Weighted Improper Colouring

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This presentation covers work by the authors[1]:

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[1] Alphabetic order

Outline

Problem overview

- We assign colours to nodes of a graph
- Nodes of the same colour interfere with each other
  - Interference is function of distance
  - In general case $f(a, b) \rightarrow \mathbb{R}_+$
- A certain amount of interference can be tolerated at each node

Motivation

- Problem introduced by Alcatel Space (now Thales Alenia Space)
  - Design of satellite antennas for multi-spot MFTDMA satellites
  - High bandwidth requirements for next-generation wireless
  - Spatial frequency reuse needed
- Initial work by joint team of Mascotte, FT and University of Tsukuba
  - Mathematical abstraction over physical and geographical aspects
  - Formulation on a grid, introduction of γ – mitigation factor
  - Relation to graph coloring
  - Linear programming solution
- Recent work by Mascotte, Université de Genève and Universidade Federal do Ceará
  - Focused on list coloring version of the problem
  - Proposed approximate results on grid subgraphs
- More abstract version this work focuses on can be applied to any cellular radio network design

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Weighted Improper Colouring

Given an edge-weighted graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+$, and a threshold $t \in \mathbb{R}_+$, we say that c is a *weighted t-improper k-colouring* of G if c is a k-colouring of the vertices of G in such a way that, for each vertex $u \in V$, the following constraint is satisfied:

$$\sum_{\{v \in N(u) | c(v) = c(u)\}} w(u, v) \leq t.$$

Given a threshold $t \in \mathbb{R}_+$, the minimum integer k such that the graph G admits a weighted t-improper k-colouring is the *weighted t-improper chromatic number* of G, denoted by $\chi_t^w(G)$.

Threshold Improper colouring

A dual of *Weighted Improper Colouring* which is, for a given edge-weighted graph $G = (V, E, w)$ and a positive integer k, to determine the minimum real t such that G admits a weighted t-improper k-colouring that is called *minimum k-threshold* of G, denoted by $\omega_k^w(G)$.

Simple distance function

We consider a simple interference function:

$$f(d) = \begin{cases} 1, & \text{if } d = 1 \\ \frac{1}{2}, & \text{if } d = 2 \\ 0, & \text{otherwise} \end{cases}$$

In other words: given a graph $G = (V, E)$ and its square $G^2 = (V, E^2)$, we study *from now on* the function $w : E \to \{1, 0.5\}$ such that $w(e) = 0.5$ if, and only if, $e \in E^2 \setminus E$.

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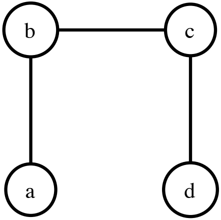
Simple distance function



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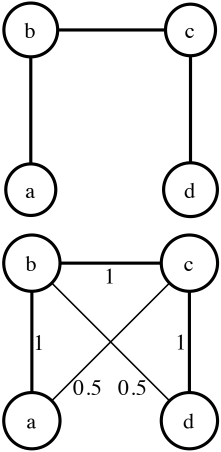
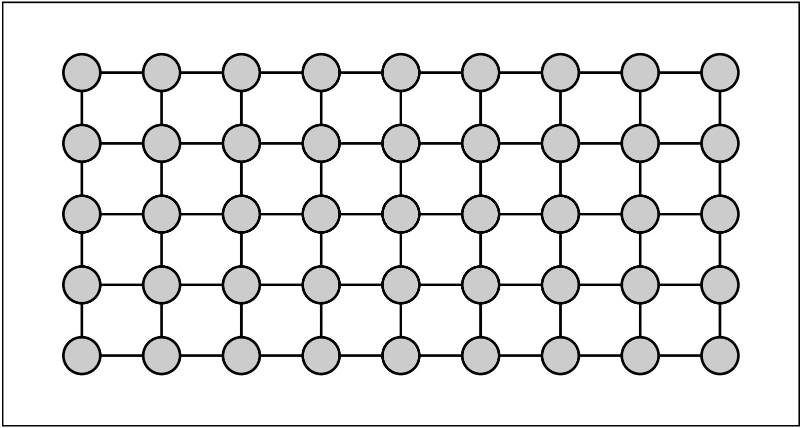
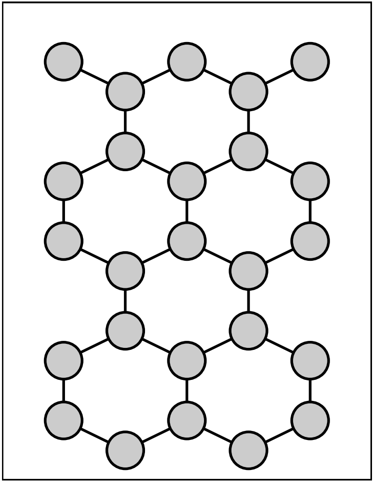


Grids 1

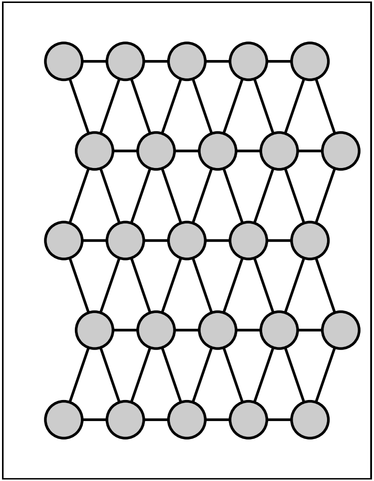


Infinite square grid

Grids 2

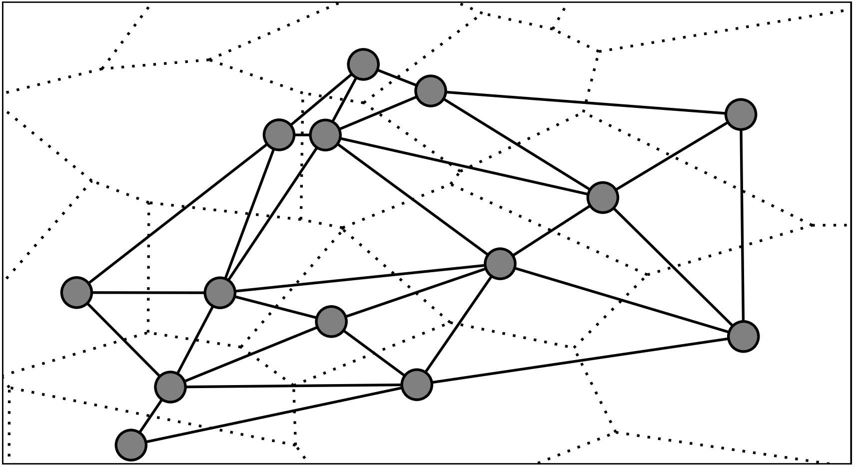


Infinite "hex" (3-regular) grid



Infinite "triangle" (6-regular) grid

Delaunay graph



Effect of Delaunay tesselation for a set of random points.
Dual of Voronoi diagram.

Outline

Upper bound for *Weighted Improper Colouring*

Theorem

Given an edge-weighted graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+$, and a threshold $t \in \mathbb{R}_+$, then the following inequality holds, for any real $\varepsilon > 0$:

$$\chi_t^w(G) \leq \left\lceil \frac{\Delta_w(G) + \varepsilon}{t + gcd(w)} \right\rceil .$$

Where:

- $\Delta_w(G) = \max_{u \in V} d_w(u)$
- $d_w(u) = \sum_{v \in N(u)} w(u, v)$

Upper bound for *Threshold Improper Colouring*

Theorem

Let $G = (V, E, w)$, $w : E \to \mathbb{R}_+$, be an edge-weighted graph and k be a positive integer. Then:

$$\omega_k^w(G) \leq \max_{v \in V} w(E_{min}^{k-1}(v))$$

Where:

- $w(E_{min}^{k-1}(v)) = \sum_{e \in E_{min}^{k-1}(v)} w(e)$
- $E_{min}^{k-1}(v)$ be the set of $d(v) - (k-1)$ least weighted edges incident to v

Paths and trees

Theorem
Let $P = (V, E)$ be an infinite path. Then,

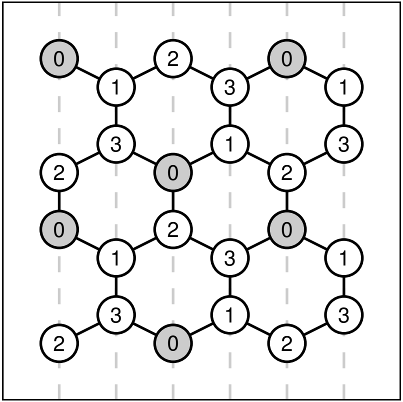
$$\chi_t^w(P^2) = \begin{cases} 1, & \textit{if } 3 \leq t; \\ 2, & \textit{if } 1 \leq t < 3; \\ 3, & \textit{if } 0 \leq t < 1. \end{cases}$$

Theorem
Let $T = (V, E)$ be a tree. Then,
$\lceil \frac{\Delta(T) - \lfloor t \rfloor}{2t+1} \rceil + 1 \leq \chi_t^w(T^2) \leq \lceil \frac{\Delta(T)-1}{2t+1} \rceil + 2.$

Hexagonal grid

If G is an infinite hexagonal grid, then

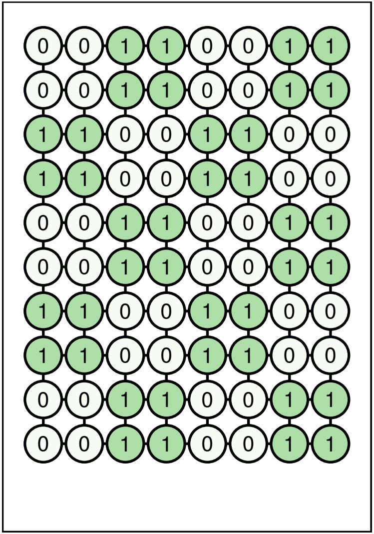
$$\chi_t^w(G^2) = \begin{cases} 4, & \text{if } 0 \le t < 1; \\ 3, & \text{if } 1 \le t < 2; \\ 2, & \text{if } 2 \le t < 6; \\ 1, & \text{if } 6 \le t. \end{cases}$$



Square grid

If G is an infinite square grid, then

$$\chi_t^w(G^2) = \begin{cases} 5, & \text{if } 0 \le t < 0.5; \\ 4, & \text{if } 0.5 \le t < 1; \\ 3, & \text{if } 1 \le t < 3; \\ 2, & \text{if } 3 \le t < 8; \\ 1, & \text{if } 8 \le t. \end{cases}$$



6-regular grid

Theorem

If G is an infinite triangular grid, then

$$\chi_t^w(G^2) = \begin{cases} \leq 7, & \textit{if } t = 0; \\ \leq 6, & \textit{if } t = 0.5; \\ \leq 5, & \textit{if } t = 1; \\ \leq 4, & \textit{if } 1.5 \leq t < 3; \\ \leq 3, & \textit{if } 3 \leq t < 5; \\ \leq 2, & \textit{if } 5 \leq t < 12; \\ 1, & \textit{if } 12 \leq t. \end{cases}$$

Outline

Levelling heuristic

- Greedy heuristic for *Threshold Improper Colouring*
- Performs local decisions to minimize immediate interference
- Enhancement: we set up an interference target t_t, bail if it's not possible to colour a vertex without raising interference over the target in any other vertex
- Outer loop:
  - Initially we set $t_t = \infty$
  - Repeat until time runs out or happy with the interference

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Levelling heuristic — inner pseudocode

$l_{v,c} \longleftarrow 0$ for $v \in V, c \in \{0, 1, \ldots, k\}$; $l\prime_v \longleftarrow 0$ for $v \in V$
$T \longleftarrow V$; possible $\longleftarrow$ **true**
while $T \neq \emptyset \wedge possible$ **do**
    $T\prime \longleftarrow \{x \in T : l\prime_x = \max l\prime\}$; $v \longleftarrow$ random from $T\prime$
    $C \longleftarrow (1, 2, \ldots, k)$ sorted to give $l_{v,i} \leq l_{v,i+1}$
    foreach $c \in C$ **do**
        if v can be coloured c **then**
            foreach $w \in N(v)$ **do**
                $l_{w,c} \longleftarrow l_{w,c} + f(v, w)$
                $l\prime_w \longleftarrow l_w + f(v, w)$
            colour v with colour c ; **break**
    if *n was coloured* **then** $T \longleftarrow T \setminus v$ **else** possible $\longleftarrow$ **false**
if *possible* **then** $t_t \longleftarrow \max l - \varepsilon$

Branch and bound

- Inspired by levelling heuristic
- Colours vertices in same order
- Considers colours in same order
- Optimal solution in finite time
- Pretty naive implementation find near-optimal solutions fast

Linear program for *Weighted Improper Colouring*

Weighted Improper Colouring solved by *integer program*:

$$
\begin{array}{cl}
\min & \sum_p c^p \\
\text{subject to:} & \\
\sum_{j \neq i} w(i,j) x_{jp} \leq t + M(1 - x_{ip}) & (\forall i \in V, \forall p \in \{1, \ldots, l\}) \\
c^p \geq x_{ip} & (\forall i \in V, \forall p \in \{1, \ldots, l\}) \\
\sum_p x_{ip} = 1 & (\forall i \in V) \\
x_{ip} \in \{0, 1\} & (\forall i \in V, \forall p \in \{1, \ldots, l\}) \\
c^p \in \{0, 1\} & (\forall p \in \{1, \ldots, l\})
\end{array}
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where M is a large integer

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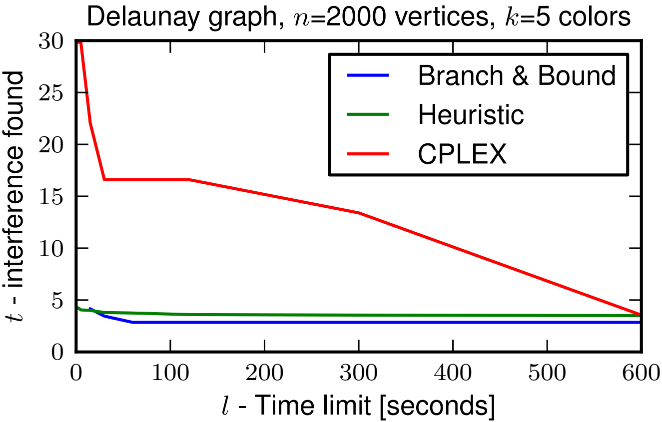
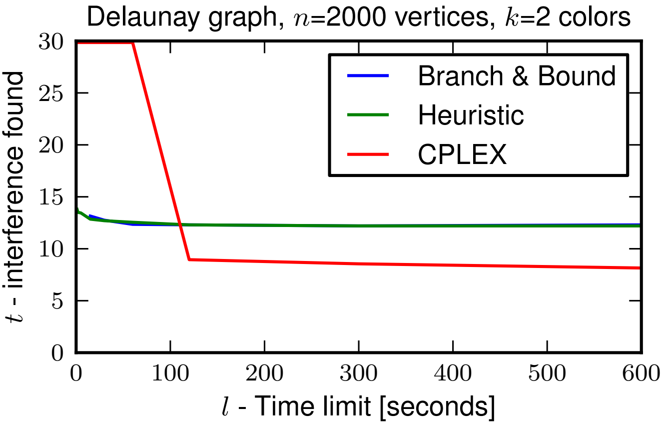
Performance comparison 1



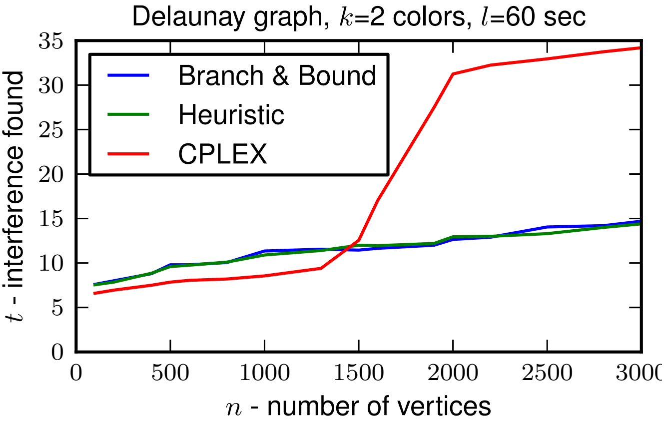
Both specific algorithms deliver results in few seconds

Performance comparison 2



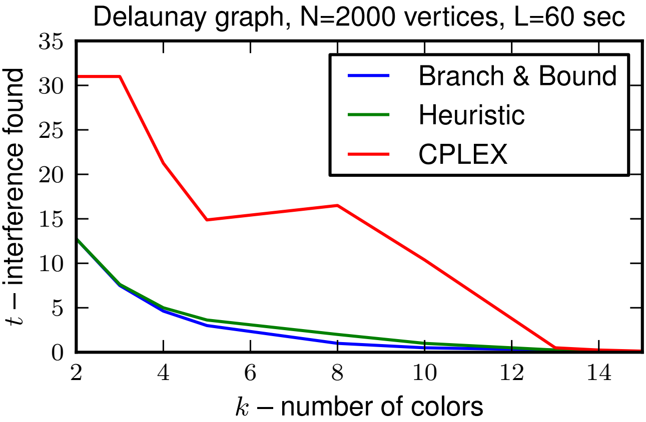
In hard cases, a good branch-and-cut implementation achieves better results

Performance comparison 3



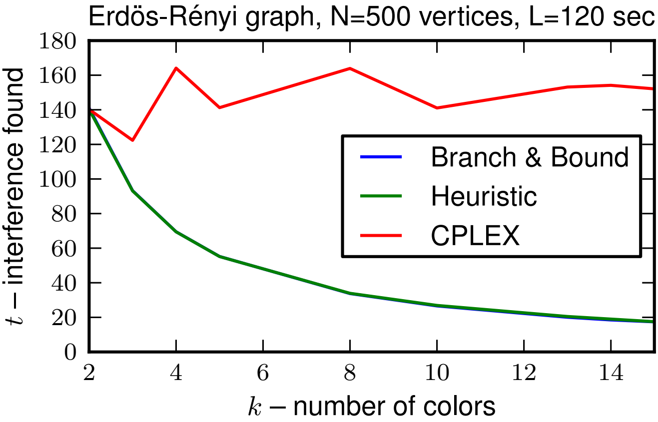
Both specific algorithms scale better with growing graphs

Performance comparison 4



Making the problem easier increases number of constraints for integer program

Performance comparison 5



In case of denser graphs, integer programming becomes pretty useless