. **138**(4-5), 780–814 (2010)
2. Alonso, R., Goudon, T., Vavasseur, A.: Damping of particles interacting with a vibrating medium. Tech. rep., Inria (2016)
3. Boldrighini, C., Bunimovich, L.A., Sinaĭ, Y.G.: On the Boltzmann equation for the Lorentz gas. J. Stat. Phys. **32**(3), 477–501 (1983). DOI 10.1007/BF01008951. URL http://dx.doi.org/10.1007/BF01008951
4. Bolley, F.: Limite de champ moyen de systèmes de particules. In: Séminaire X-EDP. Ecole Polytechnique (2010)
5. Bouchut, F., Golse, F., Pallard, C.: Nonresonant smoothing for coupled wave + transport equations and the Vlasov-Maxwell system. Rev. Mat. Iberoamericana **20**(3), 865–892 (2004)
6. Bruneau, L., De Bièvre, S.: A Hamiltonian model for linear friction in a homogeneous medium. Comm. Math. Phys. **229**(3), 511–542 (2002)
7. Caglioti, E., Golse, F.: On the Boltzmann-Grad limit for the two dimensional periodic Lorentz gas. J. Stat. Phys. **141**(2), 264–317 (2010)
8. De Bièvre, S., Goudon, T., Vavasseur, A.: Particles interacting with a vibrating medium: existence of solutions and convergence to the Vlasov–Poisson system. Tech. rep., Inria (2016)
9. De Bièvre, S., Lafitte, P., Parris, P.E.: Normal transport at positive temperatures in classical Hamiltonian open systems. In: Adventures in mathematical physics, *Contemp. Math.*, vol. 447, pp. 57–71. Amer. Math. Soc., Providence, RI (2007)
10. De Bièvre, S., Parris, P.E.: Equilibration, generalized equipartition, and diffusion in dynamical Lorentz gases. J. Stat. Phys. **142**(2), 356–385 (2011)
11. De Bièvre, S., Parris, P.E., Silvius, A.: Chaotic dynamics of a free particle interacting linearly with a harmonic oscillator. Phys. D **208**(1-2), 96–114 (2005)
12. Dobrušin, R.L.: Vlasov equations. Funktsional. Anal. i Prilozhen. **13**(2), 48–58, 96 (1979)
13. Galavotti, G.: Rigorous theory of the Boltzmann equation in the Lorentz gas. Tech. rep., Istituto di Fisica, Universitá di Roma (1973). Nota interna n. 358
14. Golse, F.: The mean-field limit for the dynamics of large particle systems. In: Journées Equations aux dérivées partielles, Forges-les-Eaux, 2-6 juin 2003 (2003)
15. Golse, F.: On the periodic Lorentz gas and the Lorentz kinetic equation. Ann. Fac. Sci. Toulouse Math. (6) **17**(4), 735–749 (2008)
16. Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus. Graduate texts in mathematics. Springer-Verlag, New York, Berlin, Heidelberg (1991). URL http://opac.inria.fr/record=b1079144. Autres tirages corrigés : 1996, 1997, 1999, 2000, 2005
17. Lafitte, P., Parris, P.E., De Bièvre, S.: Normal transport properties in a metastable stationary state for a classical particle coupled to a non-Ohmic bath. J. Stat. Phys. **132**(5), 863–879 (2008)
18. Malliavin, P.: Intégration et probabilités, analyse de Fourier et analyse spectrale. Collection Maîtrise de mathématiques pures. Masson (1982)
19. Marklof, J., Strömbergsson, A.: The Boltzmann-Grad limit of the periodic Lorentz gas. Ann. of Math. (2) **174**(1), 225–298 (2011)
20. McKean, H.P.: Propagation of chaos for a class of non-linear parabolic equations, vol. 7, pp. 41–57. Catholic Univ. (1967)
21. Méléard, S.: Asymptotic behaviour of some interacting particle systems ; McKean–Vlasov and Boltzmann models, *Lecture Notes in Math.*, vol. 1627. Springer (1996)
22. Soret, E., De Bièvre, S.: Stochastic acceleration in a random time-dependent potential. Stochastic Process. Appl. **125**(7), 2752–2785 (2015). DOI 10.1016/j.spa.2015.01.012. URL http://dx.doi.org/10.1016/j.spa.2015.01.012
23. Sznitman, A.S.: Topics in propagation of chaos. In: Ecole d'Eté de Probabilités de Saint-Flour XIX, 1989, *Lecture Notes in Mathematics*, vol. 1464, pp. 165–251. Springer (1991)
24. Villani, C.: Optimal transport, old and new, *Grundlehren der mathematischen Wissenschaften*, vol. 338. Spinger (2009)