Modeling and Simulation of Fluid-Particles Flows

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Abstract

In this paper, we review a few aspects of two phase flows where a disperse phase — the particles — interacts with a dense fluid. We are thus led to consider kinetic equations where the leading term is due to the drag force exerted by the fluid on the particles. We discuss several asymptotic questions and present a numerical scheme which is able to treat the multiscale features of the problem.

1 Introduction

This paper is concerned with models describing disperse particles interacting with a fluid. This work is motivated by the transport of pollutants [14, 39], the dispersion of smokes and dusts [18], the modeling of biomedical flows [7, 5] as well as combustion theory, with applications to Diesel engines or propulsors [1, 2, 20, 34, 40].

The basis of the models we are interested in assumes that the leading effect is due to the drag force exerted by the fluid on the particles. As a warm up, let us explain what is going on with the very simple example of a single particle, spherically shaped with radius a and mass density ρ_{p}, dropped in a fluid. The fluid is characterized by its mass density ρ_{f}, its velocity u, and its dynamic viscosity μ. The drag force is supposed to be proportional to the relative velocity between the fluid and the particle so that the motion of the particle is described by the ODE system

$$\frac{\mathrm{d}}{\mathrm{d}t}X = V, \qquad \frac{4}{3}\pi a^3 \rho_{\mathrm{p}} \ \frac{\mathrm{d}}{\mathrm{d}t}V = 6\pi\mu a \ \big(u(t,X) - V\big) + \frac{4}{3}\pi a^3 \rho_{\mathrm{p}} \mathcal{F} \quad (1.1)$$

which defines the evolution of the position X and velocity V of the particle. In (1.1), $\mathcal{F}$ represents the density of external forces applied to the particle; for instance, considering gravity and buoyancy forces, we have

$$\mathcal{F} = \mathrm{g}\Big(\frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{p}}} - 1\Big),$$

with $\mathrm{g} > 0$ the gravitational acceleration. Let us set

$$\mathcal{T}_{St} = \frac{2a^2\rho_p}{9\mu}.$$

When $u = 0$ in the gravity driven case, the velocity tends to the limit value $\mathcal{T}_{St}\mathrm{g}(\rho_{\mathrm{f}}/\rho_{\mathrm{p}} - 1)$, the so–called Stokes settling velocity, and $\mathcal{T}_{St}$, the Stokes settling time, clearly appears as a relaxation time, characterizing how the friction decelerates the particle. In complex mixtures, the disperse phase can be seen as an ensemble of particles, which is described by means of a particle distribution function $f \geq 0$ depending on time and on the phase space variable (x, v) where x stands for a space variable and v a velocity variable:

$$\int_\Omega \int_{\mathcal{V}} f(t, x, v)\,\mathrm{d}v\,\mathrm{d}x$$

is the number of particles that can be found at time $t \geq 0$, in the domain $\Omega \times \mathcal{V} \subset \mathbb{R}^N \times \mathbb{R}^N$ of the phase space. Therefore, the evolution of the density f is governed by the following Vlasov-type equation

$$\partial_t f + \nabla_x \cdot (vf) + \frac{1}{\mathcal{T}_{St}} \nabla_v \cdot \big((u - v)f\big) + \nabla_v \cdot \big(\mathcal{F} f\big) = 0. \qquad (1.2)$$

In particular, if we have P independent particles described by their position-velocity pair (X_j, V_j), $j \in \{1, ..., P\}$, obeying (1.1), then $f(t, x, v) = \sum_{j=1}^P \delta(x = X_j(t)) \otimes \delta(v = V_j(t))$ satisfies (1.2). It clarifies the connection between the statistical and the particles viewpoints. The questions that we address can be summarized as follows:

- Modeling issues are crucial for applications. A fundamental question is concerned with the drag force: the linear expression of the drag force we used above — that is the Stokes law — looks reasonable for low Reynolds numbers; otherwise, a more complex and non linear relation should be used, which will make the mathematical analysis harder. The role of the density of the fluid, which does not appear in the expression of the Stokes drag force above, should be also discussed. Furthermore, additional effects could be important depending on the flow and need to be incorporated in the model: added mass effect, Basset force, lift force... see [19, 28, 31]. The models can also account for more complex interactions between the particles due to collision effects and shape or size variation through coagulation and fragmentation phenomena (see e. g. [2, 4, 34]).

- The surrounding fluid is considered as "turbulent" which, roughly speaking, means that the velocity u has fast and high variations. Hence, one seeks some averaging procedures which allow to derive useful and simple models that account for these turbulent effects. In this spirit, we mention [13, 17, 24], and on more physical grounds [41].
- Another viewpoint consists in coupling the evolution of the particles to hydrodynamic equations that describe the behavior of the dense phase. We are thus led to nonlinear systems of PDEs and we address the questions of existence, uniqueness, stability properties of the solutions, as well as we aim at designing efficient numerical schemes able to handle the multiscale features of the problem. We refer in particular to [3, 6, 26, 27, 34, 36] for well-posedness analysis of such coupled fluid-kinetic models. Asymptotic problems and stability properties are investigated in [8, 9, 10, 21, 22, 30, 34, 37]. Concerning numerical methods, of course the key reference is [1]; we also mention [4, 34] and below we shall describe the method introduced in [11].

In this paper, we will focus on some of these questions. In Section 2, we propose a possible modeling of "turbulence" by assuming that u is a time dependent random field. Discussing the scaling of the equations of motion with respect to the relaxation time associated to the Markov properties of the velocity, we identify several relevant asymptotic regimes for which we are able to describe limit effective equations. Section 3 deals with coupled kinetic/hydrodynamic equations and we show how efficient numerical schemes can be designed, based on the dissipative and asymptotic properties of the model.

2 Particles in Turbulent Flows

In this Section, we neglect the external forces ($\mathcal{F} = 0$) and we consider the velocity field as given, but we wish to investigate the behavior of the solutions depending on parameters that characterize the flow. We will assume time randomness intended to mimic some "turbulence effects" and we will derive averaged models.

2.1 Modeling of the Turbulent Flow

The function u is thought of as the velocity field of a "turbulent" flow which is therefore modeled through a time dependent random field. We model the time dependence of the velocity by a Markov process (with, say, dimension 1) which is consistent with the rough idea of an exponentially fast decay of the time-correlations. Let us detail the model we have in mind. The velocity at time t and position x is defined by

$$u(t,x) = \mathcal{U}\ U(x/\ell, Q_{t/\mathcal{T}_M})$$

where:

- $\mathcal{U}$ is the amplitude of the velocity,
- $U : \mathbb{R}^N \times \mathbb{R}$ is a smooth, divergence free and bounded (dimensionless) function in which the modeling of turbulence is embodied, ℓ being a typical length scale of the variation of the velocity,
- $t \mapsto Q_{t/\mathcal{T}_M}$ is a stationary Markov process modeling the randomness of turbulence with typical decorrelation time $\mathcal{T}_M$.

Typically, one expects some translational invariance property of the field, i.e.:

$$\mathrm{Law}(U(x/\ell, Q_{t/\mathcal{T}_M})) = \mathrm{Law}(U(x/\ell + n, Q_{t/\mathcal{T}_M})),$$

for all $n \in \mathbb{R}^N$ or at least for all $n \in \mathbb{Z}^N$. The asymptotic regimes studied in this paper will typically lead to non-trivial evolution equations as soon as the following time auto-correlations are non-vanishing:

$$\int_0^{+\infty} \int_{(0,1)^N} \mathbb{E}\left(U(y, Q_0) \otimes U(y + vt, Q_{t/\mathcal{T}_M})\right) \, \mathrm{d}y \, \mathrm{d}t \neq 0,$$

for any $(y, v) \in \mathbb{R}^{2N}$.

To simplify the formal computations and the analysis, we restrict ourselves to the following framework:

- The velocity field satisfies

$$y \longmapsto U(y,q) \quad \text{is } (0,1)^N = \mathbb{Y}-\text{periodic}, (Y_t^\star, Q_t^\star) \tag{2.1}$$

$$\sup_{y\in\mathbb{Y},\ q\in\mathbb{R}} |U(y,q)| \le C < \infty, \tag{2.2}$$

$$\nabla_y \cdot U(y,q) = 0 \quad \text{for a. e. } q. \tag{2.3}$$

- $t \mapsto Q_t \in \mathbb{R}$ is the Markov process at hand. It is described by an operator $\mathcal{Q}$, the Markov generator. We assume long time mixing properties characterized by a stationary probability distribution $\mathcal{M}(q)\,\mathrm{d}q$, with $\mathcal{M}$ a normalized positive function:

$$\mathcal{M}(q) > 0, \qquad \int_{\mathbb{R}} \mathcal{M}\,\mathrm{d}q = 1, \qquad \mathcal{Q}(\mathbb{1}) = 0, \qquad \mathcal{Q}^\star(\mathbb{1}) = 0,$$

where, here and below, we denote by $\mathcal{Q}^\star$ the adjoint operator defined for the inner product in $L^2(\mathbb{R}, \mathcal{M}(q)\,\mathrm{d}q)$:

$$(F, G) = \int_{\mathbb{R}} F(q)G(q)\ \mathcal{M}(q)\,\mathrm{d}q.$$

The parameter $\mathcal{T}_M$ then appears as a relaxation time.

The mixing requirement on the Markov operators $\mathcal{Q}^\star/\mathcal{Q}$ can be embodied in the spectral gap assumption

$$\begin{cases} \text{There exists } \sigma > 0 \text{ such that} \\ -\displaystyle\int_{\mathbb{R}} \mathcal{Q}(F)\ F\mathcal{M}(q)\,\mathrm{d}q \geq \sigma \int_{\mathbb{R}} \Big| F(q) - \int_{\mathbb{R}} F(q')\mathcal{M}(q')\,\mathrm{d}q' \Big|^2 \mathcal{M}(q)\,\mathrm{d}q \geq 0. \end{cases} \tag{2.4}$$

A typical example uses the Fokker-Planck operator

$$\mathcal{Q}(F) = \frac{1}{\mathcal{M}(q)}\partial_q\big(\mathcal{M}\partial_q F\big) \tag{2.5}$$

with $\mathcal{M}(q) = e^{-q^2/2}/\sqrt{2\pi}$. In this case, the evolution of a particle is then governed by the following set of differential equations

$$\begin{aligned} \mathrm{d}X &= V\,\mathrm{d}t, \\ \mathrm{d}V &= \frac{1}{\mathcal{T}_{St}}\big(\mathcal{U}\ U(X/\ell, Q_{t/\mathcal{T}_M}) - V\big)\,\mathrm{d}t, \\ \mathrm{d}Q &= \partial_q\mathcal{M}(q)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}W_t \end{aligned}$$

with W_t a Brownian motion. Another example relies on a jump process, associated to the generator

$$\mathcal{Q}(F) = \int_{\mathbb{R}} F(q_\star)\mathcal{M}(q_\star)\,\mathrm{d}q_\star - F. \tag{2.6}$$

For technical purposes, we need further assumptions involving the velocity field U and the generator $\mathcal{Q}$. We will assume that U has null average

$$\int_{\mathbb{Y}\times\mathbb{R}} U(y, q)\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y = 0. \tag{2.7}$$

Occasionally, this assumption will be strengthened with the following pointwise centering condition

$$\text{For a. e. } y \in \mathbb{Y}, \qquad \int_{\mathbb{R}} U(y, q)\ \mathcal{M}(q)\,\mathrm{d}q = 0. \tag{2.8}$$

We shall also need an ergodic property which states

$$\begin{cases} \text{For a. e. } q \in \mathbb{R}, \text{ the solutions of } U(y, q)\cdot\nabla_y f = 0 \\ \text{are constants with respect to } y, \end{cases} \tag{2.9}$$

and the non-degeneracy condition

$$\begin{cases} \text{The matrix } A(y) = \int_{\mathbb{R}} U(y,q) \otimes U(y,q)\ \mathcal{M}(q)\,\mathrm{d}q \\ \text{is positive definite, and its coefficients belong to } W^{1,\infty}(\mathbb{Y}). \end{cases} \tag{2.10}$$

The expression of the time correlations of the space average of the field we obtain depend on the scales separation, that is on the ordering between the different physical parameters involved in the equation. In what follows, it will be given by the following formula:

$$\mathbb{D}_2 = \int_{\mathbb{R}} \int_{\mathbb{Y}} U(y,q)\,\mathrm{d}y \otimes (-\mathcal{Q}^\star)^{-1} \Big(\int_{\mathbb{Y}} U(y,q)\,\mathrm{d}y \Big)\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y, \tag{2.11}$$

or by the following two cases when particle transport operates at the same time scale as the random process: the first case will appear when dealing with the "Fine Particle" regime:

$$\mathbb{D}_1 = \int_{\mathbb{Y}\times\mathbb{R}} U(y,q) \otimes (U(y,q)\nabla_y - \mathcal{Q}^\star)^{-1}(U)(y,q)\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y. \tag{2.12}$$

and for the "High Inertia" regime, we get

$$\mathbb{D}_0(v) = \int_{\mathbb{Y}\times\mathbb{R}} U(y,q) \otimes (v\cdot\nabla_y - \mathcal{Q}^\star)^{-1}(U)(y,q)\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y. \tag{2.13}$$

The definition of the inverse operators involved in these formulae is an issue and the arguments, which are of Fredholm alternative type, should be discussed carefully.

Instead of considering the particle distribution function in phase space, it is therefore convenient to deal with the density in $\mathbb{R}^{2N}\times\mathbb{R}$ of the probability distribution of the random variable (X,V,Q). In other words, we introduce $F(t,x,v,q)\geq 0$ such that, at time $t\geq 0$, for any measurable sets $\Omega\subset\mathbb{R}^N$, $\mathcal{V}\subset\mathbb{R}^N$ and $\mathcal{K}\subset\mathbb{R}$, we have,

$$\int_{\Omega\times\mathcal{V}\times\mathcal{K}} F(t,x,v,q)\,\mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x = \mathrm{Proba}\big(\{(X_t,V_t,Q_t)\in\Omega\times\mathcal{V}\times\mathcal{K}\}\big).$$

Accordingly, we are led to the following evolution PDE on densities

$$\partial_t F + v\cdot\nabla_x F + \frac{1}{\mathcal{T}_{St}}\nabla_v\cdot\big((\mathcal{U}\ U(x/\ell,q) - v)F\big) = \frac{1}{\mathcal{T}_M}\mathcal{Q}^\star(F). \tag{2.14}$$

Our goal is:

- first to identify some relevant asymptotic regimes, depending on the values of the stochastic and physical parameters $\mathcal{T}_M$, $\mathcal{U}$, ℓ and $\mathcal{T}_{St}$, compared to typical time and length scales of observation,

- second, to establish the limit equations which correspond to these regimes.

These questions have been addressed with a different viewpoint in [13], as well as in [41] with a more physical insight. Our approach is also strongly inspired from [24]: there the modeling of turbulence relies on finite time decorrelations of the velocity field. It allows to perform the asymptotic analysis by using the method introduced in [38]. We revisit this analysis by replacing this finite time decorrelation by suitable mixing properties of the Markov generator. In turn, the kinetic viewpoint applied to (2.14) is based on the relaxation property of the generator, with methods reminiscient to the analysis of hydrodynamic limits in gas dynamics. Such an approach also appears in [17] where an additional variable is introduced to the usual (position-velocity) phase space in order to describe the carrier flow turbulent velocity encoutered by a particle along its path. We finally refer to the technical developments in [25] which have to be adapted to the two-phase flow context.

2.2 Dimension Analysis and Asymptotic Regimes

We introduce time and length units, denoted by T and L respectively. Then, we define dimensionless variables and unknowns as follows

$$\begin{aligned} &t \to t\mathrm{T}, \qquad x \to x\mathrm{L}, \qquad v \to \tfrac{\mathrm{L}}{\mathrm{T}} v \\ &F(t,x,v,q) \to \mathrm{L}^3 \Big(\frac{\mathrm{L}}{\mathrm{T}}\Big)^3 \; F(t,x,v,q). \end{aligned}$$

We are finally led to

$$\partial_t F + v \cdot \nabla_x F + \frac{1}{\tau} \nabla_v \cdot \big[\big(\eta U\big(x/\lambda, q\big) - v\big) F\big] = \frac{1}{\varepsilon} \mathcal{Q}^\star(F) \tag{2.15}$$

which is governed by the following four dimensionless parameters

$$\begin{aligned} &\varepsilon = \frac{\mathcal{T}_M}{\mathrm{T}}, \qquad &\tau = \frac{\mathcal{T}_{St}}{\mathrm{T}} \\ &\eta = \frac{\mathrm{UT}}{\mathrm{L}}, \qquad &\lambda = \frac{\ell}{\mathrm{L}}. \end{aligned}$$

We shall investigate the behavior of the solutions with respect to the parameters. Let us start with some a priori estimates. Since $\int Q(f) \mathcal{M} \, \mathrm{d}q = 0$, we can reproduce easily the argument in [24] which prove the following estimate on the momentum and kinetic energy.

Proposition 2.1. *Let the initial data $F_0 \geq 0$ satisfy*

$$\int_{\mathbb{R}^{2N} \times \mathbb{R}} F_0 \, \mathcal{M} \, \mathrm{d}q \, \mathrm{d}v \, \mathrm{d}x = M_0 < \infty, \qquad \int_{\mathbb{R}^{2N} \times \mathbb{R}} |v|^2 F_0 \, \mathcal{M} \, \mathrm{d}q \, \mathrm{d}v \, \mathrm{d}x = M_2 < \infty. \tag{2.16}$$

Then, for any time $t \geq 0$, *the solution* $F(t,x,v,q)$ *of* (2.15) *verifies* $\int_{\mathbb{R}^{2N}\times\mathbb{R}} F(t)\ \mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x = M_0$ *and there exists a constant* $C > 0$, *depending only on* M_0, M_2 *and* K *such that, for any* $t \geq 0$,

$$\int_{\mathbb{R}^{2N}\times\mathbb{R}} |v| F(t)\ \mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x \leq C\eta, \quad \int_{\mathbb{R}^{2N}\times\mathbb{R}} |v|^2 F(t)\ \mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x \leq C\eta^2.$$

We can also obtain estimates that uses the dissipative properties of the Markov generator. However, these estimates are useful only when τ does not go to 0.

Proposition 2.2. *Let the initial data* $F_0 \geq 0$ *satisfy* (2.16) *and*

$$\int_{\mathbb{R}^{2N}\times\mathbb{R}} |F_0|^2\ \mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x = M_e < \infty. \tag{2.17}$$

Then, for any time $0 \leq t \leq T < \infty$, *there exists a constant* $C(T/\tau) > 0$, *which blows up as* $\tau \to 0$, *such that*

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}\times\mathbb{R}} |F(t)|^2\ \mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x \leq C(T/\tau),\\
&\frac{1}{\epsilon}\int_0^t \int_{\mathbb{R}^{2N}\times\mathbb{R}} \left| F - \int F(q_\star)\ \mathcal{M}(q_\star)\,\mathrm{d}q_\star \right|^2 \mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \leq C(T/\tau).
\end{aligned}$$

Proof. By using integration by parts, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^{2N}\times\mathbb{R}} F^2\mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x - \frac{1}{\varepsilon}\int_{\mathbb{R}^{2N}\times\mathbb{R}} \mathcal{Q}^\star(F)F\mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x\\
&\quad = -\frac{1}{\tau}\int_{\mathbb{R}^{2N}\times\mathbb{R}} v\cdot\nabla_v\Big(\frac{F^2}{2}\Big)\mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x = \frac{N}{2\tau}\int_{\mathbb{R}^{2N}\times\mathbb{R}} F^2\mathcal{M}\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x.
\end{aligned}$$

We conclude by using (2.4) and the Gronwall lemma. ☐

In the spirit of [24], we shall distinguish "High Inertia Particles Regimes" where τ is kept fixed and "Fine Particles Regimes" where the drag force is the leading term within the equation. The former asymptotically lead to diffusion equations where the diffusion operates on velocity while the latter lead to asymptotic diffusion operating on position. As explained below, the Fine Particles Regimes are much harder to analyze due to concentration phenomena, see [29, 30]. Interestingly, looking at an "Over-Damped Regime" ($\tau \to 0$) yet in the "High Inertia-Particles Regimes" leads to an asymptotic diffusion evolution on position similar to the "Fine Particles Regimes", but based on a different physical background mechanism.

2.2.1 High-Inertia-Particles Regime

The high-inertia regimes investigated in [24] corresponds to assume

$$\eta = 1/\sqrt{\varepsilon}, \qquad \lambda = \varepsilon, \qquad \varepsilon \to 0, \qquad \tau \text{ fixed.} \tag{2.18}$$

Furthermore, an adpatation of [25] leads to consider the regime

$$\eta = 1/\sqrt{\varepsilon}, \qquad \lambda = \varepsilon^{\alpha}, \qquad \varepsilon \to 0, \qquad \tau \text{ fixed}, \qquad \alpha > 1. \tag{2.19}$$

We will take $\alpha = 3/2$ in the formal analysis, but the method seems to work the same for other $\alpha > 1$. Both regimes (2.18) and (2.19) yield Fokker-Planck like equations in phase space. It means that the limiting behavior as $\varepsilon \to 0$ can be described by the PDE

$$\partial_t G + v \cdot \nabla_x G = \nabla_v \cdot \left(\frac{v}{\tau} G + \frac{1}{\tau^2} \mathbb{D}_n \nabla_v G\right)$$

where the unknown $G = G(t, x, v)$ is a distribution function in phase space. For the regime (2.18), the effective diffusion matrix is $\mathbb{D}_0$ as defined in (2.13), whereas for the regime (2.19), it is $\mathbb{D}_2$ given by (2.11).

Considering classically the “Over-Damped” regime $\tau \to 0$ for regime (2.19) leads to a diffusion on position:

$$\partial_t G = \nabla_x \cdot \big(\mathbb{D}_2 \nabla_x G\big),$$

where $G = G(t, x)$ now depends only on time and space. The “Over-Damped” regime could be obtained directly from (2.19) by taking $\sqrt{\varepsilon} \ll \tau_\varepsilon \ll 1$. The case of regime (2.18) with such an over damping would also lead to a similar diffusion on position but solution of a more intricate diffusion homogeneization problem that lies beyond the scope of this paper.

2.2.2 Fine-Particles Regime

In these regimes the scaled Stokes settling time τ goes to 0, very fast. Let us first discuss the asymptotic properties of the model, without making precise the relation between the parameters; we only assume that $0 < \tau \ll 1$ sufficiently fast. We can expect that

$$f(t, x, v, q) \sim \rho(t, x, q)\ \delta_{v = \eta u(x/\lambda, q)}, \tag{2.20}$$

with the macroscopic density ρ of order 1. It fixes the dependence with respect to the variable v. It is therefore convenient to use the moment equations associated to (2.14). We set

$$\rho(t, x, q) = \int f \,\mathrm{d}v, \quad J(t, x, q) = \int v f \,\mathrm{d}v, \quad \mathbb{P}(t, x, q) = \int v \otimes v f \,\mathrm{d}v. \tag{2.21}$$

Integration of (2.14) with respect to the velocity variable yields

$$\partial_t \rho(t,x) + \mathrm{div}_x J = \frac{1}{\varepsilon}\mathcal{Q}^\star(\rho), \tag{2.22}$$

$$\partial_t J + \mathrm{Div}_x \mathbb{P} = \frac{1}{\tau}(\eta\ \rho U(x/\lambda, q) - J) + \frac{1}{\varepsilon}\mathcal{Q}^\star(J). \tag{2.23}$$

Due to (2.20), we have

$$\eta\rho U(x/\lambda, q) - J) \to 0,$$

as well as

$$\rho = \mathcal{O}(1), \qquad J = \mathcal{O}(\eta), \qquad \mathbb{P} = \mathcal{O}(\eta^2), \tag{2.24}$$

all of the above terms remaining bounded when $\tau \to 0$. Hence, we can use (2.23) to rewrite (2.22) as follows:

$$\begin{aligned}\partial_t \rho(t,x) + \mathrm{div}_x\big(\rho\eta U(x/\lambda, q)\big) - \frac{1}{\varepsilon}\mathcal{Q}^\star(\rho) \\ = \tau \mathrm{div}_x\Big[\partial_t J + \mathrm{Div}_x \mathbb{P}\Big] - \frac{\tau}{\varepsilon}\mathrm{div}_x \mathcal{Q}^\star(J).\end{aligned}$$

The scaling assumptions $\eta\tau/\varepsilon \ll 1$ and $\eta^2\tau \ll 1$ will guarantee that the right hand side goes to 0, owing to (2.24). Hence, we can expect the problem is close to the more familiar one

$$\partial_t \rho(t,x) + \mathrm{div}_x\big(\eta\rho U(x/\lambda, q)\big) - \frac{1}{\varepsilon}\mathcal{Q}^\star(\rho) = 0$$

which is a standard model describing the passive transport of tracer particles in the flow described by u, see e. g. [32]. Note that the above formal computation could also be carried out using an Hilbert expansion on the dual problem (with operators acting on test functions instead of densities).

Let us now specify the scaling. Below, the analysis will be carried out by assuming either

$$\begin{cases}\eta = 1/\sqrt{\varepsilon}, \qquad \lambda = \sqrt{\varepsilon}, \qquad \varepsilon \to 0, \\ \tau = \varepsilon^k \text{ with } k > 3/2.\end{cases} \tag{2.25}$$

or

$$\begin{cases}\eta = 1/\sqrt{\varepsilon}, \qquad \lambda = \varepsilon^\alpha, \qquad \varepsilon \to 0, \\ \tau = \varepsilon^k \text{ with } k > 3/2, \qquad \alpha > 1/2.\end{cases}, \tag{2.26}$$

As a consequence of the combination of the hydrodynamic limit to the random homogenization effects, we obtain a macroscopic diffusion equation

$$\partial_t \rho - \nabla_x \cdot (\mathbb{D}_n \nabla_x \rho) = 0$$

with $n = 1$ for (2.25) and $n = 2$ for (2.26) with $\alpha = 3/2$. Referring to [24], these scalings correspond to the "Small Stokes number-Fine particles Regimes" (the Stokes number is the ratio τ/ε). However (2.25) and (2.26) are not treated in [24] where so fast space oscillations are excluded (with $\eta = 1/\sqrt{\varepsilon}$, the result in [24] assumes λ fixed, a case which can be treated easily by the techniques exposed here).

2.3 Derivation of the Effective Equations in the Fine Particles Regimes

2.3.1 Analysis of the Regime (2.25)

Let us assume that (2.25) holds, so that we address the question of the behavior for small ε's of

$$\partial_t \rho_\varepsilon(t,x) + \frac{1}{\sqrt{\varepsilon}} \mathrm{div}_x\big(\rho_\varepsilon U(x/\sqrt{\varepsilon}, q)\big) - \frac{1}{\varepsilon} \mathcal{Q}^\star(\rho_\varepsilon) = 0. \tag{2.27}$$

We insert the ansatz

$$\rho_\varepsilon(t,x,q) = \rho_0(t,x,x/\sqrt{\varepsilon},q) + \sqrt{\varepsilon}\rho_1(t,x,x/\sqrt{\varepsilon},q) + \varepsilon\rho_2(t,x,x/\sqrt{\varepsilon},q) + ...$$

At leading order we obtain

$$U(y,q)\cdot\nabla_y\rho_0 - \mathcal{Q}^\star(\rho_0) = 0.$$

The $\mathcal{O}(1/\sqrt{\varepsilon})$ equation reads

$$U(y,q)\cdot\nabla_y\rho_1 - \mathcal{Q}^\star(\rho_1) = -U(y,q)\cdot\nabla_x\rho_0,$$

and, finally, $\mathcal{O}(1)$ terms yield

$$U(y,q)\cdot\nabla_y\rho_2 - \mathcal{Q}^\star(\rho_2) = -\partial_t\rho_0 - U(y,q)\cdot\nabla_x\rho_1.$$

We are thus led to investigate the cell equation

$$U(y,q)\cdot\nabla_y\rho - \mathcal{Q}^\star(\rho) = h$$

completed with periodic boundary conditions. Clearly, the equation can make sense only when the right hand side fulfills the compatibility condition

$$\int_{\mathbb{Y}\times\mathbb{R}} h\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y = 0.$$

This is a necessary condition; it can be shown to be also sufficient as summarized in the following claim. (We analyze the relaxation operator (2.6) only, but it is possible to extend the result to the operator (2.5).)

Lemma 2.3. *Let $\mathcal{Q}^\star$ be a bounded operator on $L^2(\mathbb{R}, \mathcal{M}\,\mathrm{d}q)$, verifying (2.4). Let U satisfy (2.1)-(2.3) and (2.9)-(2.10). Then for any $h \in L^2(\mathbb{Y}\times\mathbb{R}, \mathcal{M}\,\mathrm{d}q\,\mathrm{d}y)$ verifying $\int h\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y = 0$ there exists a unique solution $\rho \in L^2(\mathbb{Y}\times\mathbb{R}, \mathcal{M}\,\mathrm{d}q\,\mathrm{d}y)$ of $U(y,q)\cdot\nabla_y\rho - \mathcal{Q}^\star(\rho) = h$ such that $\int_{\mathbb{Y}\times\mathbb{R}} \rho\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y = 0$.*

Proof. Let us start by studying the problem when $h = 0$. Integrating with respect to both y and q, using the periodic boundary condition and (2.4), we get

$$\int_{\mathbb{Y}\times\mathbb{R}} \Big|\rho(y,q) - \int_{\mathbb{R}} \rho(y,q_\star)\mathcal{M}(q_\star)\,\mathrm{d}q_\star\Big|^2\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y = 0$$

and we infer first that the solution $\rho(y,q) = \rho(y) \in \mathrm{Ker}(\mathcal{Q}^\star)$ does not depend on q. Then, the equation simply becomes $U(y,q)\cdot\nabla_y\rho = 0$. The ergodic condition (2.9) implies that ρ does not depend on the fast variable y. Therefore imposing the solution has a null average forces $\rho = 0$.

Now we suppose $h \neq 0$ and we justify the existence of solutions by a regularization argument. Let us consider the sequence $\big(\rho_\lambda\big)_{\lambda>0}$ of solutions to

$$\lambda\rho_\lambda + U\cdot\nabla_y\rho_\lambda - \mathcal{Q}^\star(\rho_\lambda) = h. \tag{2.28}$$

If ρ_λ is bounded in $L^2(\mathbb{Y}\times\mathbb{R})$ letting $\lambda \to 0$ for a suitable (weakly) convergent subsequence yields the desired existence statement. Hence, we assume that $\|\rho_\lambda\|_{L^2(\mathbb{Y}\times\mathbb{R})} = 1$ and the ρ_λ's verifies (2.28) with a right hand side h_λ that tends to 0. We shall show that we are led to a contradiction. Indeed, let us denote $\langle\rho_\lambda\rangle = \int_{\mathbb{R}} \rho_\lambda\mathcal{M}\,\mathrm{d}q$. By using (2.4), we get

$$\int_{\mathbb{Y}\times\mathbb{R}} \Big|\rho_\lambda(y,q) - \langle\rho_\lambda\rangle(y)\Big|^2\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y \xrightarrow[\lambda\to 0]{} 0.$$

Next, since $\langle\rho_\lambda\rangle \in \mathrm{Ker}(\mathcal{Q}^\star)$, we can write

$$U\cdot\nabla_y\langle\rho_\lambda\rangle = h_\lambda - \lambda\rho_\lambda + \big(\mathcal{Q}^\star - U\cdot\nabla_y\big)(\rho_\lambda - \langle\rho_\lambda\rangle).$$

We denote by $S_\lambda(y,q)$ the right hand side and we set

$$A(y) = \langle U(y,q)\otimes U(y,q)\rangle.$$

By (2.10) this matrix is invertible and $A(y)^{-1}$ belongs to $W^{1,\infty}(\mathbb{Y})$. We deduce that

$$\nabla_y\langle\rho_\lambda\rangle = A(y)^{-1}\langle U S_\lambda(y,q)\rangle$$

belongs to a compact set of $H^{-1}(\mathbb{Y})$. Accordingly, since $\int h_\lambda\mathcal{M}\,\mathrm{d}q\,\mathrm{d}y = 0$ implies $\int\langle\rho_\lambda\rangle\,\mathrm{d}y = 0$, we deduce by a standard Fourier argument that $\langle\rho_\lambda\rangle$ belongs to a compact set in $L^2(\mathbb{Y})$. Finally, we conclude that

$\rho_\lambda = (\rho_\lambda - \langle\rho_\lambda\rangle) + \langle\rho_\lambda\rangle$ is (strongly) compact in $L^2(\mathbb{Y}\times\mathbb{R}, \mathcal{M}\,\mathrm{d}q\,\mathrm{d}y)$. Extracting a subsequence if necessary, let us denote ρ the limit of the ρ_λ's. It verifies

$$U\cdot\nabla_y\rho - \mathcal{Q}^\star(\rho) = 0, \qquad \int \rho\mathcal{M}\,\mathrm{d}q\,\mathrm{d}y = 0,$$

thus $\rho = 0$, a contradiction. □

Remark 2.4. *In fact, the above cell problem has a probabilistic interpretation that will lead to similar existence/uniqueness result but which can be extended to the space of continuous and bounded functions. Introducing*

$$t\mapsto Q_t^\star,$$

the Markov process with generator $\mathcal{Q}^\star$*, and the process solution of*

$$\dot{Y}_t^\star = -U(Y_t^\star, Q_t^\star),$$

the solution of the cell problem $U(y,q)\cdot\nabla_y\rho - \mathcal{Q}^\star(\rho) = h$ *writes down:*

$$\rho(q,y) = \int_0^{+\infty} -\mathbb{E}\left(h(Y_t^\star, Q_t^\star)|(Y_0^\star = y, Q_0^\star = q)\right)\,\mathrm{d}t,$$

which will be well defined as soon as the process $t\mapsto (Y_t^\star, Q_t^\star)$ *is mixing and* h *is centered with respect to the invariant measure* $\mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y$*. If* $\mathcal{Q}^\star$ *is a Fokker-Planck operator, a sufficient condition to get mixing is the hypoellipticity of the operator* $U(y,q)\cdot\nabla_y\rho - \mathcal{Q}^\star$ *(the usual Hörmander sense), see [35].*

This statement already proves that $\rho_0(t,x,y,q) = \rho_0(t,x)$ does not depend on q nor on y. Let us introduce $\chi = (\chi_1, ..., \chi_N)$ solution of

$$U(y,q)\cdot\nabla_y\chi - \mathcal{Q}^\star(\chi) = U(y,q)$$

which makes sense thanks to (2.7). Then we get

$$\rho_1(t,x,y,q) = -\chi(y,q)\cdot\nabla_x\rho_0(t,x).$$

Therefore, the compatibility condition for the $\mathcal{O}(1)$ equation leads to

$$\partial_t\rho_0 - \nabla_x\cdot\left(\int_{\mathbb{Y}\times\mathbb{R}} U(y,q)\otimes\chi(y,q)\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y\ \nabla_x\rho_0\right) = 0.$$

The diffusion matrix (2.12) writes down:

$$\mathbb{D}_1 = \int_{\mathbb{Y}\times\mathbb{R}} U(y,q)\otimes\chi(y,q)\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y.$$

Note it is indeed non negative since we have, for any $\xi \in \mathbb{R}^N \setminus \{0\}$,

$$\begin{aligned}\mathbb{D}_1 v \cdot v &= \int_{\mathbb{Y}\times\mathbb{R}} (U\cdot\nabla_y - \mathcal{Q}^\star)(\chi\cdot\xi)\ \chi\cdot\xi \,\mathrm{d}y\,\mathrm{d}q\\ &= -\int_{\mathbb{Y}\times\mathbb{R}} \mathcal{Q}^\star(\chi\cdot\xi)\ \chi\cdot\xi\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y\\ &\geq \sigma\int_{\mathbb{Y}\times\mathbb{R}} |\chi\cdot\xi|^2\ \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y > 0\end{aligned}$$

by (2.3) and (2.4). Moreover, it cannot vanish due to (2.10). This formal development can be made rigorous by adapting the arguments in [25] and we conclude with the following statement.

Theorem 2.5. *Let U verify (2.1)-(2.7) and (2.9)-(2.10). We assume that (2.4) is fulfilled. We suppose that the initial condition verifies*

$$\sup_{\varepsilon>0}\int_{\mathbb{R}^N\times\mathbb{R}} |\rho^\varepsilon_{\mathrm{Init}}(x,q)|^2\mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}x \leq C < \infty. \tag{2.29}$$

Then, up to a subsequence, ρ^ε solutions of (2.27) associated to $\rho^\varepsilon_{\mathrm{Init}}$ converges weakly in $L^2((0,T)\times\mathbb{R}^N\times\mathbb{R};\mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}x)$ and in $C^0([0,T], L^2(\mathbb{R}^N\times\mathbb{R},\mathcal{M}\,\mathrm{d}q\,\mathrm{d}x) - \mathrm{weak})$ to $\rho(t,x)$, where ρ is the solution of

$$\begin{cases}\partial_t\rho = \nabla_v\cdot(\mathbb{D}_1\nabla_v\rho),\\ \rho(t=0,x) = \mathrm{weak-}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\rho^\varepsilon_{\mathrm{Init}}(x,q)\mathcal{M}(q)\,\mathrm{d}q,\end{cases} \tag{2.30}$$

with $\mathbb{D}_1$ defined by (2.12).

The statement only deals with the equation (2.27), which is already an approximation of the moment system coming form (2.14). Taking into account the additional (small) terms in (2.27) leads to technical difficulties due to the functional framework: we have only the L^1 bounds at hand, see Proposition 2.1 and 2.2, and the ugly terms make derivatives with respect to space and time appear.

2.3.2 Analysis of the Regime (2.26)

Let us assume that (2.26) holds, so that we address the question of the behavior for small ε's of

$$\partial_t\rho_\varepsilon(t,x) + \frac{1}{\sqrt{\varepsilon}}\mathrm{div}_x\big(\rho_\varepsilon U(x/\varepsilon^{3/2},q)\big) - \frac{1}{\varepsilon}\mathcal{Q}^\star(\rho_\varepsilon). \tag{2.31}$$

We insert the ansatz

$$\rho_\varepsilon(t,x,q) = \rho_0(t,x,x/\varepsilon^{3/2},q) + \varepsilon^{3/2}\rho_1(t,x,x/\varepsilon^{3/2},q) + \varepsilon^3\rho_2(t,x,x/\varepsilon^{3/2},q) + ...$$

At leading order we obtain

$$U(y,q)\cdot\nabla_y\rho_0 = 0.$$

The ergodic condition (2.9) implies that $\rho_0(t,x,y,q) = \rho_0(t,x,q)$ does not depend on the fast variable. Next, we get

$$U(y,q)\cdot\nabla_y\rho_1 = \mathcal{Q}^\star(\rho_0).$$

Integrating with respect to y and bearing in mind that ρ_0 does not depend on y, we obtain $\rho_0 \in \mathrm{Ker}(\mathcal{Q}^\star)$, so that ρ_0 does not depend on q anymore. Therefore, the equation for the corrector becomes $U\cdot\nabla_y\rho_1 = 0$ implying that $\rho_1(t,x,y,q) = \rho_1(t,x,q)$. Then, we arrive at

$$U(y,q)\cdot\nabla_y\rho_2 = \mathcal{Q}^\star(\rho_1) - U(y,q)\cdot\nabla_x\rho_0(t,x).$$

Integration over $\mathbb{Y}$ yields

$$\mathcal{Q}^\star(\rho_1) = \int_{\mathbb{Y}} U(y,q)\,\mathrm{d}y\cdot\nabla_x\rho_0(t,x).$$

When U verifies the pointwise centering condition (2.8) we can appeal to the Fredholm alternative for $\mathcal{Q}^\star$ and find the auxilliary function $\chi(q)$ such that

$$\mathcal{Q}^\star(\chi) = -\int_{\mathbb{Y}} U(y,\cdot)\,\mathrm{d}y.$$

We thus write $\rho_1(t,x,q) = -\chi(q)\cdot\nabla_x\rho_0(t,x)$. We plug this formula into the $\mathcal{O}(1)$ equation, which, after integration leads to

$$\partial_t\rho_0 - \nabla_x\cdot\big(\mathbb{D}_2\nabla_x\rho_0(t,x)\big) = 0, \tag{2.32}$$

with

$$\mathbb{D}_2 = \int_{\mathbb{Y}\times\mathbb{R}} U(y,q)\otimes\chi(q)\mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}y.$$

Referring to [25] again, we obtain the following statement (note that the restriction (2.8) can be relaxed by using a fully probabilistic proof).

Theorem 2.6. *Let U verify* (2.1)–(2.3) *and* (2.8)–(2.9). *We assume that* (2.4) *is fulfilled. We suppose that the initial condition verifies* (2.29). *Then, up to a subsequence, ρ^ε solutions of* (2.31) *associated to $\rho^\varepsilon_{\mathrm{Init}}$ converges weakly in $L^2((0,T)\times\mathbb{R}^N\times\mathbb{R};\mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}x)$ and in $C^0([0,T],L^2(\mathbb{R}^N\times\mathbb{R},\mathcal{M}\,\mathrm{d}q\,\mathrm{d}x)-\mathrm{weak})$ to $\rho(t,x)$, where ρ is the solution of* (2.32) *with coefficient* (2.11).

2.4 Derivation of the Effective Equations in the High-Inertia Particles Regimes

On the same token, we can discuss the effective equations arising with the regimes (2.18) and (2.19). In these situation where we do not have concentration effect, a complete proof can be designed by adapting directly the argument in [25].

Assuming (2.18), we are concerned with the behavior as $\varepsilon \to 0$ of

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon + \nabla_v \cdot \Big[\Big(\frac{1}{\sqrt{\varepsilon}} U(x/\varepsilon, q) - v\Big) F_\varepsilon\Big] = \frac{1}{\varepsilon} \mathcal{Q}^\star(F_\varepsilon). \tag{2.33}$$

A double-scale ansatz leads to the auxilliary equation

$$v \cdot \nabla_y \chi^\star - \mathcal{Q}^\star \chi^\star = U(y, q)$$

which makes sense under the centering assumption (2.7) and we define the (non-negative) matrix by (2.11), namely

$$\mathbb{D}_0(v) = \int_{\mathbb{R}\times\mathbb{Y}} U(y, q) \otimes \chi^\star(v, y, q)\,\mathrm{d}q\,\mathrm{d}y.$$

Assuming (2.19), we are concerned with the behavior as $\varepsilon \to 0$ of

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon + \nabla_v \cdot \Big[\Big(\frac{1}{\sqrt{\varepsilon}} U(x/\varepsilon^{3/2}, q) - v\Big) F_\varepsilon\Big] = \frac{1}{\varepsilon} \mathcal{Q}^\star(F_\varepsilon). \tag{2.34}$$

Then, we are led to the auxilliary equation

$$-\mathcal{Q}^\star \chi^\star = \int_{\mathbb{Y}} U(y, \cdot)\,\mathrm{d}y$$

which makes sense under the centering assumption (2.8). We recall (2.11), in this context:

$$\mathbb{D}_2 = \int_{\mathbb{R}\times} \int_{\mathbb{Y}} U(y, q)\,\mathrm{d}y \otimes \chi^\star(q)\,\mathrm{d}q.$$

The results summarize as follows.

Theorem 2.7. *Let U verify (2.1)-(2.7). For the scaling (2.19), we assume the strengthened condition (2.8). We assume that (2.4) is fulfilled. We suppose that the initial condition verifies*

$$\sup_{\varepsilon>0} \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}} |F^\varepsilon_{\mathrm{Init}}(x, q)|^2 \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}v\,\mathrm{d}x \le C < \infty.$$

Then, up to a subsequence, F^ε solutions of (2.33) (resp. (2.34)) associated to $F^\varepsilon_{\mathrm{Init}}$ converges weakly in $L^2((0,T)\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}; \mathcal{M}(q)\,\mathrm{d}q\,\mathrm{d}x)$ and in $C^0([0,T], L^2(\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}, \mathcal{M}\,\mathrm{d}q\,\mathrm{d}x)-\mathrm{weak})$ to $G(t, x, v)$, solution of

$$\partial_t.G + v \cdot \nabla_x G = \nabla_v \cdot (vG + \mathbb{D}_n \nabla_v G)$$

with $\mathbb{D}_n$ given by (2.13) ($n = 0$) (resp. by (2.11) ($n = 2$)).

3 Numerical Schemes for Coupled Fluid/Particles Models

This Section is devoted to models which take into account the back-reaction of the particles on the dense phase. Therefore, the velocity field u involved in the drag force is defined by an evolution equation depending on the particle density. More precisely, the dense phase is described by the mass density $n(t,x)$ and the velocity field $u(t,x)$. These quantities obey the Euler or Navier-Stokes system. We take into account:

- the drag force exerted by each phase on the other; as explained in the Introduction it depends on the relative velocity $v - u(t,x)$;
- the Brownian motion of the particles which leads to diffusion with respect to the velocity variable;
- the effect of external forces embodied into a potential field $\Phi(x)$.

Accordingly, the fluid-particles flow is governed by the PDEs

$$\partial_t f + v \cdot \nabla_x f - \alpha \nabla_x \Phi \cdot \nabla_v f = \frac{9\mu}{2a^2 \rho_{\mathrm{p}}} \nabla_v \cdot \Big((v-u) f + \frac{k\theta_0}{m_{\mathrm{p}}} \nabla_v f \Big), \tag{3.1}$$

$$\partial_t n + \nabla_x \cdot (nu) = 0, \tag{3.2}$$

$$\rho_{\mathrm{f}} \Big(\partial_t (nu) + \mathrm{Div}_x (nu \otimes u) + n \nabla_x \Phi \Big) + \nabla_x p(n) = 6\pi\mu a \int_{\mathbb{R}^3} (v-u) f \, dv. \tag{3.3}$$

In (3.3), ρ_{f} is a typical mass density of the fluid, k stands for the Boltzmann constant, and $\theta_0 > 0$ denotes the temperature of the fluid, assumed to be constant; $p(n)$ is a general pressure law, for instance $p(n) = C_\gamma \, n^\gamma$, with $\gamma \geq 1$, $C_\gamma > 0$. The parameter $\alpha \in \mathbb{R}$ is a dimensionless parameter which indicates that the external force can have a different strength and direction on the two phases. Throughout this paper we have in mind the case of gravity forces where

$$\Phi(t,x) = \mathrm{g} x_3,$$
$$\alpha = (1 - \rho_{\mathrm{f}}/\rho_{\mathrm{f}}) \frac{\mathrm{U}}{\sqrt{3k\theta_0/4\pi a^3 \rho_{\mathrm{p}}}},$$

with U a typical value of the fluid velocity. Hence, α accounts for the buoyancy and gravity forces. The equation is completed by initial and boundary conditions. We suppose the standard homogeneous condition

$$u \cdot \nu(x) = 0$$

for the dense phase and for the particles the specular reflection boundary condition

$$f(t,x,v) = f(t,x,v - (v \cdot \nu(x))\nu(x))$$

for any $(x,v)\in\partial\Omega\times\mathbb{R}^N$ such that $v\cdot\nu(x)<0$, where $\nu(x)$ stands for the outer normal vector at the point $x\in\partial\Omega$. Obviously the boundary condition guarantees mass conservation.

Establishing the well-posedness of such a nonlinear system is a tough piece of analysis; we refer to [3, 6, 27, 26, 36] for such existence results in different functional framework. Next, relevant asymptotic regimes can be identified and investigated, depending on the mass density ratio, the Stokes settling time, the typical velocity of the particles compared to those of the fluid... We refer to [10, 21, 22, 37] for such discussion and analysis.

Here, we are concerned with the asymptotic regime $\varepsilon\to 0$ in the following rescaled version of (3.1)–(3.3)

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{1}{\sqrt{\epsilon}}\Big(v\cdot\nabla_x f_\varepsilon + \nabla_x\Phi\cdot\nabla_v f_\varepsilon\Big) = \dfrac{1}{\epsilon}\nabla_v\cdot\Big(\big(v-\sqrt{\epsilon}u_\varepsilon\big)f_\varepsilon + \nabla_v f_\varepsilon\Big),\\[2mm] \partial_t n_\varepsilon + \nabla_x\cdot(n_\varepsilon u_\varepsilon) = 0,\\[2mm] \partial_t(n_\varepsilon u_\varepsilon) + \mathrm{Div}_x(n_\varepsilon u_\varepsilon\otimes u_\varepsilon) + \nabla_x p(n_\varepsilon) + \eta_\epsilon\, n_\varepsilon\nabla_x\Phi = J_\varepsilon - \rho_\varepsilon u_\varepsilon, \end{cases} \tag{3.4}$$

where we use the notation

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\,\mathrm{d}v, \qquad J_\varepsilon(t,x) = \frac{1}{\sqrt{\epsilon}}\int_{\mathbb{R}^3} v\; f_\varepsilon(t,x,v)\,\mathrm{d}v.$$

It is referred to as the "Bubbling regime" in [10] and it relies on the following scaling assumptions

$$\begin{aligned} &\text{Stokes velocity} = \tfrac{2\rho_{\mathrm{p}}a^2}{9\mu}\mathrm{g}|1-\rho_{\mathrm{f}}/\rho_{\mathrm{p}}| \\ &\qquad\qquad \ll \text{Typical velocity of the fluid} \simeq \text{Thermal velocity} = \sqrt{\frac{k\theta_0}{m_{\mathrm{p}}}}, \end{aligned}$$

while the ratio $\rho_{\mathrm{p}}/\rho_{\mathrm{f}}$ is of order ϵ. Finally, we suppose that $\eta_\epsilon\to\eta_\star\in(0,\infty)$ (precisely, for gravity forces it reads $\eta_\epsilon=(1-\varepsilon)^{-1}$). The analysis of the asymptotic behavior $\epsilon\to 0$ relies on the dissipative properties of the system (3.4), which are summarized in the following claim.

Proposition 3.1 (Entropy Dissipation Property)**.** *We set*

$$\mathcal{F}_{\mathrm{p}}(t) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\Big(f\ln(f) + \frac{v^2}{2}f - \Phi f\Big)\,\mathrm{d}v\,\mathrm{d}x,$$
$$\mathcal{F}_{\mathrm{f}}(t) = \int_{\mathbb{R}^3}\Big(n\frac{|u|^2}{2} + \Pi(n) + \eta_\epsilon\Phi n\Big)\,\mathrm{d}x,$$

where $\Pi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ *is defined by* $s\Pi''(s)=p'(s)$*. Then, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathcal{F}_{\mathrm{p}}+\mathcal{F}_{\mathrm{f}}\Big) + \frac{1}{\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\big|(v-\sqrt{\varepsilon}u)\sqrt{f} + 2\nabla_v\sqrt{f}\big|^2\,\mathrm{d}v\,\mathrm{d}x \le 0. \tag{3.5}$$

Accordingly, we guess that, as ϵ goes to 0,

$$f_\varepsilon \simeq \rho(t,x)\ M(v), \qquad M(v) = (2\pi)^{-N/2} e^{-v^2/2},$$

and the asymptotic dynamics is embodied into the behavior of the macroscopic density $\rho(t,x)$. The formal expansion

$$f_\varepsilon = f^{(0)} + \sqrt{\varepsilon} f^{(1)} + \varepsilon f^{(2)} + \dots \tag{3.6}$$

allows to go a step further. Identifying terms with the same power of ϵ, we find the following equation for the corrector

$$Lf^{(1)} = v \cdot \nabla_x f^{(0)} + (u + \nabla_x \Phi) \nabla_v f^{(0)} = v\ M(v)\ \big(\nabla_x \rho - (u + \nabla_x \Phi)\rho\big),$$

with L the standard Fokker-Planck operator

$$Lf = \nabla_v \cdot (vf + \nabla_v f).$$

We obtain

$$f^{(1)}(t,x,v) = -vM(v)\ \big(\nabla_x \rho - (u + \nabla_x \Phi)\rho\big).$$

We use this information in the mass conservation relation

$$\partial_t \int f_\varepsilon \,\mathrm{d}v + \nabla_x \cdot \int \frac{v}{\sqrt{\varepsilon}} f_\varepsilon \,\mathrm{d}v = 0$$

which becomes

$$\begin{aligned} &\partial_t \int f^{(0)} \,\mathrm{d}v + \nabla_x \cdot \int v f^{(1)} \,\mathrm{d}v = 0 \\ &= \partial_t \rho + \nabla_x \cdot \big(\rho(u + \nabla_x \Phi) - \nabla_x \rho\big) = 0. \end{aligned} \tag{3.7}$$

Similarly, in the fluid equation, we get

$$J_\varepsilon - \rho_\varepsilon u_\varepsilon \simeq -(\nabla_x \rho - \rho \nabla_x \Phi).$$

Hence, in the limit system, (3.7) is completed by

$$\begin{cases} \partial_t n + \mathrm{div}_x(nu) = 0, \\ \partial_t(nu) + \mathrm{Div}_x(nu \otimes u) + \nabla_x(p(n) + \rho) + (\eta_\star n - \rho)\nabla_x \Phi = 0, \end{cases} \tag{3.8}$$

Imposing the reflection law leads to the following Robin condition

$$\big(\nabla_x \rho - (u + \nabla_x \Phi)\rho\big) \cdot \nu(x) = 0 \qquad \text{on } \partial\Omega, \tag{3.9}$$

which completes (3.7) and also preserves mass for the limit system. We wish to design a numerical scheme specifically dedicated to treat the asymptotic regime.

3.1 Asymptotic Preserving Numerical Methods

The previous discussion suggests that the solution expands as

$$f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)M(v) + \sqrt{\varepsilon}r_\varepsilon(t,x,v) \tag{3.10}$$

where Proposition 3.1 might give an estimate on the remainder r_ε. Then we rewrite (3.4) as follows

$$\partial_t f_\varepsilon + v\cdot\nabla_x r_\varepsilon + (u_\varepsilon+\nabla_x\Phi)\cdot\nabla_v r_\varepsilon = \frac{1}{\varepsilon}Lf_\varepsilon + \frac{1}{\sqrt{\varepsilon}}M(v)S_\varepsilon(t,x,v), \tag{3.11}$$

with

$$S_\varepsilon(t,x,v) = -v\cdot\nabla_x\rho_\varepsilon - v\cdot(u_\varepsilon(t,x)+\nabla_x\Phi)\rho_\varepsilon.$$

We also have

$$\begin{aligned}\partial_t r_\varepsilon = {}& \frac{1}{\varepsilon}Lr_\varepsilon + \frac{1}{\varepsilon}MS_\varepsilon \\ & -\frac{1}{\sqrt{\varepsilon}}\left[v\cdot\nabla_x r_\varepsilon + (u_\varepsilon+\nabla_x\Phi)\nabla_v r_\varepsilon - M\nabla_x\cdot\left(\int_{\mathbb{R}^3} v_\star r_\varepsilon\,\mathrm{d}v_\star\right)\right].\end{aligned} \tag{3.12}$$

To derive the numerical scheme, we use a splitting algorithm to compute the evolution of both the density f_ε and its fluctuations r_ε by using (3.11) and (3.12). This approach is inspired from [23]. More precisely, the scheme works as follows: Given n^k, u^k, f^k, r^k, approximation of n, u, f, r at time $k\Delta t$,

- *Step 0.* Solve the Euler equations for the fluid density n and velocity u. The source term is treated explicitly by plugging

  $$\int_{\mathbb{R}^3} vr^k\,\mathrm{d}v - u^k\int_{\mathbb{R}^3} f^k\,\mathrm{d}v.$$

  We use a numerical method which preserves with accuracy the shock structure of the hyperbolic system, applying directly the scheme designed in [15, 16, 33]. It defines the updated density n^{k+1} and velocity u^{k+1}.
- *Step 1.* Solve the stiff equations

  $$\partial_t f = \frac{1}{\varepsilon}Lf, \qquad \partial_t r = \frac{1}{\varepsilon}Lr + \frac{1}{\varepsilon}MS,$$

  where

  $$S = -v\cdot\nabla_x\rho + v\cdot(u^{k+1}+\nabla_x\Phi)\rho.$$

  Note that we get rid of the $\mathcal{O}(1/\sqrt{\varepsilon})$ terms in (3.11) and (3.12). The crucial point is that $\rho = \int f\,\mathrm{d}v$ is not modified during this step: $\rho^{k+1/2} = \int f^{k+1/2}\,\mathrm{d}v = \rho^k$ so that the source term in the second equation can be treated as constant in time. Accordingly, the updated quantities read

$$\begin{cases} f^{k+1/2} = e^{\Delta t L/\varepsilon} f^k, \\ r^{k+1/2} = e^{\Delta t L/\varepsilon} r^k + (1 - e^{\Delta t L/\varepsilon}) M\ S^k. \end{cases} \tag{3.13}$$

- *Step 2.* Solve the transport-like part

$$\partial_t f + v \cdot \nabla_x r + (u^{k+1} + \nabla_x \Phi) \cdot \nabla_v r = 0, \qquad \partial_t r = 0.$$

which defines f^{k+1} and $\rho^{k+1} = \int f^{k+1}\,\mathrm{d}v$. In particular, the convection term is of characteristic speed v and not $v/\sqrt{\varepsilon}$)

Let us comment further on the proposed scheme.

1. *CFL and Sub-cycling.* Since the limit equation for the particles density is a diffusion equation, it involves a different typical time scale than those of the Euler equations. Accordingly, the stability constraints in Step 0 and Steps 1-2 are different. Therefore, given the space mesh size Δx, we define a “parabolic” and a “hyperbolic” time steps, $\Delta t_p = \mathcal{O}(\Delta x^2)$ and $\Delta t_h = \mathcal{O}(\Delta x)$) respectively (with $\Delta t_p < \Delta t_h$). Then, we perform several sub-cycles (Step 1-Step 2) above on time intervals $(k\Delta t_p, (k+1)\Delta t_p)$ and only one Step 0 on the time interval $(k\Delta t_h, (k+1)\Delta t_h)$.
2. *Approximation of the Fokker-Planck semi-group.* Formulae (3.13) involve the operator e^{sL}, with L the Fokker-Planck operator, but the expression is not explicit enough to be incorporated in a numerical subroutine, and a further approximation is needed. The method we propose is based on the expression of the semi-group by means of convolution with the fundamental solution associated to the Fokker-Planck operator, see [12]. Using the fact that we are concerned with the regime $0 < \epsilon \ll 1$, we derive the following expression to be used in Step 1

$$\begin{cases} f^{k+1/2}(v) = M(v) \left(\rho^k + e^{-\Delta t/\varepsilon} v \displaystyle\int_{\mathbb{R}^3} v_\star f^k \,\mathrm{d}v_\star \right), \\ r^{k+1/2}(v) = e^{-\Delta t/\varepsilon} M(v) \left(v \displaystyle\int_{\mathbb{R}^3} v_\star r^k \,\mathrm{d}v_\star \right) + (1 - e^{-\Delta t/\varepsilon})\ M(v) S^k. \end{cases} \tag{3.14}$$

3. *Fundamental properties of the scheme.* The numerical scheme can be shown to fulfill many interesting requirements. First of all, it is Asymptotic Preserving in the sense that letting ϵ run to 0, we obtain a stable and consistent scheme for the limit system (3.7)–(3.8). Second of all, the scheme is well-balanced which means that

it preserves the equilibrium states. Finally, up to some reasonable care in the space/velocity discretization as well as in the definition of the numerical boundary conditions, the scheme conserves mass. We can check on numerics that the entropy dissipation property is also preserved.

We refer to [11] for further details and comments. We only show below a sample of the simulations, restricting ourselves to a one-dimensional situation (which is a toy-model for instance for describing the dispersion of pollutants emitted from ground sources). Initially, the dense phase is at rest $u(0,x) = 0$ with constant density $n(0,x) = 1$ while the distribution of particles is the following centered Maxwellian:

$$f(0,x,v) = 0.5\ \mathbb{1}_{[a,b]}(x)\ \frac{e^{-v^2/2}}{\sqrt{2\pi}},$$

with $0 \leq a \leq b \leq 4$. The influence of ϵ can be discussed by looking at Figures 3.1 and Figures 3.1, where in both cases the adiabatic constant is $\gamma = 1.4$. The smoothing effect of the limit $\varepsilon \to 0$ appears clearly and both the fluid unknowns (n,u) and the macroscopic density of particles are smoother for small values of ε.

In Figure 3.1, we show the solution at time $T = 20$ for different values of the adiabatic constant. The simulations illustrate the stability of sedimentation profiles, as conjectured from [10]. These profiles depend on the pressure law, with a change of convexity for the critical exponent $\gamma = 2$.

The discussion left open several important questions. In particular, assuming a constant temperature in the system (3.1)–(3.3) might be questionable. An extension of the model that includes an energy equation and energy exchanges has been proposed and studied in [8]. Another important question relies on the approximation of the Fokker-Planck semi-group, for which different approaches deserve to be discussed.

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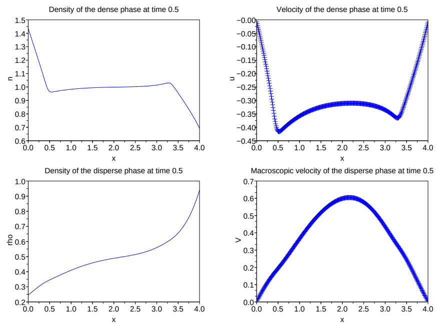

Figure 3.1: Bubbling Regime: Evolution for $\varepsilon = 0.1$. $T = 0.5$

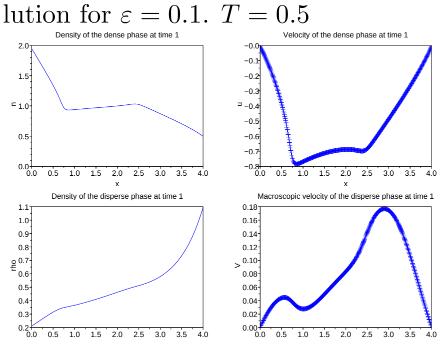

Figure 3.2: Bubbling Regime: Evolution for $\varepsilon = 0.1$. $T = 1$

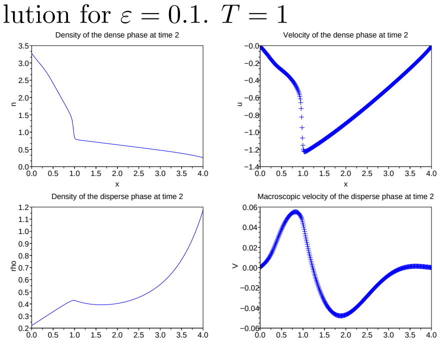

Figure 3.3: Bubbling Regime: Evolution for $\varepsilon = 0.1$. $T = 2$

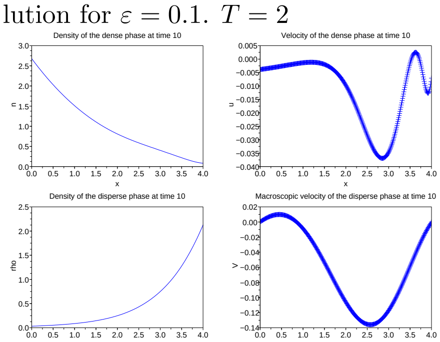

Figure 3.4: Bubbling Regime: Evolution for $\varepsilon = 0.1$. $T = 10$

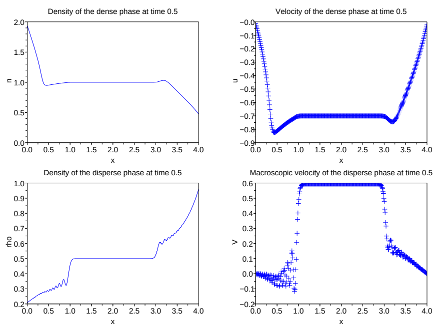

Figure 3.5: Bubbling Regime: Evolution for $\varepsilon = 0.5$. $T = 0.5$

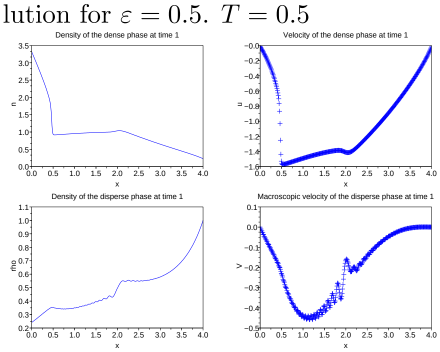

Figure 3.6: Bubbling Regime: Evolution for $\varepsilon = 0.5$. $T = 1$

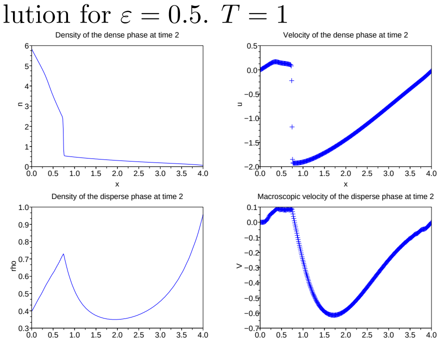

Figure 3.7: Bubbling Regime: Evolution for $\varepsilon = 0.5$. $T = 2$

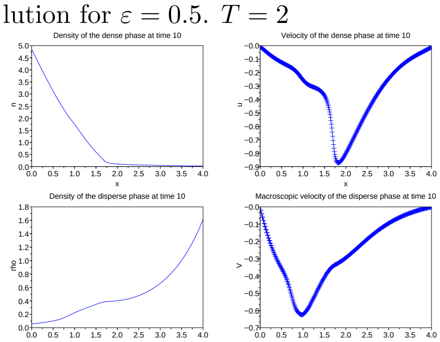

Figure 3.8: Bubbling Regime: Evolution for $\varepsilon = 0.5$. $T = 10$

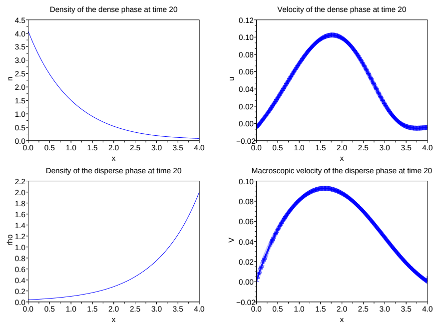

Figure 3.9: $\gamma = 1$

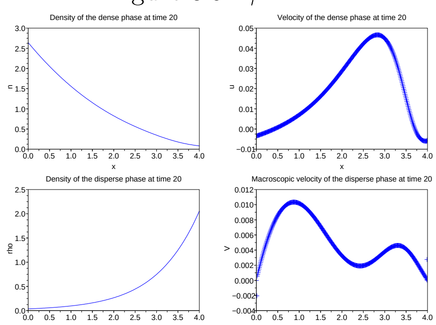

Figure 3.10: $\gamma = 1.4$

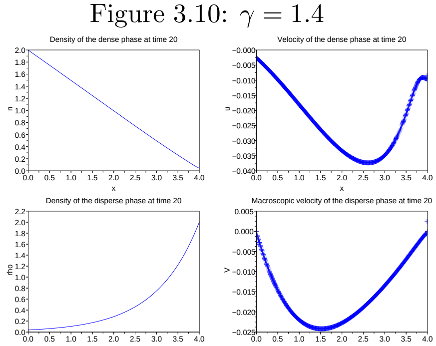

Figure 3.11: $\gamma = 2$

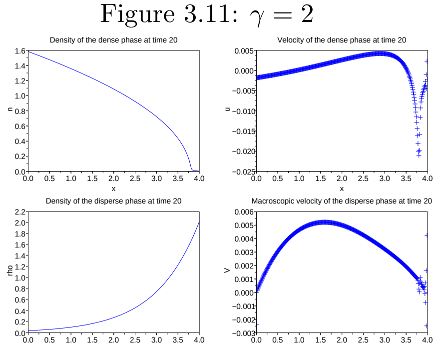

Figure 3.12: $\gamma = 3$