1.1 Collecting fridge magnets

A cereal manufacturer sells cereal boxes having each a fridge magnet. The complete set of fridge magnets represents a world map. There are a total of \(n\) distinct fridge magnets. A given family takes a random time to finish a cereal box and buy a new one. This time is exponentially distributed with parameter \(\lambda\), therefore the number of cereal boxes purchased by this family is a Poisson process with rate \(\lambda\). The children of this family are collecting the fridge magnets.

1. Let \(X\) be a random variable denoting the time between two purchases of cereal boxes. Write the cumulative distribution function of \(X\) and its probability density function.

2. Let \(N(t)\) be the number of magnets collected at time \(t\). What is the nature of the stochastic process \(\{N(t), t > 0\}\)?

3. There is in total exactly the same number of each distinct fridge magnet. What is the probability \(p_i\) that a cereal box has a fridge magnet of type \(i\)?

4. Let \(N_i(t)\) be the number of magnets of type \(i\) collected at time \(t\). What is the nature of the stochastic process \(\{N_i(t), t > 0\}\)?

5. Let \(T_i\) be the time needed to get the first magnet of type \(i\). What is the distribution of \(T_i\)?

6. Let \(T\) be the time needed to collect the entire collection of fridge magnets. What is the distribution of \(T\) (hint: write \(T\) as a function of the \(T_i\)'s)?

1.2 An unreliable computer system

Assume that a computer system is in one of 3 states: busy, idle, or undergoing repair, respectively, denoted by states 0, 1, and 2, respectively. Observing its state at 2pm each day, we believe that the system approximately behaves like a homogeneous discrete-time Markov chain with the transition matrix

\[
P = \begin{bmatrix}
0.6 & 0.2 & 0.2 \\
0.1 & 0.8 & 0.1 \\
0.6 & 0.0 & 0.4
\end{bmatrix}
\]

Prove that the chain is irreducible and aperiodic, and determine the stationary probabilities.
1.3 A simple DTMC

Consider a discrete-time Markov chain having the following transition matrix:

\[ P = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{bmatrix} \]

1. What are the limiting probabilities \( \pi = (\pi_1, \pi_2, \pi_3) = \lim_{t \to \infty} \pi(t) \) for this Markov chain?

2. What is the 2-step transition matrix for this Markov chain?

3. Let \( \pi(0) = (\frac{1}{2}, 0, \frac{1}{2}) \). What is the transient probability distribution after two time steps?