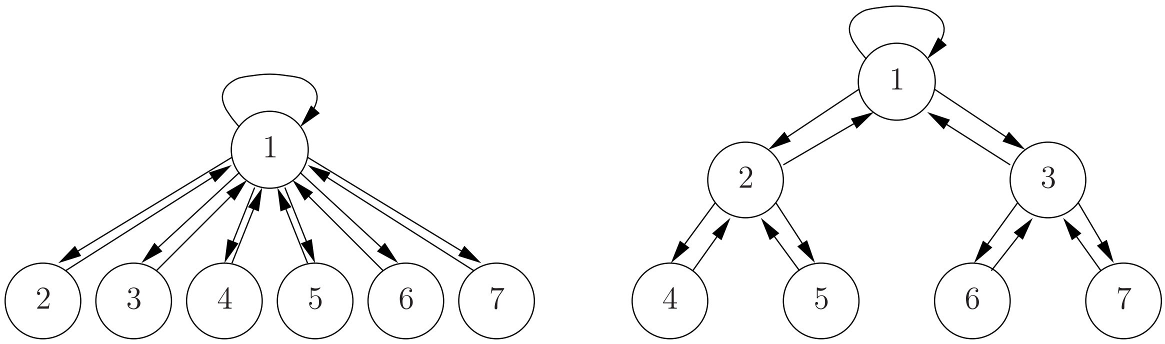
UBINET/RIF: Performance Evaluation of Networks

Homework 2
To be returned on 10 October 2017

2.1 Website structure

A website designer is considering two possible structures for a given website. Either pages can be reached only from the main page (flat structure) or pages are organized hierarchically. The objective of the website designer is to have the main page being ranked as highly as possible by search engines like Google. In the figures below, each vertex represents a web page and each arrow represents a (directed) link between two pages. If page i has $k > 0$ outgoing links then the probability to follow each of the k links is set to $1/k$.



(a) Flat structure (b) Hierarchical structure

Figure 1: Two possible website structures.

Flat structure We assume a user navigates through the website organized as in Fig. 1(a). Let $X_F(n)$ be the page visited at the nth step. The stochastic process $\{X_F(n), n = 0, 1, \dots\}$ is a discrete-time Markov chain called "random walk."

1. Write the transition matrix $\mathbf{P}^F$ for the DTMC $\{X_F(n), n = 0, 1, \dots\}$.
2. Is this chain ergodic?
3. Write the steady-state equations and compute the stationary distribution of this DTMC $\pi^F := (\pi_1^F, \pi_2^F, \pi_3^F, \pi_4^F, \pi_5^F, \pi_6^F, \pi_7^F)$.

Hierarchical structure We assume a user navigates through the website organized as in Fig. 1(b). Let $X_H(n)$ be the page visited at the nth step. The stochastic process $\{X_H(n), n = 0, 1, \dots\}$ is a discrete-time Markov chain called "random walk".

4. Write the transition matrix $\mathbf{P}^H$ for the DTMC $\{X_H(n), n = 0, 1, \dots\}$.
5. Is this chain ergodic?
6. Write the steady-state equations and compute the stationary distribution of this DTMC $\pi^H := (\pi_1^H, \pi_2^H, \pi_3^H, \pi_4^H, \pi_5^H, \pi_6^H, \pi_7^H)$.

Recommendation

7. Which structure do you recommend to improve the ranking of the main page (page 1)?

2.2 A simple CTMC

Consider a Continuous-time Markov Chain (CTMC) with state-space $\mathcal{E} = \{1, 2, 3, 4\}$ and infinitesimal generator $\mathbf{Q}$ given by

$$\mathbf{Q} = \begin{bmatrix} -4 & 2 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -3 & 3 \\ 2 & 1 & 0 & -3 \end{bmatrix}$$

1. Draw the transition rate diagram.
2. Explain why this CTMC is irreducible.
3. Compute its stationary probabilities π. Justify your calculation.
4. Compute $\overline{T}_i$, the expected time spent by the Markov chain in state $i \in \mathcal{E}$.
5. Is this CTMC time-reversible?

2.3 Students computer lab

There are 20 students in the graduate program of the Department. The students computer lab has exactly 20 machines. Each student uses a lab's computer for a time that is exponentially distributed with parameter μ. When done, the student leaves the lab to attend other matters; she returns to the computer lab after a time that is exponentially distributed with parameter λ. We assume that the students behave independently of each other. Let $X(t)$ denote how many computers are being used in the lab at time t.

1. What is the state-space $\mathcal{E}$ of the stochastic process $\{X(t), t > 0\}$?
2. When i students are in the computer lab, what is the distribution of the time until one of the $20 - i$ other students comes into the lab?
3. When i students are in the computer lab, what is the distribution of the time until one of them leaves the lab?
4. Is $\{X(t), t > 0\}$ a Markov chain?
5. Is $\{X(t), t > 0\}$ a birth and death process? If yes, determine its birth rates λ_i and its death rates μ_i and write its infinitesimal generator Q in matrix form.
6. Compute the limiting probabilities $\pi_i = \lim_{t\to\infty} P(X(t) = i)$, for $i \in \mathcal{E}$.
7. Compute N the mean number of computers being used in steady-state.