Design of fault tolerant flow networks

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1 Introduction

Modern telecommunication satellites are very complex to design and an important industrial issue is to provide robustness at the lowest possible cost. A key component of telecommunication satellites is an interconnection network which allows to redirect signals received by the satellite to a set of amplifiers from where the signals will be retransmitted. Designing such network is a complex problem that was proposed by Alcatel Space Industries. A detailed overview on the model and the motivations can be found in [2].

Informally, we are looking for a network interconnecting a set of input ports (at each of which a signal enters the network) with a set of output ports (at each of which a signal leaves after amplification). The connections are made via costly switches with 4 links, and the paths connecting inputs to outputs are link disjoint.

Here we suppose that signals are of the same kind so any signal can be routed to any output (port). In practice, amplifiers are subject to faults which cannot be repaired, but there is a value k such that the probability that more than k faults occur is practically negligible. As a first approach, given a number n of signals and a maximal number k of faults, we should design a low cost network routing n inputs and tolerating up to k faulty amplifiers. The first cost criterion is the number of amplifiers, and the second is the number of switches. So we consider networks with n inputs and $n + k$ outputs. In [2] such networks are called (n, k)-networks. An (n, k)-network is said to be *valid* if for any set of at most k faulty outputs, there exists a set of n disjoint paths interconnecting the n inputs to the n non-faulty outputs. The design problem consists of determining a valid (n, k)-network with a *minimum number of switches.*

We define our (n, k)-network as follows.

Definition 1 An (n, k)-network is a triple $N = \{(V, E), i, o\}$ where $G = (V, E)$ is a graph and i, o are integral functions defined on V called *input* and *output* functions, such that for any $v \in V$, $i(v) + o(v) + \deg(v) = 4$. The

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total number of inputs is $i(V) = \sum_{v \in V} i(v) = n$, and the total number of outputs is $o(V) = \sum_{v \in V} o(v) = n+k$, with $k \geq 0$.

Note that this definition is different but equivalent to that of [2] where inputs and outputs are introduced as vertices. Our definition enables us to apply tools of flow theory.

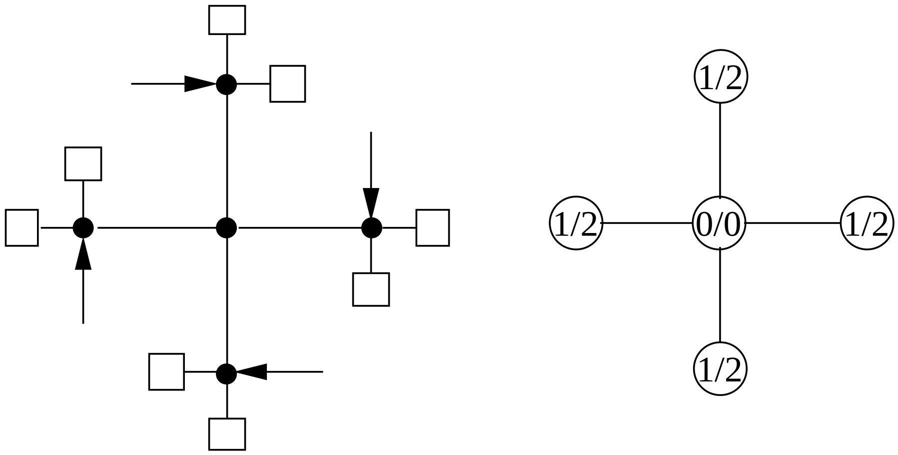


Figure 1: A valid $(4,4)$-network with 4 inputs and 8 outputs on the left, and the numeric representation of the input/output functions on the right.

Any integral function o' defined on V such that $o'(v) \leq o(v)$ for any $v \in V$, and $o'(V) = n$ is called a *faulty output function*.

Note that $o(v) - o'(v)$ is the number of faults at vertex v.

Definition 2 An (n,k)-network is *valid*, if for any faulty output function o', there are n edge-disjoint directed paths in G such that each vertex $v \in V$ is the initial vertex of $i(v)$ paths and the terminal vertex of $o'(v)$ paths.

Let us denote the minimum number of vertices in a valid (n,k)-network by $\mathcal{N}(n,k)$. A valid (n,k)-network with exactly $\mathcal{N}(n,k)$ vertices is called a *minimum* (n,k)-network.

Problem 1 *The design problem consists of determining $\mathcal{N}(n,k)$ and of constructing minimum (n,k)-networks, or at least valid (n,k)-networks with a number of vertices close to the optimal value.*

In [2], the authors proved that $\mathcal{N}(n,1) = \mathcal{N}(n,2) = n$ and gave a general construction which yields $\mathcal{N}(n+n',k) \leq \mathcal{N}(n,k) + \mathcal{N}(n',k)$, under some conditions. In particular, they proved that $\mathcal{N}(n,4) \leq n + \lceil \frac{n}{4} \rceil$.

In this paper, we present an approach which simplifies the design problem, and we apply it for practical values of k ($k \leq 12$). In a first step we derive lower bounds on $\mathcal{N}(n,k)$, then we propose effective almost optimal constructions. We also provide some asymptotic values for larger values of k. In summary we prove that

$$\begin{array}{lcl}
\mathcal{N}(n,4) & = & n + \lceil \frac{n}{4} \rceil, \\
\mathcal{N}(n,6) & = & n + \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1), \\
\mathcal{N}(n,8) & = & n + \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt[4]{n}), \\
\mathcal{N}(n,10) & = & n + \frac{3n}{8} + \Theta(\sqrt{n}), \\
\mathcal{N}(n,12) & = & n + \frac{3n}{7} + O(\sqrt{n}).
\end{array}$$

For larger values of k, we show that $\mathcal{N}(n,k) \leq n + \frac{n}{2} + O(k)$, this bound is tight as we prove also that :

$$\mathcal{N}(n,k) \geq \frac{3n}{2} - O(\frac{n}{k})$$

2 Validity and Cut-criterion

We introduce some notation that will be used throughout this paper.

Given a function f, we use the notation $f(A) = \sum_{a \in A} f(a)$ for any finite set A.

For a set $W \subset V$ of a graph $G = (V, E)$, let us denote $\Delta(W)$ the set of edges connecting W and $\overline{W} = V \setminus W$, $\delta(W)$ the cardinality of $\Delta(W)$, and $\Gamma(W)$ the set of vertices adjacent to a vertex of W.

In order to prove structural properties, we generalize the definition of (n, k)-networks. In this general definition, we impose no restriction on the degree of the vertices.

Definition 3 An (n, k)-graph is a triple $N = \{(V, E), i, o\}$ where $G = (V, E)$ is a multi-graph and i, o are integral functions defined on V, where, for any $v \in V$, $i(v)$ (resp. $o(v)$) denotes the number of inputs (resp. outputs) at v. The total number of inputs is $i(V) = n$, and the total number of outputs is $o(V) = n + k$, with $k \geq 0$.

Any integral function o' defined on V such that $o'(v) \leq o(v)$ for any $v \in V$, and $o'(V) = n$ is called a *faulty output function*.

A crucial fact is that the validity of a (n, k)-graph is nicely expressed in term of a supply/demand flow problem (see [3, 1] for a definition).

Definition 4 An (n, k)-graph $N = \{(V, E), i, o\}$ is *valid* if and only if for any faulty output function o' the following supply/demand flow problem is feasible in the graph $G = (V, E)$: *for any $v \in V$ the demand is $demand(v) = o'(v) - i(v)$ [1] and the capacity of every edge of E is one.*

As integral flow problems admit integral solutions, this definition is equivalent to state that, for any set of k faults, there exist n pairwise edge-disjoint paths connecting inputs to non faulty outputs.

2.1 A cut-criterion for (n, k)-graphs

As any input can be routed to any output, the validity problem of a (n, k)-graph reduces to a simple flow problem once the faulty outputs are identified (more formally when the faulty output function is fixed as in definition 4). The next property characterises valid (n, k)-graphs and is a direct consequence of the Ford-Fulkerson theorem [3].

Property 5 (min cut & max flow criterion) *An (n, k)-graph $\{(V, E), i, o\}$ is valid if and only if, for any subset of vertices $W \subset V$ the* excess *of W,*

$$\mathcal{E}(W) = \delta(W) + o(W) - i(W) - \min\{k, o(W)\}$$

satisfies $\mathcal{E}(W) \geq 0$.

Proof. Let o' be a fixed faulty output function, then a supply/demand flow problem is defined by an integral (not necessarily positive) demand at each node v. In our case, the demand of a node $v \in V$ is $demand(v) = o'(v) - i(v)$. (Note that $demand(V) = 0$, which is always the case for supply/demand problems.) A variant of the Ford-Fulkerson Theorem states that the supply/demand problem is feasible if and only if

$$\forall W \subset V: \quad \delta(W) \geq demand(\overline{W}) = o'(\overline{W}) - i(\overline{W}) = i(W) - o'(W).$$

It follows that the (n, k)-graph is valid if and only if

$$\forall W \subset V: \quad \delta(W) \geq i(W) - \min\{o'(W) \mid \ o' \text{ a faulty output function}\} \tag{1}$$

[1] If $o'(v) - i(v) < 0$, then this means that v supplies $i(v) - o'(v)$ units of flow; remark also that the problem is correctly defined as $demand(V) = o'(V) - i(V) = 0$.

By definition, $\min\{o'(W) \mid o' \text{ a faulty output function}\}$ is the minimum number of non-faulty outputs in W. This minimum is attained either by choosing all the outputs in W to be faulty when $o(W) \leq k$, or by choosing k outputs in W to be faulty when $o(W) \geq k$.

Hence, $\min\{o'(W) \mid o' \text{ a faulty output function}\} = o(W) - \min\{o(W), k\}$. The property follows then from equation 1. $\square$

Remark 6 $\min\{o(W), k\}$ is the maximum number of faults that can occur in W, so the cut-criterion simply states that the capacity of the border of W is larger than the difference $i(W) - o(W)$, plus the maximum number of faults in W.

3 Reduction of (n, k)-networks to their Kernels

In what follows, **we suppose** $k \geq 1$.

We will see that we can apprehend all the interesting properties of an (n, k)-network by considering a bipartite graph, called its *kernel*, associated to it.

The reduction is based on the following observations (see also [2]).

Property 7 *In a minimal valid (n, k)-network with $k \geq 1$, $i(v) \leq 1$ for all vertices v.*

Proof. The *cut criterion* applied to any single vertex v implies that $i(v) \leq 2$. If $i(v) = 2$, then $o(v) \leq 0$, since $k \geq 1$. Supposing $i(v) = 2$ and $0(v) = 0$, we have $\delta(v) = 2$. We can remove v and connect the two inputs directly to the two neighbors of v, and obtain a smaller valid (n, k)-network. $\square$

This implies that there are two kinds of vertices in an (n, k)-network, those with $i(v) = 1$ and those with $i(v) = 0$.

Property 8 *In a minimal valid (n, k)-network with $k \geq 1$, vertices with $i(v) = 1$ do not form a circuit.*

Proof. Let C be a minimal circuit formed by vertices with $i(v) = 1$. If C has ℓ vertices, then $i(C) = \ell$ and $\delta(C) + o(C) = \ell$ (as there are ℓ edges inside C we have $4\ell = \delta(C) + i(C) + o(C) + 2\ell$). By the *cut criterion*, $o(C) = 0$, since $k \geq 1$. Therefore, we can remove C and connect the ℓ inputs directly to the neighbors of C, and get a smaller valid (n, k)-network. $\square$

Intuitively the kernel will be a bipartite (n, k)-graph with two vertex classes: the "blocks" and "S-switches". Most of the blocks will correspond to the connected components formed by vertices with $i(v) = 1$. S-switches will be in correspondence with vertices with $i(v) = 0$. In fact for simplicity and technical purposes, we have to introduce "special blocks".

Special blocks of type α, Consider a vertex v of an (n, k)-network with $i(v) = 0$. If $o(v) = 0$, then it is an S-switch. If $o(v) \geq 1$, then associate an S-switch S to v with $i(S) = 0, o(S) = 0$, and $deg(S) = deg(v) + o(v) = 4$, joined to $o(v)$ new blocks B each with $i(B) = 0$, $o(B) = 1$, and $deg(B) = 1$. (called blocks of type α)

Now we are almost done, except that there can be adjacent S-switches in our present (n, k)-network. In order to guarantee the kernel to be a bipartite graph, we introduce a second type of special blocks.

Special blocks of type β, If there is a link between two S-switches, then we subdivide the link by inserting a block B with $i(B) = o(B) = 0$, and $deg(B) = 2$ (called blocks of type β).

Let us summarise what we have done.

Definition 9 To every (n, k)-network N, we can associate a bipartite (n, k)-graph $K(N)$, called (n, k)-kernel with two classes of vertices, blocks and S-switches.

- Blocks are either maximal connected components of the (n,k)-network formed by vertices with $i(v)=1$, or special blocks of type α or β.

- The S-switches are in one-to-one correspondence with the vertices of N with $i(v)=0$.

Denote by $\mathcal{B}$ (resp. $\mathcal{S}$) the set of blocks (resp. S-switches) of an (n,k)-graph, and put $s=|\mathcal{S}|$.

The functions i and o are interpreted in $K(N)$ as follows : for every vertex u of $K(N)$ representing a set U of N we set $i(u)=i(U)$, $o(u)=o(U)$.

Property 10

(i) $K(N)$ *is bipartite,*

(ii) $i(V_N)=i(V_{K(N)})=n$ *and* $o(V_N)=o(V_{K(N)})=n+k$*,*

(iii) for every S*-switch* S *of* $K(N)$*,* $i(S)=o(S)=0$*,*

(iv) for every block B *of* $K(N)$*,* $\deg(B)=i(B)+2-o(B)$*,*

(v) $|V(N)|=n+s$*.*

Proof. (i), (ii), and (iii) are true by definition.

(iv) holds for the special blocks of type α and β by definition. Suppose that the block B is formed by a connected component B' of b vertices v with $i(v)=1$, so that $i(B)=b$. For every $v\in B'$, $deg(v)=4-i(v)-o(v)=3-o(v)$, hence $\sum_{v\in B'}deg(v)=3b-o(B)$. Because B' is connected and circuit free, there are exactly $b-1$ edges inside B' and $\deg(B)=\delta(B')=\sum_{v\in B'}deg(v)-2(b-1)=b+2-o(B)$

For (v), observe that $i(\mathcal{S})=0$, $\sum_{B\in\mathcal{B}}i(B)=n$, and there is a one-to-one correspondence between the S-switches of $K(N)$ and the vertices with $i(v)=0$ of N. $\square$

4 The reduced problem

($\mathcal{E}(W)$ is defined in property 5)

Lemma 11 *Let* W *be a set of vertices of a* (n,k)*-graph , and assume that* $\Gamma(W)$ *contain a vertex* v *such that* $d(v)\le i(v)-o(v)+2$ *then the follwoing relation holds :*

$$\mathcal{E}(W\cup\{v\})\le\mathcal{E}(W)$$

Proof. Let $W_1=W\cup\{v\}$, $i(W_1)=i(W)+i(v)$, $o(W_1)=o(W)+o(v)$, and $\delta(W_1)\le\delta(W)+\deg(v)-2=\delta(W)+i(v)-o(v)$. Hence, by definition we have $\mathcal{E}(W_1)=\delta(W_1)+o(W_1)-i(W_1)-min(k,o(W_1))\le\delta(W)+o(W)-i(W)-\min\{k,o(W)+o(v)\}\le\mathcal{E}(W)$. $\square$

Remark 12 Note that the above lemma can be applied either to a **block** v **of** $K(N)$ (as we have $d(v)=i(v)-o(v)+2$), or to **a vertex** v' **of** N **such that** $i(v')=1$ (as then $d(v')=3-o(v')=i(v')-o(v')+2$.

Lemma 13 N *is a valid* (n,k)*-network if and only if then* $K(N)$ *is a valid* (n,k)*-graph.*

Proof. First assume that $K(N)$ is non valid; equivalently it exists a subgraph W of $K(N)$ with $\mathcal{E}_{K(N)}(W)<0$. According to lemma 11 and remark 12 we can assume that W contains all the blocks adjacent to it. We associate to W a subset W' of N obtained as follows :

- to a S-switch and to the blocks α, β adjacent to it we associate the corresponding vertex of N;
- to a normal block B we associate the connected component B' of N.

We have $i(W) = i(W'), o(W) = o(W'), \delta(W) = \delta(W')$, so $\mathcal{E}_N(W') = \mathcal{E}(W) < 0$; so N is non valid.

Conversely, assume that N is not valid, or suppose equivalently that it exists a connected subgraph W' with $\mathcal{E}_N(W') < 0$. According to lemma 11 and remark 12 we can also assume that W' is such that for any connected component C formed by vertices such that $i(v) = 1$ either $C \subset W'$ or $W' \cap C = \emptyset$.

So, according to the definition of the kernel $K(N)$, we can associate to W' a subset W of $K(N)$ (by associating to a connected component B' of N the corresponding normal block B of $K(N)$, and to a vertex with $i(v) = 0$ the corresponding S-switch plus possibly some special blocks α, β). We have $\mathcal{E}(W) = \mathcal{E}(W') < 0$. Hence $K(N)$ is not valid. $\square$

The properties proved in Property 10 allow to formulate a simpler version of the design problem.

Problem 2 *Denoting $\mathcal{N}'(n,k)$ the minimum number of S-switches in a valid (n,k)-kernel, the design problem consists of finding an (n,k)-kernel having a number of S-switches equal to (or close to) $\mathcal{N}'(n,k)$.*

In fact, Problem 2 is equivalent to Problem 1.

Theorem 14 $\mathcal{N}(n,k) = \mathcal{N}'(n,k) + n$

Proof.

According to Property 10/(v) and Lemma 13 it is sufficient to prove that any (n,k)-kernel is the kernel of some (n,k)-network. To do this we provide an inverse of the algorithm described in Definition 9. Note that the inverse operation is not deterministic (the result is not unique), which is the consequence of the fact that several (n,k)-networks has the same (n,k)-kernel. Apply the following operations on an (n,k)-kernel $K = \{(\mathcal{S} \cup \mathcal{B}, E), i, o\}$.

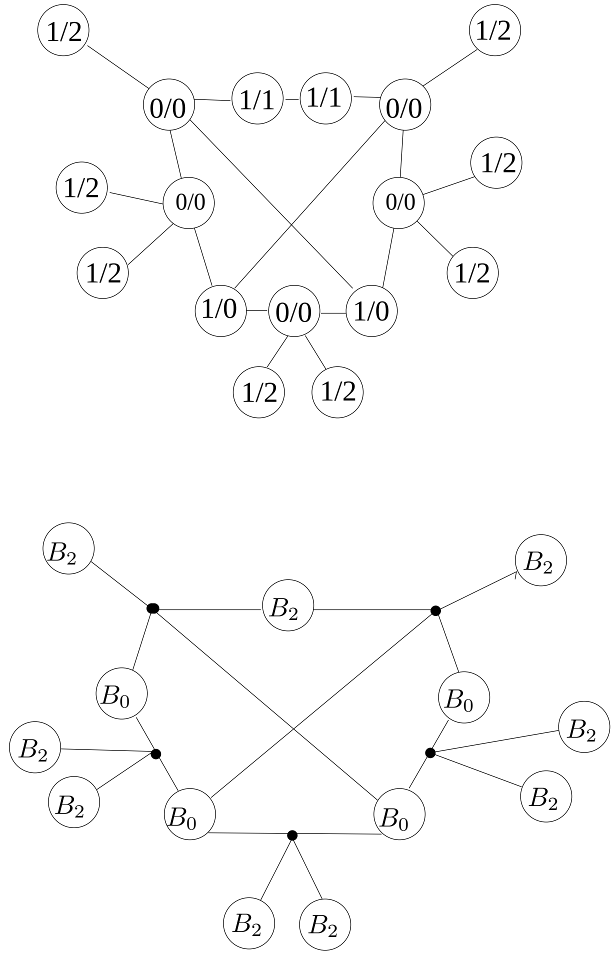
(a) As long as K contains a block B with $i(B) = 0$, $o(B) = 0$, $deg(B) = 2$ connected to two S-switches S_1 and S_2, replace it by an edge $[S_1, S_2]$.

(b) As long as K contains a block B with $i(B) \geq 2$, replace B by a chain of $i(B)$ vertices each with $i(v) = 1$. Distribute $o(B)$ among the vertices so that $\sum_1^{i(B)} o(v) = o(B)$ respecting the condition $o(v) + i(v) + deg(v) = 4$, and connect the remaining $deg(B)$ edges of the chain to the neighbors of B in an arbitrary order. This can always be done, since $deg(B) = i(B) - o(B) + 2$.

(c) As long as K contains a block B with $i(B) = 0$, $o(B) = 1$, $deg(B) = 1$ connected to a vertex S, delete B and let $o(S) := o(S) + 1$.

$\square$

For an example see Figure 2.



$(12, 6)$-kernel

Figure 2: Equivalence between (n, k)-networks and (n, k)-kernels. The indices of the blocks indicate the number of outputs of the blocks. In the bottom kernel S-switches are represented by dots, while blocks are represented by circles.

5 Basic properties for kernels

First an immediate property.

Property 15 *For $k \geq 3$, the blocks of a valid (n, k)-kernel contains at most 2 outputs.*

Proof. Given a block B, we compute the excess on the one-element set $\{B\}$. $\mathcal{E}(\{B\}) = deg(B) + o(B) - i(B) - \min(k, o(B)) = 2 - \min(k, o(B))$, because $deg(B) = 2 + i(B) - o(B)$. So, in a valid network $\min(k, o(B)) \leq 2$, and for $k > 2$, $o(B) \leq 2$. $\square$

From now one **we will always suppose that** $k \geq 3$; according to Property 15, we will distinguish three types of blocks :

Definition 16

- For $i = 0, 1, 2$, denote by $\mathcal{B}_i$ the set of blocks containing i outputs (i.e. $o(B) = i$). Let b_i be the number of blocks of $\mathcal{B}_i$, let n_i the total number of inputs of the blocks of $\mathcal{B}_i$, and let $e_i = \sum_{B \in \mathcal{B}_i} deg(B)$.

- Clearly $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ (see Proposition 15);

Lemma 17 *The blocks of $\mathcal{B}_i$ are adjacent to $n_i + (2-i)b_i$ edges, in other words,*

$$e_0 = n_0 + 2b_0, e_1 = n_1 + b_1, e_2 = n_2 \tag{2}$$

Proof. Recall that by Property 10, the degree of a block $B \in \mathcal{B}_i$ is exactly $i(B) + 2 - o(B) = i(B) + 2 - i$. The statement of Lemma 17 follows by summing over all blocks of $\mathcal{B}_i$. □

We have also the following set of equations.

Theorem 18 *In a valid (n, k)-kernel, with $k \geq 3$*

$$\begin{aligned} n &= n_0 + n_1 + n_2 && (3)\\ n + k &= b_1 + 2b_2 && (4)\\ 4s &= n + 2b_0 + b_1 && (5)\\ 4s &= 2n + k + 2b_0 - 2b_2 && (6) \end{aligned}$$

Proof. Equation (3) follows from the fact that the total number of inputs is n. Equation (4) is obtained by counting the outputs $\sum_{B \in \mathcal{B}} o(B) = \sum_{i=0,1,2} ib_i$. For equation (5), consider the number of edges in the bipartite graph; on the one hand we have $\sum_{i=0,1,2} e_i = n + 2b_0 + b_1$ (from 2); on the other hand there are $4s$ edges. The last equation is simply (5) - (4). □

6 Lower bounds for $k \geq 3$

As k increases, the optimal value $\mathcal{N}'(n, k)$ gets larger and larger. We will successively introduce counting arguments for $k = 4, 6, \ldots, 12$. The claims obtained for a given value of k are always valid for larger k. The main idea is to use several counting arguments and to determine which patterns are forbidden in a valid kernel for a given value of k.

Before deriving additional properties of kernel, we first consider the cases $k = 3, 4$ which can be solved immediately thanks to the reduction of the initial problem to kernel problem.

6.1 Cases $k = 3, 4$.

Note that the next result was first conjectured in [**?**] where the upper bound was proved.

Theorem 19 $\mathcal{N}'(n, 3) = \mathcal{N}'(n, 4) = \left\lceil \frac{n}{4} \right\rceil$.

Proof. $\mathcal{N}'(n, 3), \mathcal{N}'(n, 4) \geq \left\lceil \frac{n}{4} \right\rceil$ is an immediate consequence of equation (5),

$$4s = n + 2b_0 + b_1 \geq n.$$

For the upper bound, it is enough to construct a valid $(n, 4)$-kernel. Constructions are presented in section 9 but as it is the *minimum* exactly and extremely simple, we present here some constructions for $k = 4, n = 4p$. According to equation (5), the lower bound can only be attained, if $b_0 = b_1 = 0$. That is when all the blocks are in $\mathcal{B}_2$. To proceed with the construction, we need the following definition (that will be of use for larger values of k, too)

Definition 20 A *twin* is a connected subgraph of an (n, k)-network formed by an S-switch adjacent two blocks of $\mathcal{B}_2$ with one input each.

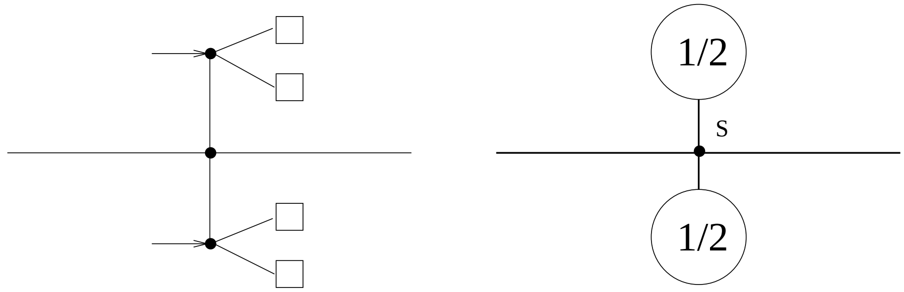
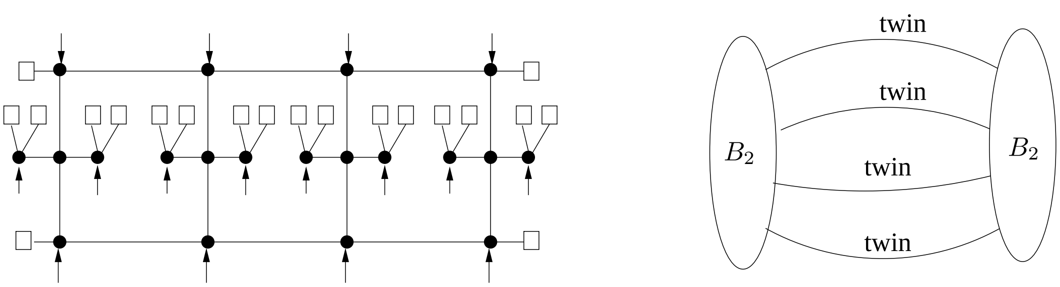


Figure 3: A twin

A twin contains 2 inputs and 4 outputs. A twin can be considered as linking the 2 blocks which are adjacent to the S-switch contained in the twin. One construction is obtained by connecting two large blocks of $\mathcal{B}_2$ both of size p by p twins (see as example figure 4). We have then $2p$ inputs in the two large blocks of B_2 and $2p$ inputs in the p twins, 4 outputs in the two large blocks and $4p$ outputs in the p twins, p S-switches each in a twin.

Figure 4 shows a minimum $(16, 4)$-kernel. There are $2 \cdot 4 = 8$ inputs and $2 \cdot 2 = 4$ outputs in the two large blocks of $\mathcal{B}_2$, each small blocks of $\mathcal{B}_2$ contains 1 input and 2 outputs. In total, there are 16 inputs, 20 outputs and 4 S-switches.

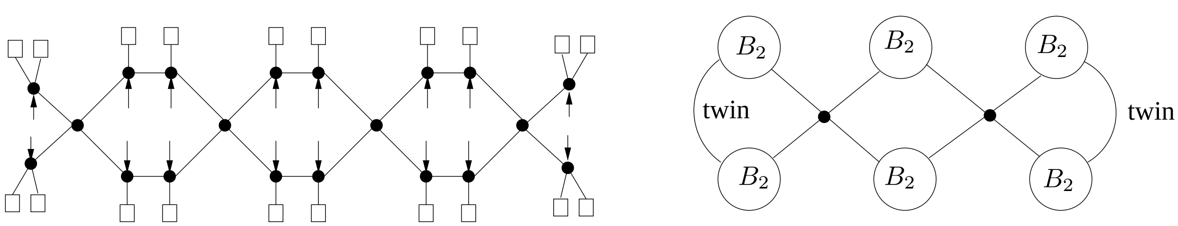


$(16, 4)$-network

$(16, 4)$-kernel

Figure 4: One minimum $(16, 4)$-kernel, with 4 S-switches.

The construction given in [2] (see Figure 5) is attained by connecting $p - 1$ pairs of blocks of $\mathcal{B}_2$ with $i(B) = 2$. The pairs form a chain where the extremal pairs are connected by a twin and all the others pairs are connected to S-switches.



$(16, 4)$-network

$(16, 4)$-kernel

Figure 5: Another minimum $(16, 4)$-kernel.

The validity of these constructions can be easily verified by checking the cut-criterion. (See Section 9 for a short proof.) $\square$

7 Cut-criterion for (n,k)-kernels

In this section, we refine the general cut-criterion obtained in Property 5 for the case of (n,k)-kernels. For $X \subset \mathcal{S}$

Property 21 (cut-criterion) *An (n,k)-kernel $K = \{(\mathcal{S} \cup \mathcal{B}, E), i, o\}$ is valid if and only if for every non-empty $X \subset \mathcal{S}$,*

$$2|\Gamma(X)| \geq 4|X| + \min(o(\Gamma(X)), k). \tag{7}$$

Proof. First we prove that the criterion (5) implies (7). Let $X \subset \mathcal{S}$ and let $W = X \cup \Gamma(X)$. As $K(N)$ is bipartite, there are $4|X|$ edges inside W. So $\delta(W) = \delta(\Gamma(X)) - 4|X|$, $o(W) = o(\Gamma(X))$, and $i(W) = i(\Gamma(X))$. By Property 10/(iv)

$$\delta(\Gamma(X)) = i(\Gamma(X)) - o(\Gamma(X)) + 2|\Gamma(X)|,$$

and so $\mathcal{E}(W) = \delta(W) + o(W) - i(W) - \min(k, o(W)) = 2|\Gamma(X)| - 4|X| - \min(k, o(\Gamma(X)))$.

If $K(N)$ is valid, then, by Property 5 $\mathcal{E}(W) \geq 0$ for all W. Applying it for $W = X \cup \Gamma(X)$, we obtain (7).

Conversely, suppose that condition (7) is satisfied, but Property 5 is violated. We show that there is a set $X \subset \mathcal{S}$ such that $\mathcal{E}(X \cup \Gamma(X)) < 0$, contradicting condition (7).

If property 5 is violated, it exists a set $W \subset \mathcal{S} \cup \mathcal{B}$ such that $\mathcal{E}(W) < 0$. Without loss of generality, we may assume that W is connected, otherwise take as W a connected component with strictly negative excess.

If $W \subset \mathcal{B}$, then W is reduced to a block B, so $deg(B) = i(B) - o(B) + 2$ but from Property **(??)** $o(B) \leq 2$, so $\mathcal{E}(W) \geq 0$. If W intersects $\mathcal{S}$, let $X = W \cap \mathcal{S}$. As $K(N)$ is bipartite and W is connected, $W \subset X \cup \Gamma(X)$. By lemma 11 and remark12 , for every $B \in \Gamma(X) \setminus W$, $\mathcal{E}(W \cup \{B\}) \leq \mathcal{E}(W)$ and so $\mathcal{E}(W \cup \{B\}) < 0$ and $\mathcal{E}(X \cup \Gamma(X)) < 0$, contradicting (7). $\square$

In the example of Figure 6, there are 4 blocks adjacent to an S-switch S, two blocks of $\mathcal{B}_2$ containing 5 and 3 inputs, one of $\mathcal{B}_1$ containing 1 input, and one of $\mathcal{B}_0$ containing 3 inputs. Putting $X = \{S\}$, and apply the cut-criterion 7 with $|\Gamma(X)| = 4$, $o(\Gamma(X)) = 5$, and $4|X| = 4$. We obtain $4 \geq \min(5, k)$, so, whenever $k \geq 5$, no (n,k)-kernel can contain the depicted sub-graph.

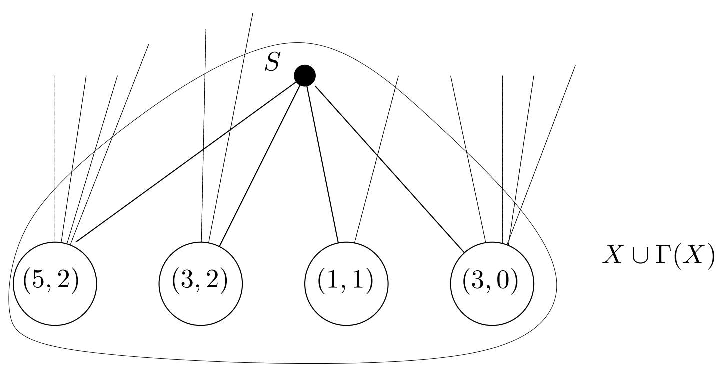


Figure 6: An S-switch S and its four adjacent blocks.

Note that the cut-criterion is due to parity the same for $k = 2p+1$ and $k = 2p+2$. So a valid (n,k)-kernel for $k = 2p+1$ also satisfies the cut-criterion for $k = 2p+2$, but one output is missing. Hence it is enough to add to the network one output, without violating the cut-criterion of such a network. It seems very likely that such a transformation can always be done with a bounded number of extra S-switches. This justify the next conjecture

Conjecture 21.1 $\mathcal{N}'(n, 2p+1) = \mathcal{N}'(n, 2p+2) + O(1)$.

7.1 Cases $k = 5, 6$

In order to derive an accurate bound, we need to classify the set of S-switches adjacent to the blocks of $\mathcal{B}_2$.

Definition 22

- Denote $\mathcal{S}_{i,j}, i + j \leq 4$ the set of S-switches which are adjacent to i blocks of $\mathcal{B}_2$ and to j blocks of $\mathcal{B}_0$ (and hence to $4 - i - j$ blocks in $\mathcal{B}_1$) Let $s_{i,j} = |\mathcal{S}_{i,j}|$.
- Let $\mathcal{S}_2 = \cup_j \mathcal{S}_{2,j}$, $\mathcal{S}_1 = \cup_j \mathcal{S}_{1,j}$ and $\mathcal{S}_0 = \cup_j \mathcal{S}_{0,j}$. As usual, let $s_2 = |\mathcal{S}_2|, s_1 = |\mathcal{S}_1|, s_0 = |\mathcal{S}_0|$.

In the same way we define for $A \subset \mathcal{S}$, $\mathcal{S}_{i,j}(A) = \mathcal{S}_{i,j} \cap A$, and $s_{i,j}(A) = |\mathcal{S}_{i,j}(A)|$.

Lemma 23 *In a valid (n, k)-kernel with $k \geq 5$, $\mathcal{S}_{i,j} = \emptyset$ whenever $j < i$.*

Proof. Consider an S-switch $S \in \mathcal{S}$ and apply the cut criterion (property 21) to the set $X = \{S\}$. $2|\Gamma(X)| \geq 4 + \min(o(\Gamma(X)), k)$. As $|\Gamma(X)| = 4$, we obtain $\min(o(\Gamma(X)), k) \leq 4$. Whenever $k \geq 5$, necessarily $o(\Gamma(X)) \leq 4$.

Finally observe that for an $X = \{S\}$ with S an S-switch in $\mathcal{S}_{i,j}$, $o(\Gamma(X)) = 2i + (4 - (i + j)) = 4 + i - j$, so whenever $k \geq 5$, $o(\Gamma(X)) \leq 4$ implies $i \leq j$. $\square$

Note that Lemma 23 implies that every S-switch has more neighbors in $\mathcal{B}_0$ than in $\mathcal{B}_2$. In particular, $\mathcal{S}_2 = \mathcal{S}_{2,2}$.

In the case $k = 4$, the minimum network that we constructed in 6.1 is such that all the S-switches are elements of $\mathcal{S}_{4,0}$.

Lemma 24 *In a valid (n,k)$-network$ with $k \geq 5$,*

$$s_2 \leq \binom{b_0}{2}.$$

Proof. We say that two (not necessarily distinct) blocks of $\mathcal{B}_0$ *share an S-switch* if they are adjacent to the same S-switch of $\mathcal{S}_2$. Associate an auxiliary multi-graph $H(K)$ to any valid (n, k)-kernel K as follows.

- The nodes of $H(K)$ are the blocks of $\mathcal{B}_0$.

- Connect B_a and B_b by j edges, if B_a and B_b share j S-switches of $\mathcal{S}_2$.

The graph $H(K)$ has b_0 vertices and s_2 edges. It suffice to prove that H is a simple graph if the associated kernel is valid, as $s_2 \leq \binom{b_0}{2}$ holds for any simple graph with b_0 vertices and s_2 edges.

First we show that $H(K)$ does not contain loops. Consider an S-switch $S \in \mathcal{S}_2$, and apply the cut-criterion (property 21) to $X = \{S\}$. We have $|X| = 1$, and $o(\Gamma(X)) = 4$ (since $X \in \mathcal{S}_2$). So $\min(o(\Gamma(X)), k) \geq 4$, and the cut-criterion implies $|\Gamma(X)| \geq 4$, consequently, X is adjacent to 4 distinct blocks.

Next we prove that $H(K)$ does not have double edges. Consider two S-switches $S_a, S_b \in \mathcal{S}_2$ and let $X = \{S_a, S_b\}$. The cut-criteria implies that $2|\Gamma(X)| \geq 8 + \min(o(\Gamma(X)), k)$.

As $o(\Gamma(X)) \geq 4$, we have $2|\Gamma(X)| \geq 12$, so $|\Gamma(X)| \geq 6$. Either $|\Gamma(X)| = 6$, but $o(\Gamma(X)) = 4$, which means that S_a and S_b are adjacent to the same pair of blocks of $\mathcal{B}_2$, and therefore the pairs of blocks of $\mathcal{B}_0$ adjacent to S_a and S_b are distinct. Otherwise $|\Gamma(X)| \geq 7$ which means that S_a and S_b have at most one block of $\mathcal{B}_0$ as common neighbor. $\square$

Theorem 25 *For $k = 6$, $\mathcal{N}'(n, 6) \geq \frac{n}{4} + \sqrt{\frac{n}{8} + \frac{3}{2}}$.*

Proof. Equation (6) can be written as

$$2b_2 = 2n - 4s + k + 2b_0. \tag{8}$$

By definition of $\mathcal{S}_2$, the number of blocks of $\mathcal{B}_2$ is $b_2 \le 2s_2 + (s - s_2) = s + s_2$, that is $b_2 - s \le s_2$. Combining it with (8) we get

$$2s_2 \ge 2(b_2 - s) \ge 2n - 6s + k + 2b_0.$$

Assume that $s < \frac{n}{4} + \sqrt{\frac{n}{8} + \frac{3}{2}}$, then

$$2s_2 > \frac{n}{2} + k - 6\sqrt{\frac{n}{8} + \frac{3}{2}} + 2b_0.$$

Applying Lemma 24, we deduce

$$b_0(b_0 - 1) \quad > \quad \tfrac{n}{2} + k - 3\sqrt{\tfrac{n}{2} + 6} + 2b_0.$$

For $k = 6$, this implies

$$b_0(b_0 - 3) > \frac{n}{2} + 6 - 3\sqrt{\frac{n}{2} + 6} = \left(\sqrt{\frac{n}{2} + 6}\right)\left(\sqrt{\frac{n}{2} + 6} - 3\right),$$

so $b_0 > \sqrt{\frac{n}{2} + 6}$. Putting this in equality (5), we obtain a contradiction. It follows that $s \ge \frac{n}{4} + \sqrt{\frac{n}{8} + \frac{3}{2}}$, as required. □

7.2 Case $k = 7, 8$.

For $k \ge 7$, it is simpler to use a specific case of the cut criteria (property 21) for specific subgraphs that we call *patterns*.

Definition 26 A pattern P is a subset $P \subset \mathcal{S}_1 \cup \mathcal{S}_2$ such that $P \cup (\Gamma(P) \cap \mathcal{B}_0)$ is a connected subgraph of the kernel.

Lemma 27 (pattern condition) *In a valid (n, k)-kernel, a pattern P with $o(\Gamma(P)) \le 2\lceil k/2 \rceil$ satisfies :*

$$s_{1,2}(P) + 2(s_2(P) + s_{1,1}(P)) \le 2$$

Proof. Let denote $m_i(P), i = 0, 1, 2$, the number of edges between P and $\Gamma(P) \cap \mathcal{B}_i$

Remark that as $o(\Gamma(P)) \le 2\lceil k/2 \rceil$ the cut criteria for P simplifies to $4|P| \le 2|\Gamma(P)| - o(\Gamma(P))$ Note that $o(\Gamma(P)) = |\Gamma(P) \cap \mathcal{B}_1| + 2|\Gamma(P) \cap \mathcal{B}_2|$, so

$$2|\Gamma(P)| - o(\Gamma(P)) = |\Gamma(P) \cap \mathcal{B}_1| + 2|\Gamma(P) \cap \mathcal{B}_0|$$

P being a pattern, we have

$$|\Gamma(P) \cap \mathcal{B}_0| \le m_0(P) - |P| + 1$$

Combining these two equations leads to $4|P| \le 2(m_0(P) - |P|) + m_1(P) + 2$. Now, $4|P| = m_0(P) + m_1(P) + m_2(P)$, so $m_2(P) \le m_0(P) - 2|P| + 2$; equivalently

$$m_2(P) - m_0(P) + 2|P| \le 2$$

For a S-switch S in $\mathcal{S}_{i,j}$ we have $m_2(S) - m_0(S) + 2 = i - j + 2$ (that is 2 for $S \in \mathcal{S}_2 \cup \mathcal{S}_{1,1}$, 1 for $S \in \mathcal{S}_{1,2}$ and 0 for $S \in \mathcal{S}_{1,3}$). So, $m_2(P) - m_0(P) + 2|P| = 2(s_2(P) + s_{1,1}(P)) + s_{1,2}(P)$. □

Now, we derive new equations that will be used for the cases $k = 7, 8$ and $k = 9, 10$. Both equations hold, however, for every valid minimal (n, k)-kernel with $k \geq 3$.

Lemma 28

$$4s + 4s_2 + 2s_1 \geq 2n + k + 2b_0 + 2(e_2 - b_2) \geq 2n + k + 2b_0 \tag{9}$$

$$4s - \sum (j - i)s_{i,j} \geq n + k + n_1 + 2(e_2 - b_2) \geq n + k + n_1 \tag{10}$$

Proof. Let us show the following inequality.

$$3e_2 + e_1 + e_0 \geq 2n + k + 2b_0 + 2(e_2 - b_2). \tag{11}$$

To see this, recall that by lemma 17 $e_2 = n_2, e_1 = n_1 + b_1$, and $e_0 = n_0 + 2b_0$. So $e_2 + 2e_2 + e_1 + e_0 = n_2 + 2b_2 + 2(e_2 - b_2) + n_1 + b_1 + n_0 + 2b_0 = 2n + k + 2b_0 + 2(e_2 - b_2)$ by (3) and (4).

Inequality (9) is a combination of (11) and $3e_2 + e_1 + e_0 = (e_2 + e_1 + e_0) + 2e_2 = 4s + 4s_2 + 2s_1$. Note that, as $e_2 \geq b_2$, this implies left part of inequality 9.

For inequality (10), we simply evaluate $e_1 + 2e_2 = \sum ((4 + i - j) \cdot s_{i,j}) = 4s - \sum (j - i)s_{i,j}$ which is also $b_1 + n_1 + 2n_2 = b_1 + n_1 + 2b_2 + 2(n_2 - b_2) = n + k + n_1 + 2(n_2 - b_2)$. □

We will now focus on the S-switches of $\mathcal{S}' = \mathcal{S}_2 \cup \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2}$. The blocks of $\mathcal{B}_0$ adjacent to $\mathcal{S}'$ will be of importance so *we define $\mathcal{B}_0'$ as the set of blocks of $\mathcal{B}_0$ adjacent to some S-switch of $\mathcal{S}'$.*

Lemma 29 *In a valid (n,k)$-kernel$ with $k \geq 7$, if a block of $\mathcal{B}_0'$ is adjacent to two S-switches of $\mathcal{S}'$, then necessarily both S-switches are in $\mathcal{S}_{1,2}$.*

Proof. Note that we simply consider a pattern P with 2 S-switches. We have $o(\Gamma(P)) \leq 8$ and we can apply the pattern condition 27 stating that $2(s_2(P) + s_{1,1}(P)) + s_{1,2}(P) \leq 2$. This last equation is equivalent to the lemma. □

Lemma 30 *In a valid (n, k)-kernel with $k \geq 7$,*

$$b_0 \geq 2s_2 + s_{1,1} + \sqrt{4s_{1,2}} \tag{12}$$

Proof. Let $\mathcal{B}_0''$ be the set of blocks adjacent to $\mathcal{S}_{1,2}$, and let $b_0'' = |\mathcal{B}_0''|$. . According to Lemma 29, the blocks of $\mathcal{B}_0'$ adjacent to $\mathcal{S}_2 \cup \mathcal{S}_{1,1}$ are all distinct (that is $|\Gamma(\mathcal{S}_2 \cup \mathcal{S}_{1,1})| = 2s_2 + s_{1,1}$ blocks) moreover they are also distinct from the blocks of $\mathcal{B}_0''$, so

$$b_0 \geq 2s_2 + s_{1,1} + b_0'' \tag{13}$$

It remains to show that $b_0'' \geq \sqrt{4s_{1,2}}$; for that we associate an auxiliary graph $H(K)$ to every valid (n, k)-kernel as follows.

- The vertices are the blocks of $\mathcal{B}_0''$.

- We put an edge between two blocks if they are adjacent to the same S-switch of $\mathcal{S}_{1,2}$. (Note that an S-switch of $\mathcal{S}_{1,2}$ is adjacent to 2 blocks of $\mathcal{B}_0''$.)

We show that the graph $H(K)$ is simple and triangle free. First, lemma 29 assures that $H(K)$ is simple. Assume now that the graph $H(K)$ contains a triangle where the three edges correspond to S-switches $A_1, A_2, A_3 \in \mathcal{S}_{1,2}$ then an easy computation shows that the cut-criteria 21 is not satisfied for the set $X = \{A_1, A_2, A_3\}$ (contradiction).

As $H(K)$ is triangle free, it contains at most $(\frac{b_0''}{2})^2$ edges, hence $s_{1,2} \leq (\frac{b_0''}{2})^2$ and $b_0'' \geq \sqrt{4s_{1,2}}$. The lemma follows then from equation 13. □

Theorem 31 *For* $k \geq 7, \mathcal{N}'(n,k) \geq \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(1)$.

Proof. Combining (9) with (12), we get $4s + 4s_2 + 2s_1 \geq 2n + k + 4s_2 + 2s_{1,1} + 2\sqrt{4s_{1,2}} + 2(e_2 - b_2)$ or equivalently

$$4s + 2(s_{1,2} + s_{1,3}) \geq 2n + k + 2\sqrt{4s_{1,2}} + 2(e_2 - b_2). \tag{14}$$

This implies

$$6s \geq 2n + k + 2\sqrt{4s_{1,2}}. \tag{15}$$

So, $s \geq \frac{n}{3} + \frac{k}{6} + \frac{\sqrt{4s_{1,2}}}{3}$. Assume that $s = \frac{n}{3} + O(\sqrt{n})$. Then by inequality (14) we must have $s_{1,2} + s_{1,3} = s + O(\sqrt{n})$, (i.e. almost all the S-switches are in $\mathcal{S}_{1,2}$, and $\mathcal{S}_{1,3}$). So $s_{1,2} + s_{1,3} = \frac{n}{3} + O(\sqrt{n})$, Using equation (10), we have $4s - s_{1,2} - 2s_{1,3} \geq n + k + O(\sqrt{n})$.

$$3s_{1,2} + 2s_{1,3} \geq n + k + O(\sqrt{n}).$$

So $s_{1,3} = O(\sqrt{n})$ and $s_{1,2} = \frac{n}{3} + O(\sqrt{n})$, and $\sqrt{\mathcal{S}_{1,2}} = \sqrt{n/3} + O(1)$. Taking again equation (15), we obtain $6s \geq 2n + 8 + 2\sqrt{4n/3} + O(1)$. Finally $s \geq \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(1)$. $\square$

Remark 32 Consider a $(n,7)$-kernel with $\frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + o(\sqrt{n})$ S-switches, then it satisfies all the equations in a tight way. So in such a network we have :

- All the blocks B_2 except $o(\sqrt{n})$ have degree 1;
- $b_2 = e_2 = s + o(\sqrt{n}) = \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + o(\sqrt{n})$.
- $s = s_{1,2} + s_{1,3} + o(\sqrt{n})$.
- $b_1 = n + 7 - 2b_2 = \frac{n}{3} - \frac{4}{3}\sqrt{\frac{n}{3}} + o(\sqrt{n})$.
- $b_0 = 2\sqrt{\frac{n}{3}} + o(\sqrt{n})$
- $e_0 = 2s_{1,2} + 3s_{1,3} = n + 2\sqrt{\frac{n}{3}} - s_{1,2} + o(\sqrt{n})$
- $n_1 = s_{1,2} - b_1 = s_{1,2} - \frac{n}{3} + \frac{4}{3}\sqrt{\frac{n}{3}} + o(\sqrt{n})$

Note that it implies $s_{1,2} \geq \frac{n}{3} - \frac{4}{3}\sqrt{\frac{n}{3}} + o(\sqrt{n})$ and $n_1 \in [0, 2\sqrt{\frac{n}{3}}]$. So that the network is mostly build from $\frac{n}{3} + O(\sqrt{n})$ "groups T_7" made of 1 block $\mathcal{B}_1$, one block $\mathcal{B}_2$ both with degree 1 and two blocks of $\mathcal{B}_0$ altogether adjacent to one S-switch of $\mathcal{S}_{1,2}$. (see Figure 7).

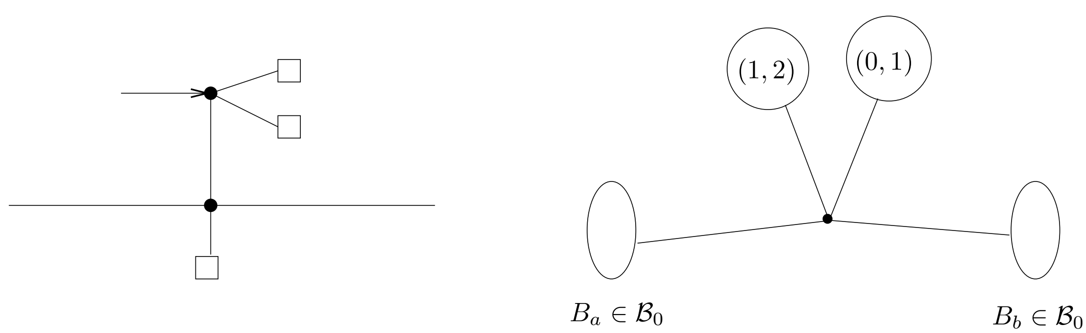


Figure 7: The group T_7 contains 1 input and 2 outputs, the group can be considered as an "edge" linking two blocks of $\mathcal{B}_0$

7.3 Case $k = 9, 10$

For $k \geq 9$, we define $\hat{\mathcal{B}_0} = \mathcal{B}_0 \setminus B'_0$ *(blocks of* $\mathcal{B}_0$ *not adjacent to* $\mathcal{S}'$*).*

Lemma 33 *In a valid* (n, k)*-kernel with* $k \geq 9$*, there is no pattern made of* 3 S*-switches in* $\mathcal{S}_{1,2}$

Proof. Remark that such a pattern P would satisfy $o(\Gamma(P)) \leq 10$; so the pattern condition 27 can be applied, and $s_{1,2}(P) \leq 2$. □

Lemma 34 *In a valid* (n, k)*-kernel with* $k \geq 9$*,*

$$b_0 \geq \hat{b_0} + 2s_2 + s_{1,1} + \frac{3}{2}s_{1,2} \tag{16}$$

Proof. As $b_0 = \hat{b_0} + b'_0$, it suffices to consider b'_0. According to Lemma 29, a block $\mathcal{B}'_0$ adjacent to two S-switches of $\mathcal{S}'$ is adjacent to two S-switches of $\mathcal{S}_{1,2}$. Counting the blocks of $\mathcal{B}'_0$ we find 2 distinct blocks for each S-switch in $\mathcal{S}_2$, 1 distinct block for each S-switch in $\mathcal{S}_{1,1}$, and at least $3/2$ blocks for each S-switch in $\mathcal{S}_{1,2}$ (in fact either there are 2 distinct blocks for one S-switch in $\mathcal{S}_{1,2}$, or 3 distinct blocks for 2 S-switches sharing a block of $\mathcal{B}'_0$ by lemma 33). This implies $b'_0 \geq 2s_2 + s_{1,1} + \frac{3}{2}s_{1,2}$. □

Lemma 35 *In a valid* (n, k)*-kernel with* $k \geq 9$*, an* S*-switch* S *of* $\mathcal{S}_{1,3}$ *cannot be adjacent to two blocks of* $\mathcal{B}'_0$*.*

Proof. Let S be a S-switch of $\mathcal{S}_{1,3}$ adjacent to two blocks $B_a, B_b \in \mathcal{B}'_0$, and let S_a (resp. S_b) be a S-switch of $\mathcal{S}'$ adjacent to B_a (resp. B_b). Let $X = S, S_a, S_b$ then one can easily check that the cut-criteria 21 is not satisfied for X. □

Theorem 36 *For* $k \geq 9$*,* $\mathcal{N}'(n, k) \geq \frac{3}{8}n + \frac{\sqrt{n}}{4} + O(1)$*.*

Proof. Combining inequalities (16) and (9), we obtain

$$4s + 2s_{1,3} \geq 2n + k + 2\hat{b_0} + s_{1,2} + 2(e_2 - b_2). \tag{17}$$

Summing with equation (10) ($4s - s_{1,2} - 2s_{1,3} - (s_0 - s_{0,0}) \geq n + k + n_1$), we get

$$8s \geq 3n + 2k + 2\hat{b_0} + 2s_{1,2} + n_1 + (s_0 - s_{0,0} + 2(e_2 - b_2)). \tag{18}$$

So we have already proven that $s \geq \frac{3n+2k}{8}$. Suppose now $s = \frac{3}{8}n + O(\sqrt{n})$, then every inequality used to establish inequality (18) is tight within an $O(\sqrt{n})$ error. Particularly from inequality (18), $n_1 = O(\sqrt{n})$, $s_{1,2} = O(\sqrt{n})$, and $s_{0,0} = s_0 + O(\sqrt{n})$, and $n_2 = e_2 = b_2 + O(\sqrt{n})$.

As $n_1 = O(\sqrt{n})$, almost all the blocks of $\mathcal{B}_1$ have degree 1 and no input, similarly, $n_2 = b_2 + O(\sqrt{n})$ implies that almost all blocks of $\mathcal{B}_2$ have degree 1 and 1 input. Note that no S-switch S in $\mathcal{S}_{1,1} \cup \mathcal{S}_{0,0}$ can be adjacent to blocks in $\mathcal{B}_1 \cup \mathcal{B}_2$ having all degree one (otherwise the set $W = \mathcal{S} \cup \Gamma(S)$ would satisfy $o(W) - i(W) \geq 3$ and $|\Gamma(W)| = 1$). So, $s_{1,1} + s_0 = s_{1,1} + s_{0,0} + O(\sqrt{n}) = O(\sqrt{n})$. Globally we get

$$s = s_2 + s_{1,3} + O(\sqrt{n}) = \frac{3n}{8} + O(\sqrt{n})$$

Now, $n + k = 2b_2 + b_1 = 2e_2 + e_1 + O(\sqrt{n}) = 4s_2 + 2s_{1,3} + O(\sqrt{n}) = \frac{3n}{2} - 2s_{1,3} + O(\sqrt{n})$. It follows that $s_{1,3} = \frac{n}{4} + O(\sqrt{n}), s_2 = \frac{n}{8} + O(\sqrt{n})$.

Consequently, the (n, k)-kernel is mostly made of $\frac{n}{8}$ S-switches of $\mathcal{S}_2$ and approximately $2\left(\frac{n}{8}\right) = \frac{n}{4}$ S-switches of $\mathcal{S}_{1,3}$. From Lemma 35, an S-switch $S \in \mathcal{S}_{1,3}$ cannot be adjacent to two blocks of $\mathcal{B}'_0$. As there is at least

$2s_2 = \frac{n}{4} + O(\sqrt{n})$ blocks of $\mathcal{B}'_0$ and $\frac{n}{4} + O(\sqrt{n})$ S-switches of $\mathcal{S}_{1,3}$, we conclude that each S-switch of $\mathcal{S}_{1,3}$ shares exactly one block with an S-switch of $\mathcal{S}_2$.

Moreover most of the common blocks $\mathcal{B}'_0$ must have only degree 2 and no input.

It follows that the (n,k)-kernel is mostly made of T_{10} groups defined below.

Definition 37 A group T_{10} is a subgraph of an (n,k)-graph formed by one S-switch of $\mathcal{S}_2$ and two S-switches of $\mathcal{S}_{1,3}$ sharing 2 blocks of B'_0 with no inputs (see Figure 8 for illustrations). Such a group is adjacent to four blocks of $\hat{\mathcal{B}}_0$.

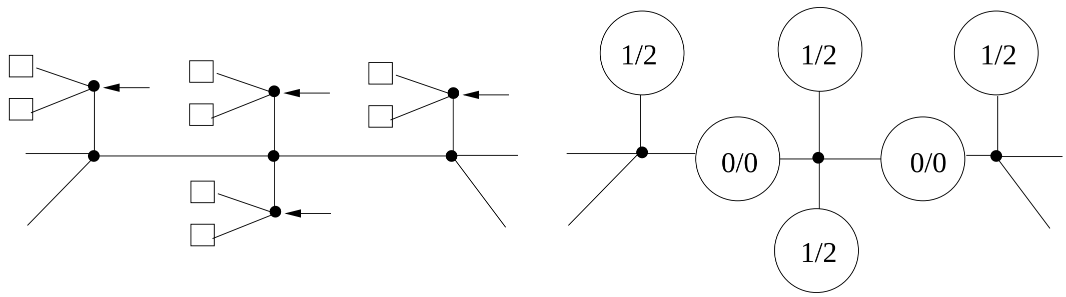


Figure 8: a group T_{10} in two representations.

Let t_{10} be the number of T_{10} groups, we just proved that supposing $s \leq \frac{3n}{8} + O(\sqrt{n})$ implies that

$$t_{10} = \frac{n}{8} + O(\sqrt{n})$$

We define an auxiliary graph $G(K)$ as follows. Let $V(G(K)) = \hat{\mathcal{B}}_0$, and connect two blocks of $\hat{\mathcal{B}}_0$ by an edge in $G(K)$, if they are adjacent to one S-switch of $\mathcal{S}_{1,3}$ of a T_{10}-group), as defined $G(K)$ contains $2t_{10}$ edges and $\hat{b_0}$ vertices.

Easy Applications of The cut-criteria (Property21), implies that $G(K)$ is simple and without triangle, therefore $2t_{10} \leq (\frac{\hat{b_0}}{2})^2$; which means that $\hat{b_0} \geq \sqrt{n} + O(1)$ Substituting this last value into inequality (18), we obtain $s \geq \frac{3}{8}n + \frac{\sqrt{n}}{4} + O(1)$ as required. $\square$

8 Asymptotics

Lemma 38

$$\forall k \geq 9,\ \ \mathcal{N}'(n,k) \geq \frac{n}{2} - \frac{2n}{k}$$

Proof. Let us assume $k = 2p+8$. The idea is to prove that $b_0 \geq b_2 - \frac{2n}{k}$, and to use Equation (6) which implies that $s \geq \frac{n}{2} + \frac{b_0 - b_2}{2}$. Let $\mathcal{S}^*$ be the set of S-switches adjacent to at least one block of $\mathcal{B}_2$ (i.e $\mathcal{S}^* = \mathcal{S}_2 \cup \mathcal{S}_{1,1} \cup \mathcal{S}_{1,2} \cup \mathcal{S}_{1,3}$). We evaluate the number of blocks in $\mathcal{B}_0$ adjacent to S-switches in $\mathcal{S}^*$, so let $b_0^* = |\Gamma(\mathcal{S}^*) \cap \mathcal{B}_0|$ be this number. Let us call *adjacent* two S-switches of $\mathcal{S}^*$ adjacent to the same block of $\mathcal{B}_0$, and consider the set of connected components $\mathcal{K} = K_0, K_1, \ldots, K_{p-1}$ of the graph that this relation induces on the vertex set $\mathcal{S}^*$. Then, from construction $\cup_{i=0,p-1} K_i = \mathcal{S}^*$ and $\Gamma(K_i) \cap \mathcal{B}_0 \cap \Gamma K_j = \emptyset$; hence we have :

$$b_0^* = \Gamma(\mathcal{S}^*) \cap \mathcal{B}_0 = \sum_{i=0,p-1} |\Gamma(K_i) \cap \mathcal{B}_0| \tag{19}$$

We partition $\mathcal{K}$ into two sets $\mathcal{K}_{small} \cup \mathcal{K}_{big}$, where $\mathcal{K}_{small}$ contains *small* components (such that $o(\Gamma(K_i)) \leq k$) and K_{big} contains the *big* ones (such that $o(\Gamma(K_i)) > k$). Applying cut-criteria to small component leads too :

$2|\Gamma(K)| \geq 4|K| + 2|\Gamma(K) \cap \mathcal{B}_2| + |\Gamma(K) \cap \mathcal{B}_1|$; aternatively :

$$K \in K_{small}, |\Gamma(K) \cap \mathcal{B}_0| \geq |\Gamma(K) \cap \mathcal{B}_2| + (4|K| - |\Gamma(K)|) \geq |\Gamma(K) \cap \mathcal{B}_2| \quad (20)$$

Now we evaluate $|\Gamma(K) \cap \mathcal{B}_0|$ for K a big component. Consider a subset $J \subset K$ where J is connected and such that $|J| < p$, the cut criteria implies $o(\Gamma(J)) \leq 2|J| + 4$; so J is a pattern satisfying condition 27. So we have

$$2(s_{1,1}(J) + s_2(J)) + s_{1,2}(J) \leq 2$$

If K can be written as the union of at most q sub-components J, we have

$$2(s_{1,1}(K) + s_2(K)) + s_{1,2}(K) \leq 2q \quad (21)$$

The cut-criteria for K is $2|\Gamma(K)| \geq k + 4|K|$; as $|\Gamma(K) \cap \mathcal{B}_1| + |\Gamma(K) \cap \mathcal{B}_2|| \leq |K| + s_2(K) + 2s_{1,1}(K) + s_{1,2}(K)$ this implies $2|\Gamma(K) \cap \mathcal{B}_0| \geq k + 4|K| - (2|K| + 2s_2(K) + 4s_{1,1}(K) + 2s_{1,2}(K))$. So we get the next equation :

$$|\Gamma(K) \cap \mathcal{B}_0| \geq |K| + \frac{k}{2} - (2s_{1,1}(K) + s_2(K) + s_{1,2}(K)) \quad (22)$$

$$(23)$$

Using the equation

$$|\Gamma(K) \cap \mathcal{B}_2| \leq s_2(K) + |K|$$

We obtain

$$|\Gamma(K) \cap \mathcal{B}_0| \geq |\Gamma(K) \cap \mathcal{B}_2| + \frac{k}{2} - 2(s_{1,1}(K) + s_2(K)) - s_{1,2}(K)$$

According to 21 this implies :

$$|\Gamma(K) \cap \mathcal{B}_0| \geq |\Gamma(K) \cap \mathcal{B}_2| - (2q - \frac{k}{2})$$

We choose to split K into sets J of size $p, p+1$, so we have $q \leq \frac{|K|}{p}$.

Now the value $\sum_K$ big component $\left(\frac{2|K|}{p} - \frac{k}{2}\right)$ is maximum when there is only one big component that is $\sum_K$ big component $\leq \frac{2n}{p}$.

Now, summing over all the components (the big and the small) we obtain :

$$b_0^* = \sum_{K \in \mathcal{K}} |\Gamma(K) \cap \mathcal{B}_0| \geq \sum_{K \in \mathcal{K}} |\Gamma(K) \cap \mathcal{B}_2| - \frac{2n}{p} \geq b_2 - \frac{2n}{p}$$

Hence, as $k = 2p + 8$, $b_0 \geq b_2 - 4\frac{n}{k-8}$ and $s \geq \frac{n}{2} - \frac{2n}{k}$

□

9 Upper bounds, Constructions

In this section, we give constructions of valid (n, k)-graphs for $k = 6, 8, 10$. We will see that the validity of the constructions can be easily checked using the cut-criteria 21.

9.1 Case $k = 3, 4$

Proposition 39 *The construction of Figure 4 for $k = 4$ is valid.*

Proof. Note that every $S \in \mathcal{S}$ is a twin S-switch, so for all $S \in \mathcal{S}$, $o(\Gamma(\{S\})) \geq 4$. So, byt the cut criterion it suffices to verify :

$$|\Gamma(X)| \geq 2|X| + 2, \quad \forall X \subset \mathcal{S},\ X \neq \emptyset. \tag{24}$$

Now, any set $X \subset S$ is adjacent to the 2 large block of $\mathcal{B}_2$ and to $2|X|$ distinct blocks of $\mathcal{B}_2$ each having degree 1. So, $\Gamma(X) = 2|X| + 2$. And property 24 is satisfied. □

9.2 Case $k = 5, 6$

Theorem 40

$$\mathcal{N}'(n, 6) \leq \frac{n}{4} + \sqrt{\frac{n}{8}} + O(1).$$

According to the lower bound (Theorem 25), an optimal solution must be mainly build of twins connecting blocks of $\mathcal{B}_0$, in such a way that the auxiliary graph $H(K)$ is simple.

Proof.

First we describe the network

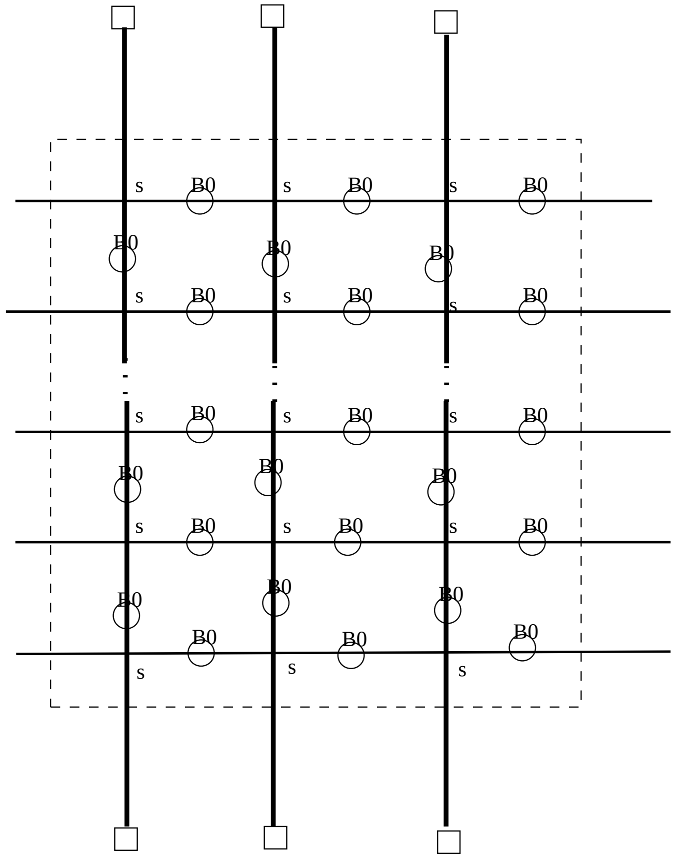


Figure 9: Generic (n, k)-kernel for $k = 5, 6$

Consider a toroidal grid of 3 vertical line segments and $\ell, \ell/geq2$ horizontal cycles. Each of the 3ℓ vertices of the grid represents an S-switch, and each of the $6\ell - 3 = b_0$ edges joining these vertices represents a block of $\mathcal{B}_0$.

- Each of the 6 S-switches of the first and last horizontal cycles is adjacent to 3 blocks B_0; we add 6 blocks B_1 with degree 1 and no input and connect each of them to one of these 6 S-switches. Note that the 6 S-switches are in $\mathcal{S}_{0,3}$.
- The remaining $3\ell - 6$ S-switches are adjacent to 4 blocks B_0 and hence belong to $\mathcal{S}_0 4$.
- Finally we add a twin (see definition 20) connecting any pair of blocks B_0 not sharing a S-switch.

The network contains $b_0 = 6\ell - 3$ blocks in $\mathcal{B}_0$. Among them $b_0 - 12$ share a S-switch with 6 others blocks, 6 share a S-switch with 5 blocks (corresponding to the 6 vertical edges of the top and bottom), and 6 share a S-switch with 4 blocks (corresponding to 6 horizontal edges of the top and bottom). Let t be the number of twins, a twin being adjacent to 2 blocks

$$2t = (b_0 - 12)(b_0 - 6) + 6(b_0 - 7) + 6(b_0 - 4) = b_0(b_0 - 7) + 18$$

Note that each block B_0 is adjacent to 2 S-switches in the grid and to twins, it follows that $n_0 = \sum_{B \in c B_0}(deg(B) - 2) = 2t$; moreover $n_1 = 0, n_2 = 2t$. Hence

$$n = 4t$$

Now, the number of outputs is $b_1 + 2b_2 = 4t + 6 = n + 6$. The number of S-switches is

$$s = t + 3\ell = \frac{n}{4} + \frac{b_0 + 3}{2}$$

As $t = \frac{b_0(b_0-7)+18}{2}$, $b_0 = \frac{7}{2} + \sqrt{\frac{n}{2} - \frac{23}{4}}$, and $s \leq \frac{n}{4} + \sqrt{\frac{n}{8} - \frac{23}{16}} + 3 + \frac{1}{4}$.

For example with $\ell = 2$ we get $b_0 = 9, t = 18$, $s = t + 3\ell = 24$, $n = 4t = 72$. Hence $s = 18 + + \sqrt{9 - 23/16} + 3 + 1/4$.

Now, we prove the validity of the kernel.

For this Note that the cut-criterion is now :

$$\Gamma(X) - 2|X| \geq min(o(\Gamma(X))/2, 3)$$

To prove the statement, let denote $X' \subset X$ be the set of switches in the grid, and note t the number of twins in X then $|\Gamma(X)| - 2|X| \geq \Gamma(X') + 2t - 2(t + |X'|) = \Gamma(X') - 2|X'|$. Remark now that $\Gamma(X')$ is the number of edges adjacent to a subset of vertices of the grid, and one check easily that $\Gamma(X') - 2|X'| \geq 3$ as soon $|X'| \geq 2$, so if $|X'| \geq 2$ the cut-critetion is ensured.

Note also that if $t \geq 3$, one can remove on twin decreasing $\Gamma(X)$ by at least 2, and keeps $min(o(\Gamma(X))/2, 3)$ unchanged, hence we can restrict ourselves too cuts with at most 2 twins.

The only cases that still need to be considered are $|X'| \leq 1, t \leq 2$. Those can be checked immediatly.

□

9.3 Case $k = 7, 8$

Theorem 41 *For $k \leq 8$,*

$$\mathcal{N}'(n,k) \leq \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(\sqrt[4]{n}).$$

Proof. The network is constructed as follows :

Recall that according to the lower bound blocks of $\mathcal{B}_0$ must be connected either by edges corresponding to T_7 groups or by triangles corresponding to $\mathcal{S}_{1,3}$ S-switches whose adjacent block B_2 has degree 1 (triangle group). We will denote t_7 (resp. y) the number of T_7 (resp. triangle) groups.

- First we use two sets $\mathcal{C} = \{C_1, C_2, \ldots, C_p\}$ and $\mathcal{D} = \{D_1, D_2, \ldots, D_p\}$ of p blocks of $\mathcal{B}_0$ with $deg(C_i) = deg(D_i) = p + 1$; these blocks are connected into a bipartite graph $K_{p,p}$, where edges are T_7 groups, we have

  $$t_7 = p^2$$

  Note that then each block of $\mathcal{C} \cup \mathcal{D}$ must still be connected to a S-switch (see Figure 7 for an illustration).

- Secondly we add four sets $\mathcal{E} = \{E_1, E_2, \ldots, E_{\sqrt{p}}\}$, $\mathcal{F} = \{F_1, F_2, \ldots, F_{\sqrt{p}}\}$, $\mathcal{I} = \{I_1, I_2, \ldots, I_{\sqrt{p}}\}$, and $\mathcal{J} = \{J_1, J_2, \ldots, J_{\sqrt{p}}\}$ of $\sqrt{p}$ blocks of $\mathcal{B}_0$ with $deg(E_i) = deg(F_i) = deg(I_i) = deg(J_i) = \sqrt{p} + 2$.

  As $|\mathcal{E}||\mathcal{F}| = \mathcal{C}$ (resp. $|\mathcal{I}||\mathcal{J}| = |\mathcal{D}|$) we can associate to each block $C \in \mathcal{C}$ (resp. $D \in \mathcal{D}$) to one pair (E_C, F_C) (resp. (I_D, J_D) resp.) so that the mapping is one to one. Then, the triple $\{C, E_C, F_C\}$ (resp. $\{D, I_D, J_D\}$ is connected by a triangle. Note that :

  $$y = 2p.$$

  and that each block of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{I} \cup \mathcal{J}$ must still be connected to 2 S-switches and this is performed using a grid like network (see figure).

Let us count the number of inputs. outputs and S-switches.

- the number of inputs is $t_7 + y + \sum_{B\in\mathcal{B}_0}(deg(B) - 2) = 3t_7 + 2y = n = 3p^2 + 4p$.
- the number of outputs is $b_1 + 2b_2 = 3t_7 + 2y + 8 = n + 8$.
- there are t_7 S-switches in $\mathcal{S}_{1,2}$ plus y S-switches in $\mathcal{S}_{1,3}$ plus $8\sqrt{p}$ S-switches in $\mathcal{S}_{0,4}$ (in the grid-like network).

Finally $n = 3p^2 + 4p$, $\sqrt{n/3} = p + O(1)$, $s = p^2 + 2p + 8\sqrt{p} = \frac{3p^2+4p}{3} + \frac{2p}{3} + O(n^{1/4}) = \frac{n}{3} + \frac{2}{3}\sqrt{\frac{n}{3}} + O(n^{1/4})$.

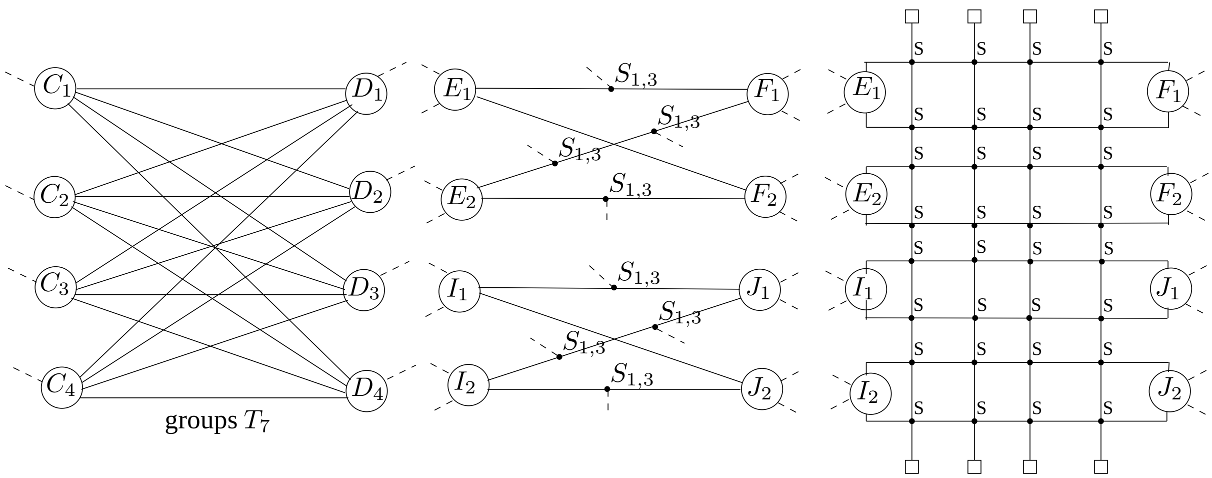


Figure 10: The three layers of the construction for $k = 8$ and $p = 4$.

Now we prove the validity of such a network.
Call a *bad set* a set $X \subset \mathcal{S}$ violating the cut-criteria, and assume that it exist a bad set. First, remark that there exists a bad set A such that $A \cap \mathcal{S}_{0,4} = \emptyset$
For $X \subset \mathcal{S}$, let $\Gamma^0(X) = \Gamma(X) \cap \mathcal{B}_0$ and let $X_{t_7} = X \cap \mathcal{S}_{1,2}$ and $X_y = X \cap \mathcal{S}_{1,3}$.

Note that $o(\Gamma(A)) = 3|A_{t_7}| + 2|A_Y|$, $|\Gamma(A)| = 2|A_{t_7}| + |A_y| + |\Gamma^0(A)|$. So, whenever $o(\Gamma(A)) \geq 7$ the cut-criteria reduces to $2|\Gamma^0(A)| \geq 8 + 2|A_y|$, in other words :

$$\Gamma^0(A) \geq 4 + |A_y| \tag{25}$$

Assume that it does not exist a bad set with $o(\Gamma(A)) < 10$. Consider a bad set B. We can assume $B_{t_7} = \emptyset$ (otherwise removing any T_7 S-switch leads also to a bad set as it decrease $\Gamma(B) \cap \mathcal{B}_0$ and let A_y unchanged). Hence B contains only S-switches in $\mathcal{S}_{1,3}$, moreover as $o(\Gamma(B)) \geq 10$ we have $|B| = |B_y| \geq 5$. As 5 edges of a bipartite graph are adjacent to a least 5 nodes B is adjacent to at least 5 blocks of $\mathcal{E} \cup \mathcal{F} \cup \mathcal{I} \cup \mathcal{J}$ and to $|B|$ blocks of $\mathcal{C} \cup \mathcal{D}$, hence $|\Gamma^0(B)| \geq |B| + 5$.

Now we prove that no bad set with $o(\Gamma(A)) < 10$ exists.

For $o(\Gamma(A)) \in [7, 9]$ it suffice to show that $\Gamma^0(A) \geq 4 + |A_y|$. in this case either $A_y = 0, 1, 2$; $A_{t_7} = 3 - |A_y|$; or $A_y = 3$ and $A_{t_7} = 1$ (these cases can be checked immediately).

For $o(\Gamma(A)) < 7$, the criteria becomes $2\Gamma^0(B) \geq 4|B_y| + 3|B_{t_7}|$ there are 3 cases :

- $|B_{t_7}| = 0, |B_y| \leq 3, \Gamma^0(B) \geq 2|B_y|$;
- $|B_{t_7}| = 1, |B_y| \leq 1, \Gamma^0(B) \geq 2|B_y| + 2$ for
- $|B_{t_7}| = 2, |B_y| = 0, \Gamma^0(B) \geq 3$

All can be checked immediately. □

9.4 Case $k = 9, 10$

Theorem 42 *For $k \leq 10$,*

$$\mathcal{N}'(n,k) \leq \frac{3n}{8} + \frac{\sqrt{n}}{4} + O(n^{1/4})$$

Proof. The construction is the same that to the one given for $k = 7, 8$ (see section 9.3) except that in the first bipartite graph edges corresponding when $k = 7, 8$ to T_7 groups are now replaced by T_{10} groups; indeed a T_{10} group replace $2T_7$ groups. Precisely a pair of edges is replaced by a t_{10} group, note to do so edges are paired arbitrarily.

Let us count the number of inputs. outputs and S-switches.

- $t_{10} = p^2/2$
- $y = 2p$
- the number of inputs is $4t_{10} + y + \sum_{B \in \mathcal{B}_0}(deg(B) - 2) = 8t_{10} + 2y = n = 4p^2 + 4p$.
- the number of outputs is $b_1 + 2b_2 = 8t_{10} + 2y + 8 = n + 8$.
- there are $3t_{10}$ S-switches in the T_{10} groups plus y S-switches in the isolated Y groups plus $10\sqrt{p}$ S-switches in the grid-like network. So $s = \frac{3p^2}{2} + 2p + 10\sqrt{p}$

Finally $n = 4p^2 + 4p$, $\frac{\sqrt{n}}{4} = \frac{p}{2} + O(1)$, $s = \frac{3n}{8} + \frac{p}{2} + O(\sqrt{p}) = \frac{3n}{8} + \frac{\sqrt{n}}{4} + O(n^{1/4})$.

Now we prove the validity of such a network.
Call a *bad set* a set $X \subset \mathcal{S}$ violating the cut-criteria, and assume that it exist a bad set. First, remark that it exists a bad set A such that $A \cap \mathcal{S}_{0,4} = \emptyset$

Assume that $B \cap \mathcal{S}_2 = \emptyset$, then $o(\Gamma(B)) = 2|B|$ and the criteria becomes $|\Gamma(X) \cap \mathcal{B}_|\ \geq 3|X|$, if $|X| \leq 5$ and $|\Gamma(X) \cap \mathcal{B}_|\ \geq 2|X| + 5$ for $|X| \geq 5$. Note that all S-switches can be considered as edges of the two bipartite graphs where vertices are block of $\mathcal{B}_0$, if X denotes

with e edges in the large one and e' in the small one ($e + e' = |X|$), so $\Gamma(X) \cap \mathcal{B}_{\geq} 2(e + e') + 2\sqrt{e} + 2sqrte' = 2|X| + 2\sqrt{X}$. This is larger than $3|X|$ for $X \leq 4$. For $X = 5$, we get at least 5 vertices and the criteria is satisfied.

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First we show that if a bad set exists it contains less than 14 inputs. To do this we check the cut-criteria for *large* sets S (i.e such that $o(\Gamma(S)) \geq 10$), for such sets, the criteria reduces to $\Gamma(B) \cap \mathcal{B}_{\geq} 2|B| + 5$. Consider a bad set B with $o(\Gamma(B)) \geq 10$ and such that $|B|$ is minimum. Then

- either $o(\Gamma(B)) \geq 14$ and $\mathcal{S}_2(B) = \emptyset$ (otherwise removing any $\mathcal{S}_{1,3}$ S-switch leads also to a smaller bad set as it decrease B by 1 and $\Gamma(B) \cap \mathcal{B}_2$ by 2). Moreover, B do not contains any $\mathcal{S}_{1,3}$ block located inside a T_{10} group (such a block would be adjacent to 2 blocks (1 B_2 and 1 B_0 with zero input) that are otherwise isolated, so it could be removed to obtain a smaller large bad set.

  So, B contains only S-switches of $\mathcal{S}_{1,1}3$ corresponding to the edges of the second bipartite graph, then $|\Gamma(B) \cap \mathcal{B}_0| \geq 2|B| + 2\sqrt{|B|}$, As $|B| \geq 7$, $|\Gamma(B) \cap \mathcal{B}_0| \geq 2|B| + 7$ (contradiction). Hence $o(\Gamma(B)) \leq 14$
- or $o(\Gamma(B)) \leq 14$.

It remains to check the criteria for set with less than 14 inputs.

□

9.5 Asymptotic

Lemma 43

$$\forall k \in \mathbf{N}, \qquad \lim_{n\to\infty} \mathcal{N}'(n,k) \le \frac{1}{2}n + O(k).$$

Proof. We describe a general construction for a valid (n,k)-kernel ($k \ge 6$, $2|k$) with $n/2 + O(k)$ S-switches. It is built almost exclusively of Z-groups.

A Z-group is defined as follows. A block B_0 with $i(B_0) = 1$, $o(B_0) = 0$ and a block B_2 with $i(B_2) = 1$, $o(B_2) = 2$ connected to an S-switch (see Figure 11.) It is in fact a combination of a group Y and block of $\mathcal{B}_0$.

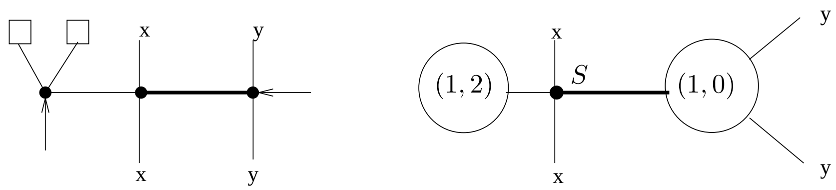

Figure 11: A Z-group

For a Z-group Z, $\Delta(Z)$ consists of four links. The role of these links is not symmetric, they are labeled x, x, y, y.

To describe the network, it is enough to define the interconnection between links of Z-groups. For $k \ge 6$ and $n = k\ell$ ($2 \nmid \ell$), consider $k\ell/2$ Z-groups indiced as $Z_{i,j}$ ($i \in [0, \frac{k}{2} - 1], j \in [1, \ell]$). Now connect the two links labeled x in $Z_{i,j}$ to the links labeled y in $Z_{i+1,j+2^j k}$ and $Z_{i+1,j-2^j k}$ (arithmetics on indices is considered in $(\bmod\ \frac{k}{2})$ and $(\bmod\ \ell)$ resp.).

In this way, every extremal link is connected. The edges between the sets of groups $\{Z_{i,j} : j \in [1,\ell]\}$ and $\{Z_{i+1,j} : j \in [1,\ell]\}$) form a cycle denoted by K_i. At each cycle K_i, insert an S-switch with two exits in the middle of one arbitrarily chosen edge. So the number of outputs exceeds the number of inputs by exactly k.

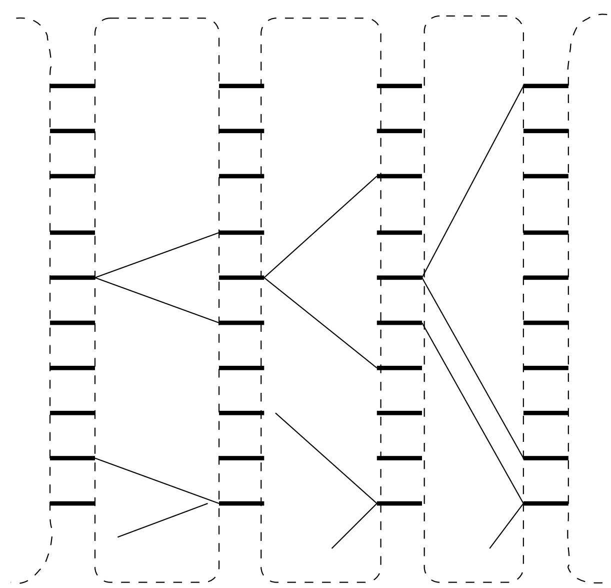

Figure 12: interconnection of extremal links

Let $X \subset \mathcal{S}$, so that $\Gamma^*(X)$ is connected. (For the moment we ignore the $\frac{k}{2}$ S-switches inserted ultimately). Considering the auxiliary 4-regular graph H formed by Z-groups. The girth of H is at least k. To see this, observe that a cycle in the network represent a solution of the equation

$$\sum_{i=0}^{k-1} x_i 2^{ip} = 0 \mod \ell.$$

Which means that

$$\sum_{i=0}^{k-1} |x_1| \geq k.$$

If $|X| < k$, then X induces a tree X' in H, and $\delta(\Gamma^*(X)) = \delta(X') = 2|X'| + 2 = |X| + 2 = i(X) + o(X) + 2$.

If X intersect a Z-group of every cycle K_i, then at each K_i it either contains two edges of K_i, or it contains two extra outputs inserted to K_i. So $\mathcal{G}(X) \geq 0$.

If X is not a tree and does not intersect a Z-group of every cycle K_i, then $\delta(\Gamma^*(X)) \geq k$, and hence $\mathcal{G}(X) \geq 0$. □

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