UBINET: Performance Evaluation of Networks

Homework 2
To be returned on 4 October 2016

2.1 The transient distribution of a DTMC

Let $\{X(n), n \geq 0\}$ be a discrete time homogeneous Markov chain with state space $\mathcal{E} = \{0, 1\}$ and transition matrix:

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}, \quad \text{with } 0 \leq \alpha, \beta \leq 1 \ .$$

Define, for all steps $n = 0, 1, ...$, the vector $\pi(n) = \big(\pi_0(n), \pi_1(n)\big) = \big(P(X(n) = 0), P(X(n) = 1)\big)$ giving the transient distribution of the DTMC at step n. The initial distribution $\pi(0) = (\pi_0(0), \pi_1(0))$ is assumed to be known.

1. Diagonalize the matrix P.
2. Compute P^n in function of α, β, n.
3. Give an expression for $\pi_0(n)$ and $\pi_1(n)$ in function of $\alpha, \beta, n, \pi_0(0)$ and $\pi_1(0)$.
4. Specify the conditions under which the limiting/stationary distributions exist and compute them when they do (use the answer to the previous question).

2.2 A simple CTMC

Consider a Continuous-time Markov Chain (CTMC) with state-space $\mathcal{E} = \{1, 2, 3\}$ and infinitesimal generator $\mathbf{Q}$ given by

$$\mathbf{Q} = \begin{pmatrix} -5 & 2 & 3 \\ 1 & -2 & 1 \\ 3 & 0 & -3 \end{pmatrix}.$$

1. Draw the transition rate diagram.
2. Explain why this CTMC is irreducible.
3. Compute its stationary probabilities (denoted as π_1, π_2, π_3). Justify your calculation.
4. Let $\overline{T}_i$ be the expected time spent by the Markov chain in state $i \in \mathcal{E}$. Compute $\overline{T}_1$, $\overline{T}_2$ and $\overline{T}_3$.

2.3 A Web traffic model

The objective of this model is to predict the requests rate generated by a Web user. We assume that a user surfing on the Web can be in one of three distinct states:

L: the user is taking a *long* time to read a Web page,
S: the user is taking a *short* time to read a Web page,
W: the user is *waiting* while a new page is being downloaded.

The behavior of the user can be described as follows.

- Once a page is read, the user requests another page that will be downloaded. With probability p this new page is interesting and the user will spend a long time reading it. With probability $1-p$, the new page will be read quickly.
- The long reading time is a rv that is exponentially distributed with parameter $\mu_L > 0$. The short reading time is a rv that is exponentially distributed with parameter $\mu_S > 0$. The download time is a rv that is exponentially distributed with parameter $\mu_W > 0$.

1. Explain why the state of the Web user is a CTMC on the state-space $\mathcal{E} = \{L, S, W\}$.
2. Draw the transition diagram and write the infinitesimal generator.
3. Under which conditions is the chain irreducible?
4. Write the balance eqautions and find the steady-state distribution $\pi = (\pi_L, \pi_S, \pi_W)$.
5. Find the rate θ with which a Web user generates requests in terms of the probabilities π_L, π_S, π_W. Explain your reasoning.
6. Numerical application: let $p = 0.3$, $\frac{1}{\mu_L} = 2$ minutes, $\frac{1}{\mu_S} = 10$ seconds, $\frac{1}{\mu_W} = 5$ seconds. Compute π and θ (express it in requests per minute).