UBINET: Performance Evaluation of Networks

Homework 2
To be returned on 6 October 2015

2.1 Transient analysis of a continuous time Markov chain

Exercise borrowed from Alain Jean-Marie
Let $\{X(t), t \geq 0\}$ be a continuous time homogeneous Markov chain with state space $\mathcal{E} = \{0, 1\}$ and infinitesimal generator:

$$\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \qquad \lambda, \mu > 0 \ .$$

Define, for all $t \geq 0$, the vector $\pi(t) = (\pi_0(t), \pi_1(t)) = \big(P(X(t) = 0), P(X(t) = 1)\big)$; $\pi(t)$ is the transient distribution of the CTMC.

1. Diagonalize the matrix $\mathbf{Q}$.
2. Compute $e^{\mathbf{Q}t}$ as a function of λ, μ, t.
3. Give an expression for $\pi_0(t)$ and $\pi_1(t)$.
4. Does the limiting distribution π exist? Compute π when it exists. How fast does $\pi(t)$ converge to π?

2.2 Collisions

We consider a wireless channel used for communications. Messages to be sent over this wireless channel are generated according to a Poisson process with rate λ. Each message occupies the wireless channel for a random period of time that is exponentially distributed with parameter μ. If a message is generated while the channel is busy with a previous message, there is a collision and both messages are instantly lost (the wireless channel is then instantly *free*). A lost message is not retransmitted. We define as system state the number of messages in the wireless channel *seen by an arriving message* (i.e. it is the number of messages in the channel just before a message is generated). Let $X(t)$ be this number at time $t > 0$.

1. Say why $\{X(t), t \geq 0\}$ is a homogenous CTMC; write its state space $\mathcal{E}$ and the transition rates.
2. Write the infinitesimal generator matrix and draw the transition diagram.
3. Write the balance equations and find the stationary distribution.
4. What is the message loss probability in this channel?
5. What is the message loss rate?