Empirical Risk Minimization and Zero-Sum Games with Noisy Observations

Samir M. Perlaza
Join work with **Ke Sun** and **Alain Jean-Marie.**
Inria, Centre d'Université Côte d'Azur

Industrial Engineering and Operations Research Department
Indian Institute of Technology
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Samir M. Perlaza, Iñaki Esnaola, and H. Vincent Poor. "**Sensitivity of the Gibbs Algorithm to Data Aggregation in Supervised Machine Learning**". Research Report, INRIA, No. RR-9474, Sophia Antipolis, France, Jun., 2022.

Samir M. Perlaza, Gaetan Bisson, Iñaki Esnaola, Alain Jean-Marie, Stefano Rini, "**Empirical Risk Minimization with Relative Entropy Regularizations**". Research Report, INRIA, No. RR-9454, Sophia Antipolis, France, Feb., 2022.

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The Problem of Supervised Learning

Notation and Definitions

Consider the following **supervised learning** scenario:

- three sets $\mathcal{X}$ (patterns), $\mathcal{Y}$ (labels) and $\mathcal{M} \subset \mathbb{R}^d$ (models), with $d \in \mathbb{N}$.
- a function $f : \mathcal{M} \times \mathcal{X} \to \mathcal{Y}$ (explicit expression **is known**)

Statistical Assumptions

Two random variables X and Y satisfy

$$Y = f(\boldsymbol{\theta}^\star, X), \tag{1}$$

for some specific model $\boldsymbol{\theta}^\star$ (optimal model or hypothesis).

- model $\boldsymbol{\theta}^\star$ is **unknown**
- a dataset $\boldsymbol{z} = \big((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\big) \in (\mathcal{X} \times \mathcal{Y})^n$ **is available**

The Problem of Supervised Learning

Notation and Definitions

Consider the following **supervised learning** scenario:

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Statistical Assumptions

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for some specific model $\boldsymbol{\theta}^\star$ (optimal model or hypothesis).

Objective: Model Selection

Given a dataset $\boldsymbol{z} \in (\mathcal{X} \times \mathcal{Y})^n$, find the model $\boldsymbol{\theta}^\star$ in (1)

- Let $\ell : \mathcal{Y} \times \mathcal{Y} \rightarrow [0, +\infty)$ be a **risk (or loss or cost)** function.

Risk

Given a data point $(x, y) \in \mathcal{X} \times \mathcal{Y}$, the model $\boldsymbol{\theta} \in \mathcal{M}$ induces the **risk** $\ell\left(f(\boldsymbol{\theta}, x), y\right)$.

Empirical Risk

Given a dataset $\boldsymbol{z} = \big((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n) \big) \in (\mathcal{X} \times \mathcal{Y})^n$, the *empirical risk* induced by the model $\boldsymbol{\theta} \in \mathcal{M}$ is

$$\mathsf{L}_{\boldsymbol{z}}(\boldsymbol{\theta}) = \frac{1}{n} \sum_{i=1}^{n} \ell\left(f(\boldsymbol{\theta}, x_i), y_i\right). \tag{2}$$

- Let $\ell : \mathcal{Y} \times \mathcal{Y} \to [0, +\infty)$ be a **risk (or loss or cost)** function.

Risk

Given a data point $(x, y) \in \mathcal{X} \times \mathcal{Y}$, the model $\boldsymbol{\theta} \in \mathcal{M}$ induces the **risk** $\ell\left(f(\boldsymbol{\theta}, x), y\right)$.

Empirical Risk

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Problem Formulation: Empirical Risk Minimization (ERM)

Given the dataset $\boldsymbol{z}$, solve $\min_{\boldsymbol{\theta} \in \mathcal{M}} \mathsf{L}_{\boldsymbol{z}}(\boldsymbol{\theta})$.

The Problem of Supervised Learning: **Empirical Risk Minimization**

Problem Formulation: Empirical Risk Minimization (ERM)

Given a **noisy** dataset $\tilde{\boldsymbol{z}}$, $\min_{\boldsymbol{\theta} \in \mathcal{M}} \mathsf{L}_{\tilde{\boldsymbol{z}}}(\boldsymbol{\theta})$.

Strong connection with other problems:

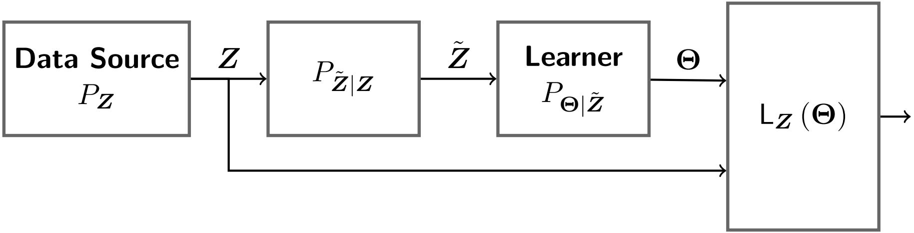
- M-Estimation
- minimum contrast estimation
- sample average approximation

Appears in: machine learning, statistical physics, statistics, operations research, decision making, game theory, information theory, stochastic optimization, ...

The Problem of Supervised Learning: **Empirical Risk Minimization**

Problem Formulation: Empirical Risk Minimization (ERM)

Given a **noisy** dataset $\tilde{z}$, $\min_{\boldsymbol{\theta} \in \mathcal{M}} \mathsf{L}_{\tilde{z}}(\boldsymbol{\theta})$.



The Problem of Supervised Learning: **Empirical Risk Minimization**

Problem Formulation: Empirical Risk Minimization (ERM)

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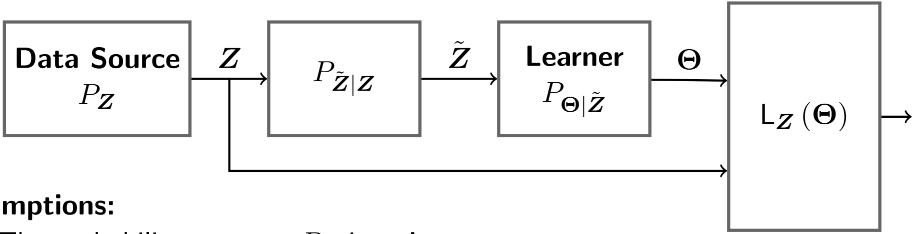
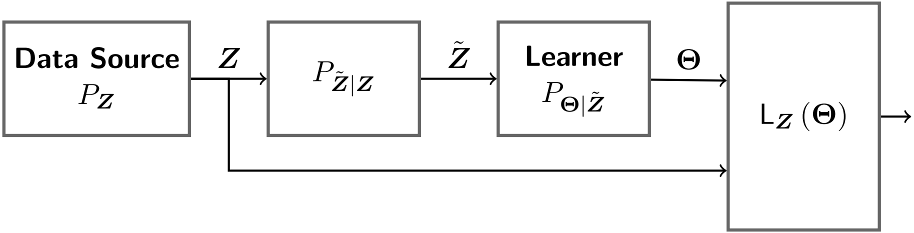


Assumptions:

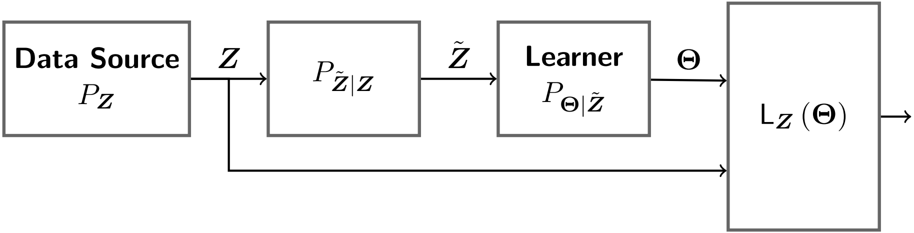
- The probability measure $P_{\boldsymbol{Z}}$ is **unknown**.
- A **channel estimation** can be obtained with **arbitrary precision** by the Learner.
- A **prior** $Q_{\boldsymbol{Z}}$ on the data might be available:
  - Perfect prior: $D(P_{\boldsymbol{Z}} \| Q_{\boldsymbol{Z}}) = 0$ (ideal case)
  - **Mismatch:** $D(P_{\boldsymbol{Z}} \| Q_{\boldsymbol{Z}}) > 0$ (practical case)
- Consider $P_{\boldsymbol{Z}}$ is the measure that **maximizes the expected empirical risk**

The Problem of Supervised Learning: **Empirical Risk Minimization**



Key Question:

How to **model** this interaction ?



Key Question:

How to **model** this interaction ?

is this a **Zero-Sum Game**?

Consider a 2×2 ZSG in normal form, denoted by $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$, with payoff matrix

$$\underline{\boldsymbol{u}} = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}.$$

- **Two actions** for Player 1: $\mathcal{A}_1 \triangleq \{a_1, a_2\}$; and for Player 2: $\mathcal{A}_2 \triangleq \{b_1, b_2\}$.
- $\forall (i,j) \in \{1,2\}$, when Player 1 plays a_i and Player 2 plays b_j, the **payoff** is $u_{i,j}$

Player 1 **chooses lines** and **Player** 2 **chooses columns**

- $\forall k \in \{1,2\}$, a **strategy** for Player k is a probability measure $P_{A_k} \in \triangle\left(\mathcal{A}_k\right)$.
- **Expected Payoff** determined by the function $u : \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right) \rightarrow \mathbb{R}$:

$$u\left(P_{A_1}, P_{A_2}\right) = \sum_{(i,j)\in\{1,2\}^2} P_{A_1}\left(a_i\right) P_{A_2}\left(b_j\right) u_{i,j}, \tag{3}$$

Player 1 **maximizes**, while Player 2 **minimizes** the payoff.

Relevant solution concept when **actions are chosen simultaneously**: Nash Equilibrium.

Nash Equilibrium

The strategies $P^{\star}_{A_1} \in \triangle(\mathcal{A}_1)$ and $P^{\star}_{A_2} \in \triangle(\mathcal{A}_2)$ form an NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ if:

- For all $Q \in \triangle(\mathcal{A}_1)$,

$$u(P^{\star}_{A_1}, P^{\star}_{A_2}) \geqslant u(Q, P^{\star}_{A_2}); \text{ and} \quad (4)$$

- For all $Q \in \triangle(\mathcal{A}_2)$,

$$u(P^{\star}_{A_1}, P^{\star}_{A_2}) \leqslant u(P^{\star}_{A_1}, Q). \quad (5)$$

Lemma (Nash Equilibria)

Let $P^\star_{A_1} \in \triangle(\mathcal{A}_1)$ and $P^\star_{A_2} \in \triangle(\mathcal{A}_2)$ form one NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$. If the matrix $\underline{\boldsymbol{u}}$ satisfies

$$(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) > 0 \text{ and } (u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) > 0, \tag{6}$$

then, the NE is unique and

$$P^\star_{A_1}(a_1) = \frac{u_{2,2} - u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \in (0,1) \tag{7}$$

$$P^\star_{A_2}(a_1) = \frac{u_{2,2} - u_{1,2}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \in (0,1), \text{ and} \tag{8}$$

$$u(P^\star_{A_1}, P^\star_{A_2}) = \frac{u_{1,1}u_{2,2} - u_{1,2}u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}}. \tag{9}$$

If the matrix $\underline{\boldsymbol{u}}$ satisfies

$$(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) \leqslant 0 \text{ or } (u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) \leqslant 0. \tag{10}$$

then, there exist either a unique NE or infinitely many NEs. Moreover,

$$u(P^\star_{A_1}, P^\star_{A_2}) = \min_{i \in \{1,2\}} \max_{j \in \{1,2\}} u_{i,j} = \max_{j \in \{1,2\}} \min_{i \in \{1,2\}} u_{i,j}. \tag{11}$$

Solution concept with **irrevocable and public commitments**: Stackelberg Equilibrium.

- Player 2 (the leader) commits to choose its action by sampling a strategy $P_{A_2} \in \triangle(\mathcal{A}_2)$.
- Player 1 (the follower) observes P_{A_2} **but not the action chosen by Player** 2.
- Player 1 (the follower) chooses its strategy $P_{A_1} \in \triangle(\mathcal{A}_1)$ as a best response to P_{A_2}.

Definition (Stackelberg Equilibrium)

The strategies $P^{\bullet}_{A_1} \in \triangle(\mathcal{A}_1)$ and $P^{\bullet}_{A_2} \in \triangle(\mathcal{A}_2)$ form an SE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$ if:

$$P^{\bullet}_{A_2} \in \arg \min_{Q \in \triangle(\mathcal{A}_2)} \max_{P \in \triangle(\mathcal{A}_1)} u(P, Q); \text{ and} \quad (12)$$

$$P^{\bullet}_{A_1} \in \arg \max_{P \in \triangle(\mathcal{A}_1)} u(P, P^{\bullet}_{A_2}). \quad (13)$$

Solution concept with **irrevocable and public commitments**: Stackelberg Equilibrium.

- Player 2 (the leader) commits to choose its action by sampling a strategy $P_{A_2} \in \triangle(\mathcal{A}_2)$.
- Player 1 (the follower) observes P_{A_2} **but not the action chosen by Player** 2.
- Player 1 (the follower) chooses its strategy $P_{A_1} \in \triangle(\mathcal{A}_1)$ as a best response to P_{A_2}.

Lemma

Let $P^{\star}_{A_1} \in \triangle(\mathcal{A}_1)$ and $P^{\star}_{A_2} \in \triangle(\mathcal{A}_2)$ form one NE of the game $\mathscr{G}(\underline{\boldsymbol{u}})$. Let also $P^{\bullet}_{A_1} \in \triangle(\mathcal{A}_1)$ and $P^{\bullet}_{A_2} \in \triangle(\mathcal{A}_2)$ form one Stackelberg equilibrium of the game $\mathscr{G}(\underline{\boldsymbol{u}})$. Then,

$$u(P^{\bullet}_{A_1}, P^{\bullet}_{A_2}) = u(P^{\star}_{A_1}, P^{\star}_{A_2}). \tag{12}$$

Solution concept with **irrevocable and public commitments in pure strategies**

- Player 2 (the leader) **constrained** to commit to a strategy $P_{A_2} \in \triangle(\mathcal{A}_2)$ such that

$$P_{A_2}(b_1) \in \{0, 1\}. \tag{13}$$

- The follower observes P_{A_2}, which is equivalent to **observing the action** w.p. one.
- The follower chooses its strategy knowing the action played by Player 2.

Lemma

If the matrix $\underline{\boldsymbol{u}}$ satisfies

$$(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) > 0 \text{ and } (u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) > 0, \tag{14}$$

then,

$$\max_{j \in \{1,2\}} \min_{i \in \{1,2\}} u_{i,j} \leqslant \frac{u_{1,1}u_{2,2} - u_{1,2}u_{2,1}}{u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2}} \leqslant \min_{i \in \{1,2\}} \max_{j \in \{1,2\}} u_{i,j}. \tag{15}$$

Zero-Sum Games

Summary

- Two popular solution concepts: Nash and Stackelberg equilibria
- Commitments are immaterial as long as **the follower observes the actions**
- Commitments in pure strategies are **equivalent to perfect observation of the actions**
- No solution concept for **noisy observation of the actions**

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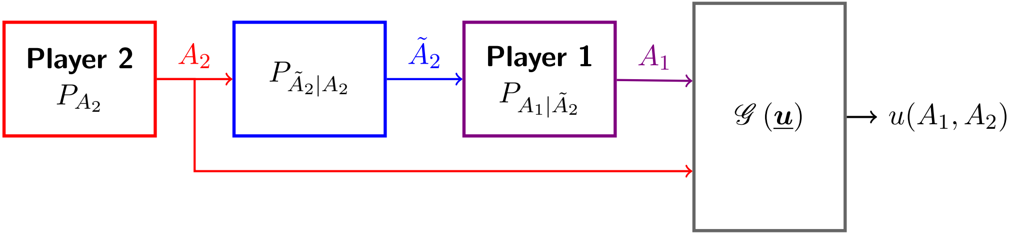
Zero-Sum Games with **Noisy Observations**

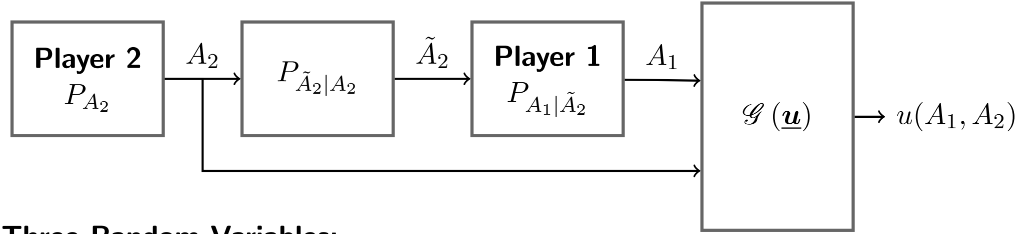
Game Formulation

Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.



- **Assumptions:**
  - Player 2 **publicly and irrevocably** commits to use a strategy;
  - Player 2 plays an action, which is observed by the Player 1 through a **binary channel**;
  - Player 1 chooses its strategy knowing the strategy and the leader's action **up to some noise**.

Consider the 2×2 ZSG in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$.



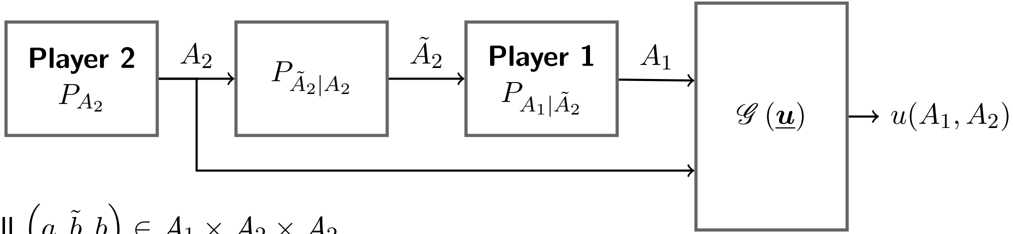
- **Three Random Variables:**
  - Action of Player 1: A_1
  - Action of Player 2: A_2
  - Noisy Observation of the Action of Player 2: $\tilde{A}_2$.
  - Joint Probability Distribution: For all $\left(a, \tilde{b}, b\right) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$,

$$P_{A_1 \tilde{A}_2 A_2}\left(a, \tilde{b}, b\right) = P_{A_2}\left(b\right) P_{\tilde{A}_2|A_2=b}\left(\tilde{b}\right) P_{A_1|\tilde{A}_2=\tilde{b}}\left(a\right). \quad (16)$$

Zero-Sum Games with **Noisy Observations**

Game Formulation

Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.

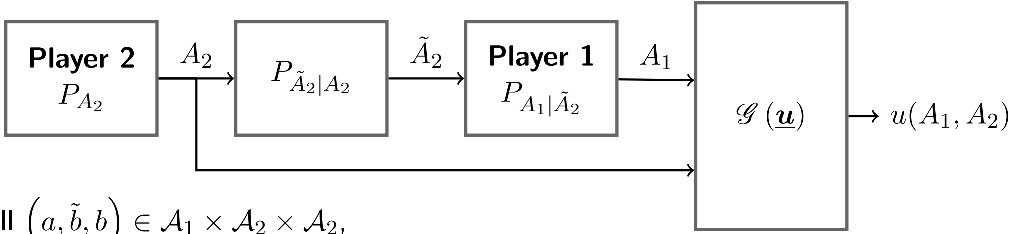


For all $\left(a, \tilde{b}, b\right) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$,

$$P_{A_1 \tilde{A}_2 A_2}\left(a, \tilde{b}, b\right) = P_{A_2}(b) \, P_{\tilde{A}_2|A_2=b}\left(\tilde{b}\right) P_{A_1|\tilde{A}_2=\tilde{b}}(a) \, . \tag{16}$$

- **Strategy of Player** 1**:** $\forall b \in \mathcal{A}_2$: $P_{A_1|\tilde{A}_2=b} \in \triangle(\mathcal{A}_1)$
- **Strategy of Player** 2**:** $P_{A_2} \in \triangle(\mathcal{A}_2)$
- Binary **Channel:** $\forall b \in \mathcal{A}_2$: $P_{\tilde{A}_2|A_2=b} \in \triangle(\mathcal{A}_2)$

Consider the 2×2 ZSG in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$.



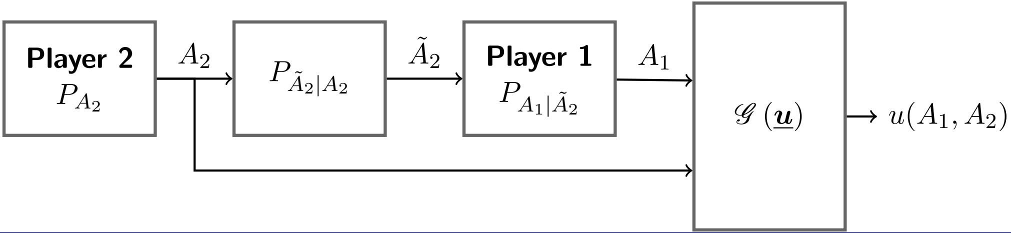
For all $\left(a, \tilde{b}, b\right) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$,

$$P_{A_1 \tilde{A}_2 A_2}\left(a, \tilde{b}, b\right) = P_{A_2}\left(b\right) P_{\tilde{A}_2|A_2=b}\left(\tilde{b}\right) P_{A_1|\tilde{A}_2=\tilde{b}}\left(a\right). \tag{16}$$

The **expected payoff** is determined by $v : \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right) \to \mathbb{R}$,

$$v\left(P_{A_1|\tilde{A}_2=b_1}, P_{A_1|\tilde{A}_2=b_2}, P_{A_2}\right) = \sum_{(i,j)\in\{1,2\}^2} u_{i,j} P_{A_1 A_2}\left(a_i, a_j\right). \tag{17}$$

Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.



Definition (Equilibrium)

The tuple $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_2)$ forms an equilibrium if

$$P^{\dagger}_{A_2} \in \arg \min_{P \in \triangle(\mathcal{A}_2)} \left(\max_{(Q_1,Q_2) \in \triangle(\mathcal{A}_1)^2} v(Q_1, Q_2, P) \right) \text{ and } \quad (16)$$

$$\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}\right) \in \max_{(Q_1,Q_2) \in \triangle(\mathcal{A}_1)^2} v(Q_1, Q_2, P). \quad (17)$$

- Alternative representation of the **channel**: $\underline{\boldsymbol{w}} \triangleq \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_1) & P_{\tilde{A}_2|A_2=a_2}(a_1) \\ P_{\tilde{A}_2|A_2=a_1}(a_2) & P_{\tilde{A}_2|A_2=a_2}(a_2) \end{pmatrix}$.
- For all $i \in \{1,2\}$, let $\underline{\boldsymbol{u}}^{(i)}$ be a 2×2 matrix $\underline{\boldsymbol{u}}^{(i)} \triangleq \underline{\boldsymbol{u}} \begin{pmatrix} P_{\tilde{A}_2|A_2=a_1}(a_i) & 0 \\ 0 & P_{\tilde{A}_2|A_2=a_2}(a_i) \end{pmatrix}$.
- Let the function $\hat{v} : \triangle(\mathcal{A}_2) \rightarrow \mathbb{R}$ be

$$\hat{v}(P) = \max_{(Q_1,Q_2)\in \mathrm{BR}_1(P)} v(Q_1, Q_2, P). \tag{18}$$

- Let the function $\hat{u} : \triangle(\mathcal{A}_2) \rightarrow \mathbb{R}$ be

$$\hat{u}(P) = \max_{Q\in\triangle(\mathcal{A}_1)} u(Q, P). \tag{19}$$

The set of best responses of Player 1 is determined by $\mathrm{BR}_1 : \triangle\left(\mathcal{A}_2\right) \rightarrow 2^{\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)}$,

$$\mathrm{BR}_1\left(P_{A_2}\right)=\arg \max_{(Q_1,Q_2)\in\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)} v(Q_1, Q_2, P_{A_2}). \tag{20}$$

Lemma

For all $(Q_1, Q_2) \in \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_1\right)$ *and for all* $P \in \triangle\left(\mathcal{A}_2\right)$,

$$v\left(Q_1, Q_2, P\right)=\begin{pmatrix} Q_1\left(a_1\right) \\ Q_1\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(1)} \begin{pmatrix} P\left(a_1\right) \\ P\left(a_2\right) \end{pmatrix} + \begin{pmatrix} Q_2\left(a_1\right) \\ Q_2\left(a_2\right) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(2)} \begin{pmatrix} P\left(a_1\right) \\ P\left(a_2\right) \end{pmatrix}. \tag{21}$$

The set of best responses of Player 1 is determined by $\mathrm{BR}_1 : \triangle(\mathcal{A}_2) \rightarrow 2^{\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)}$,

$$\mathrm{BR}_1(P_{A_2}) = \arg\max_{(Q_1,Q_2)\in\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)} v(Q_1, Q_2, P_{A_2}). \quad (20)$$

Corollary

For all $Q \in \triangle(\mathcal{A}_1)$ and for all $P \in \triangle(\mathcal{A}_2)$,

$$v(Q, Q, P) = u(Q, P). \quad (21)$$

Zero-Sum Games with **Noisy Observations**

Best Response of the Follower

The set of best responses of Player 1 is determined by $\mathrm{BR}_1 : \triangle(\mathcal{A}_2) \rightarrow 2^{\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)}$,

$$\mathrm{BR}_1(P_{A_2}) = \arg \max_{(Q_1,Q_2)\in\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)} v(Q_1, Q_2, P_{A_2}). \quad (20)$$

Corollary

For all $P \in \triangle(\mathcal{A}_2)$,

$$\mathrm{BR}_1(P) = \mathrm{BR}_{1,1}(P) \times \mathrm{BR}_{1,2}(P), \quad (21)$$

where for all $i \in \{1, 2\}$, *the correspondence* $\mathrm{BR}_{1,i} : \triangle(\mathcal{A}_2) \rightarrow 2^{\triangle(\mathcal{A}_1)}$ *is such that*

$$\mathrm{BR}_{1,i}(P) = \arg \max_{Q\in\triangle(\mathcal{A}_1)} \begin{pmatrix} Q(a_1) \\ Q(a_2) \end{pmatrix}^T \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P(a_1) \\ P(a_2) \end{pmatrix}. \quad (22)$$

The set of best responses of Player 1 is determined by $\mathrm{BR}_1 : \triangle\left(\mathcal{A}_2\right) \rightarrow 2^{\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)}$,

$$\mathrm{BR}_1\left(P_{A_2}\right)=\mathrm{BR}_{1,1}\left(P\right) \times \mathrm{BR}_{1,2}\left(P\right). \tag{20}$$

Lemma

For all $P \in \triangle\left(\mathcal{A}_2\right)$ *and for all* $i \in \{1, 2\}$,

$$\mathrm{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } s_i > 0, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } s_i < 0, \\ \triangle\left(\mathcal{A}_1\right), & \text{if } s_i = 0, \end{cases} \tag{21}$$

where $s_i \in \mathbb{R}$ *is given by*

$$s_i \triangleq \left(u_{1,1} - u_{2,1}\right) P\left(a_1\right) P_{\tilde{A}_2|A_2=a_1}\left(a_i\right) + \left(u_{1,2} - u_{2,2}\right) P\left(a_2\right) P_{\tilde{A}_2|A_2=a_2}\left(a_i\right). \tag{22}$$

The set of best responses of Player 1 is determined by $\mathrm{BR}_1 : \triangle\left(\mathcal{A}_2\right) \rightarrow 2^{\triangle(\mathcal{A}_1)\times\triangle(\mathcal{A}_1)}$,

$$\mathrm{BR}_1\left(P_{A_2}\right)=\mathrm{BR}_{1,1}\left(P\right)\times\mathrm{BR}_{1,2}\left(P\right). \tag{20}$$

Lemma

For all $P \in \triangle\left(\mathcal{A}_2\right)$ and for all $i \in \{1,2\}$,

$$\mathrm{BR}_{1,i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } s_i > 0, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } s_i < 0, \\ \triangle\left(\mathcal{A}_1\right), & \text{if } s_i = 0, \end{cases} \tag{21}$$

where $s_i \in \mathbb{R}$ is given by

$$s_i \triangleq \left(u_{1,1} - u_{2,1}\right) P\left(a_1\right) P_{\tilde{A}_2|A_2=a_1}\left(a_i\right) + \left(u_{1,2} - u_{2,2}\right) P\left(a_2\right) P_{\tilde{A}_2|A_2=a_2}\left(a_i\right) \tag{22}$$

- Consider the ZSGs in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}^{(1)}\right)$ and $\mathscr{G}\left(\underline{\boldsymbol{u}}^{(2)}\right)$ (a.k.a **constituent games**).
- For all $i \in \{1, 2\}$, let $P^{(i)} \in \mathbb{R}$ satisfy

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P^{(i)} \\ 1 - P^{(i)} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix} P^{(i)} \\ 1 - P^{(i)} \end{pmatrix}. \tag{23}$$

Zero-Sum Games with **Noisy Observations**

Constituent Games

- For all $i \in \{1,2\}$, let $P^{(i)} \in \mathbb{R}$ satisfy

$$\begin{pmatrix}1\\0\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix}. \quad (23)$$

Lemma

If $P^{\star}_{A_1}(a_1) \in (0,1)$ and $P^{\star}_{A_2}(a_1) \in (0,1)$ form the unique NE of the game $\mathscr{G}\left(\underline{\mathbf{u}}\right)$, then

(a) *For all $i \in \{1,2\}$,*

$$P^{(i)} = \frac{P^{\star}_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i)}{P^{\star}_{A_2}(a_2) P_{\tilde{A}_2|A_2=a_1}(a_i) + P^{\star}_{A_2}(a_1) P_{\tilde{A}_2|A_2=a_2}(a_i)} \in [0,1]\,; \text{and} \quad (24)$$

(b) $P^{(i)} \in \{0,1\}$ *if and only if* $|\det \underline{\boldsymbol{w}}| = 1$

- For all $i \in \{1,2\}$, let $P^{(i)} \in \mathbb{R}$ satisfy

$$\begin{pmatrix}1\\0\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix}. \quad (23)$$

Lemma

If $P^{\star}_{A_1}(a_1) \in (0,1)$ *and* $P^{\star}_{A_2}(a_1) \in (0,1)$ *form the unique NE of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$, *then*

$$0 \leqslant \min\left\{P^{(1)}, P^{(2)}\right\} \leqslant P^{\star}_{A_2}\left(a_1\right) \leqslant \max\left\{P^{(1)}, P^{(2)}\right\} \leqslant 1. \quad (24)$$

The equality $P^{(1)} = P^{\star}_{A_2}\left(a_1\right) = P^{(2)}$ *holds if and only if* $\det \underline{\boldsymbol{w}} = 0$.

- For all $i \in \{1,2\}$, let $P^{(i)} \in \mathbb{R}$ satisfy

$$\begin{pmatrix}1\\0\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix}. \quad (23)$$

Lemma

If $P^{\star}_{A_1}(a_1) \in (0,1)$ *and* $P^{\star}_{A_2}(a_1) \in (0,1)$ *form the unique NE of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$*, then*

$$\min\left\{P^{(1)}, P^{(2)}\right\} = \begin{cases} P^{(1)} & \text{if} \quad \det \underline{\boldsymbol{w}} > 0 \\ P^{(2)} & \text{if} \quad \det \underline{\boldsymbol{w}} < 0 \\ P^{(1)} = P^{(2)} = P^{\star}_{A_2}(a_1) & \text{if} \quad \det \underline{\boldsymbol{w}} = 0, \end{cases} \quad (24)$$

Zero-Sum Games with **Noisy Observations**

Constituent Games

- For all $i \in \{1,2\}$, let $P^{(i)} \in \mathbb{R}$ satisfy

$$\begin{pmatrix}1\\0\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(i)} \begin{pmatrix}P^{(i)}\\1-P^{(i)}\end{pmatrix}. \quad (23)$$

Lemma

Assume that $P^{\star}_{A_1}(a_1) \in (0,1)$ *and* $P^{\star}_{A_2}(a_1) \in (0,1)$ *form the unique NE of the game* $\mathscr{G}(\underline{\boldsymbol{u}})$. *If* $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$, *then it holds that for all* $i \in \{1,2\}$ *and for all* $P \in \triangle(\mathcal{A}_2)$,

$$\mathrm{BR}_{1i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0,1]\}, & \text{if } P(a_1) = P^{(i)}. \end{cases} \quad (24)$$

If $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} \leq 0$, *then it holds that for all* $i \in \{1,2\}$ *and for all* $P \in \triangle(\mathcal{A}_2)$,

$$\mathrm{BR}_{1i}(P) = \begin{cases} \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 1\}, & \text{if } P(a_1) < P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = 0\}, & \text{if } P(a_1) > P^{(i)}, \\ \{Q \in \Delta(\mathcal{A}_1) : Q(a_1) = \beta, \beta \in [0,1]\}, & \text{if } P(a_1) = P^{(i)}. \end{cases} \quad (25)$$

- Let the function $\hat{v} : \triangle(\mathcal{A}_2) \to \mathbb{R}$ be

$$\hat{v}(P) = \max_{(Q_1,Q_2)\in \mathrm{BR}_1(P)} v(Q_1, Q_2, P). \tag{26}$$

- Let the function $\hat{u} : \triangle(\mathcal{A}_2) \to \mathbb{R}$ be

$$\hat{u}(P) = \max_{Q\in\triangle(\mathcal{A}_1)} u(Q, P). \tag{27}$$

- Let the function $\hat{v} : \triangle(\mathcal{A}_2) \rightarrow \mathbb{R}$ be

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- Let the function $\hat{u} : \triangle(\mathcal{A}_2) \rightarrow \mathbb{R}$ be

$$\hat{u}(P) = \max_{Q\in\triangle(\mathcal{A}_1)} u(Q, P). \tag{27}$$

Lemma

Let $P^\star_{A_1} \in \triangle(\mathcal{A}_1)$ and $P^\star_{A_2} \in \triangle(\mathcal{A}_2)$ form an NE in the game $\mathscr{G}(\underline{\boldsymbol{u}})$. For all $P \in \triangle(\mathcal{A}_2)$,

$$u(P^\star_{A_1}, P^\star_{A_2}) \leqslant \hat{u}(P) \leqslant \hat{v}(P) \leqslant \sum_{k\in\{1,2\}} P(a_k)\left(\max_{i\in\{1,2\}} u_{i,k}\right). \tag{28}$$

- Let the function $\hat{v} : \triangle(\mathcal{A}_2) \rightarrow \mathbb{R}$ be

$$\hat{v}(P) = \max_{(Q_1,Q_2)\in \mathrm{BR}_1(P)} v(Q_1, Q_2, P). \tag{26}$$

Lemma

Assume that $P^{\star}_{A_1}(a_1) \in (0,1)$ *and* $P^{\star}_{A_2}(a_1) \in (0,1)$ *form the unique NE of the game* $\mathscr{G}(\underline{\boldsymbol{u}})$ *and* $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$.

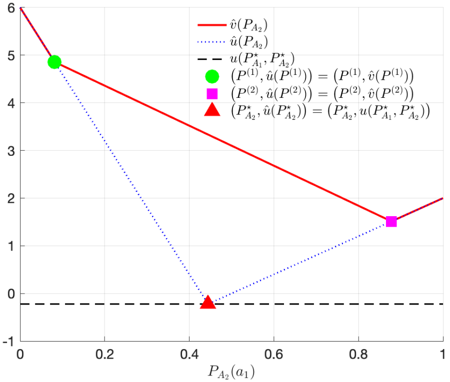
- *For all* $P \in \triangle(\mathcal{A}_2)$ *such that* $P(a_1) \geqslant \max\{P^{(1)}, P^{(2)}\}$, $\hat{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2)$.
- *If* $\det \underline{\boldsymbol{w}} > 0$, *then for all* $P \in \triangle(\mathcal{A}_2)$ *such that* $P^{(1)} < P(a_1) < P^{(2)}$,
$$\hat{v}(P) = \left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_1) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_2)\right) P(a_1) + \left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_1) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_2)\right) P(a_2).$$
- *If* $\det \underline{\boldsymbol{w}} \leqslant 0$, *then for all* $P \in \triangle(\mathcal{A}_2)$ *such that* $P^{(2)} < P(a_1) < P^{(1)}$,
$$\hat{v}(P) = \left(u_{1,1}P_{\tilde{A}_2|A_2=a_1}(a_2) + u_{2,1}P_{\tilde{A}_2|A_2=a_1}(a_1)\right) P(a_1) + \left(u_{1,2}P_{\tilde{A}_2|A_2=a_2}(a_2) + u_{2,2}P_{\tilde{A}_2|A_2=a_2}(a_1)\right) P(a_2).$$
- *For all* $P \in \triangle(\mathcal{A}_2)$ *such that* $P(a_1) \leqslant \min\{P^{(1)}, P^{(2)}\}$, $\hat{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2)$.

- Let the function $\hat{v} : \triangle(\mathcal{A}_2) \rightarrow \mathbb{R}$ be

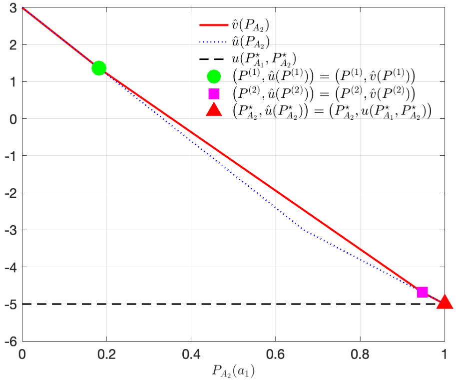
$$\hat{v}(P) = \max_{(Q_1, Q_2) \in \mathrm{BR}_1(P)} v(Q_1, Q_2, P). \quad (26)$$

Corollary

The function $f : [0,1] \rightarrow \mathbb{R}$ such that $f(x) = \hat{v}(P)$, with $P(a_1) = x$, is piece-wise linear.



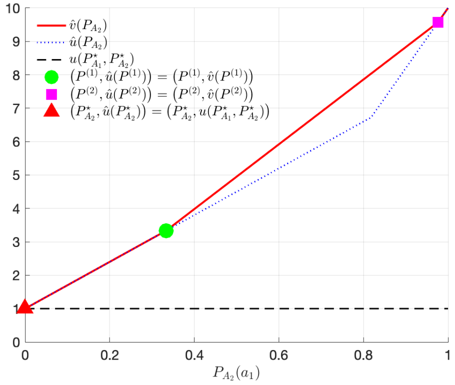
(a) Payoff matrix $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$ and channel matrix $\underline{\boldsymbol{w}} = (0.9, 0.1; 0.1, 0.9)$.



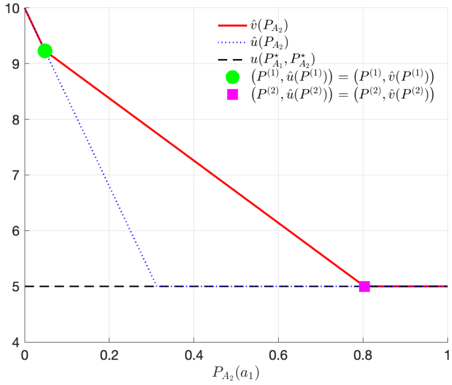
(b) Payoff matrix $\underline{\boldsymbol{u}} = (-5, 1; -6, 3)$ and channel matrix $\underline{\boldsymbol{w}} = (0.9, 0.1; 0.1, 0.9)$.

Zero-Sum Games with **Noisy Observations**

Best Commitments



(a) Payoff matrix $\underline{\boldsymbol{u}} = (2, 9; -9, 5)$ and channel matrix $\underline{\boldsymbol{w}} = (0.9, 0.1; 0.1, 0.9)$.



(b) Payoff matrix $\underline{\boldsymbol{u}} = (5, 5; -6, 10)$ and channel matrix $\underline{\boldsymbol{w}} = (0.9, 0.1; 0.1, 0.9)$.

Definition (Equilibrium)

The tuple $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right)$ forms an equilibrium if

$$P^{\dagger}_{A_2} \in \arg \min_{P \in \triangle(\mathcal{A}_2)} \left(\max_{(Q_1, Q_2) \in \triangle(\mathcal{A}_1)^2} v\left(Q_1, Q_2, P\right) \right) \text{ and } \quad (27)$$

$$\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}\right) \in \max_{(Q_1, Q_2) \in \triangle(\mathcal{A}_1)^2} v\left(Q_1, Q_2, P\right). \quad (28)$$

Definition (Equilibrium)

The tuple $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_2)$ forms an equilibrium if

$$P^{\dagger}_{A_2} \in \arg \min_{P \in \triangle(\mathcal{A}_2)} \left(\max_{(Q_1,Q_2) \in \triangle(\mathcal{A}_1)^2} v(Q_1, Q_2, P) \right) \text{ and } \quad (27)$$

$$\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}\right) \in \max_{(Q_1,Q_2) \in \triangle(\mathcal{A}_1)^2} v(Q_1, Q_2, P). \quad (28)$$

Theorem (Existence)

The game $\mathscr{G}(\underline{\mathbf{u}}, \underline{\mathbf{w}})$ always possesses an equilibrium.

Theorem (Equilibrium Payoff)

Let the tuple $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_2)$ *be an equilibrium of the game* $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}})$. *Let* $P^{\star}_{A_1}(a_1)$ *and* $P^{\star}_{A_2}(a_1)$ *be an NE of the game* $\mathscr{G}(\underline{\boldsymbol{u}})$.

- *If* $P^{\star}_{A_1}(a_1) \in (0,1)$, $P^{\star}_{A_2}(a_1) \in (0,1)$, *and* $(P^{\star}_{A_1}, P^{\star}_{A_2})$ *forms the unique NE, then*
$$\hat{v}\left(P^{\dagger}_{A_2}\right) = \min\{\hat{v}(P_1), \hat{v}(P_2)\}, \tag{27}$$
where, for all $i \in \{1,2\}$, $P_i(a_1) = P^{(i)}$.
- *Otherwise,*
$$\hat{v}\left(P^{\dagger}_{A_2}\right) = \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}. \tag{28}$$

Lemma

Let $\mathcal{S} \subseteq \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_2)$ *be the set of equilibria of the game* $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}})$ *and let* $P^{\star}_{A_1} \in \triangle(\mathcal{A}_1)$ *and* $P^{\star}_{A_2} \in \triangle(\mathcal{A}_2)$ *be an NE of the game* $\mathscr{G}(\underline{\boldsymbol{u}})$. *If* $P^{\star}_{A_1}(a_1) \in (0,1)$, $P^{\star}_{A_2}(a_1) \in (0,1)$, *and* $(P^{\star}_{A_1}, P^{\star}_{A_2})$ *forms the unique NE, then, there exists a tuple* $(Q_1, Q_2, P) \in \mathcal{S}$ *such that* $P(a_1) \in \{P^{(1)}, P^{(2)}\}$. *Furthermore, if* $P(a_1) = P^{(i)}$, *with* $i \in \{1,2\}$, *then,*

$$P_{A_2|\tilde{A}_2=a_i}(a_1) = P^{\star}_{A_2}(a_1). \tag{27}$$

Lemma

Let $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right)$ *be an equilibrium of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}\right)$*. Let also the tuple* $\left(P^{\star}_{A_1}, P^{\star}_{A_2}\right) \in \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right)$ *be an NE of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$*. Then,*

$$v\left(P^{\dagger}_{A_1|\tilde{A}_2}, P^{\dagger}_{A_2}\right) = u(P^{\star}_{A_1}, P^{\star}_{A_2}), \tag{28}$$

if and only if,

(a) $\left(u_{1,1} - u_{1,2}\right)\left(u_{2,2} - u_{2,1}\right) \leqslant 0$ *or* $\left(u_{1,1} - u_{2,1}\right)\left(u_{2,2} - u_{1,2}\right) \leqslant 0$*; or*

(b) $\left(u_{1,1} - u_{1,2}\right)\left(u_{2,2} - u_{21,}\right) > 0$ *and* $\left(u_{1,1} - u_{2,1}\right)\left(u_{2,2} - u_{1,2}\right) > 0$ *and* $\det \underline{\boldsymbol{w}} = 0$.

Zero-Sum Games with **Noisy Observations**

Relevance of Noisy Observations

Lemma

Let $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right)$ *form an equilibrium of the game* $\mathscr{G}\left(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}\right)$*. If* $|\det \underline{\boldsymbol{w}}| = 1$*, then*

$$\hat{v}\left(P^{\dagger}_{A_2}\right) = \min\{\max\{u_{1,1}, u_{2,1}\}, \max\{u_{1,2}, u_{2,2}\}\}. \tag{28}$$

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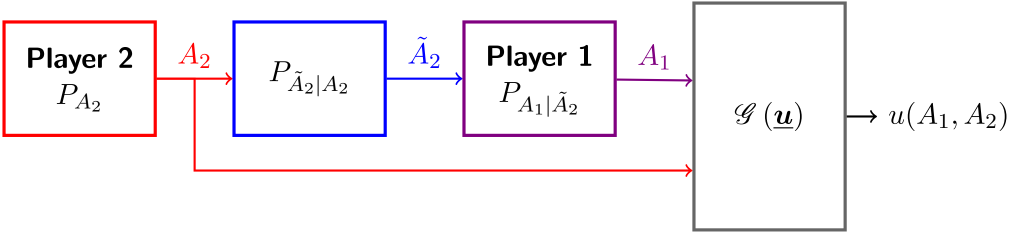
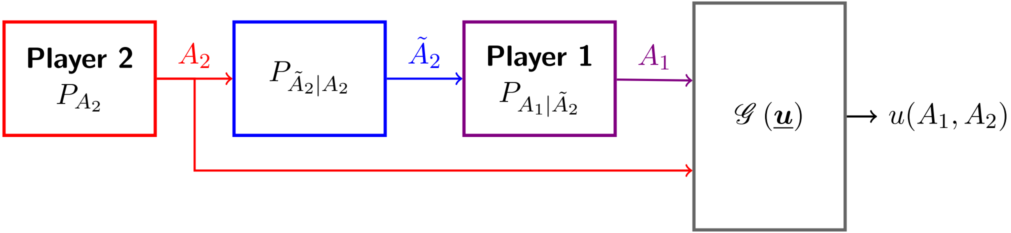
Consider the 2×2 ZSG in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$.



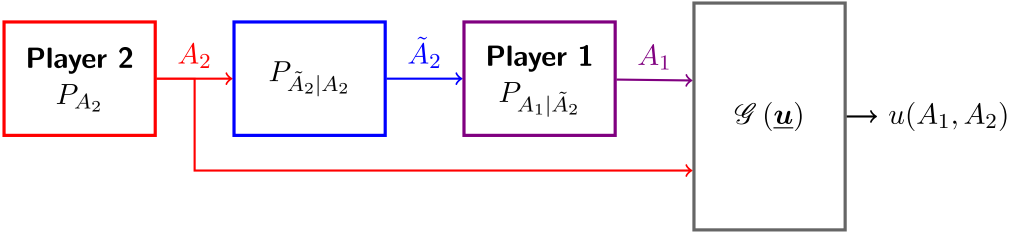
- **Assumptions:**
  - Player 2 **publicly and irrevocably** commits to use a strategy and chooses an action;

Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.



- **Assumptions:**
  - Player 2 **publicly and irrevocably** commits to use a strategy and chooses an action;
  - Player 1 observes the commitment subject to a **distorsion**;

Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.

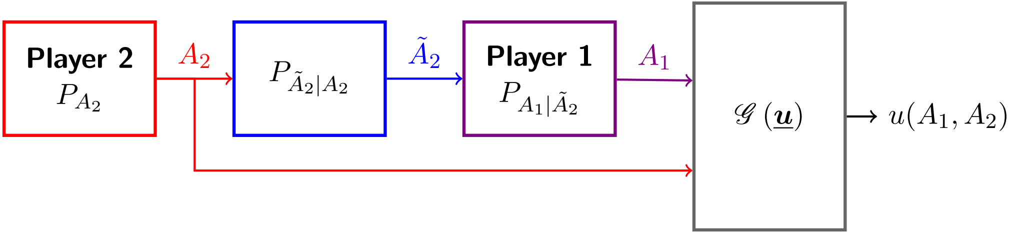


- **Assumptions:**
  - Player 2 **publicly and irrevocably** commits to use a strategy and chooses an action;
  - Player 1 observes the commitment subject to a **distorsion**;
  - Player 1 observes the action through a **binary channel** ; and

Zero-Sum Games with **Noisy Observations**

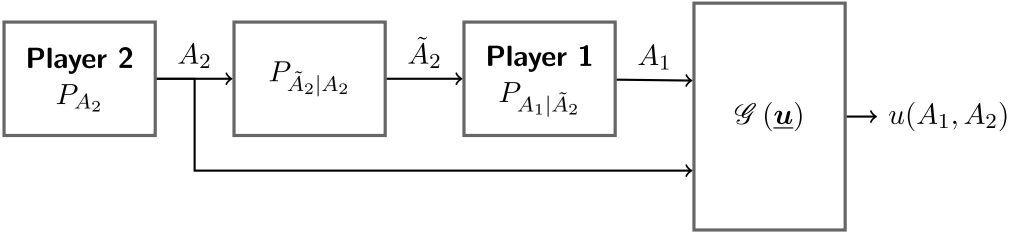
Commitment **Mismatch** and **Noisy Observation** of the Actions

Consider the 2×2 ZSG in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$.



- **Assumptions:**
  - Player 2 **publicly and irrevocably** commits to use a strategy and chooses an action;
  - Player 1 observes the commitment subject to a **distorsion**;
  - Player 1 observes the action through a **binary channel** ; and
  - Player 1 chooses its strategy.

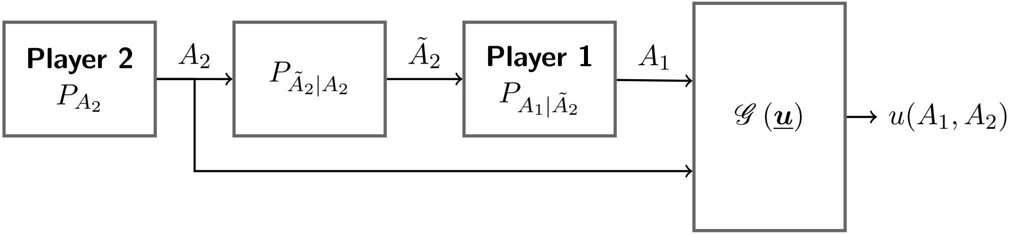
Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.



- **Distorsion of the Commitment**
  - Let $\underline{\boldsymbol{t}}$ be a given 2×2 **nonsingular stochastic matrix**.
  - Let $P_{A_2} \in \triangle(\mathcal{A}_2)$ be the commitment announced by Player 2.
  - The commitment **observed** by Player 1 is

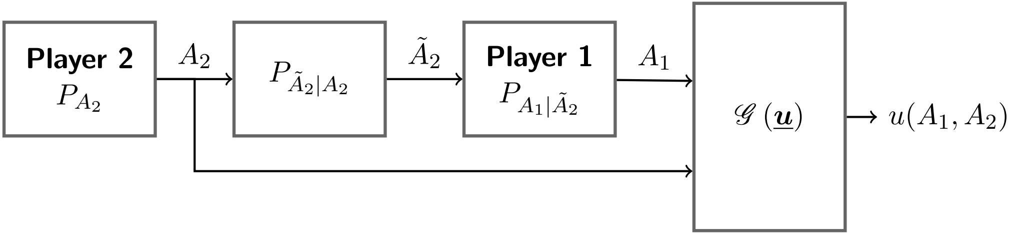
$$\begin{pmatrix} \tilde{P}_{A_2}(a_1) \\ \tilde{P}_{A_2}(a_2) \end{pmatrix} = \underline{\boldsymbol{t}} \begin{pmatrix} P_{A_2}(a_1) \\ P_{A_2}(a_2) \end{pmatrix}. \quad (29)$$

Consider the 2×2 ZSG in normal form $\mathscr{G}(\underline{\boldsymbol{u}})$.



- The follower assumes that $\tilde{P}_{A_2}$ is the actual commitment of leader; and
- The leader is aware of assumption of the follower.

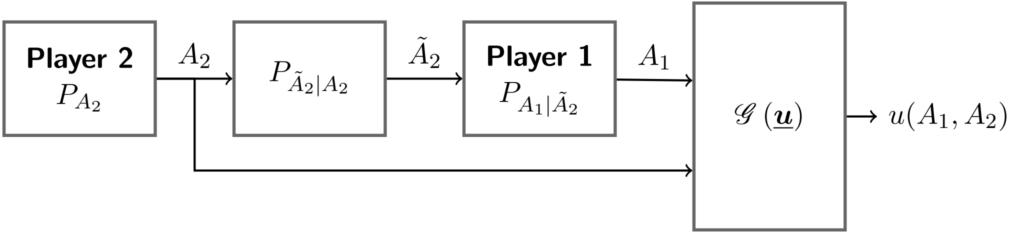
Consider the 2×2 ZSG in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$.



Let the correspondence $\tilde{v} : \triangle\left(\mathcal{A}_2\right) \to \mathbb{R}$ be such that

$$\tilde{v}\left(P_{A_2}\right) = \max_{(Q_1,Q_2)\in \mathrm{BR}_1\left(\tilde{P}_{A_2}\right)} v\left(Q_1, Q_2, P_{A_2}\right). \tag{29}$$

Consider the 2×2 ZSG in normal form $\mathscr{G}\left(\underline{\boldsymbol{u}}\right)$.



Definition (Equilibrium)

The tuple $\left(P_{A_1|\tilde{A}_2=a_1}, P_{A_1|\tilde{A}_2=a_2}, P_{A_2}\right) \in \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_1\right) \times \triangle\left(\mathcal{A}_2\right)$ is an equilibrium of the game $\mathscr{G}\left(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}}\right)$ if

$$P_{A_2} \in \arg \min_{P \in \triangle(\mathcal{A}_2)} \tilde{v}\left(P\right) \text{ and } \tag{29}$$

$$\left(P_{A_1|\tilde{A}_2=a_1}, P_{A_1|\tilde{A}_2=a_2}\right) \in \mathrm{BR}_1\left(\tilde{P}_{A_2}\right). \tag{30}$$

Zero-Sum Games with **Noisy Observations**

A Fundamental Observation

Observation:

- **Assume** that $(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) > 0$ and $(u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) > 0$
- **If** Player 2 commits to $P_{A_2} \in \triangle(\mathcal{A}_2)$ such that $P_{A_2}(a_1) = \tilde{P}^{(i)}$, with

$$\begin{pmatrix} \tilde{P}^{(i)} \\ 1 - \tilde{P}^{(i)} \end{pmatrix} = \underline{\boldsymbol{t}}^{-1} \begin{pmatrix} P^{(i)} \\ 1 - P^{(i)} \end{pmatrix}, \qquad (31)$$

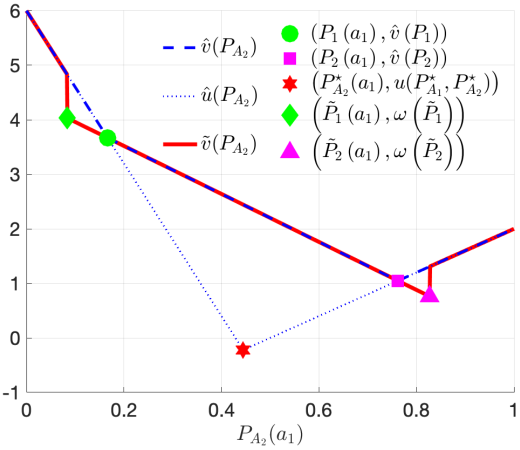
  for some $i \in \{1, 2\}$.
- **Then**, Player 1 observes the commitment $\tilde{P}_{A_2} \in \triangle(\mathcal{A}_2)$ such that $\tilde{P}_{A_2}(a_1) = P^{(i)}$
- **And**
  - $\mathrm{BR}_{1,i}\left(\tilde{P}_{A_2}\right) = \triangle(\mathcal{A}_1)$
  - Given $(Q_1, Q_2) \in \mathrm{BR}_1\left(\tilde{P}_{A_2}\right)$ and $(P_1, P_2) \in \mathrm{BR}_1\left(\tilde{P}_{A_2}\right)$,

$$v(Q_1, Q_2, P_{A_2}) \neq v(P_1, P_2, P_{A_2}). \qquad (32)$$

Lemma

Assume that $(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) > 0$, $(u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) > 0$, *and* $u_{1,1} - u_{1,2} - u_{2,1} + u_{2,2} > 0$. *Assume also that* $\det \underline{\boldsymbol{w}} > 0$ *and* $\det \underline{\boldsymbol{t}} > 0$. *For all* $P \in \triangle(\mathcal{A}_2)$,

- *If* $P(a_1) > \tilde{P}^{(2)}$, *then* $\tilde{v}(P) = u_{1,1}P(a_1) + u_{1,2}P(a_2)$.
- *If* $P(a_1) = \tilde{P}^{(2)}$, *then* $\tilde{v}(P) = \left\{ (u_{1,1}P(a_1) + u_{1,2}P(a_2))\beta + \left(\begin{pmatrix}1\\1\end{pmatrix}^{T} (\underline{\boldsymbol{u}} \circ \underline{\boldsymbol{w}}) \begin{pmatrix}P(a_1)\\P(a_2)\end{pmatrix}\right)(1-\beta) : \beta \in [0,1] \right\}$.
- *If* $\tilde{P}^{(1)} < P(a_1) < \tilde{P}^{(2)}$, *then it follows that* $\tilde{v}(P) = \left(\begin{pmatrix}1\\1\end{pmatrix}^{T} (\underline{\boldsymbol{u}} \circ \underline{\boldsymbol{w}}) \begin{pmatrix}P(a_1)\\P(a_2)\end{pmatrix}\right)$.
- *If* $P(a_1) = \tilde{P}^{(1)}$, *then* $\tilde{v}(P) = \left\{ (u_{2,1}P(a_1) + u_{2,2}P(a_2))(1-\beta) + \left(\begin{pmatrix}1\\1\end{pmatrix}^{T} (\underline{\boldsymbol{u}} \circ \underline{\boldsymbol{w}}) \begin{pmatrix}P(a_1)\\P(a_2)\end{pmatrix}\right)\beta : \beta \in [0,1] \right.$
- *If* $P(a_1) < \tilde{P}^{(1)}$, *then it follows that* $\tilde{v}(P) = u_{2,1}P(a_1) + u_{2,2}P(a_2)$.



$\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$.

Lemma

Consider the following assumptions:

(a) $(u_{1,1} - u_{1,2})(u_{2,2} - u_{2,1}) > 0$, *and* $(u_{1,1} - u_{2,1})(u_{2,2} - u_{1,2}) > 0$;

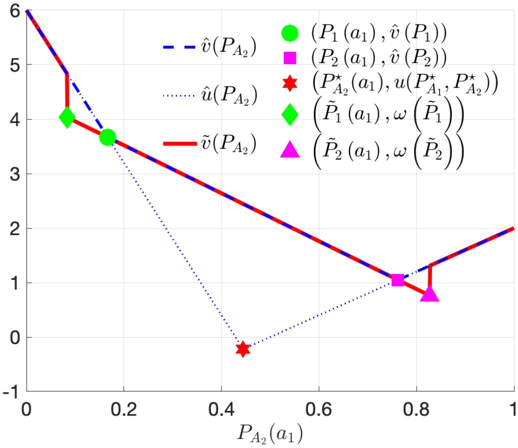
(b) *For all* $i \in \{1, 2\}$, *the probability measures* $Q_i \in \triangle(\mathcal{A}_2)$ *such that* $Q_i(a_1) = P^{(i)}$, *satisfy* $\hat{u}(Q_1) \neq \hat{u}(Q_2)$; *and*

(c) *The tuple* $\left(P^{\dagger}_{A_1|\tilde{A}_2=a_1}, P^{\dagger}_{A_1|\tilde{A}_2=a_2}, P^{\dagger}_{A_2}\right) \in \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_1) \times \triangle(\mathcal{A}_2)$ *is an equilibrium of the game* $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}})$.

If $\det \underline{\boldsymbol{t}} \notin \{0, 1\}$, *then, there exists a strategy* $P \in \triangle(\mathcal{A}_2)$ *such that*

$$\tilde{v}(P) < \hat{v}(P^{\dagger}_{A_2}). \qquad (33)$$

The game $\mathscr{G}\left(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}}\right)$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.



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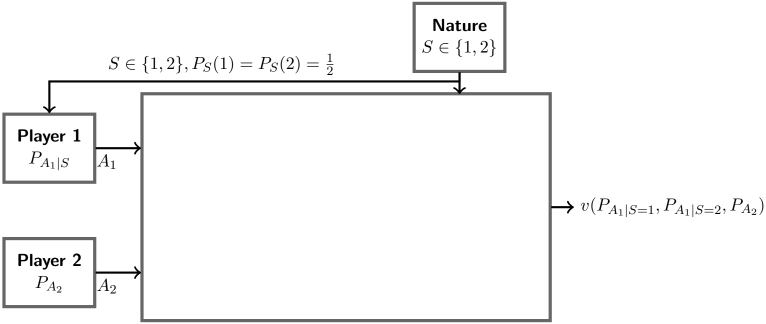
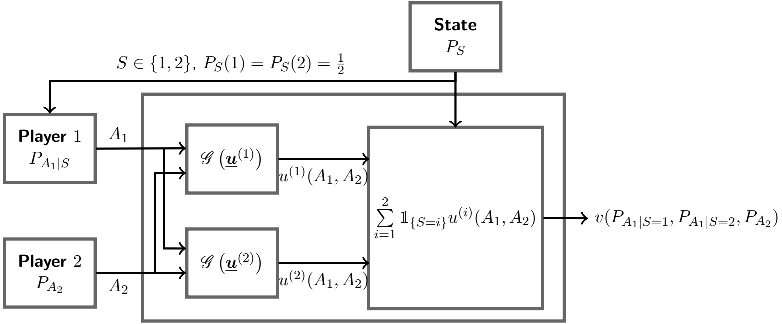
Connections with Existing Results

Aumann's **Games with Incomplete Information**



Consider a ZSG with **incomplete information:**

- Both players know the probability distribution P_S;
- Player 1 **observes** the realization of the state S; and
- Player 2 **ignores** the realization of the state S.



Expected Payoff:

$$v\left(P_{A_1|S=1}, P_{A_1|S=2}, P_{A_2}\right)$$
$$= \frac{1}{2}\begin{pmatrix} P_{A_1|S=1}(a_1) \\ P_{A_1|S=1}(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(1)} \begin{pmatrix} P_{A_2}(a_1) \\ P_{A_2}(a_2) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} P_{A_1|S=2}(a_1) \\ P_{A_1|S=2}(a_2) \end{pmatrix}^{\mathsf{T}} \underline{\boldsymbol{u}}^{(2)} \begin{pmatrix} P_{A_2}(a_1) \\ P_{A_2}(a_2) \end{pmatrix},$$

Connections with Existing Results

Aumann's **Games with Incomplete Information**

A ZSG with noisy observation of the actions and **perfect observation** of the commitment can be modelled by an **Aumann's game with incomplete information.**

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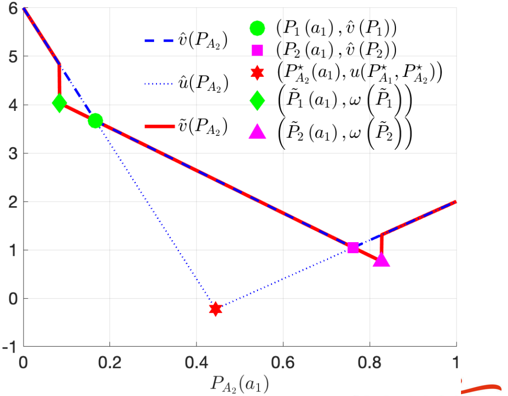
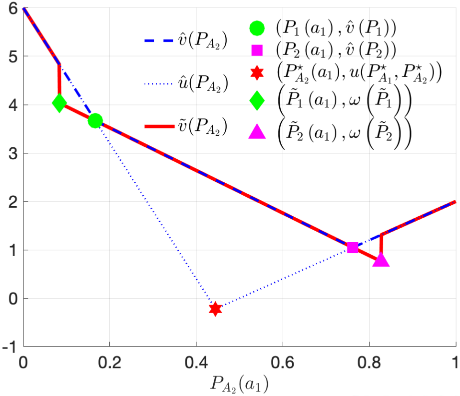
The game $\mathscr{G}\left(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}}\right)$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.

- An equilibrium exists if Player 1 plays the action that benefits the leader when it is **indifferent to play any of its actions**.
- Player 1 aims to **maximize the expected payoff**.
- How to make Player 1 play the action that **minimizes the expected payoff** ?



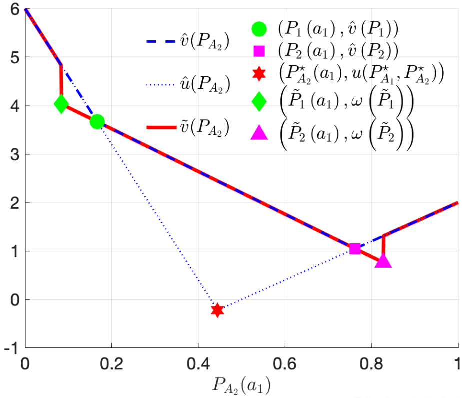
The game $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}})$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.

- An equilibrium exists if Player 2 adopts a commitment that is δ-suboptimal.



The game $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}})$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.

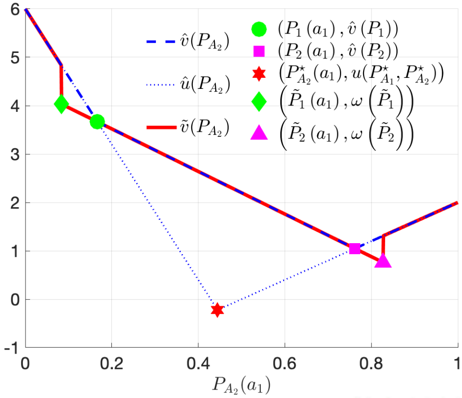
- An equilibrium exists if Player 2 adopts a commitment that is δ-suboptimal.
- A **suboptimal commitment** forces a **unique and predictable** best response from Player 1.



The game $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}})$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.

- An equilibrium exists if Player 2 adopts a commitment that is δ-suboptimal.
- A **suboptimal commitment** forces a **unique and predictable** best response from Player 1.
- **Suboptimal commitment:** Given $\epsilon > 0$, $P_{A_2} \in \triangle(\mathcal{A}_2)$:

$$P_{A_2}(a_1) = \tilde{P}^{(2)} - \epsilon. \quad (34)$$



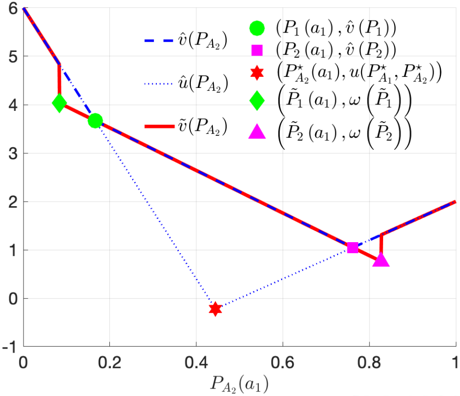
The game $\mathscr{G}\left(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}}\right)$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.

- How to determine a suboptimal commitment P?

$$\min_{P \in \triangle(\mathcal{A}_2)} \tilde{v}(P) + \lambda D(P \| Q), \qquad (34)$$

  with $Q \in \triangle\left(\mathcal{A}_2\right)$, such that $Q(a_1) = \tilde{P}^{(2)}$.

- Type-I Empirical Risk Minimization with **Relative Entropy Regularization**



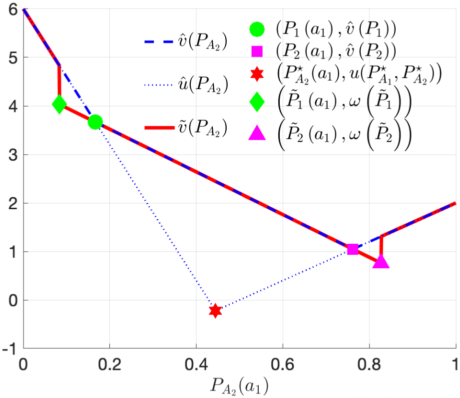
The game $\mathscr{G}(\underline{\boldsymbol{u}}, \underline{\boldsymbol{w}}, \underline{\boldsymbol{t}})$, with $\underline{\boldsymbol{u}} = (-8, 6; 2, -2)$, $\underline{\boldsymbol{w}} = (0.8, 0.2; 0.2, 0.8)$ and $\underline{\boldsymbol{t}} = (0.9, 0.1; 0.1, 0.9)$ **does not possess an equilibrium**.

- How to determine **another** suboptimal commitment P?

$$\min_{P \in \triangle(\mathcal{A}_2)} \tilde{v}(P) + \lambda D(Q \| P) \,, \qquad (34)$$

with $Q \in \triangle(\mathcal{A}_2)$, such that $Q(a_1) = \tilde{P}^{(2)}$.

- Type-II Empirical Risk Minimization with **Relative Entropy Regularization**



Final Remarks

- Often **Robust** ERM is **neither a Nash nor a Stackelberg equilibria** of a ZSG.
  - **Why**? Because of observation of datasets and priors on the datasets.
- A new game formulation is proposed to incorporate:
  - **noisy observations** of the actions; and
  - **Distorted** Commitments
- Priors are associated to commitment mismatches in ZSG with Noise Observations
- Channel Model: **Binary Channels**. More elaborate Channel models:
  - **Erasures** on data points: Erasure Channel
  - **Additive Noise** due to data acquisition: AWGN Channel

Thank you for your attention.

Ke Sun, Samir M. Perlaza, and Alain Jean-Marie. "**Zero-Sum Games with Noisy Observations**". Preprint arXiv:2211.01703 [cs.GT].

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