Jcalm 42-AM/2015

Counting : Algorithms and Complexity

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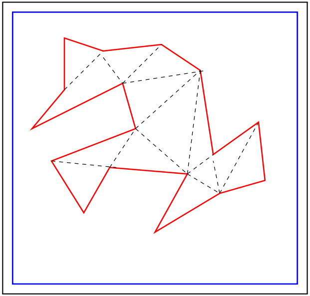
Outline

Counting, Generating why ?

- Mesure probabilities, volumes : Area of a shape.



  - Math solution : Triangulate, divide into simplexes decompose into shapes for which we have a formula.
  - High dimension ? $\Rightarrow$ Even a polytope do have exponential number of vertices, non practical.
  - Can we get an efficient algorithm ?

- **Example:** Tree Polytope of $G = (V, E)$

$$w : E \Longrightarrow \mathbf{R}^+$$
$$\forall V' \subset V, w([V', V']) \leq |V'| - 1$$
$$w([V, V]) = |V| - 1$$

Polytope Vertices = The spanning trees

(How \ Can) we deal with that ?

Counting to Evaluate probabilities

(Discretes|finites Probabilities) $\Longleftrightarrow$ Counting

Question

- *Network $N = (V, E)$, edges i.i.d failures ($p = \frac{1}{2}$).*
  Compute $B(N) = Prob(N$ gets disconnected $)$?

$$B(N) = Prob[\cup_{S \in V, S \notin \{\emptyset, V\}} [S, \overline{S}] \text{ fails }] \sim \cup_{S \in V, S \notin \{\emptyset, V\}} 2^{-|[S, \overline{S}]|}$$

- Random N with a fixed support $\rightarrow$ Probability of a property ?
- Erdũs Reynii : (trivial) Case $N = K_n \rightarrow$ Formulas, theorems ...
- Other Random distributions: degree sequences, Euclidian, planar, whatever ... $\rightarrow$ Again formulas.

Generating and Sampling

Typical Questions

- Generate a random tree of G/
- Generate a random simple path from u to v.
- Generate a random failure scenario .
- Generate complex items for simulation or testing, but in a fair way.
- Make experiment on a random graph that looks like a typical case.

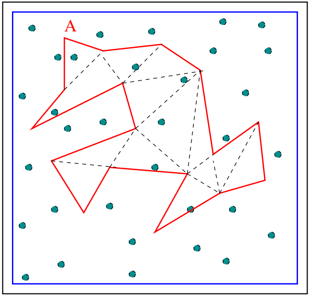
What about Enumerative Combinatorics?

- Usually about a fixed object (K_n or the Hypercube . . .)
- Fibbonacci, # partitions, "usual stuff" $\rightarrow$ 95 % of the time when we count we build bijections.
- Many (most?) inductive counting argument $\rightarrow$ inductive constructions.
- **Often :** Can count $\rightarrow$ Can Generate.

Maths Sucks method (Naive Monte Carlo)

Monte Carlo Uber Generator

- Throw random points in the square (we know how to sample it)
- point belongs to A ? $\rightarrow$ returns it.
- Great generator, works for any **NP**problem



Super Counting Algorithm

- Throw n random points in the square, $a(n)$ points lie in A
- Return $Vol(P) = \frac{a(n)}{n} Vol(Square)$

Works great if $\frac{Vol(A)}{Vol(Square)}$ is not too small (ie polynomial)

Math Reason, empirical mean often works . . .

Chernoff Bound

Z : Sum of n i.i.d Bernouilli variables $X_1, X_2, \ldots X_n$ (random $(A, 1-A)$ biased coins) $\mu = E[Z] = n \cdot A$,

$$Pr[|Z(\omega) - \mu| \geq \delta\mu] \leq 2e^{\delta^2 \mu/3}$$

Almost good and quite sure ?

- We want $\mu = \frac{1}{k}$
- and $2e^{-\delta^2 \mu/3} \leq \frac{1}{2}$

So we want $\mu \geq 3 \ln 4\delta^2$ So

$n \sim$ Who care 1 barbu c'est un barbu et le deuxième momemnt suffit zzz

We need n of order $vol(A)/Vol(Square)$, up to polylog things , indeed we simply need to see events of A happening.

Some formalism (sorry for that)

Counting Algo.

Input : A ground set X and some predicate $I(\in \mathbf{P})$

output : an estimation of $\sharp\{x \in X | I(x) = True\} \stackrel{\text{def}}{=} \sharp x$

e.g. (Ham. cycle) : $X = \mathcal{P}(E)$, and $f(x)$ is true if x is a Hamilton[a] cycle

[a] According to Majus, people from Newcatle say An (H)amiltonian cycle

Random Algorithm

Uses fair independent random bits (don't ask me how we get them)

Qualities

$$\begin{array}{lll} \text{Approximation} & e^{-\rho}\sharp x \leq output(x) \leq e^{\rho}\sharp x & (\varepsilon - Approx) \\ \text{Success} \geq \tau & Prob[\text{to be } \rho - approximated] \geq \tau & (1) \end{array}$$

Sampling Algo.

Input : A finite Probability measure $P(X)$, over X

output : Algorithm returns X with proba :

Counting $\rightarrow$ Sampling ?

We need a bit more than couting

we Assume that we can still count when we fix a part of the solution

Classical usage of conditional expectations

Example

- $\mu(G, F_1, F_0) =$ Number of matching in G containing F_1 discarding F_0
- Nothing more than $\mu(H = f(G, F_1, F_0))$ (here $H =$ remove from G vertices appearing in F and remove F_0 from E)

  Algo:

  - Compute $N = \mu(G, \emptyset)$ and $p(e) = \mu(G, \{e\}, \{\})$ and $1 - p(e) = \mu(G, \{\}, \{e\})$
  - Pick e with probability $p(e)$, otherwise discard it
  - Procede inductively, either with $(G, \{e\}, \{\})$ or $(G, \{\}, \{e\})$

  If $\mu()$ is exact $\rightarrow$ Perfect Random Generator of matchings.
- Error e^{ε} on $\mu \rightarrow$ Drift of $e^{t\varepsilon}$ (t steps) (gen. $exp(\sum_{i=0,\ldots t} \varepsilon_i)$)

Sampling $\rightarrow$ Counting ?(Prince Albert Revenge)

Monte Carlo on nested areas (often works)

$$\mu(A_n) = \frac{\mu(A_n)}{\mu(A_{n-1})} \ldots \times \frac{\mu(A_1)}{\mu(A_0)} \mu(A_0)$$

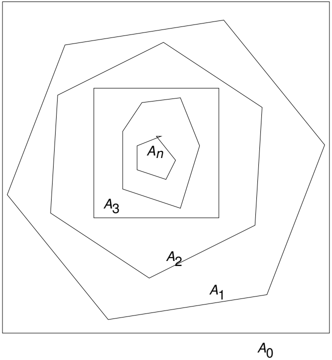
$$Pr[A_{i+1}|A_i] = \mu(A_{i+1})/\mu(A_i) = \alpha_i$$

$(1+\varepsilon_0)$ approx of α_i takes likes $\frac{1}{\alpha_i \varepsilon_0}$

n steps $\varepsilon = \sum \varepsilon_i$, $\varepsilon_i = \frac{\varepsilon}{n}$

if $\alpha_i \geq \beta \rightarrow$ around $\frac{n^2}{\beta\varepsilon}$

Direct : Pay like β^n to observe one A_0 in A_n



examples Matchings (add more and more edges) $\mu(G + \{e\}) \leq 2\mu(G)$, forests, colorings with more than Δ colors, knapsacks with cost less than C ...

Some formalism : The $\sharp P$ Class

What contains $\sharp P$?

Unformaly :

> *Any Counting problem that can be associated to successful computations of a Non Deterministic Turing Machine (in Polynomial time)*

Counting Prob in $\sharp P$

Elements of a Set $S(x)$	Bijection	$\{y \mid TM(x,y)$ says ok $\}$
Elements of a Set $S(x)$	Bijection	Correct proofs that $(x,y) \in \mathcal{S}$

Example

Ham. Cycle : $x = (V, E)$, $S(x) = \{$Ham. Cycles of $(V, E)\}$, the proof is the cycle itself. For *SAT* where x is the instance (the graph), y is the variable assignement (set of edges) and the machine checks that it works.

Cook theorem and $\sharp P$-completeness of $3SAT$

Theorem (Fake)

$$\sharp 3SAT \text{ is } \sharp P - complete.$$

Proof. Almost a tautology.

Correctness of a NdetTM computation can be captured by a (big) $3SAT$ formula.

It's Cook's Theorem, mostly says computation is local

$3SAT$ variables	bijection	(Random) Choices of the NdetTM
$3SAT$ Solutions	bijection	Sucessfull Choices of the NdetTM

∎

Remarque

Indeed One says that Cook reduction is parsimonious.

Counting Solutions of NP-hard problems ?

- Not really interesting, Almost immediately $\sharp P - complete$
- No approximation theory (deciding 0 or 1 is hard, ∞ ratio).
- Easy to amplify the number of solutions (add k fake binary clauses $\times 2^k$)

Counting exactly is way too strong and complicated

Theorem (Toda 25-AM/1998)

Any problem in the Polynomial hierarchy can be solved using a counter. Fancy Madmen notation is

$$PH \subset P^{\sharp P}$$

madness pays off

Let us be silly and get the godel prize !

Valiant Vazirani, isolation lemma [11AM]

Theorem

If you can solve problem when they have unique solution you can solve SAT (up to some randomization)

Detecting unique solutions

- 0 solution $\rightarrow$ says 0
- 1 solution $\rightarrow$ says 1
- $>$ 1 output garbage, anything.

Idea :

- Add linear constraints (see prob. in Z_2^n).
- Dichotomy, one contraint $\rightarrow$ Should divide the solution state by 2
- turn linear constraints into extra clauses (silly but needed)

Isolation Lemma (2)

Theorem (Isolation)

- *S any set of Z_2^n*
- *pick constraints $H_i = \{x \mid v_i.x = 0\}$ randomly,*
- *let $S_0 = S, S_{i+1} = S_i \cap H_i$.*

Then with probability $P \geq \frac{1}{4}$ we have $\exists i, |S_i| = 1$.

So with positive probability once can construct a SAT instance that is stonger (more constrained) than the original one and that admits a single solution.

Toda proof 3

Sorry No Godel Prize for you !

Let's try something simpler

We may still do something for . . .

- simple Path, trees
- Matchings
- Polytopes

About Matchings ? Fun situation

Counting exactly Matching is $\sharp P$-complete [Valiant 6-AM/79]

One can approx count (and generate) Matchings [Jerrum 21-AM/95]

Permanent, One factor, Matchings

Definition (Permanent, determinant)

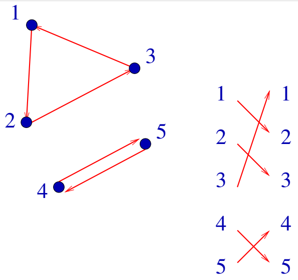
$$Perm(A) = \sum_{\pi \in \mathfrak{S}_n} a_{i,\pi(i)} \quad \Big| \quad Det(A) = \sum_{\pi \in \mathfrak{S}_n} sign(\pi) a_{i,\pi(i)}$$

Term of the sum $= 0 \iff$ some edge $(i, \pi(i))$ does not exist.

Term of the sum $= 1$ if all the edges $(i, \pi(i))$ exist

$\Rightarrow$ The permanent counts *One Factors* of G

It also counts Matchings in $[G, G]$.



Weigthed version : Instead of 1 we count $\Pi_{e \in F} w(e)$ for a factor (a matching) $F \subset E$

Formal version : Multivariate Generating serie of the Matchings

Proof Structure

Proof Organisation

- # Weigthed Matchings $\overset{\text{regular reduction}}{\Longleftrightarrow}$ # Exact Covers by Triples
- # Weighted Perfect Matchings $\rightarrow$ Exact couting for Weigthed Matchings
- Emulating integral weights.
- #Perfect Matchings int. weigths $\rightarrow$ Can count with any weights.

A small Gadget

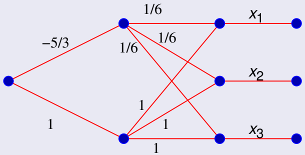
Main property

$$Perm(G(x_2, x_2, x_3)) = \frac{1 + x_1 x_2 x_3}{3}$$

No term with degrees $1, 2 \rightarrow \forall A \subset \{x_1, x_2, x_3\}$

$\rightarrow$ $\sharp\{M \in Matching | M \cap \{e_1, e_2, e_3\} = A\} = 0$ unless $A = \{x_1, x_2, x_3\}$ or $A = \emptyset$

The Gadget (uses negative weigth)



Graph $G(x_1, x_2, x_3)$

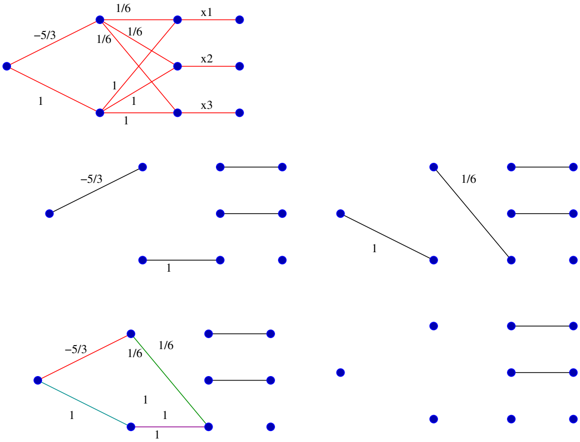
Just Checking $Perm(G(1,1,0)$.

- 1) M contains two edges, two cases : $\frac{-5}{3} \times 1$ and $\frac{1}{6} \times 1$ (tot. $\frac{-9}{6}$)
- 2) M contains one edge (4 cases) : $-\frac{5}{3} + 1 + 1 + \frac{1}{6}$ (tot. $\frac{3}{6}$)
- 3) M is empty 1 (tot. $+1$)

total contribution is zero

similarly:

$P(1,1,1) = \frac{-5}{3} + 1 + 1 = \frac{1}{3}$



Consequence

Property

When we attach the gadget H with its 3 ends to a graph, computing $Perm(G+H)$ compute the "number" of matchings that either contain $\{e_1, e_2, e_3\}$ or do not intersect it.

Gadgets behave like a triple

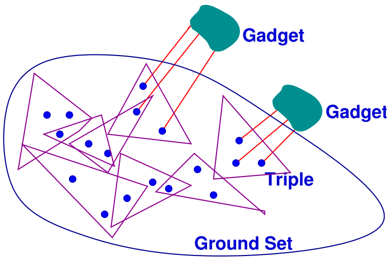
S: Instance of cover-with-triples , $3m$ elements (ground set).

Triples and gadgets (Bijection)

What do we count ? The Exact covers ? No! We count triple-disjoint partial cover

k disjoints triple (+stuff): $\left(\frac{1}{3}\right)^k$

$Perm(H(S)) = \sum \frac{N(k)}{3^k}$, $N(k)$ number of disjoint k covers.



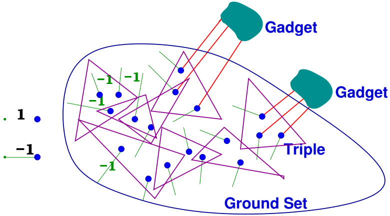
Getting ride of partial Covers

Add pending leaves to vertices of the ground set.

Edge weight is $-1 \Rightarrow$ Graph $H'(S)$

We count now 0 for a partial Cover.

We still count $\frac{1}{3^m}$ for a perfect cover.

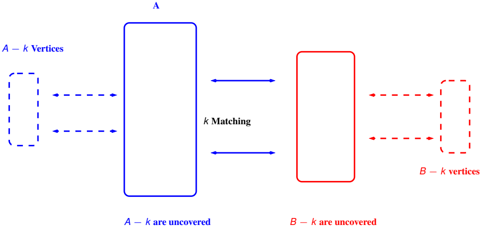


$$Perm(H'(S)) = \frac{\text{number of exact covers}}{3^m}$$

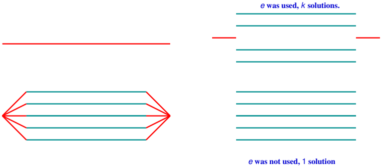
Some More gadgets

- Matching $\rightarrow$ Perfect Matching

$\forall k$ We count each k matching $(A-k)!(B-k)!$ times



- Simulating Integral weigths



- At the end 4 bad weights $x=\frac{1}{6}, y=\frac{5}{3}, z-1$ Polynomial on a bounded number $(k=4)$ of variables, degree n (polynomial), n^4 coefficients $\rightarrow$ can be computed.

General framework

Ideas

- Move randomly in the state space (form Use Markov Chains)
- Ensure that moves are fair (the station. distribution is uniform)
- Want to be random fast (Ensure Rapid-Mixing)

Theorem (Perron Frobenius + some Folks)

A stochastic matrix M admits a unique fixed point (eigenvector with eigenvalue 1*) and everything else decays fast. i.e if* $u.1 = 0$ *(noise), then* $M^t u \to 0$

More or less: eigenvalues $1 = \lambda_1, \lambda_2, \ldots \lambda_n$

$$|M^t(u) - u_0| \leq (1 - \lambda_2)^t$$

Where $\lambda_2 < 1$ depends on the structure of the chain M.
State space S it converges in $\frac{\log |S|}{\log_2(1-\lambda_2(M))}$

Limitation, Difficulties

Design a Good Enough Chain ...

Prove that it converges fast

No way to compute λ_2 numerically

State Space is of exponential size

Works only for symmetric chains (but you design it)

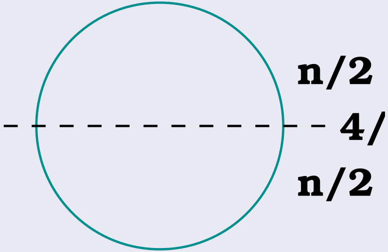
Stupid condition (non bipartite), solved making chain Lazy, loop half of the time

Typical Fake-chain

- Pick $\frac{|V|}{2}$ edges,
  if they form a matching return it
  else play again
- Mathematically sound, return unbiased matching
- Mixes slowly (loops forever)

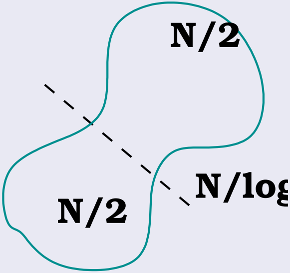
Bad Guy: Cycle

- Totally random moves, takes $\theta(n^2)$ to get random (unbiased random walk with t steps move away from zero lie $\theta(\sqrt{t})$.
- actual time to mix is $\frac{n^2}{2\pi^2}$.
- Very bad expansion, $\frac{2}{n}$



Good Girl: De Bruijn

- Binary chains, length n, shift and inject a new bit.
- random moves, takes n to exactly anywhere with probably $\frac{1}{2^n}$
- actual time to perfectly mix is $\log_2 |S| = n$.
- Good expansion, $\sim \frac{1}{log_2 n}$



Why do Liner Algebra Matter

How does the discrepancy evolve ?

- Look how non uniform is a distribution $X \to \delta(X) = \sum_{e=(u,v)\in E} |X_u - X_v|^2$
- I an Incidence matrix of the graph
- $\delta(X) = |IX|^2 = XI^t IX$
- $\mathcal{L} = II^t$ is the Laplacian of G
- $\mathcal{L} = \Delta(G) - M$ (M adjacency matrix, Δ diagonal of the degrees)
- G is regular : $\mathcal{L} = \Delta Id - II^t$.
- Normalisation : divide by Δ
- $1 - \lambda_2$ is the largest eigenvalue of $\frac{Id-M}{\Delta}$ which is a SDP matrix.

Link with congested cuts

- X indicator vector for S : $X_u = 1, u \in S, X_u = -1 u \in \overline{S}$
- $\rightarrow I^t IX = 4|[S, \overline{S}]$

Definition (isoperimetric constant, conductance)

$$\phi = Min \frac{[S, \overline{S}]}{|S|}$$

High Conductance = Rpid Mixing

$$\frac{\phi^2}{2} \leq 1 - \lambda_2 \leq 2\phi \qquad \text{(Cheeger inequality)}$$

To prove rapid-mixing $\rightarrow$ Prove that conductance is high.

Hum ? Need to have an idea of the **??**

still complicated

The cannonical Path Idea

Define a cannonical path between any pair of states

Hum ? just a routing indeed

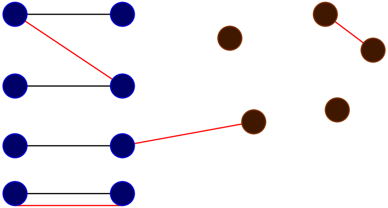
Get low congestion of the edges of G

Here low means logarithmic in the state space size $|S|$ (i.e indeed polynomial).

A “fast” Chain for matchings

M curent solution, select $e \in E$ Randomly.

1) No extremity covered $\rightarrow M \cup \{e\}$
2) 1 extremity cov. (by f) $\rightarrow M \setminus \{f\} \cup \{e\}$
3) 2 extremities cov., $e \notin M \rightarrow M$
4) $e \in M \rightarrow M \setminus \{e\}$

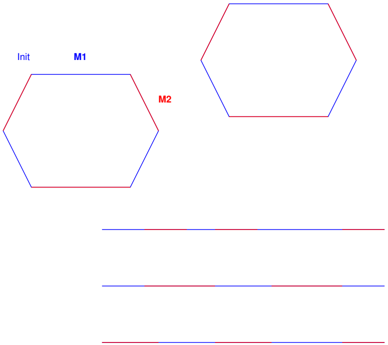


A good routing

Take the symetric difference of the two matchings.

Order the vertices, induce order on the components

Process by component : for each start from the “first” vertex and do the augmenting path thing.



Case of Trees

Cayley formula : labeled trees on K_n(comp.graph)

Induction is $\forall e = (u, v) N(G) = N(G \setminus \{e\}) + N(G[u = v])$.
Generalizes as a determinant for general G.

Markov Chain

- Potentialy Rapidly Mixing Chain : Take an edge and flip it (like when you look for the Min Cost Spanning Tree)
- Prob. Mixes fast (need to check)
- But there is Better ...

Super Smart Generator

Move in G randomly add edges to your tree unless it makes a cycle.

Mixes perfectly

Sampling from inside a Polytope (Lovasz & Simonovits)

We are given a "Nice" Polytope (the solutions of a linear program)

Random Walk inside P

discrete : Divide into cells, make a discrete randomwalk.

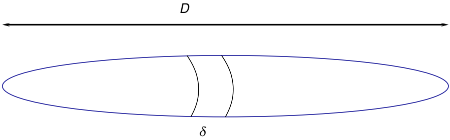
conti: $x \to$ Move randomly inside $B(x, \rho) \cap P$

Complicated :

If ρ big we haven't done anything

ρ small $\to$ No move (mixes slowly)

Continuous space, uniformity ?



Mixing time :

Poincaré Inequality

Up to some conditon, for a convex body diameter D :

$$Congestion \leq \Theta(\frac{D^2 n}{\delta})$$