UBINET/RIF: Performance Evaluation of Networks

Homework 4
To be returned on 24 October 2017

4.1 Collecting fridge magnets (again)

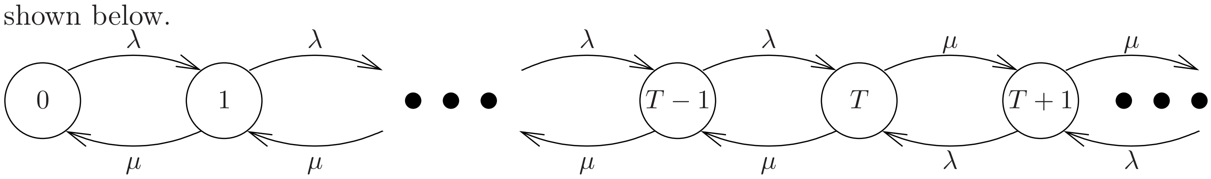
The children of a family collect the fridge magnets that come in cereal boxes. There are a total of n distinct fridge magnets. The probability p_i that a cereal box has a fridge magnet of type i is $p_i = 1/n$ (uniform case). Let $N_i(t)$ be the number of magnets of type i collected at time t. $\{N_i(t), t > 0\}$ is a Poisson process with rate λ/n.

Let $X(t)$ denote the number of *distinct* fridge magnets collected at time t, with $X(0) = 0$. The state space of $X(t)$ is $\mathcal{E} = \{0, 1, \ldots, n\}$.

1. What is the sojourn time in state 0 (think of the time needed to get the first fridge magnet)? How is it distributed?
2. What is the sojourn time in state i, for $1 \leq i \leq n-1$ (think of the time needed to get a *new* fridge magnet when having already i distinct ones)? How is it distributed?
3. What is the nature of the stochastic process $\{X(t), t > 0\}$?
   Be exhaustive and justify your answer.
4. Draw the transition diagram of $\{X(t), t > 0\}$.
5. What is the expected time until the collection is complete?

4.2 A threshold queue

We define a threshold queue with parameter T as follows: When the number of jobs is $< T$, jobs arrive according to a Poisson process with rate λ and their service time is exponentially distributed with rate μ. However, when the number of jobs is $> T$, jobs arrive according to a Poisson process with rate μ and their service time is exponentially distributed with rate λ, as shown below.



1. Write the *global* balance equations; see Exercise 3.1 (only two equations are needed).
2. Derive the stationary distribution as a function of T and $\rho := \frac{\mu}{\lambda}$.
   What is the stability condition?
3. Compute the mean number of jobs, $\overline{N}$, in this threshold queue as a function of T and ρ.
4. What happens to π_0 and $\overline{N}$ when $T = 0$?
   Does this answer make sense?

4.3 The infinite helpdesk

Consider the $M/M/\infty$ queueing system, where arrivals form a Poisson process with rate λ and service times are exponentially distributed with parameter μ.

1. Write the state space.
2. Draw the state diagram of the queue size $X(t)$.
3. Derive the limiting probabilities in closed form.
4. What is the stability condition?
5. According to the limiting probabilities derived, what is the distribution of the queue size?
6. Compute the expected queue size in closed form.
7. Compute the system's mean reponse time.
   Explain why the result found makes sense.