Asymptotic analysis of precoded small cell networks

Sreenath Ramanath[1], Merouane Debbah[2], Eitan Altman[1], Vinod Kumar[3]
[1]INRIA, Sophia-Antipolis, France; [2]SUPELEC, Paris, France; [3]Alcatel-lucent Bell-labs, Paris, France;

***Abstract*—In this paper, we study precoded MIMO based small cell networks. We derive the theoretical sum-rate capacity, when multi-antenna base stations transmit precoded information to its multiple single-antenna users in the presence of inter-cell interference from neighboring cells. Due to an interference limited scenario, increasing the number of antennas at the base stations does not yield necessarily a linear increase of the capacity. We assess exactly the effect of multi-cell interference on the capacity gain for a given interference level. We use recent tools from random matrix theory to obtain the ergodic sum-rate capacity, as the number of antennas at the base station, number of users grow large. Simulations confirm the theoretical claims and also indicate that in most scenarios the asymptotic derivations applied to a finite number of users give good approximations of the actual ergodic sum-rate capacity.**

***Index Terms*—Cellular networks; MIMO; Small cells; random matrix theory; linear precoding.**

I. Introduction

Small cell based wireless networks are gaining wide popularity to provide the end user with uniform coverage, symmetry and throughput [15], [14]. Existing cellular networks like GSM and WiMAX do not achieve expected throughput to ensure seamless mobile broadband, owing to large coverage area and inability to reach indoor users. For a given radio architecture, dividing a large (macro) cell into number of small (Pico) cells is one of the most effective ways to increase both system capacity [14] and coverage to bring the user a step closer to any-place, any-time, any-device mobile broadband access.

While, dividing a macro-cell into multiple small cells enhances the capacity, the spatial dimension has been exploited in the recent past to enhance the capacity further. It is now well established that Multiple antenna at the transmitter (N_t) and the receiver (N_r) achieve capacity gains which grow linearly as $\min(N_t, N_r)$.

Recently, the MIMO broadcast channel [13], [6], [7], where, a multi-antenna base station, transmitting on M antennas to K single antenna users is shown to achieve capacity gains which grow linearly as $min(M, K)$, provided the transmitter and receivers all know the channel [9]. To achieve this, several methods have been proposed among which linear precoders offer a good compromise between complexity and performance trade-off [1],[8].

Further, MIMO based systems have been studied in the framework of multi-cell networks. In a multi-cell scenario, the achievable sum-rate in the downlink, diminishes due to interference from neighboring base stations. Thus increasing the number of antennas at the base-stations does not necessarily yield a linear increase in capacity. Frequency reuse and various forms of interference co-ordination [3], [5] have been proposed to achieve linear growth in capacity.

System model

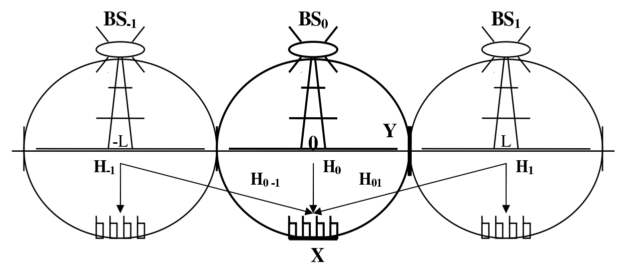

Fig. 1. System model: multi-cell network. BS with M antennas, serving K users. Users at X experience nominal interference and users at Y experience high interference

In our contribution, we want to asses exactly the effect of multi-cell interference in MIMO based small cell networks. Small cells being in close proximity experience higher levels of interference, which would bring down the capacity gains significantly. We want to study the impact of multi-cell interference when base-stations employ linear precoding techniques, such as channel inversion (CI) at the base station.

As mentioned before, linear precoding techniques such as channel inversion(CI) and regularized channel inversion (RCI) offer a convenient trade-off between complexity and achievable sum-rate performance [7], [8]. The behavior of CI in uncorrelated MIMO broadcast channels (MIMO-BC) has already been studied in [7], [8] for i.i.d. Gaussian channels. In particular, the authors in [7] showed that CI achieves linear growth in multiplexing-gain. Further, authors in [2], extended the case to include antenna correlations due to dense packing of the antennas at the transmitter. The analysis carried out considers single cell systems and they show that for the case of CI, the sum-rate is maximized when the number of antennas M on the BS is equal to the number of users K.

For the multi-cell case, the problem of interference coordination in uplink has been discussed at length in [4]. In [5], authors address downlink macro-diversity in cellular systems. They study the potential benefit of base-station (BS) cooperation for downlink transmission in a modified Wyner-type [19] multicell model. They compare various precoders and obtain analytical sum rate expressions for both the fading and the non-fading case. They demonstrate via monte-carlo simulations the effectiveness of linear precoding. Authors in [13] suggests that asymptotically, equal power allocation is optimal when the channel is i.i.d. Gaussian.

In our work, we are interested in studying the impact of interference from adjacent base stations, which is more pronounced in MIMO based small cell networks on the achievable sum-rate capacity. We consider multiple-input multiple-output (MIMO) multi-cell systems, each cell composed of a transmitter equipped with M antennas and K single-antenna receivers. We consider Wyner-type cellular models in our study. We neglect the effects of channel correlation due to densely packed antennas at the base-station transmitter, with a view to keep the analysis tractable.

The analytic expressions of the sum-rates for CI are derived by applying recent tools from random matrix theory (RMT). These expressions are independent of the specific channel realizations.

In our study, we find that

- The achievable sum-rate is significantly diminished by the effect of multi-cell interference in MIMO based small cell networks.
- The sum-rate capacity tends to grow sub-linearly with respect to the number of base-station antennas as long as the interference is non-zero.
- Also, there is an optimal number of users for a given number of antennas at the transmitter, which maximizes the sum-capacity. This depends on the interference level and the transmit power at the base-station.

The remainder of this paper is organized as follows: Section II briefly reviews various tools of random matrix theory which will be used in later derivations. Section III introduces the multi-cell system model. In Section IV we study channel inversion precoding. Section V provides simulation results which are shown to corroborate the theoretical derivations. Finally in Section VI we provide our conclusions.

Notations: In the following, boldface lower-case symbols represent vectors, capital boldface characters denote matrices ($\mathbf{I}_N$ is the $N \times N$ identity matrix). The Hermitian transpose is denoted $(\cdot)^{\mathsf{H}}$. The operator $\mathrm{tr}[\mathbf{X}]$ represents the trace of matrix $\mathbf{X}$. The eigenvalue distribution of an Hermitian random matrix $\mathbf{X}$ is $\mu_{\mathbf{X}}(x)$. The symbol $\mathrm{E}[\cdot]$ denotes expectation. The derivative of a function $f(x)$ of a single variable x is denoted $f'(x)$. All logarithms are base-2 logarithms.

II. Random Matrix Theory Tools

In this work, we are interested in the behavior of large random Hermitian matrices, and particularly in the asymptotic distribution of their eigenvalues. Specifically, the eigenvalue distribution of large Hermitian matrices converges, in many practical cases, to a definite probability distribution, hereafter called the *empirical distribution* of the random matrix, when the matrix dimensions grow to infinity.

A tool of particular interest in this work is the *Stieltjes transform* $\mathcal{S}_{\mathbf{X}}$ of a large Hermitian non-negative definite matrix $\mathbf{X}$, defined on the half the space $\mathbb{C}^+ = \mathbb{C} \setminus R^+ = \{z \in \mathbb{C}, \mathrm{Re}(z) < 0\}$, as

$$\mathcal{S}_{\mathbf{X}}(z) = \int_0^{+\infty} \frac{1}{\lambda - z} \mu_{\mathbf{X}}(\lambda) \mathrm{d}\lambda \tag{1}$$

where $\mu_{\mathbf{X}}$ is the empirical distribution of $\mathbf{X}$.

Couillet et al. [10] derived a fixed-point expression of the Stieltjes transform for Gaussian matrices with correlations in the following theorem,

Theorem 1: Let the entries of the $K \times M$ matrix $\mathbf{W}$ be i.i.d. Gaussian with zero mean and variance $1/M$. Let $\mathbf{X}$ and $\mathbf{Q}$ be respectively $K \times K$ and $M \times M$ Hermitian non-negative definite matrices with eigenvalue distributions $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{Q}}$. We impose further that the largest eigenvalues of $\mathbf{X}$ and $\mathbf{Q}$ are bounded independently of K, M. Let $\mathbf{Y}$ be an $K \times K$ Hermitian matrix with the same eigenvectors as $\mathbf{X}$ and let f be some function mapping the eigenvalues of $\mathbf{X}$ to those of $\mathbf{Y}$. Let $z \in \mathbb{C}^+ = \mathbb{C} \setminus \mathbb{R}^+$. Then, for M, K large with $K/M = 1/\beta$, the Stieltjes transform $\mathcal{S}_{\mathbf{H}}(z)$ of $\mathbf{H} = \mathbf{X}^{1/2}\mathbf{W}\mathbf{Q}\mathbf{W}^{\mathsf{H}}\mathbf{X}^{1/2} + \mathbf{Y}$ is given

$$\mathcal{S}_{\mathbf{H}}(z) = \int \left(f(x) + x \int \frac{q \cdot \mu_{\mathbf{Q}}(q)\mathrm{d}q}{1 + \frac{1}{\beta} q \mathcal{T}_{\mathbf{H}}(z)} - z \right)^{-1} \mu_{\mathbf{X}}(x)\mathrm{d}x \tag{2}$$

where $\mathcal{T}_{\mathbf{H}}$ is a solution of the fixed-point equation

$$\mathcal{T}_{\mathbf{H}}(z) = \int x \left(f(x) + x \int \frac{q \cdot \mu_{\mathbf{Q}}(q)\mathrm{d}q}{1 + \frac{1}{\beta} q \mathcal{T}_{\mathbf{H}}(z)} - z \right)^{-1} \mu_{\mathbf{X}}(x)\mathrm{d}x \tag{3}$$

An immediate corollary, when only right-correlation is considered, unfolds naturally as follows,

Corollary 2: [11] Let the entries of the $K \times M$ matrix $\mathbf{W}$ be i.i.d. Gaussian with zero mean and variance $1/M$. Let $\mathbf{Y}$ be an $K \times K$ Hermitian non-negative matrix with eigenvalue distribution $\mu_{\mathbf{Y}}(x)$. Moreover, let $\mathbf{Q}$ be a $M \times M$ non-negative definite matrix with eigenvalue distribution $\mu_{\mathbf{Q}}(x)$, such that the eigenvalues of $\mathbf{Q}$ are bounded irrespectively of M. Then, for large K, M, such that $K/M = \alpha$, the Stieltjes transform on $\mathbb{C}^+$ of the matrix

$$\mathbf{H} = \mathbf{W}\mathbf{Q}\mathbf{W}^{\mathsf{H}} + \mathbf{Y} \tag{4}$$

verifies

$$\mathcal{S}_{\mathbf{H}}(z) = \mathcal{S}_{\mathbf{Y}} \left(z - \int \frac{q}{1 + \alpha q \mathcal{S}_{\mathbf{H}}(z)} \mu_{\mathbf{Q}}(q)\mathrm{d}q \right) \tag{5}$$

III. System model and assumptions

We discuss the system model in this section. We consider a multi-cell Wyner-type model, for example as shown in figure (1). For simplicity and to be able to keep the analysis tractable, we consider a three-cell network. The cell at the center is our reference. The users in this cell experience interference from the neighboring base stations as shown. Each cell serves K users from a base-station with M antennas. We assume that the base station antennas are uncorrelated. The information from the base-station to its user set is precoded assuming perfect channel state information at the transmitter (CSIT). i.e, each base station knows perfectly the channel towards the users in its cell, but not the interfering channels. Users receive desired signal plus interference signals from adjacent base stations. We assume channel inversion (CI) precoding at the transmitter. The transmitted signals from the base stations undergo Rayleigh fading and path-loss. Further, we assume that the channel is constant for some interval long enough for the transmitter to learn and use it until it changes to a new value. We are interested in the behavior of the system and its sum-rate capacity. Many of our results are obtained for large limits, because the limiting results are often tractable. Nevertheless, we often consider M, K small in our simulation examples. Further, all users are assumed to have the same average (but not instantaneous) received signal power, so our model assumes that the users are similar distances from the base station and are not in deep shadow fades.

IV. Channel inversion precoding

Channel inversion precoding, also referred to as zero-forcing (ZF) precoding, annihilates all the inter-user interference by performing an inversion of the channel matrix $\mathbf{H}$ at the transmitter. We begin our analysis with the single cell case, which is discussed in detail in [2], [7], and further we shall consider the multi-cell case.

A. Single cell

Without loss of generality, we consider cell 0. The signal received by users in this cell is

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}. \tag{6}$$

where, $\mathbf{H}$ is the $K \times M$ channel matrix with zero-mean unit-variance i.i.d complex Gaussian entries, $\mathbf{x} = \mathbf{G}\mathbf{s}$ is the transmit vector obtained by linear precoding of the symbol vector $\mathbf{s}$ with the precoding matrix $\mathbf{G}$. Symbol $s_k \in \mathbf{s}$ for any user k is complex Gaussian with zero mean and unit variance. The $M \times K$ linear precoding matrix is defined as

$$\mathbf{G} = \alpha \mathbf{H}^{\mathsf{H}} \left(\mathbf{H}\mathbf{H}^{\mathsf{H}}\right)^{-1}. \tag{7}$$

where α is chosen appropriately to satisfy the total transmit power constraint $\mathrm{tr}(E[\mathbf{x}\mathbf{x}^H]) \leq \mathrm{tr}(\mathbf{G}\mathbf{G}^H) \leq P$.

Now the received vector in Cell 0

$$\mathbf{y} = \alpha \mathbf{s} + \mathbf{n}. \tag{8}$$

The parameter α which satisfies the transmit power constraint and depends only on the channel realization $\mathbf{H}$ is given by

$$\alpha^2 = \frac{P}{\mathrm{tr}\left((\mathbf{H}\mathbf{H}^{\mathsf{H}})^{-1}\right)} \tag{9}$$

The SNR (signal to noise ratio) for any user k is defined as

$$\eta_k = \frac{E_s\left[|\alpha s_k|^2\right]}{E|n_k|^2} = \frac{\alpha^2}{\sigma^2}. \tag{10}$$

is independent of the selected user. σ^2 is the noise variance.

The ergodic capacity for user k is

$$C_k = \log(1+\eta_k). \tag{11}$$

and the sum-rate is

$$R_{ci} = \sum_{k=1}^{K} \log(1+\eta_k). \tag{12}$$

B. Asymptotic analysis for a single-cell

α is a function of $\mathbf{H}$ and as $M, K \to \infty$, α tends to a constant. Thus the sum-rate can be written as

$$\mathcal{R}_{\mathrm{ci}} = K \log\left(1+\eta_k\right) \tag{13}$$

Let us denote $\mathbf{H}' = \frac{1}{\sqrt{M}}\mathbf{H}$. It follows from (9) that When M is large with $M/K = \beta$,

$$\begin{aligned}
\frac{1}{M}\mathrm{tr}\left(\mathbf{H}'\mathbf{H}'^{\mathsf{H}}\right)^{-1} &= \frac{1}{M}\sum_{i=1}^{K}\frac{1}{\lambda_i} \\
&= \frac{K}{M}\left(\frac{1}{K}\sum_{i=1}^{K}\frac{1}{\lambda_i}\right) \\
&= \frac{K}{M}\int\frac{1}{\lambda}\left(\frac{1}{K}\sum_{i=1}^{K}\delta(\lambda-\lambda_i)\right)d\lambda \\
&= \frac{1}{\beta}\int\frac{1}{\lambda}\mu^{K}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(\lambda)\mathrm{d}\lambda \\
&= \frac{1}{\beta}\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)
\end{aligned}$$

As a consequence, for large (K, M)

$$\frac{\alpha^2}{\sigma^2} \to \frac{\rho\beta}{\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)},\ \text{where } \rho = P/\sigma^2 \tag{14}$$

and the sum-rate is

$$\mathcal{R}_{\mathrm{ci}} = K\log\left(1+\frac{\rho\beta}{\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)}\right) \tag{15}$$

According to Corollary 2, $\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)$ is the solution of[1]

$$\begin{aligned}
\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0) &= \left(\int\frac{\lambda}{1+\frac{\lambda}{\beta}\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)}\mu(\lambda)\mathrm{d}\lambda\right)^{-1} \\
&= \left(\int\frac{\lambda\ \delta(\lambda-1)}{1+\frac{\lambda}{\beta}\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)}\right)^{-1} \\
&= \left(1+\frac{\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)}{\beta}\right)
\end{aligned} \tag{16}$$

Solving for $\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)$ yields,

$$\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0) = \frac{\beta}{(\beta-1)} \tag{17}$$

and the sum-rate is re-written as

$$\mathcal{R}_{\mathrm{ci}} = K\log\left(1+\rho(\beta-1)\right)\ \text{ for } \beta \geq 1 \tag{18}$$

The rate-per-antenna is

$$\frac{\mathcal{R}_{\mathrm{ci}}}{M} = \frac{1}{\beta}\log\left(1+\rho(\beta-1)\right). \tag{19}$$

As $\beta \to 1$, $\mathcal{R}_{\mathrm{ci}}/M \to 0$, which implies that the sum rate of channel inversion does not increase linearly with M (or K)

[1] it is important to note here that we slightly misapply Corollary 2 since the result is only proven valid outside for any $z > 0$.

C. Optimizer β^* for the single cell

Following [7] we now look for a value $\beta^\star$ of the ratio M/K such that, for a fixed number of transmit antennas M, the sum-rate $\mathcal{R}_{\text{ci}}(\beta)$ is maximized. By differentiating eqn (19) with respect to β and setting the derivative to zero, β^* is the solution of the implicit equation

$$\rho\beta^* = (1 + \rho(\beta^* - 1))\, log\, (1 + \rho(\beta^* - 1)) \tag{20}$$

D. Multi-cell

In this section, we study the effect of multi-cell interference. Without loss of generality, we consider users in Cell 0 affected by interference from adjacent base-stations. We consider a 3-cell Wyner-type model as shown in figure 1. Cell C_0 is at the center. Adjacent cells are designated Cell C_1 and Cell C_{-1}.

Following our analysis of the single cell case, the received vector for users of cell C_0, is

$$\mathbf{y} = \mathbf{H}_0\mathbf{G}_0\mathbf{s}_0 + \sqrt{\gamma}\mathbf{H}_{01}\mathbf{G}_1\mathbf{s}_1 + \sqrt{\gamma}\mathbf{H}_{0-1}\mathbf{G}_{-1}\mathbf{s}_{-1} + \mathbf{n}. \tag{21}$$

As before, $\mathbf{H}_0$ is the channel matrix from base station in cell $\mathbb{C}_0$ to its users. $\mathbf{H}_{01}$ and $\mathbf{H}_{0-1}$ are interfering channels from cell $\mathbb{C}_1$ and $\mathbb{C}_{-1}$, respectively. $\mathbf{G}_1$ and $\mathbf{G}_{-1}$ are precoding matrices for users in cell $\mathbb{C}_1$ and $\mathbb{C}_{-1}$, respectively. γ is the signal (interference) attenuation.

As stated earlier, all users in cell $\mathbb{C}_0$ are assumed to have the same average received signal power, so our model assumes that the users are similar distances from the base station and are not in deep shadow fades.

The precoding matrices in cell i can be written as

$$\mathbf{G}_i = \alpha_i\mathbf{H}_i^{\mathsf{H}}(\mathbf{H}_i\mathbf{H}_i^{\mathsf{H}})^{-1} \tag{22}$$

The ergodic capacity for user k in cell $\mathbb{C}_0$ is expressed as

$$C_k = \log\left(1 + \frac{\alpha_0^2}{\mathrm{E}[|n_k|^2]}\right) \tag{23}$$

Where, n_k is the k^{th} element of the covariance matrix $\mathbf{n}$. The expectation of this matrix can be written as

$$\begin{aligned}\mathrm{E}[\mathbf{n}\mathbf{n}^{\mathsf{H}}] &= \gamma\mathbf{H}_{01}\mathbf{G}_1\mathbf{G}_1^{\mathsf{H}}\mathbf{H}_{01}^{\mathsf{H}} \\ &+ \gamma\mathbf{H}_{0-1}\mathbf{G}_{-1}\mathbf{G}_{-1}^{\mathsf{H}}\mathbf{H}_{0-1}^{\mathsf{H}} + \sigma^2\mathbf{I}\end{aligned} \tag{24}$$

Expanding and simplifying,

$$\begin{aligned}\mathrm{E}[\mathbf{n}\mathbf{n}^{\mathsf{H}}] &= \gamma\alpha_1^2\mathbf{H}_{01}\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-2}\mathbf{H}_1\mathbf{H}_{01}^{\mathsf{H}} \\ &+ \gamma\alpha_{-1}^2\mathbf{H}_{0-1}\mathbf{H}_{-1}^{\mathsf{H}}(\mathbf{H}_{-1}\mathbf{H}_{-1}^{\mathsf{H}})^{-2}\mathbf{H}_{-1}\mathbf{H}_{0-1}^{\mathsf{H}} \\ &+ \sigma^2\mathbf{I}\end{aligned} \tag{25}$$

Since,

$$\mathrm{E}[|n_1|^2] = \mathrm{E}[|n_2|^2] \ldots = \mathrm{E}[|n_k|^2] \tag{26}$$

We can write,

$$\begin{aligned}\mathrm{E}[|n_k|^2] &\rightarrow \frac{1}{K}\sum_{k=1}^{K}\mathrm{E}[|n_i|^2] \\ &= \frac{1}{K}\mathrm{tr}\left(E[\mathbf{n}\mathbf{n}^{\mathsf{H}}]\right)\end{aligned} \tag{27}$$

$$\begin{aligned}\mathrm{E}[|n_k|^2] = \frac{1}{K}\mathrm{tr}\,\big(&\gamma\alpha_1^2\mathbf{H}_{01}\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-2}\mathbf{H}_1\mathbf{H}_{01}^{\mathsf{H}} \\ &+ \gamma\alpha_{-1}^2\mathbf{H}_{0-1}\mathbf{H}_{-1}^{\mathsf{H}}(\mathbf{H}_{-1}\mathbf{H}_{-1}^{\mathsf{H}})^{-2}\mathbf{H}_{-1}\mathbf{H}_{0-1}^{\mathsf{H}}\big) \\ &+\sigma^2\mathbf{I}\end{aligned} \tag{28}$$

E. Asymptotic analysis for the multi-cell

Lemma 3: As $K, M \to \infty$

$$\frac{1}{K}\mathrm{tr}\left(\mathbf{H}_{01}\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-2}\mathbf{H}_1\mathbf{H}_{01}^{\mathsf{H}}\right) \to \frac{1}{\beta - 1}$$

Proof: Denote

$$\mathbf{A} = \mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-2}\mathbf{H}_1$$

Now,

$$\begin{aligned}\frac{1}{K}\mathrm{tr}\left(\mathbf{H}_{01}\mathbf{A}\mathbf{H}_{01}^{\mathsf{H}}\right) &= \frac{1}{K}\mathrm{E}\left[\mathrm{tr}\left(\mathbf{H}_{01}\mathbf{A}\mathbf{H}_{01}^{\mathsf{H}}\right)\right] \\ &= \frac{1}{K}\mathrm{tr}\left(\mathrm{E}\left[\mathbf{H}_{01}\mathbf{A}\mathbf{H}_{01}^{\mathsf{H}}\right]\right) \\ &= \frac{1}{K}\mathrm{tr}\left(\mathrm{E}\left[\mathrm{tr}(\mathbf{A})\right]\mathbf{I}_{K\times K}\right) \\ &= \mathrm{E}\left[\mathrm{tr}(\mathbf{A})\right] \\ &= \mathrm{E}\left[\mathrm{tr}\left(\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-2}\mathbf{H}_1\right)\right] \\ &= \mathrm{E}\left[\mathrm{tr}\left(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-1}\right)\right]\end{aligned}$$

If $K \times M$ matrix $\mathbf{H}_1$ is zero-mean, i.i.d. Gaussian, then $\mathbf{W} = \mathbf{H}_1\mathbf{H}_1^{\mathsf{H}}$ is a *Wishart* matrix. For a Wishart matrix [2],

$$\mathrm{E}\left[\mathrm{tr}\left(\mathbf{W}\right)^{-1}\right)\right] = \frac{K}{M - K}$$

$$\mathrm{E}\left[\mathrm{tr}\left(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-1}\right)\right] = \frac{K}{M - K} = \frac{1}{\beta - 1}$$

and hence,

$$\frac{1}{K}\mathrm{tr}\left(\mathbf{H}_{01}\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-2}\mathbf{H}_1\mathbf{H}_{01}^{\mathsf{H}}\right) \to \frac{1}{\beta - 1} \qquad \square$$

Thus the expectation in eq. (28) reduces to,

$$\mathrm{E}[|n_k|^2] \to \alpha_1^2\gamma\frac{1}{\beta - 1} + \alpha_{-1}^2\gamma\frac{1}{\beta - 1} + \sigma^2 \tag{29}$$

And hence, the sum-rate is

$$\mathcal{R}_{\text{ci}} = K\log\left(1 + \frac{\alpha_0^2(\beta - 1)}{\alpha_1^2\gamma + \alpha_{-1}^2\gamma + \sigma^2(\beta - 1)}\right) \tag{30}$$

Following (14), for large (K, M),

$$\frac{\alpha_0^2}{\sigma^2} = \frac{\alpha_1^2}{\sigma^2} = \frac{\alpha_{-1}^2}{\sigma^2} \to \frac{\rho\beta}{\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)},\ \text{where } \rho = P/\sigma^2 \tag{31}$$

Thus the above sum-rate expression can be simplified as

$$\mathcal{R}_{\text{ci}} = K\log\left(1 + \frac{\rho\beta(\beta - 1)}{(\beta - 1)\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0) + 2\gamma\rho\beta}\right) \tag{32}$$

[2] Refer section 2.1.6, equation (2.9) of [16] and the references there-in ([17], [18])

Substituting for $\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathrm{H}}}(0)$,

$$\mathcal{R}_{\mathrm{ci}} = K \log\left(1 + \frac{\rho(\beta - 1)}{1 + 2\gamma\rho}\right) \tag{33}$$

Re-writing,

$$\frac{\mathcal{R}_{\mathrm{ci}}}{M} = \frac{1}{\beta} \log\left(1 + \frac{\rho(\beta - 1)}{1 + 2\gamma\rho}\right) \tag{34}$$

We observe that when $\gamma = 0$, that is when there is no interference, the capacity formula is that of the single-cell case.

As $\beta \to 1$, $\mathcal{R}_{\mathrm{ci}}/M \to 0$, which implies that the sum rate of channel inversion does not increase linearly with M (or K)

F. Optimizer β^* for the multi-cell

Following on similar lines of the single-cell case, we now look for a value $\beta^\star$ of the ratio M/K such that, for a fixed number of transmit antennas M, the sum-rate $\mathcal{R}_{\mathrm{ci}}(\beta)$ is maximized. By differentiating eqn (34) with respect to β and setting the derivative to zero, β^* is the solution of the implicit equation

$$\rho\beta^* = [\rho(\beta^* - 1) + (1 + 2\gamma\rho)] \log\left[1 + \frac{\rho(\beta^* - 1)}{1 + 2\gamma\rho}\right] \tag{35}$$

One can observe that by setting $\gamma = 0$, we fall back to the implicit equation (20) of the single cell case.

G. Some observations:

Following our single cell and multi-cell analysis, we plot in figure 2, the optimal β, i.e, β^* (refer equation 35), which maximizes the sum rate and in figure 3 the corresponding optimal number of users $K^* = M/\beta^*$ for M = 16 and different SNR. We observe that,

1) With increasing SNR more and more users should be served to maximize the sum rate.

2) Also, the number of users required to maximize the sum rate tends to saturate with an increase in the interference factor γ.

Next, we plot the optimal sum rate (refer equation 34), i.e, the sum rate achieved when $\beta = \beta^*$ in figure 4. We compare this for example with $\beta = 2$, shown in figure 5. We obtain the sum-rate by computing the rate per user in the asymptotic regime and then multiplying this with a finite number of antennas M at the BS. For this example we have used $M = 16$.

There are some interesting observations here:

1) The sum-rate tends to increase at a constant rate when $\beta = \beta^*$, when there is no interference ($\gamma = 0$).

2) The sum-rate tends to saturate with interference and the saturation occurs sooner when the interference is higher.

3) The sum-rate with interference for any other β, for example $\beta = 2$ (fig 5), is not much different from $\beta = \beta^*$ (fig 4) in the presence of interference. The rate per transmit antenna tends to saturate with interference.

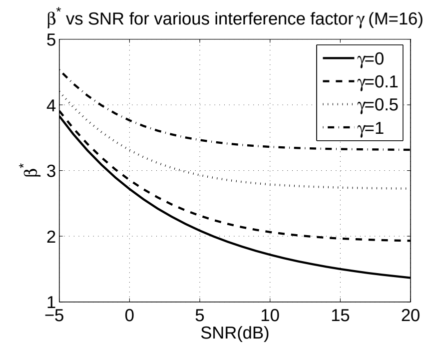

Fig. 2. β^* vs SNR for various interference factors ($M = 16$)

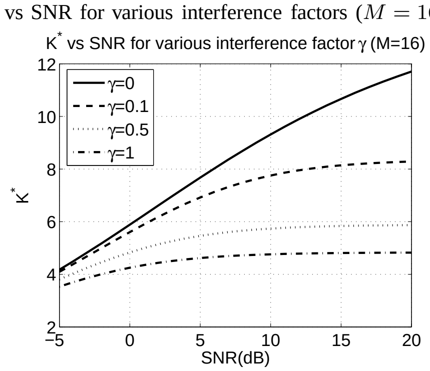

Fig. 3. K^* vs SNR for various interference factors ($M = 16$)

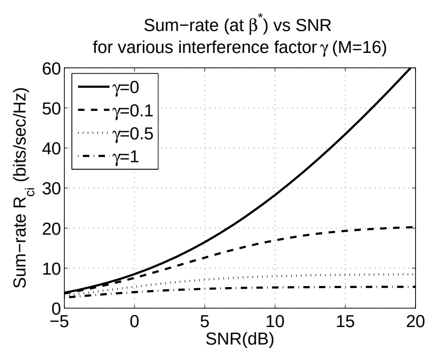

Fig. 4. Sum rate at β^* for various interference factors ($M = 16$)

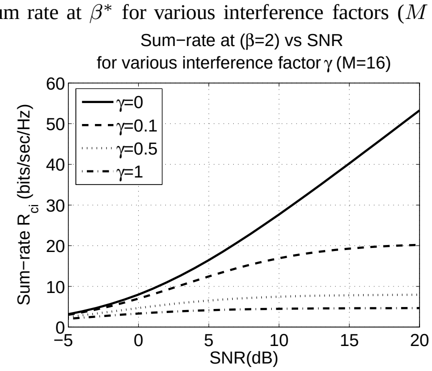

Fig. 5. Sum rate at $\beta = 2$ for various interference factors ($M = 16$)

H. Single cell and multi-cell with unequal power

We re-define the power-constraint as

$$\mathrm{tr}[\mathbf{x}\mathbf{P}\mathbf{x}^{\mathsf{H}}] \leq \mathrm{tr}[\mathbf{G}\mathbf{P}\mathbf{G}^{\mathsf{H}}] \leq P \tag{36}$$

such that the k^{th} diagonal element of $\mathbf{P}$ represents power p_k for user k with $\sum_{k=1}^{K} p_k = \mathrm{tr}(\mathbf{P})$.

Expanding $\mathrm{tr}[\mathbf{G}\mathbf{P}\mathbf{G}^{\mathsf{H}}]$,

$$\begin{aligned} \mathrm{tr}\left(\mathbf{G}\mathbf{P}\mathbf{G}^{\mathsf{H}}\right) &= \mathrm{tr}\left(\mathbf{H}^{\mathsf{H}}(\mathbf{H}\mathbf{H}^{\mathsf{H}})^{-1}\mathbf{P}(\mathbf{H}\mathbf{H}^{\mathsf{H}})^{-1}\mathbf{H}\right) \\ &= \mathrm{tr}\left((\mathbf{H}\mathbf{H}^{\mathsf{H}})^{-1}\mathbf{P}\right) \\ &= \mathrm{tr}(\mathbf{H}\mathbf{H}^{\mathsf{H}})^{-1} \ \frac{1}{K}\mathrm{tr}(\mathbf{P}) \\ &= \frac{1}{M}\mathrm{tr}(\mathbf{H}'\mathbf{H}'^{\mathsf{H}})^{-1} \ \frac{1}{K}\mathrm{tr}(\mathbf{P}) \\ &= \frac{1}{\beta}\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0) \ \frac{1}{K}\mathrm{tr}(\mathbf{P}) \end{aligned} \tag{37}$$

From eqn (36) and (37), we see that

$$\frac{1}{K}\mathrm{tr}(\mathbf{P}) \leq \frac{P\beta}{\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)}$$

With $\rho = P/\sigma^2$,

$$\frac{1}{\sigma^2} \leq \frac{\rho\beta}{\frac{1}{K}\mathrm{tr}(\mathbf{P}) \ \mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0)} \tag{38}$$

Substituting $\mathcal{S}_{\mathbf{H}'\mathbf{H}'^{\mathsf{H}}}(0) = \beta/(\beta-1)$, the ergodic capacity for user k in the single-cell case is

$$C_k = \log\left(1 + \frac{p_k}{\sigma^2}\right) = \log\left(1 + \frac{p_k\rho(\beta-1)}{\frac{1}{K}\mathrm{tr}(\mathbf{P})}\right). \tag{39}$$

and the sum-rate is

$$R_{ci} = \sum_{k=1}^{K} \log\left(1 + \frac{p_k\rho(\beta-1)}{\frac{1}{K}\mathrm{tr}(\mathbf{P})}\right). \tag{40}$$

We can easily see that with equal power for all users, $\frac{1}{K}\mathrm{tr}(\mathbf{P}) = p = p_k$ and the above expression will reduce to the expressions derived for the single-cell case with equal power constraint (eqn 18).

The rate per antenna is

$$\frac{R_{ci}}{M} = \frac{1}{\beta}\frac{1}{K}\sum_{k=1}^{K} \log\left(1 + \frac{p_k\rho(\beta-1)}{\frac{1}{K}\mathrm{tr}(\mathbf{P})}\right) \tag{41}$$

For the multi-cell case, the ergodic capacity eqn (23) for user k is

$$C_k = \log\left(1 + \frac{p_k}{\mathrm{E}[|n_k|^2]}\right) \tag{42}$$

Where,

$$\begin{aligned} \mathrm{E}[\mathbf{n}\mathbf{n}^{\mathsf{H}}] &= \gamma\mathbf{H}_{01}\mathbf{G}_1\mathbf{P}_1\mathbf{G}_1^{\mathsf{H}}\mathbf{H}_{01}^{\mathsf{H}} \\ &+ \gamma\mathbf{H}_{0-1}\mathbf{G}_{-1}\mathbf{P}_{-1}\mathbf{G}_{-1}^{\mathsf{H}}\mathbf{H}_{0-1}^{\mathsf{H}} \\ &+ \sigma^2\mathbf{I} \end{aligned} \tag{43}$$

After suitable simplification similar to the multi-cell analysis in the previous section, we can re-write the above expression as

$$\begin{aligned} \mathrm{E}[|n_k|^2] = \frac{1}{K}\mathrm{tr}\Big(&\mathbf{H}_{01}\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-1}\mathbf{P}_1(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-1}\mathbf{H}_1\mathbf{H}_{01}^{\mathsf{H}} \\ &+\mathbf{H}_{0-1}\mathbf{H}_{-1}^{\mathsf{H}}(\mathbf{H}_{-1}\mathbf{H}_{-1}^{\mathsf{H}})^{-1}\mathbf{P}_{-1}(\mathbf{H}_{-1}\mathbf{H}_{-1}^{\mathsf{H}})^{-1}\mathbf{H}_{-1}\mathbf{H}_{0-1}^{\mathsf{H}}\Big) \\ &+\sigma^2\mathbf{I} \end{aligned} \tag{44}$$

As $M, K \to \infty$,

$$\begin{aligned} \mathrm{tr}\left(\mathbf{H}_{01}\mathbf{H}_1^{\mathsf{H}}(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-1}\mathbf{P}_i(\mathbf{H}_1\mathbf{H}_1^{\mathsf{H}})^{-1}\mathbf{H}_1\mathbf{H}_{01}^{\mathsf{H}}\right) \\ \to \frac{1}{\beta-1}\mathrm{tr}(\mathbf{P}_i) \end{aligned} \tag{45}$$

Therefore, the expectation can be written as,

$$\begin{aligned} \mathrm{E}[|n_k|^2] = \gamma\frac{1}{\beta-1}\frac{\mathrm{tr}(\mathbf{P}_1)}{K} + \gamma\frac{1}{\beta-1}\frac{\mathrm{tr}(\mathbf{P}_{-1})}{K} \\ +\sigma^2\mathbf{I}. \end{aligned} \tag{46}$$

The capacity of user k is

$$C_k = \log\left(1 + \frac{(\beta-1) \ p_k}{\gamma\frac{\mathrm{tr}(\mathbf{P}_1)}{K} + \gamma\frac{\mathrm{tr}(\mathbf{P}_{-1})}{K} + \sigma^2(\beta-1)}\right) \tag{47}$$

Substituting eqn (38) for $1/\sigma^2$,

$$C_k = \log\left(1 + \frac{\rho(\beta-1) \ p_k}{\gamma\rho \ \frac{\mathrm{tr}(\mathbf{P}_1)}{K} + \gamma\rho \ \frac{\mathrm{tr}(\mathbf{P}_{-1})}{K} + \frac{\mathrm{tr}(\mathbf{P})}{K}}\right) \tag{48}$$

and the sum-rate is expressed as

$$\mathcal{R}_{\mathrm{ci}} = \sum_{k=1}^{K} \log\left(1 + \frac{\rho(\beta-1) \ p_k}{\gamma\rho \ \frac{\mathrm{tr}(\mathbf{P}_1)}{K} + \gamma\rho \ \frac{\mathrm{tr}(\mathbf{P}_{-1})}{K} + \frac{\mathrm{tr}(\mathbf{P})}{K}}\right) \tag{49}$$

Notice that if $\frac{1}{K}\mathrm{tr}(\mathbf{P}_i) = p = p_k$, the above expression will reduce to the expressions derived for the multi-cell case with equal power constraint (eqn 33).

The sum-rate per antenna is

$$\frac{\mathcal{R}_{\mathrm{ci}}}{M} = \frac{1}{\beta}\frac{1}{K}\sum_{k=1}^{K} \log\left(1 + \frac{\rho(\beta-1) \ p_k}{\gamma\rho \ \frac{\mathrm{tr}(\mathbf{P}_1)}{K} + \gamma\rho \ \frac{\mathrm{tr}(\mathbf{P}_{-1})}{K} + \frac{\mathrm{tr}(\mathbf{P})}{K}}\right) \tag{50}$$

We observe two things here. 1) One can come up with an optimal power allocation policy (for ex. based on the channel characteristics) which maximizes the sum-capacity in the unequal power allocation scheme. 2) If some of the users in the adjacent base stations are not being serviced, i.e, their respective antenna at the transmitter is switched off, the interference comes down (for ex. if one or more user links are inactive in cell 1, then $\mathrm{tr}(\mathbf{P}_1) < P_1$) and hence the sum-capacity scales up.

V. Simulation results

In this section we evaluate by simulation how interference from neighboring base stations impacts the behavior of the sum-rate of linearly precoded MIMO small cell networks when the antenna array at the transmitter are large. We compare numerical results obtained by Monte-Carlo simulations with our previously derived asymptotic expressions for finite (K, M). In particular, we have the following cases.

1) We fix the SNR ($\rho = 20$ dB) and calculate rate achieved per antenna as we vary $\beta = M/K$ (refer equation (34)). We plot this in figure (6) for various interference factors γ. We observe that the rate per antenna is maximized for a certain $\beta = \beta^*$. This matches with the β^* computed by solving the implicit eqn (35). It is also interesting to observe that β^* increases with increasing interference. Also, beyond β^*, the capacity growth is not in proportion to the growth in number of antennas M at the base station.

2) We fix the SNR ($\rho = 20$ dB) and the ratio $M/K = \beta = 2$. We compute the rate achieved per antenna as we vary the interference factor γ. We compare asymptotic results via monte-carlo simulations. We plot this in figure 7. We observe that the achievable rate is very sensitive to interference. The drop in rate is very steep in the beginning and tends to saturate for higher interference. Thus, the rate per antenna saturates with γ. This seems to indicate that the high amount of interference envisaged in small cells might not be as harmful. Many of the proposed interference management and co-ordination schemes might work well even in the case of small cells.

3) Next we show how the sum-rate increases with increasing number of base-station antennas M at SNR ($\rho = 0, 20dB$) for various interference factors γ, when $\beta = 2$. We compute the rate per antenna from equation (34) for the asymptotic part to compare it with monte-carlo simulations. The observations are plotted in figures (8), (9). We observe that the increase in sum-rate is linear when interference is nil. The increase is sub-linear for other interference factors. Since the number of antennas at the base station and number of users are increasing simultaneously, the capacity is expected to grow in proportion to $\min(M, K)$, scaled by a factor, that depends on the interference factor γ and the SNR ρ.

In all the cases, we observe that in all simulations the asymptotic results closely match the numerical results even for small values of (K, M).

VI. Conclusions

We looked at the problem of inter-cell interference in MIMO based small cell networks. We started our analysis with a single cell, where multi-antenna base station employ channel inversion precoding to communicate with multiple single-antenna users. We extended the case to multi-cell scenario, using a simple wyner-type model. We derived the sum-rate capacity in the asymptotic regime, i.e, when the number of antennas at the base station and number of user grow large, but, with a fixed ratio. We used recent tools from random matrix theory, which have proven to give reliable results even when the quantities involved are practical and finite. We further derived β^*, the ratio of number of transmit antennas to users, which maximizes the achievable sum-rate. The asymptotic analysis was validated with monte-carlo simulations in the finite regime.

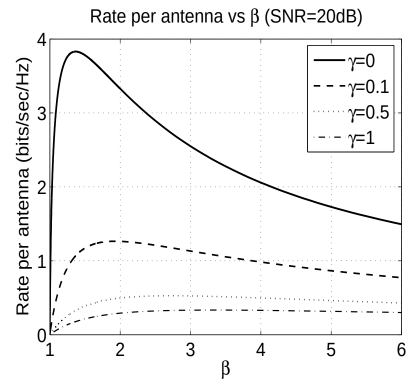

Fig. 6. Rate per antenna vs β at SNR of 20 dB for various interference factors γ

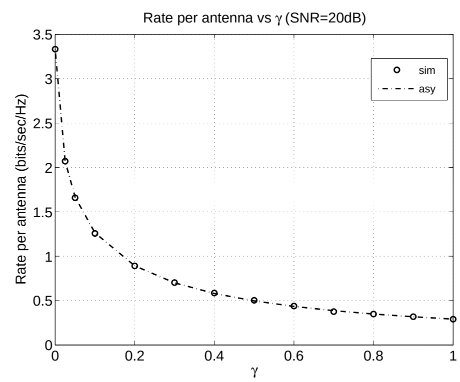

Fig. 7. Rate per antenna vs γ, when, $\beta = 2$, SNR $\rho = 20$ dB for various interference factors γ

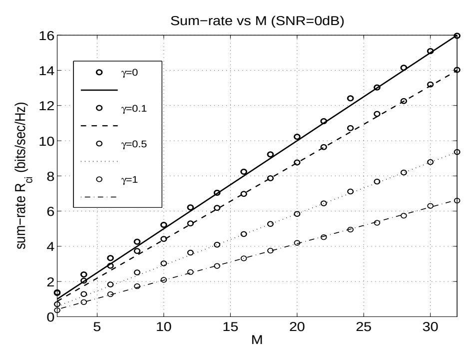

Fig. 8. Sum rate per antenna as a function of M for $\beta = 2$ at SNR of 0 dB for various interference factors

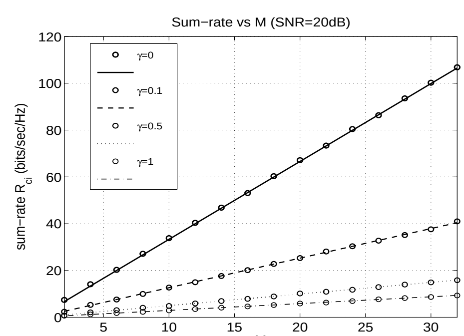

Fig. 9. Sum rate per antenna as a function of M for $\beta = 2$ at SNR of 20 dB for various interference factors

We conclude that the achievable sum-rate is significantly diminished by the effect of multi-cell interference in MIMO based small cell networks. The sum-rate capacity tends to grow sub-linearly with increasing interference. Also, there is an optimal number of users for a given number of antennas at the transmitter, which maximizes the sum-capacity. This depends on the interference level and the transmit power at the base-station. For a given number of transmit antenna, moving away from the optimal, β^*, tends to saturate the capacity growth at high SNR. The saturation occurs sooner with higher interference.

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