UBINET: Performance Evaluation of Networks

Homework 2
To be returned on 4 October 2016

2.1 The transient distribution of a DTMC

Let \{X(n), n \geq 0\} be a discrete time homogeneous Markov chain with state space \(E = \{0, 1\}\) and transition matrix:

\[
P = \begin{bmatrix} \alpha & 1 - \alpha \\ 1 - \beta & \beta \end{bmatrix}, \quad \text{with } 0 \leq \alpha, \beta \leq 1.
\]

Define, for all steps \(n = 0, 1, ..., \) the vector \(\pi(n) = (\pi_0(n), \pi_1(n)) = (P(X(n) = 0), P(X(n) = 1))\) giving the transient distribution of the DTMC at step \(n\). The initial distribution \(\pi(0) = (\pi_0(0), \pi_1(0))\) is assumed to be known.

1. Diagonalize the matrix \(P\).
2. Compute \(P^n\) in function of \(\alpha, \beta, n\).
3. Give an expression for \(\pi_0(n)\) and \(\pi_1(n)\) in function of \(\alpha, \beta, n, \pi_0(0)\) and \(\pi_1(0)\).
4. Specify the conditions under which the limiting/stationary distributions exist and compute them when they do (use the answer to the previous question).

2.2 A simple CTMC

Consider a Continuous-time Markov Chain (CTMC) with state-space \(E = \{1, 2, 3\}\) and infinitesimal generator \(Q\) given by

\[
Q = \begin{pmatrix} -5 & 2 & 3 \\ 1 & -2 & 1 \\ 3 & 0 & -3 \end{pmatrix}.
\]

1. Draw the transition rate diagram.
2. Explain why this CTMC is irreducible.
3. Compute its stationary probabilities (denoted as \(\pi_1, \pi_2, \pi_3\)). Justify your calculation.
4. Let \(T_i\) be the expected time spent by the Markov chain in state \(i \in E\). Compute \(T_1, T_2\) and \(T_3\).
2.3 A Web traffic model

The objective of this model is to predict the requests rate generated by a Web user. We assume that a user surfing on the Web can be in one of three distinct states:

- $L$: the user is taking a long time to read a Web page,
- $S$: the user is taking a short time to read a Web page,
- $W$: the user is waiting while a new page is being downloaded.

The behavior of the user can be described as follows.

- Once a page is read, the user requests another page that will be downloaded. With probability $p$ this new page is interesting and the user will spend a long time reading it. With probability $1 - p$, the new page will be read quickly.

- The long reading time is a rv that is exponentially distributed with parameter $\mu_L > 0$. The short reading time is a rv that is exponentially distributed with parameter $\mu_S > 0$. The download time is a rv that is exponentially distributed with parameter $\mu_W > 0$.

1. Explain why the state of the Web user is a CTMC on the state-space $\mathcal{E} = \{L, S, W\}$.

2. Draw the transition diagram and write the infinitesimal generator.

3. Under which conditions is the chain irreducible?

4. Write the balance equations and find the steady-state distribution $\pi = (\pi_L, \pi_S, \pi_W)$.

5. Find the rate $\theta$ with which a Web user generates requests in terms of the probabilities $\pi_L, \pi_S, \pi_W$. Explain your reasoning.

6. Numerical application: let $p = 0.3$, $\frac{1}{\mu_L} = 2$ minutes, $\frac{1}{\mu_S} = 10$ seconds, $\frac{1}{\mu_W} = 5$ seconds. Compute $\pi$ and $\theta$ (express it in requests per minute).