Weighted Improper Colouring.[☆]

J. Araujo[a,b,*], J-.C. Bermond[a], F. Giroire[a], F. Havet[a], D. Mazauric[a], R. Modrzejewski[a]

[a]*MASCOTTE Project, I3S (CNRS & UNS) and INRIA, INRIA Sophia-Antipolis, 2004 route des Lucioles, BP 93, 06902 Sophia-Antipolis Cedex France.*
[b]*ParGO Research Group - Universidade Federal do Ceará - UFC - Campus do Pici, Bloco 910. 60455-760 - Fortaleza, CE - Brazil.*

Abstract

In this paper, we study a colouring problem motivated by a practical frequency assignment problem and, up to our best knowledge, new. In wireless networks, a node interferes with other nodes, the level of interference depending on numerous parameters: distance between the nodes, geographical topography, obstacles, etc. We model this with a weighted graph (G, w) where the weight function w on the edges of G represents the noise (interference) between the two end-vertices. The total interference in a node is then the sum of all the noises of the nodes emitting on the same frequency. A weighted t-improper k-colouring of (G, w) is a k-colouring of the nodes of G (assignment of k frequencies) such that the interference at each node does not exceed the threshold t. We consider here the Weighted Improper Colouring problem which consists in determining the weighted t-improper chromatic number defined as the minimum integer k such that (G, w) admits a weighted t-improper k-colouring. We also consider the dual problem, denoted the Threshold Improper Colouring problem, where, given a number k of colours, we want to determine the minimum real t such that (G, w) admits a weighted t-improper k-colouring. We first present general upper bounds for both problems; in particular we show a generalisation of Lovász's Theorem for the weighted t-improper chromatic number. We then show how to transform an instance of the Threshold Improper Colouring problem into another equivalent one where the weights are either one or M, for a sufficiently large M. Motivated by the original application, we then study a special interference model on various grids (square, triangular, hexagonal) where a node produces a noise of intensity 1 for its neighbours and a noise of intensity 1/2 for the nodes at distance two. We derive the weighted t-improper chromatic number

[☆]This work was partially supported by région PACA, ANR Blanc AGAPE (ANR-09-BLAN-0159) and ANR International Taiwan GRATEL (ANR-09-BLAN-0373).

[*]Corresponding author

Email addresses: julio.araujo@inria.fr (J. Araujo), jean-claude.bermond@inria.fr (J-.C. Bermond), frederic.giroire@inria.fr (F. Giroire), frederic.havet@inria.fr (F. Havet), dorian.mazauric@inria.fr (D. Mazauric), remigiusz.modrzejewski@inria.fr (R. Modrzejewski)

for all values of t. Finally, we model the problem using integer linear programming, propose and test heuristic and exact Branch-and-Bound algorithms on random cell-like graphs, namely the Poisson-Voronoi tessellations.

Keywords: graph colouring, improper colouring, interference, radio networks, frequency assignment.

1. Introduction

Let $G = (V, E)$ be a graph. A *k-colouring* of G is a function $c : V \to \{1, \ldots, k\}$. The colouring c is *proper* if $uv \in E$ implies $c(u) \neq c(v)$. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum integer k such that G admits a proper k-colouring. The goal of the Vertex Colouring problem is to determine $\chi(G)$ for a given graph G. It is a well-known NP-hard problem [16].

A k-colouring c is *l-improper* if $|\{v \in N(u) \mid c(v) = c(u)\}| \leq l$, for all $u \in V$ (as usual in the literature, $N(u)$ stands for the set $\{v \mid uv \in E(G)\}$). Given a non-negative integer l, the *l-improper chromatic number* of a graph G, denoted by $\chi_l(G)$, is the minimum integer k such that G admits an l-improper k-colouring. Given a graph G and an integer l, the Improper Colouring problem consists in determining $\chi_l(G)$ and is also NP-hard [19, 8]. Indeed, if $l = 0$, observe that $\chi_0(G) = \chi(G)$. Consequently, Vertex Colouring is a particular case of Improper Colouring.

In this work we define and study a new variation of the Improper Colouring problem for edge-weighted graphs. An edge-weighted graph is a pair (G, w) where $G = (V, E)$ is a graph and $w : E \to \mathbb{R}_+^*$. Given an edge-weighted graph (G, w) and a colouring c of G, the *interference* of a vertex u in this colouring is defined by

$$I_u(G, w, c) = \sum_{\{v \in N(u) \mid c(v) = c(u)\}} w(u, v).$$

For any non-negative real number t, called *threshold*, we say that c is a *weighted t-improper k-colouring* of (G, w) if c is a k-colouring of G such that $I_u(G, w, c) \leq t$, for all $u \in V$.

Given a threshold $t \in \mathbb{R}_+^*$, the minimum integer k such that the graph G admits a weighted t-improper k-colouring is the *weighted t-improper chromatic number* of (G, w), denoted by $\chi_t(G, w)$. Given an edge-weighted graph (G, w) and a threshold $t \in \mathbb{R}_+^*$, determining $\chi_t(G, w)$ is the goal of the Weighted Improper Colouring problem. Note that if $t = 0$ then $\chi_0(G, w) = \chi(G)$, and if $w(e) = 1$ for all $e \in E$, then $\chi_l(G, w) = \chi_l(G)$ for any positive integer l. Therefore, the Weighted Improper Colouring problem is clearly NP-hard since it generalises Vertex Colouring and Improper Colouring.

On the other hand, given a positive integer k, we define the *minimum k-threshold* of (G, w), denoted by $T_k(G, w)$ as the minimum real t such that (G, w) admits a weighted t-improper k-colouring. Then, for a given edge-weighted graph (G, w) and a positive integer k, the THRESHOLD IMPROPER COLOURING problem consists in determining $T_k(G, w)$. The THRESHOLD IMPROPER COLOURING problem is also NP-hard. This fact follows from the observation that determining whether $\chi_l(G) \leq k$ is NP-complete, for every $l \geq 2$ and $k \geq 2$ [10, 9, 8]. Consequently, in particular, it is a NP-complete problem to decide whether a graph G admits a weighted t-improper 2-colouring when all the weights of the edges of G are equal to one, for every $t \geq 2$.

1.1. Motivation

Our initial motivation to these problems was the design of satellite antennas for multi-spot MFTDMA satellites [2]. In this technology, satellites transmit signals to areas on the ground called *spots*. These spots form a grid-like structure which is modelled by an hexagonal cell graph. To each spot is assigned a radio channel or colour. Spots are interfering with other spots having the same channel and a spot can use a colour only if the interference level does not exceed a given threshold t. The level of interference between two spots depends on their distance. The authors of [2] introduced a factor of mitigation γ and the interference of remote spots are reduced by a factor $1-\gamma$. When the interference level is too low, the nodes are considered to not interfere anymore. Considering such types of interference, where nodes at distance at most i interfere, leads to the study of the i-th power of the graph modelling the network and a case of special interest is the power of grid graphs (see Section 3).

1.2. Related Work

Our problems are particular cases of the FREQUENCY ASSIGNMENT problem (FAP). FAP has several variations that were already studied in the literature (see [1] for a survey). In most of these variations, the main constraint to be satisfied is that if two vertices (mobile phones, antennas, spots, etc.) are close, then the difference between the frequencies that are assigned to them must be greater than some function which usually depends on their distance.

There is a strong relationship between most of these variations and the $L(p_1, \ldots, p_d)$-LABELLING problem [20]. In this problem, the goal is to find a colouring of the vertices of a given graph G, in such a way that the difference between the colours assigned to vertices at distance i is at least p_i, for every $i = 1, \ldots, d$.

In some other variants, for each non-satisfied interference constraint a penalty must be paid. In particular, the goal of the MINIMUM INTERFERENCE FREQUENCY ASSIGNMENT problem (MI-FAP) is to minimise the total penalties that must be paid, when the number of frequencies to be assigned is given. This problem can also be studied for only *co-channel interference*, in which the penalties are applied only if the two vertices have the same frequency. However, MI-FAP under these constraints does not correspond to WEIGHTED IMPROPER COLOURING, because we consider the co-channel interference, i.e. penalties, just between each vertex and its neighbourhood.

The two closest related works we found in the literature are [18] and [11]. However, they both apply penalties over co-channel interference, but also to the *adjacent channel interference*, i.e. when the colours of adjacent vertices differ by one unit. Moreover, their results are not similar to ours. In [18], they propose an enumerative algorithm for the problem, while in [11] a Branch-and-Cut method is proposed and applied over some instances.

1.3. Results

In this article, we study both parameters $\chi_t(G, w)$ and $T_k(G, w)$. We first present general bounds; in particular we show a generalisation of Lovász's Theorem for $\chi_t(G, w)$. We after show how to transform an instance of Threshold Improper colouring into an equivalent one where the weights are either one or M, for a sufficiently large M.

Motivated by the original application, we then study a special interference model on various grids (square, triangular, hexagonal) where a node produces a noise of intensity 1 for its neighbours and a noise of intensity 1/2 for the nodes that are at distance two. We derive the weighted t-improper chromatic number for all possible values of t.

Finally, we propose a heuristic and a Branch-and-Bound algorithm to solve Threshold Improper colouring for general graphs. We compare them to an integer linear programming formulation on random cell-like graphs, namely Voronoi diagrams of random points of the plan. These graphs are classically used in the literature to model telecommunication networks [5, 13, 14].

2. General Results

In this section, we present some results for Weighted Improper colouring and Threshold Improper colouring for general graphs and general interference models.

2.1. Upper bounds

Let (G, w) be an edge-weighted graph with positive real weights given by $w : E(G) \to \mathbb{Q}^*_+$. For any vertex $v \in V(G)$, its *weighted degree* is $d_w(v) = \sum_{u \in N(v)} w(u, v)$. The *maximum weighted degree* of G is $\Delta(G, w) = \max_{v \in V} d_w(v)$.

Given a k-colouring $c : V \to \{1, \ldots, k\}$ of G, we define, for every vertex $v \in V(G)$ and colour $i = 1, \ldots, k$, $d^i_{w,c}(v) = \sum_{\{u \in N(v) | c(u) = i\}} (u, v)$. Note that $d^{c(v)}_{w,c}(v) = I_v(G, w, c)$. We say that a k-colouring c of G is *w-balanced* if c satisfies the following property:

$$\text{For any vertex } v \in V(G), \;\; I_v(G, w, c) \leq d^j_{w,c}(v), \;\; \text{for every } j = 1, \ldots, k.$$

We denote by $\gcd(w)$ the greatest common divisor of the weights of w (observe that $\gcd(w) > 0$ because we just consider positive weights). We use here the generalisation of the gcd to non-integer numbers (e.g. in $\mathbb{Q}$) where a number x is said to divide a number y if the fraction y/x is an integer. The important property of $\gcd(w)$ is that the difference between two interferences is a multiple of $\gcd(w)$; in particular, if for two vertices v and u, $d^i_{w,c}(v) > d^j_{w,c}(u)$, then $d^i_{w,c}(v) \geq d^j_{w,c}(u) + \gcd(w)$.

If t is not a multiple of the $\gcd(w)$, that is, there exists an integer $a \in \mathbb{Z}$ such that $a \gcd(w) < t < (a+1)\gcd(w)$, then $\chi^w_t(G) = \chi^w_{a \gcd(w)}(G)$.

Proposition 1. *Let (G, w) be an edge-weighted graph. For any $k \geq 2$, there exists a w-balanced k-colouring of G.*

Proof. Let us colour $G = (V, E)$ arbitrarily with k colours and then repeat the following procedure: if there exists a vertex v coloured i and a colour j such that $d^i_{w,c}(v) > d^j_{w,c}(v)$, then recolour v with colour j. Observe that this procedure neither increases (we just move a vertex from one colour to another) nor decreases (a vertex without neighbour on its colour is never moved) the number of colours within this process. Let W be the sum of the weights of the edges having the same colour in their end-vertices. In this transformation, W has increased by $d^j_{w,c}(v)$ (edges incident to v that previously had colour j in its endpoint opposite to v), but decreased by $d^i_{w,c}(v)$ (edges that previously had colour i in both of their end-vertices). So, W has decreased by $d^i_{w,c}(v) - d^j_{w,c}(v) \geq \gcd(w)$. As $W \leq |E| \max_{e \in E} w(e)$ is finite, this procedure finishes and produces a w-balanced k-colouring of G. □

The existence of a w-balanced colouring gives easily some upper bounds on the weighted t-improper chromatic number and the minimum k-threshold of an edge-weighted graph (G, w). It is a folklore result that $\chi(G) \leq \Delta(G) + 1$, for any graph G. Lovász [17] extended this result for Improper Colouring problem using w-balanced colouring. He proved that $\chi_l(G) \leq \lceil \frac{\Delta(G)+1}{l+1} \rceil$. In what follows, we extend this result to weighted improper colouring.

Theorem 2. *Let (G, w) be an edge-weighted graph with $w : E(G) \to \mathbb{Q}^*_+$, and t a multiple of* $\gcd(w)$*. Then*

$$\chi_t(G, w) \leq \left\lceil \frac{\Delta(G, w) + \gcd(w)}{t + \gcd(w)} \right\rceil .$$

Proof. If t, ω, and G are such that $\chi_t(G, \omega) = 1$, then the inequality is trivially satisfied. Thus, consider that $\chi_t(G, \omega) > 1$.

Observe that, in any w-balanced k-colouring c of a graph G, the following holds:

$$d_w(v) = \sum_{u \in N(v)} w(u, v) \geq k d^{c(v)}_{w,c}(v). \tag{1}$$

Let $k^* = \left\lceil \frac{\Delta(G,w)+\gcd(w)}{t+\gcd(w)} \right\rceil \geq 2$ and c^* be a w-balanced k^*-colouring of G. We claim that c^* is a weighted t-improper k^*-colouring of (G, w).

By contradiction, suppose that there is a vertex v in G such that $c^*(v) = i$ and that $d^i_{w,c}(v) > t$. Since c^* is w-balanced, $d^j_{w,c}(v) > t$, for all $j = 1, \ldots, k^*$. By the definition of $\gcd(w)$ and as t is a multiple of $\gcd(w)$, it leads to $d^j_{w,c}(v) \geq$

$t + \gcd(w)$ for all $j = 1, \ldots, k^*$. Combining this inequality with Inequality (1), we obtain:

$$\Delta(G, w) \geq d_w(v) \geq k^*(t + \gcd(w)),$$

giving

$$\Delta(G, w) \geq \Delta(G, w) + \gcd(w),$$

a contradiction. The result follows.

□

Note that when all weights are unit, we obtain the bound for the improper colouring derived in [17]. Brooks [7] proved that for a connected graph G, $\chi(G) = \Delta(G)+1$ if, and only if, G is complete or an odd cycle. One could wonder for which edge-weighted graphs the bound we provided in Theorem 2 is tight. However, Correa *et al.* [8] already showed that it is NP-complete to determine if the improper chromatic number of a graph G attains the upper bound of Lovász, which is a particular case of Weighted Improper colouring, i.e. of the bound of Theorem 2.

We now show that w-balanced colourings also yield upper bounds for the minimum k-threshold of an edge-weighted graph (G, w). When $k = 1$, then all the vertices must have the same colour, and $T_1(G, w) = \Delta(G, w)$. This may be generalised as follows, using w-balanced colourings.

Theorem 3. *Let (G, w) be an edge-weighted graph with $w : E(G) \to \mathbb{R}_+^*$, and let k be a positive integer. Then*

$$T_k(G, w) \leq \frac{\Delta(G, w)}{k}.$$

Proof. Let c be a w-balanced k-colouring of G. Then, for every vertex $v \in V(G)$:

$$kT_k(G, w) \leq kd_{w,c}^{c(v)}(v) \leq d_w(v) = \sum_{u \in N(v)} w(u, v) \leq \Delta(G, w)$$

□

Because $T_1(G, w) = \Delta(G, w)$, Theorem 3 may be restated as $kT_k(G, w) \leq \ldots \leq T_1(G, w)$. This inequality may be generalised as follows.

Theorem 4. *Let (G, w) be an edge-weighted graph with $w : E(G) \to \mathbb{R}_+$, and let k and p be two positive integers. Then*

$$T_{kp}(G, w) \leq \frac{T_p(G, w)}{k}.$$

Proof. Set $t = T_p(G, w)$. Let c be a t-improper p-colouring of (G, w). For $i = 1, \ldots, p$, let G_i be the subgraph of G induced by the vertices coloured i by c. By definition of improper colouring $\Delta(G_i, w) \leq t$ for all $1 \leq i \leq p$. By Theorem 3, each (G_i, w) admits a t/k-improper k-colouring c_i with colours $\{(i-1)k+1, \ldots, ik\}$. The union of the c_i's is then a t/k-improper kp-colouring of (G, w). □

Theorem 4 and its proof suggest that to find a kp-colouring with small impropriety, it may be convenient to first find a p-colouring with small impropriety and then to refine it. In addition, such a strategy allows to adapt dynamically the refinement. In the above proof, the vertex set of each part G_i is again partitioned into k parts. However, sometimes, we shall get a better kp-colouring by partitioning each G_i into a number of k_i parts, with $\sum_{i=1}^{p} k_i = kp$. Doing so, we obtain a T-improper kp-colouring of (G, w), where $T = \max\{\frac{\Delta(G_i,w)}{k_i}, 1 \leq i \leq p\}$.

One can also find an upper bound on the minimum k-threshold by considering first the $k-1$ edges of largest weight around each vertex. Let (G, w) be an edge-weighted graph, and let $v_1, \ldots, v_n$ be an ordering of the vertices of G. The edges of G may be ordered in increasing order of their weight. Furthermore, to make sure that the edges incident to any particular vertex are totally ordered, we break ties according to the label of the second vertex. Formally, we say that $v_i v_j \leq_w v_i v_{j'}$ if either $w(v_i v_j) < w(v_i v_{j'})$ or $w(v_i v_j) = w(v_i v_{j'})$ and $j < j'$. With such a partial order on the edge set, the set $E_w^k(v)$ of $\min\{|N(v)|, k-1\}$ greatest edges (according to this ordering) around a vertex is uniquely defined. Observe that every edge incident to v and not in $E_w^k(v)$ is smaller than an edge of $E_k(v)$ for $\leq_w$.

Let G_w^k be the graph with vertex set $V(G)$ and edge set $\bigcup_{v \in V(G)} E_w^k(v)$. Observe that every vertex of $E_w^k(v)$ has degree at least $\min\{|N(v)|, k-1\}$, but a vertex may have an arbitrarily large degree. For if any edge incident to v has a greater weight than any edge not incident to v, the degree of v in G_w^k is equal to its degree in G. However we now prove that at least one vertex has degree $k-1$.

Proposition 5. *If (G, w) is an edge-weighted graph, then G_w^k has a vertex of degree at most $k-1$.*

Proof. Suppose for a contradiction, that every vertex has degree at least k, then for every vertex x there is an edge xy in $E(G_w^k) \setminus E_w^k(x)$, and so in $E_w^k(y) \setminus E_w^k(x)$. Therefore, there must be a cycle $(x_1, \ldots, x_r)$ such that, for all $1 \leq i \leq r$, $x_i x_{i+1} \in E_w^k(x_{i+1}) \setminus E_w^k(x_i)$ (with $x_{r+1} = x_1$). It follows that $x_1 x_2 \leq_w x_2 x_3 \leq_w \cdots \leq_w x_r x_1 \leq_w x_1 x_2$. Hence, by definition, $w(x_1 x_2) = w(x_2 x_3) = \cdots = w(x_r x_1) = w(x_1 x_2)$. Let m be the integer such that x_m has maximum index in the ordering $v_1, \ldots, v_n$. Then there exists j and j' such that $x_m = v_j$ and $x_{m+2} = v_{j'}$. By definition of m, we have $j > j'$. But this contradicts the fact that $x_m x_{m+1} \leq_w x_{m+1} x_{m+2}$. □

Corollary 6. *If (G, w) is an edge-weighted graph, then G_w^k has a proper k-colouring.*

Proof. By induction on the number of vertices. By Proposition 5, G_w^k has a vertex x of degree at most $k-1$. Trivially, $G_w^k - x$ is a subgraph of $(G-x)_w^k$. By the induction hypothesis, $(G-x)_w^k$ has a proper k-colouring, which is also a proper k-colouring of $G_w^k - x$. This colouring can be extended in a proper k-colouring of G_w^k, by assigning to x a colour not assigned to any of its $k-1$ neighbours. □

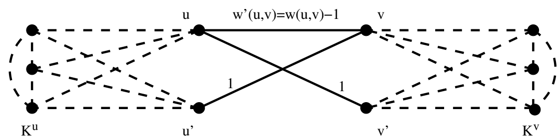


Figure 1: Construction of G' from G using edge $uv \in E(G)$ and $k = 4$ colours. Dashed edges represent edges of weight M.

Corollary 7. *If (G, w) is an edge-weighted graph, then $T_k(G, w) \leq \Delta(G \setminus E(G^k_w), w)$.*

2.2. Transformation

In this section, we prove that the Threshold Improper Colouring problem can be transformed into a problem mixing proper and improper colouring. More precisely, we prove the following:

Theorem 8. *Let (G, w) be an edge-weighted graph where w is an integer-valued function, and let k be a positive integer. We can construct an edge-weighted graph (G^*, w^*) such that $w^*(e) \in \{1, M\}$ for any $e \in E(G^*)$, satisfying $T_k(G, w) = T_k(G^*, w^*)$, where $M = 1 + \sum_{e \in E(G)} w(e)$.*

Proof. Consider the function $f(G, w) = \sum_{\{e \in E(G) | w(e) \neq M\}} (w(e) - 1)$.

If $f(G, w) = 0$, all edges have weight either one or M and G has the desired property. In this case, $G^* = G$. Otherwise, we construct a graph G' and a function w' such that $T_k(G', w') = T_k(G, w)$, but $f(G', w') = f(G, w) - 1$. By repeating this operation $f(G, w)$ times we get the required edge-weighted graph (G^*, w^*).

In case $f(G, w) > 0$, there exists an edge $e = uv \in E(G)$ such that $2 \leq w(e) < M$. G' is obtained from G by adding two complete graphs on $k - 1$ vertices K^u and K^v and two new vertices u' and v'. We join u and u' to all the vertices of K^u and v and v' to all the vertices of K^v. We assign weight M to all these edges. Note that, u and u' (v and v') always have the same colour, namely the remaining colour not used in K^u (resp. K^v).

We also add two edges uv' and $u'v$ both of weight 1. The edges of G keep their weight in G', except the edge $e = uv$ whose weight is decreased by one unit, i.e. $w'(e) = w(e) - 1$. Thus, $f(G', \omega') = f(G, \omega) - 1$ as we added only edges of weights 1 and M and we decreased the weight of e by one unit.

Now consider a weighted t-improper k-colouring c of (G, w). We produce a weighted t-improper k-colouring c' of (G', w') as follows: we keep the colours of all the vertices in G, we assign to u' (v') the same colour as u (resp. v), and we assign to K^u (resp. K^v) the $k - 1$ colours different from the one used in u (resp. v).

Conversely, from any weighted improper k-colouring c' of (G', w'), we get a weighted improper k-colouring c of (G, w) by just keeping the colours of the vertices that belong to G.

For such colourings c and c' we have that $I_x(G, w, c) = I_x(G', w', c')$, for any vertex x of G different from u and v. For $x \in K^u \cup K^v$, $I_x(G', w', c') = 0$. The neighbours of u with the same colour as u in G' are the same as in G, except possibly v' which has the same colour of u if, and only if, v has the same colour of u. Let $\epsilon = 1$ if v has the same colour as u, otherwise $\epsilon = 0$. As the weight of uv decreases by one and we add the edge uv' of weight 1 in G', we get $I_u(G', w', c') = I_u(G, w, c) - \epsilon + w'(u, v')\epsilon = I_u(G, w, c)$. Similarly, $I_v(G', w', c') = I_v(G, w, c)$. Finally, $I_{u'}(G', w', c') = I_{v'}(G', w', c') = \epsilon$. But $I_u(G', w', c') \geq (w(u, v) - 1)\epsilon$ and so $I_{u'}(G', w', c') \leq I_u(G', w', c')$ and $I_{v'}(G', w', c') \leq I_v(G', w', c')$. In summary, we have

$$\max_x I_x(G', w', c') = \max_x I_x(G, w, c)$$

and therefore $T_k(G, w) = T_k(G', w')$. □

In the worst case, the number of vertices of G^* is $n + m(w_{max} - 1)2k$ and the number of edges of G^* is $m + m(w_{max} - 1)[(k + 4)(k - 1) + 2]$ with $n = |V(G)|$, $m = |E(G)|$ and $w_{max} = \max_{e \in E(G)} w(e)$.

In conclusion, this construction allows to transform the Threshold Improper Colouring problem into a problem mixing proper and improper colouring. Therefore the problem consists in finding the minimum l such that a (non-weighted) l-improper k-colouring of G^* exists with the constraint that some subgraphs of G^* must admit a proper colouring. The equivalence of the two problems is proved here only for integers weights, but it is possible to adapt the transformation to prove it for rational weights.

3. Squares of Particular Graphs

As mentioned in the introduction, Weighted Improper colouring is motivated by networks of antennas similar to grids [2]. In these networks, the noise generated by an antenna undergoes an attenuation with the distance it travels. It is often modelled by a decreasing function of d, typically $1/d^\alpha$ or $1/(2^{d-1})$.

Here we consider a simplified model where the noise between two neighbouring antennas is normalised to 1, between antennas at distance two is $1/2$ and 0 when the distance is strictly greater than two. Studying this model of interference corresponds to study the Weighted Improper colouring of the square of the graph G, that is the graph G^2 obtained from G by joining every pair of vertices at distance two, and to assign weights $w_2(e) = 1$, if $e \in E(G)$, and $w_2(e) = 1/2$, if $e \in E(G^2) \setminus E(G)$. Observe that in this case the interesting threshold values are the non-negative multiples of $1/2$.

Figure 2 shows some examples of colouring for the square grid. In Figure 2(b), each vertex x has neither a neighbour nor a vertex at distance two coloured with its own colour, so $I_x(G^2, w_2, c) = 0$ and G^2 admits a weighted 0-improper 5-colouring. In Figure 2(c), each vertex x has no neighbour with its colour and at most one vertex of the same colour at distance 2. So $I_x(G^2, w_2, c) = 1/2$ and G^2 admits a weighted 0.5-improper 4-colouring.

For any $t \in \mathbb{R}_+$, we determine the weighted t-improper chromatic number for the square of infinite paths, square grids, hexagonal grids and triangular grids under the interference model w_2. We also present lower and upper bounds for $\chi_t(T^2, w_2)$, for any tree T and any threshold t.

3.1. Infinite paths and trees

In this section, we characterise the weighted t-improper chromatic number of the square of an infinite path, for all positive real t. Moreover, we present lower and upper bounds for $\chi_t(T^2, w_2)$, for a given tree T.

Theorem 9. *Let $P = (V, E)$ be an infinite path. Then,*

$$\chi_t(P^2, w_2) = \begin{cases} 3, & \text{if } 0 \le t < 1; \\ 2, & \text{if } 1 \le t < 3; \\ 1, & \text{if } 3 \le t. \end{cases}$$

Proof. Let $V = \{v_i \mid i \in \mathbb{Z}\}$ and $E = \{(v_{i-1}, v_i) \mid i \in \mathbb{Z}\}$. Each vertex of P has two neighbours and two vertices at distance two. Consequently, the equivalence $\chi_t(P^2, \omega_2) = 1$ if, and only if, $t \ge 3$ holds trivially.

There is a 2-colouring c of (P^2, w_2) with maximum interference 1 by just colouring v_i with colour $(i \bmod 2) + 1$. So $\chi_t(P^2, w_2) \le 2$ if $t \ge 1$. We claim that there is no weighted 0.5-improper 2-colouring of (P^2, w_2). By contradiction, suppose that c is such a colouring. If $c(v_i) = 1$, for some $i \in \mathbb{Z}$, then $c(v_{i-1}) = c(v_{i+1}) = 2$ and $c(v_{i-2}) = c(v_{i+2}) = 1$. This is a contradiction because v_i would have interference 1.

Finally, the colouring $c(v_i) = (i \bmod 3) + 1$, for every $i \in \mathbb{Z}$, is a feasible weighted 0-improper 3-colouring. □

Theorem 10. *Let $T = (V, E)$ be a (non-empty) tree. Then,* $\left\lceil \frac{\Delta(T) - \lfloor t \rfloor}{2t+1} \right\rceil + 1 \le \chi_t(T^2, w_2) \le \left\lceil \frac{\Delta(T) - 1}{2t+1} \right\rceil + 2.$

Proof. The lower bound is obtained by two simple observations. First, $\chi_t(H, w) \le \chi_t(G, w)$, for any $H \subseteq G$. Let T be a tree and v be a node of maximum degree in T. Then, observe that the weighted t-improper chromatic number of the subgraph of T^2 induced by v and its neighbourhood is at least $\lceil \frac{\Delta(T) - \lfloor t \rfloor}{2t+1} \rceil + 1$. Indeed, the colour of v can be assigned to at most $\lfloor t \rfloor$ vertices on its neighbourhood. Any other colour used in the neighbourhood of v cannot appear in more than $2t + 1$ vertices because each pair of vertices in the neighbourhood of v is at distance two.

Let us look now at the upper bound. Choose any node $r \in V$ to be the root of T. Colour r with colour 1. Then, by a breadth-first traversal in the tree, for each visited node v colour all the children of v with the $\lceil \frac{\Delta(T) - 1}{2t+1} \rceil$ colours different from the ones assigned to v and to its parent in such a way that at most $2t+1$ nodes have the same colour. This is a feasible weighted t-improper k-colouring of T^2, with $k \leq \lceil \frac{\Delta(T)-1}{2t+1} \rceil + 2$, since each vertex interferes with at most $2t$ vertices at distance two which are children of its parent. □

For a tree T and the weighted function w^2, Theorem 10 provides upper and lower bounds on $\chi_t(T^2, w_2)$, but we do not know the computational complexity of determining $\chi_t(T^2, w_2)$.

3.2. Grids

In this section, we show the optimal values of $\chi_t(G^2, w_2)$, whenever G is an infinite square, hexagonal or triangular grid, for all the possible values of t.

3.2.1. Square Grid

The square grid is the graph $\mathfrak{G}$ in which the vertices are all integer linear combinations $ae_1 + be_2$ of the two vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, for any $a, b \in \mathbb{Z}$. Each vertex (a, b) has four neighbours: its *down neighbour* $(a, b-1)$, its *up neighbour* $(a, b+1)$, its *right neighbour* $(a+1, b)$ and its *left neighbour* $(a-1, b)$ (see Figure 2(a)).

Theorem 11.

$$\chi_t(\mathfrak{G}^2, w_2) = \begin{cases} 5, & \textit{if } t = 0; \\ 4, & \textit{if } t = 0.5; \\ 3, & \textit{if } 1 \leq t < 3; \\ 2, & \textit{if } 3 \leq t < 8; \\ 1, & \textit{if } 8 \leq t. \end{cases}$$

Proof. If $t = 0$, then the colour of vertex (a, b) must be different from the ones used on its four neighbours. Moreover, all the neighbours have different colours, as each pair of neighbours is at distance two. Consequently, at least five colours are needed. The following construction provides a weighted 0-improper 5-colouring of $(\mathfrak{G}^2, w_2)$: for $0 \leq j \leq 4$, let $A_j = \{(j, 0) + a(5e_1) + b(2e_1 + 1e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 4$, assign the colour $j+1$ to all the vertices in A_j (see Figure 2(b)).

When $t = 0.5$, we claim that at least four colours are needed to colour $(\mathfrak{G}^2, w_2)$. The proof is by contradiction. Suppose that there exists a weighted 0.5-improper 3-colouring of it. Let (a, b) be a vertex coloured 1. None of its neighbours is coloured 1, otherwise (a, b) has interference 1. If three neighbours have the same colour, then each of them will have interference 1. So two of its neighbours have to be coloured 2 and the two other ones 3 (see Figure 3(a)). Now consider the four nodes $(a-1, b-1)$, $(a-1, b+1)$, $(a+1, b-1)$ and $(a+1, b+1)$. For all configurations, at least two of these four vertices have to be coloured 1 (the ones indicated by a * in Figure 3(a)). But then (a, b) will have interference at least 1, a contradiction. A weighted 0.5-improper 4-colouring of $(\mathfrak{G}^2, w_2)$ can be obtained as follows (see Figure 2(c)): for $0 \leq j \leq 3$, let $B_j = \{(j, 0) + a(4e_1) + b(3e_1 + 2e_2) \mid \forall a, b \in \mathbb{Z}\}$ and $B'_j = \{(j+1, 2) + a(4e_1) +$

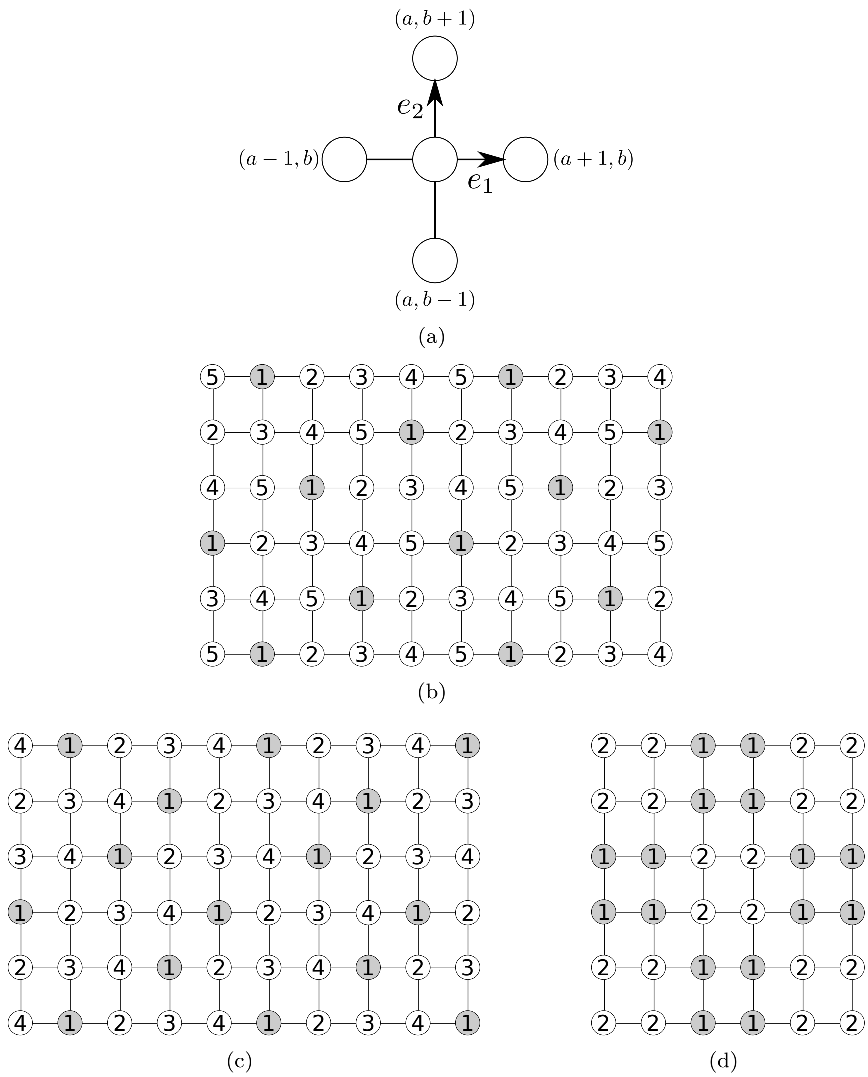


Figure 2: Optimal colourings of $(\mathfrak{S}^2, w_2)$: (b) weighted 0-improper 5-colouring of $(\mathfrak{S}^2, w_2)$, (c) weighted 0.5-improper 4-colouring of $(\mathfrak{S}^2, w_2)$, and (d) weighted 3-improper 2-colouring of $(\mathfrak{S}^2, w_2)$.

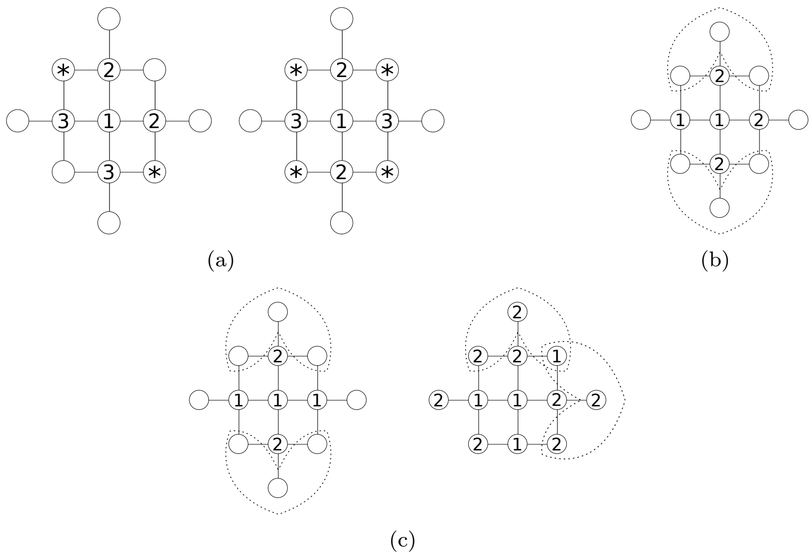


Figure 3: Lower bounds for the square grid: (a) if $t \leq 0.5$ and $k \leq 3$, there is no weighted t-improper k-colouring of $(\mathfrak{G}^2, w_2)$; (b) the first case when $t \leq 2.5$ and $k \leq 2$, and (c) the second case.

$b(3e_1+2e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 3$, assign the colour $j+1$ to all the vertices in B_j and in B'_j.

If $t = 1$, there exists a weighted 1-improper 3-colouring of $(\mathfrak{G}^2, w_2)$ given by the following construction: for $0 \leq j \leq 2$, let $C_j = \{(j, 0) + a(3e_1) + b(e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 2$, assign the colour $j+1$ to all the vertices in C_j.

Now we prove by contradiction that for $t = 2.5$ we still need at least three colours in a weighted 2.5-improper colouring of $(\mathfrak{G}^2, w_2)$. Consider a weighted 2.5-improper 2-colouring of $(\mathfrak{G}^2, w_2)$ and let (a, b) be a vertex coloured 1. Vertex (a, b) has at most two neighbours of colour 1, otherwise it will have interference 3. We distinguish three cases:

1. Exactly one of its neighbours is coloured 1; let $(a-1, b)$ be this vertex. Then, the three other neighbours are coloured 2 (see Figure 3(b)). Consider the two sets of vertices $\{(a-1, b-1), (a+1, b-1), (a, b-2)\}$ and $\{(a-1, b+1), (a+1, b+1), (a, b+2)\}$ (these sets are surrounded by dotted lines in Figure 3(b)); each of them has at least two vertices coloured 1, otherwise the vertex $(a, b-1)$ or $(a, b+1)$ will have interference 3. But then (a, b) having four vertices at distance two coloured 1 has interference 3, a contradiction.

2. Two neighbours of (a, b) are coloured 1.

(a) These two neighbours are opposite, say $(a-1, b)$ and $(a+1, b)$ (see Figure 3(c) left). Consider again the two sets $\{(a-1, b-1), (a+1, b-1), (a, b-2)\}$ and $\{(a-1, b+1), (a+1, b+1), (a, b+2)\}$ (these sets are surrounded by dotted lines in Figure 3(c) left); they both contain at least one vertex of colour 1 and therefore (a, b) will have interference 3, a contradiction.

(b) The two neighbours of colour 1 are of the form $(a, b-1)$ and $(a-1, b)$ (see Figure 3(c) right). Consider the two sets of vertices $\{(a+1, b-1), (a+1, b+1), (a+2, b)\}$ and $\{(a+1, b+1), (a-1, b+1), (a, b+2)\}$ (these sets are surrounded by dotted lines in Figure 3(c) right); these two sets contain at most one vertex of colour 1, otherwise (a, b) will have interference 3. Moreover, each of these sets cannot be completely coloured 2, otherwise $(a+1, b)$ or $(a, b+1)$ will have interference at least 3. So vertices $(a+1, b-1)$, $(a+2, b)$, $(a, b+2)$ and $(a-1, b+1)$ are of colour 2 and the vertex $(a+1, b+1)$ is of colour 1. But then $(a-2, b)$ and $(a-1, b-1)$ are of colour 2, otherwise (a, b) will have interference 3. Thus, vertex $(a-1, b)$ has exactly one neighbour coloured 1 and we are again in Case 1.

3. All neighbours of (a, b) are coloured 2. If one of these neighbours has itself a neighbour (distinct from (a, b)) of colour 2, we are in Case 1 or 2 for this neighbour. Therefore, all vertices at distance two from (a, b) have colour 1 and the interference in (a, b) is 4, a contradiction.

A weighted 3-improper 2-colouring of $(\mathfrak{S}^2, w_2)$ can be obtained as follows: a vertex of the grid (a, b) is coloured with colour $(\lfloor \frac{a}{2} \rfloor + \lfloor \frac{b}{2} \rfloor \mod 2) + 1$, see Figure 2(d).

Finally, since each vertex has four neighbours and eight vertices at distance two, there is no weighted 7.5-improper 1-colouring of $(\mathfrak{S}^2, w_2)$ and, whenever $t \geq 8$, one colour suffices. □

3.2.2. Hexagonal Grid

There are many ways to define the system of coordinates of the hexagonal grid. Here, we use grid coordinates as shown in Figure 4. The hexagonal grid graph is then the graph $\mathfrak{H}$ whose vertex set consists of pairs of integers $(a, b) \in \mathbb{Z}^2$ and where each vertex (a, b) has three neighbours: $(a-1, b)$, $(a+1, b)$, and $(a, b+1)$ if $a+b$ is odd, or $(a, b-1)$ otherwise.

Theorem 12.

$$\chi_t(\mathfrak{H}^2, w_2) = \begin{cases} 4, & \text{if } 0 \leq t < 1; \\ 3, & \text{if } 1 \leq t < 2; \\ 2, & \text{if } 2 \leq t < 6; \\ 1, & \text{if } 6 \leq t. \end{cases}$$

Proof. Note first, that when $t = 0$, at least four colours are needed to colour the grid, because a vertex and its neighbourhood in $\mathfrak{H}$ form a clique of size four in $\mathfrak{H}^2$. The same number of colours are needed if we allow a threshold $t = 0.5$. To prove this fact, let A be a vertex (a, b) of $\mathfrak{H}$ and $B = (a-1, b)$, $C = (a, b-1)$ and $D = (a+1, b)$ be its neighbours in $\mathfrak{H}$. Denote by $G = (a-2, b)$, $E = (a-1, b-1)$, $F = (a-2, b-1)$, $H = (a+1, b-1)$, $I = (a+2, b-1)$ and $J = (a+1, b-2)$ (see Figure 6(a)). By contradiction, suppose there exists a weighted 0.5-improper

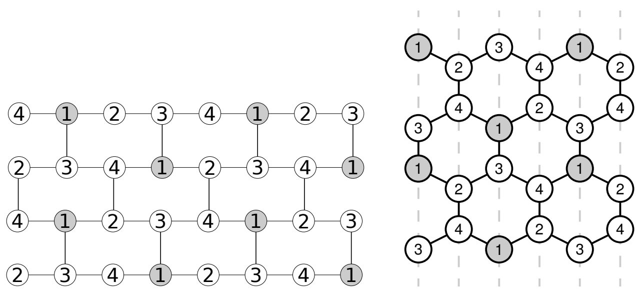
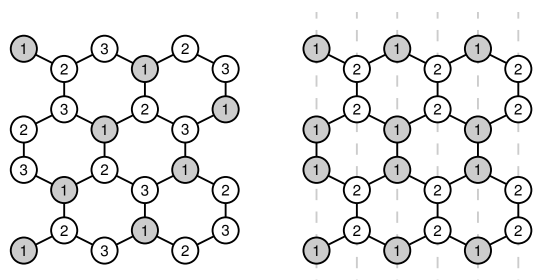


Figure 4: Weighted 0-improper 4-colouring of $(\mathfrak{H}^2, w_2)$. Left: Graph with coordinates. Right: Corresponding hexagonal grid in the euclidean space.



(a) $t = 1$, $k = 3$ (b) $t = 2$, $k = 2$

Figure 5: (a) weighted 1-improper 3-colouring of $(\mathfrak{H}^2, w_2)$ and (b) weighted 2-improper 2-colouring of $(\mathfrak{H}^2, w_2)$.

3-colouring of $\mathfrak{H}^2$. Consider a node A coloured 1. Its neighbours B, C, D cannot be coloured 1 and they cannot all have the same colour. W.l.o.g., suppose that two of them B and C have colour 2 and D has colour 3. Then E, F and G cannot be coloured 2 because of the interference constraint in B and C. If F is coloured 3, then G and E are coloured 1, creating interference 1 in A. So F must be coloured 1 and G and E must be coloured 3. Then, H can be neither coloured 2 (interference in C) nor 3 (interference in E). So H is coloured 1. The vertex I is coloured 3, otherwise the interference constraint in H or in C is not satisfied. Then, J can receive neither colour 1, because of the interference in H, nor colour 2, because of the interference in C, nor colour 3, because of the interference in I.

There exists a construction attaining this bound and the number of colours, i.e. a 0-improper 4-colouring of $(\mathfrak{H}^2, w_2)$ as depicted in Figure 4. We define for $0 \leq j \leq 3$ the sets of vertices $A_j = \{(j,0) + a(4e_1) + b(2e_1 + e_2) | \forall a, b \in \mathbb{Z}\}$. We then assign the colour $j+1$ to the vertices in A_j. This way no vertex experiences any interference as vertices of the same colours are at distance at least three.

For $t = 1.5$ it is not possible to colour the grid with less than three colours. By contradiction, suppose that there exists a weighted 1.5-improper 2-colouring. Consider a vertex A coloured 1. If all of its neighbours are coloured 2, they have already interference 1, so all the vertices at distance two from A need to be coloured 1; this gives interference 3 in A. Therefore one of A's neighbours, say D, has to be coloured 1 and consider that the other two neighbours B and C are coloured 2. B and C have at most one neighbour of colour 2. It implies that A has at least two vertices at distance two coloured 1. This is a contradiction, because the interference in A would be at least 2 (see Figure 6(b)).

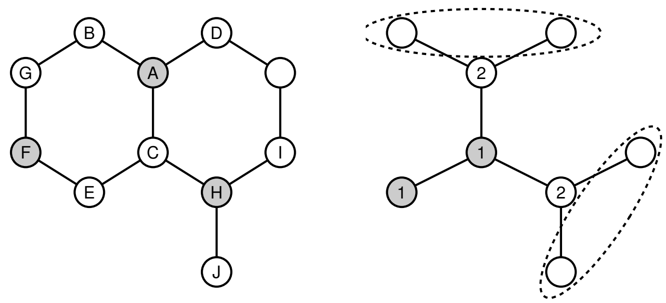
Figure 5(a) presents a weighted 1-improper 3-colouring of $(\mathfrak{H}^2, w_2)$. To obtain this colouring, let $B_j = \{(j,0) + a(3e_1) + b(e_1 + e_2) \mid \forall a, b \in \mathbb{Z}\}$, for $0 \leq j \leq 2$. Then, we colour all the vertices in the set B_j with colour $j+1$, for every $0 \leq j \leq 2$.

For $t < 6$, it is not possible to colour the grid with one colour. As a matter of fact, each vertex has three neighbours and six vertices at distance two in $\mathfrak{H}$. Using one colour leads to an interference equal to 6. There exists a 2-improper 2-colouring of the hexagonal grid as depicted in Figure 5(b). We define for $0 \leq j \leq 1$ the sets of vertices $C_j = \{(j,0) + a(2e_1) + be_2 | \forall a, b \in \mathbb{Z}\}$. We then assign the colour $j+1$ to the vertices in C_j.

□

3.2.3. Triangular Grid

The triangular grid is the graph $\mathfrak{T}$ whose vertices are all the integer linear combinations $af_1 + bf_2$ of the two vectors $f_1 = (1,0)$ and $f_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with the ordered pairs (a,b) of integers. Each vertex $v = (a,b)$ has six neighbours: its *right neighbour* $(a+1,b)$, its *right-up neighbour* $(a,b+1)$, its *left-up neighbour* $(a-1,b+1)$, its *left neighbour* $(a-1,b)$, its *left-down neighbour* $(a,b-1)$ and its *right-down neighbour* $(a+1,b-1)$ (see Figure 8(a)).



(a) (b)

Figure 6: Lower bounds for the hexagonal grid. (a) when $t \leq 0.5$ and $k \leq 3$, there is no weighted t-improper k-colouring of $(\mathfrak{H}^2, w_2)$; (b) vertices coloured 2 force a vertex coloured 1 in each ellipse, leading to interference 2 in central node.

Theorem 13.

$$\chi_t(\mathfrak{T}^2, w_2) = \begin{cases} 7, & \textit{if } t = 0; \\ 6, & \textit{if } t = 0.5; \\ 5, & \textit{if } t = 1; \\ 4, & \textit{if } 1.5 \leq t < 3; \\ 3, & \textit{if } 3 \leq t < 5; \\ 2, & \textit{if } 5 \leq t < 12; \\ 1, & \textit{if } 12 \leq t. \end{cases}$$

Proof. If $t = 0$, there is no weighted 0-improper 6-colouring of $(\mathfrak{T}^2, w_2)$, since in $\mathfrak{T}^2$ there is a clique of size seven induced by each vertex and its neighbourhood. There is a weighted 0-improper 7-colouring of $(\mathfrak{T}^2, w_2)$ as depicted in Figure 7(a). This colouring can be obtained by the following construction: for $0 \leq j \leq 6$, let $A_j = \{(j, 0) + a(7f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 6$, assign the colour $j + 1$ to all the vertices in A_j.

In what follows, we denote by V_0 a vertex coloured 1; by N_0, N_1, N_2, N_3, N_4, N_5 the six neighbours of V_0 in $\mathfrak{T}$ be in a cyclic order. Let Γ^2 be the set of twelve vertices at distance two of V_0 in $\mathfrak{T}$; more precisely $N_{i(i+1)}$ denotes the vertex of Γ^2 adjacent to both N_i and N_{i+1} and by N_{ii} the vertex of Γ^2 joined only to N_i, for every $0 \leq i \leq 5$, $i + 1$ is taken modulo 6 (see Figure 8(b)) and we denote by N_{ijk} the vertex at distance three from V_0 adjacent to both N_{ij} and N_{jk} .

We claim that there is no weighted 0.5-improper 5-colouring of $(\mathfrak{T}^2, w_2)$. We prove it by contradiction, thus let c be such a colouring. No neighbour of V_0 can be coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$. As two consecutive neighbours are adjacent, they cannot have the same colour. Furthermore, there cannot be three neighbours with the same colour (each of them will have an interference at least 1). As there are four colours different from 1, exactly two of them, say 2 and 3, are repeated twice among the six neighbours. So, there exists a sequence

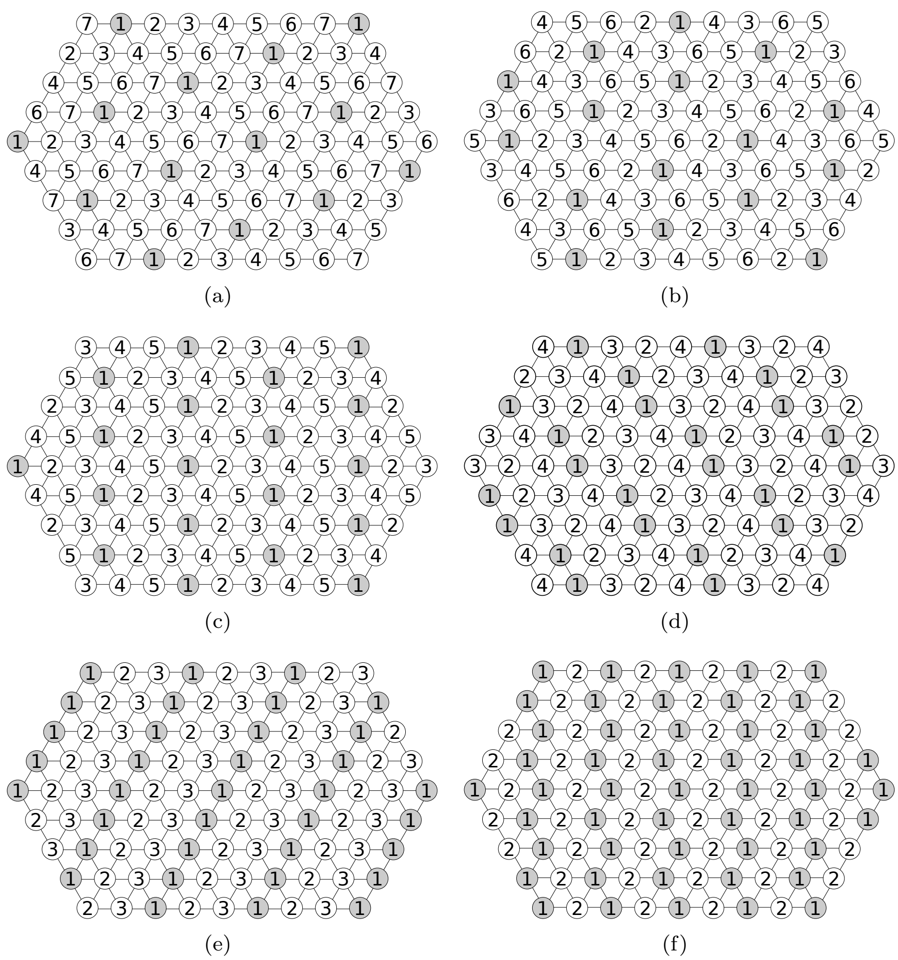
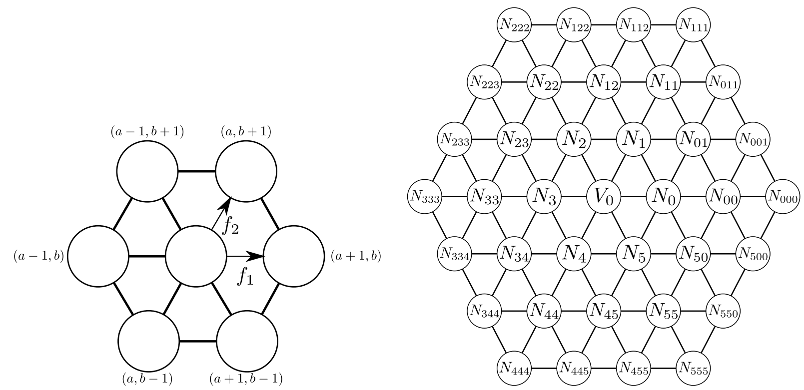


Figure 7: Optimal colourings of $(\mathfrak{T}^2, w_2)$: (a) weighted 0-improper 7-colouring, (b) weighted 0.5-improper 6-colouring, (c) weighted 1-improper 5-colouring, (d) weighted 1.5-improper 4-colouring, (e) weighted 3-improper 3-colouring, and (f) weighted 5-improper 2-colouring.



(a) (b)

Figure 8: Notations used in proofs: (a) of existence, and (b) of non-existence; of weighted improper colourings of $(\mathfrak{T}^2, w_2)$.

of three consecutive neighbours the first one with a colour different from 2 and 3 and the two others coloured 2 and 3. W.l.o.g., let $c(N_5) = 4$, $c(N_0) = 2$, $c(N_1) = 3$.

Note that the vertices coloured 2 and 3 have already an interference of 0.5, and so none of their vertices at distance two can be coloured 2 or 3. In particular, let $A = \{N_{50}, N_{00}, N_{01}, N_{11}, N_{12}\}$; the vertices of A cannot be coloured 2 or 3. At most one vertex in Γ^2 can be coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$. If there is no vertex coloured 1 in A, we have a contradiction as we cannot have a sequence of five vertices uniquely coloured 4 and 5 (indeed colours should alternate and the vertex in the middle N_{01} will have interference at least 1). Suppose N_4 is coloured 3, then N_{45} and N_{55} can only be coloured 1 and 5; but, as they have different colours, one is coloured 1 and so there is no vertex coloured 1 in A. So the second vertex coloured 3 in the neighbourhood of V_0 is necessarily N_3 (it cannot be N_2 neighbour of N_1 coloured 3). Then, N_4 cannot be also coloured 5, otherwise N_{45} is coloured 1 and again there is no vertex coloured 1 in A. In summary $c(N_4) = 2$, $c(N_3) = 3$ and the vertex of Γ^2 coloured 1 is in A. But then the five consecutive vertices $A' = \{N_{23}, N_{33}, N_{34}, N_{44}, N_{45}\}$ can only be coloured 4 and 5. A contradiction as $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 1$.

A weighted 0.5-improper 6-colouring of $(\mathfrak{T}^2, w_2)$ can be obtained by the following construction (see Figure 7(b)): for $0 \leq j \leq 11$, let $B_j = \{(j, 0) + a(12f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 5$, assign the colour $j + 1$ to all the vertices in B_j, B_6 with colour 2, B_7 with colour 1, B_8 with colour 4, B_9 with colour 3, B_{10} with colour 6 and B_{11} with colour 5.

Now we prove that $(\mathfrak{T}^2, w_2)$ does not admit a weighted 1-improper 4-colouring. Again, by contradiction, suppose that there exists a weighted 1-improper 4-colouring c of $(\mathfrak{T}^2, w_2)$. We analyse some cases:

1. There exist two adjacent vertices in $\mathfrak{T}$ with the same colour.

Let V_0 and one of its neighbours be both coloured 1. Note that no other neighbour of V_0, nor the vertices at distance two from V_0 are coloured 1 (otherwise, $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$). We use intensively the following facts:

Fact 1. *There do not exist three consecutive vertices with the same colour (otherwise the vertex in the middle would have interference at least 2).*

Fact 2. *In a path of five vertices there cannot be four of the same colour (otherwise the second or the fourth vertex in this path would have interference at least 1.5).*

One colour other than 1 should appear at least twice in the neighbourhood of V_0. Let this colour be denoted 2 (the other colours being denoted 3 and 4).

(a) Two neighbours of V_0 coloured 2 are consecutive, say N_0 and N_1. By Fact 1, N_2 is coloured 3 w.l.o.g. None of N_{05}, N_{00}, N_{01}, N_{11}, N_{12}, N_{22} and N_{23} can be coloured 2, otherwise $I_{N_1}(\mathfrak{T}^2, w_2, c) > 1$. One of N_{12}, N_{22} and N_{23} is coloured 3, otherwise we contradict Fact 1 with colour 4 and at most one of N_{01}, N_{11}, N_{12}, N_{22} and N_{23} is coloured 3, otherwise $I_{N_2}(\mathfrak{T}^2, w_2, c) > 1$; but we have a contradiction with Fact 2.

(b) Two neighbours of V_0 coloured 2 are at distance two, say N_0 and N_2. Then N_{50}, N_{00} and N_{01} (respectively N_{12}, N_{22} and N_{23}) are not coloured 2, otherwise $I_{N_0}(\mathfrak{T}^2, w_2, c) > 1$ (respectively $I_{N_2}(\mathfrak{T}^2, w_2, c) > 1$). One of N_3 and N_5 is not coloured 1, say N_3. It is not coloured 2, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) > 1$. Let $c(N_3) = 3$. If N_4 or N_{11} is coloured 2, then N_{33} and N_{34} are not coloured 2, otherwise $I_{N_2}(\mathfrak{T}^2, w_2, c) > 1$ and we have a sequence of five vertices N_{12}, N_{22}, N_{23}, N_{33} and N_{34} contradicting Fact 2 as four are of colour 4 (indeed, at most one is coloured 3 due to interference in colour 3 with N_3 or N_{22}). So N_{11} is coloured 3 or 4. If N_1 also is coloured 3 or 4, we have a contradiction with Fact 2 applied to the five vertices N_{00}, N_{01}, N_{11}, N_{12} and N_{22}, by the same previous argument. So $c(N_1) = 1$; furthermore N_4 is not coloured 1 (at most one neighbour coloured 1), nor 2 as we have seen above, nor 3, otherwise we are in the case (a). Therefore $c(N_4) = 4$ and $c(N_5) = 3$, by the same reason. But then $c(N_{23}) = 4$, otherwise the interference in V_0 or N_2 or N_3 is greater than 1. N_{33} and N_{34} can be only coloured 2, otherwise V_0, N_3, N_4 or N_{23} will have interference strictly greater than 1, but N_{33} has interference greater than 1, a contradiction.

(c) Two neighbours of V_0 coloured 2 are at distance three say N_0 and N_3. Then N_{50}, N_{00} and N_{01} (respectively N_{23}, N_{33} and N_{34}) are not coloured 2, otherwise $I_{N_0}(\mathfrak{T}^2, w_2, c) > 1$ (respectively $I_{N_3}(\mathfrak{T}^2, w_2, c) > 1$). W.l.o.g., let N_1 be the vertex coloured 1. Among the four vertices N_{12}, N_{22}, N_{44} and N_{45} at most one is coloured 2, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) > 1$. So, w.l.o.g, we can suppose N_{44} and N_{45} are coloured 3 or 4; but we have a set of five consecutive vertices N_{23}, N_{33}, N_{34}, N_{44}, N_{45}, contradicting Fact 2 (indeed at most one can be of the colour of N_4).

2. No colour appears in two adjacent vertices of $\mathfrak{T}$.

   Let V_0 be coloured 1. No colour can appear four or more times among the neighbours of V_0, otherwise there are two adjacent neighbours with the same colour.

(a) One colour appears three times among the neighbours of V_0, say $c(N_0) = c(N_2) = c(N_4) = 2$. W.l.o.g., let $c(N_1) = 3$. No vertex at distance two can be coloured 2. N_{01}, N_{11} and N_{12} being neighbours of N_1 cannot be coloured 3 and they cannot be all coloured 4. So one of N_{01}, N_{11}, N_{12} is coloured 1. Similarly one of N_{23}, N_{33}, N_{34} is coloured 1 (same reasoning with N_3 instead of N_1) and one of N_{45}, N_{55}, N_{50} is coloured 1, so $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$.

(b) The three colours appear each exactly twice in the neighbourhood of V_0.

   i. The same colour appears in some N_i and N_{i+2}, $0 \leq i \leq 3$. W.l.o.g., let $c(N_0) = c(N_2) = 2$ and $c(N_1) = 3$. Then, $c(N_3) = c(N_5) = 4$ and $c(N_4) = 3$. Then, $c(N_{50}) = 1$ or 3, $c(N_{01}) = 1$ or 4. If $c(N_{50}) = 3$ and $c(N_{01}) = 4$, then $c(N_{00}) = 1$. Among N_{50}, N_{00}, N_{01}, at least one has colour 1. Similarly one of N_{12}, N_{22}, N_{23} has colour 1. So $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 1$ and $c(N_{34}) = c(N_{45}) = 2$. Consequently, no matter the colour of N_{44} some vertex will have interference greater than 1.

   ii. We have $c(N_0) = c(N_3) = 2$, $c(N_1) = c(N_4) = 3$ and $c(N_2) = c(N_5) = 4$. Here we find in each of the sets $\{N_{50}, N_{00}, N_{01}\}$,$\{N_{12}, N_{22}, N_{23}\}$ and $\{N_{34}, N_{44}, N_{45}\}$ a vertex coloured 1. Therefore $I_{V_0}(\mathfrak{T}^2, w_2, c) > 1$, a contradiction.

To obtain a weighted 1-improper 5-colouring of $(\mathfrak{T}^2, w_2)$, for $0 \leq j \leq 4$, let $C_j = \{(j, 0) + a(5f_1) + b(2f_1 + f_2) \mid \forall a, b \in \mathbb{Z}\}$. For $0 \leq j \leq 4$, assign the colour $j + 1$ to all the vertices in C_j. See Figure 7(c).

$(\mathfrak{T}^2, w_2)$ has a weighted 1.5-improper 4-colouring as depicted in Figure 7(d). Formally, this colouring can be obtained by the following construction: for $0 \leq j \leq 3$, let $D_j = \{(j, 0) + a(4f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$; then assign colour 4 to all the vertices in D_0, 1 to all the vertices in D_1, 3 to all the vertices in D_2 and 2 to all the vertices in D_3. Now, for $0 \leq j \leq 3$, let $D'_j = \{(j, 1) + a(4f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 3$, assign colour $j + 1$ to all the vertices in D'_j.

Let us prove that $(\mathfrak{T}^2, w_2)$ does not admit a weighted 2.5-improper 3-colouring. Suppose, by contradiction, that there exists a weighted 2.5-improper 3-colouring c of $(\mathfrak{T}^2, w_2)$. A vertex can have at most two neighbours of the same colour as it. Suppose again, w.l.o.g., that $c(V_0) = 1$. We use the following facts:

Fact 3. *No vertex has three neighbours of the same colour.*

Fact 4. *If a vertex has two neighbours of the same colour, then it has at most one vertex at distance two with its colour.*

Fact 5. *There is no path of five vertices of the same colour.*

We say that a vertex v is *saturated*, if we know that $I_v(\mathfrak{T}^2, w_2, c) \geq 2.5$.

Let us analyse now each of these cases.

Case: V_0 **has exactly two neighbours coloured 1**.

We assume, w.l.o.g., that N_0 is coloured 1. We subdivide this case into three subcases according to the position of the second neighbour of V_0 coloured 1. Due to the symmetry, we analyse the three possible cases where respectively N_1, N_2 or N_3 is coloured 1.

1. **Subcase** $c(N_1) = 1$.

We now show that no colouring is feasible, for all possible different colourings of the vertices N_2, N_3, N_4 and N_5 (up to symmetries). We can have all these vertices of the same colour (Case 1a) or three of the same colour, say 2, and the other of colour 3 (Cases 1b and 1c) and two of colour 2 and two of colour 3 (Cases 1d, 1e and 1f).

(a) Suppose that $c(N_2) = c(N_3) = c(N_4) = c(N_5) = 2$. Observe that $c(N_{12}) = c(N_{50}) = 3$, thanks to Facts 3 and 5. Since N_3 and N_4 are saturated, we get that all the vertices N_{22}, N_{23}, N_{33}, N_{34}, N_{44}, N_{45} and N_{55} cannot be coloured 2. At most one of these vertices is coloured 1, due to the interference in V_0. W.l.o.g, we can then consider that $c(N_{22}) = c(N_{23}) = c(N_{33}) = 3$. But then, since N_{23} and N_3 are saturated, we conclude that N_{223}, N_{233}, N_{333}, N_{334} and N_{34} must be all coloured 1. This is a contradiction to Fact 5.

(b) Now consider the case in which $c(N_2) = c(N_3) = c(N_4) = 2$ and $c(N_5) = 3$. Observe that N_{12} cannot be coloured 1. Let us study the two other cases:

i. Now consider the case in which N_{12} is coloured 2. We observe that N_2 and N_3 are saturated.
In case N_{44} is coloured 1, we also have that V_0 is saturated and thus all the vertices N_{22}, N_{23}, N_{33} and N_{34} must be coloured 3. Consequently, as N_{23} and N_{33} are saturated, we reach a contradiction to Fact 5 as all the vertices N_{222}, N_{223}, N_{233}, N_{333} and N_{334} must be coloured 1. Thus, N_{44} is coloured 3 (it cannot be coloured 2 due to Fact 5).
In case N_{33} is coloured 1, we have that V_0 is saturated and all the vertices N_{23}, N_{34} and N_{45} are coloured 3. As N_{34} is saturated, the vertices N_{233}, N_{333} and N_{334} must be coloured 1. This contradicts Fact 3. Consequently, N_{33} is coloured 3. N_{34} cannot be coloured 3, because it would imply that $c(N_{45}) = 1$ and, consequently, V_0 is saturated and the vertices N_{22} and N_{23} should be coloured 3 and we would have a contradiction to Fact 5. Thus, N_{34} is coloured 1. Consequently, N_{22}, N_{23} and N_{45} are coloured 3. The vertices N_{334} and N_{344} must then be coloured 1 due to the interference constraints on the vertices N_3, N_{33} and N_{44}. However, we reach a contradiction as no colour is feasible to vertex N_{233} (and N_{333}).

ii. So we conclude that $c(N_{12}) = 3$.

- Consider first the case $c(N_{22}) = 1$ (and thus V_0 is saturated). We have that N_{23}, N_{33} and N_{34} must be coloured 3, thanks to the Facts 3 and 4 and V_0 being saturated. N_{44} cannot be coloured 3 as we would have $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$. Since V_0 is also saturated, it implies that $c(N_{44}) = 2$. Therefore, N_4 is saturated and so $c(N_{45}) = c(N_{55}) = c(N_{50}) = 3$, but then $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

- Thus, consider the case $c(N_{22}) = 2$. Then, N_2 and N_3 are saturated. One of the vertices N_{33}, N_{34}, N_{44} and N_{45} is coloured 1, thanks to Fact 5. So V_0 is saturated and $c(N_{01}) = c(N_{11}) = c(N_{23}) = 3$. Then, N_{112} and N_{122} cannot be coloured 3, otherwise $I_{N_{12}}(\mathfrak{T}^2, w_2, c) \geq 3$; they cannot be coloured 2 as N_2 is saturated; so $c(N_{112}) = c(N_{122}) = 1$, but then we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.
- We then conclude that $c(N_{22}) = 3$. Due to Facts 3 and 5, at least one of the vertices N_{23}, N_{33} and N_{34} is coloured 1 and the two others are coloured 3. Consequently, V_0 is saturated. In case N_{44} is coloured 2, then N_4 is saturated and the vertices N_{45}, N_{55} and N_{50} are coloured 3, contradicting Fact 3. Thus, $c(N_{44}) = 3$.
  - If N_{45} is coloured 2, N_3 and N_4 are saturated and then, N_{55} and N_{50} are coloured 3 and it implies that N_5 is saturated. Consequently, N_{34} is coloured 1 and N_{23} and N_{33} are coloured 3.

    Thus, N_{23} is saturated and the vertices N_{223}, N_{233}, N_{333} and N_{334} are coloured 1, contradicting Fact 5.
  - Thus, N_{45} is also coloured 3 and we obtain $c(N_{55}) = 2$. N_{23} cannot be coloured 1, otherwise N_{33} and N_{34} being coloured 3, we would contradict Fact 5. If N_{34} is coloured 3, N_{44} is saturated and then N_{50} must be coloured 2 and N_4 is saturated. In this case, we get a contradiction to Fact 5 because all the vertices N_{334}, N_{344}, N_{444} and N_{445} must be coloured 1.

    So $c(N_{23}) = c(N_{33}) = 3$, $c(N_{34}) = 1$ and $c(N_{11}) = 2$.

    If N_{01} is coloured 2, we have that N_2 is saturated and, since N_{22} is saturated, we have that the vertices N_{112}, N_{122}, N_{222}, N_{223} and N_{233} must be all coloured 1, contradicting Fact 5. Thus, N_{01} is coloured 3 and then N_{50} must be coloured 2, due to the interference constraint in N_5.

    Consequently, N_4 is saturated and all the vertices N_{344}, N_{444}, N_{445} and N_{455} must be coloured 1, due to the interference constraints in N_4, N_{44} and N_{45}. This contradicts Fact 5.

(c) Let us consider now the case $c(N_2) = c(N_3) = c(N_5) = 2$ and $c(N_4) = 3$. Recall that N_{12}, N_{11}, N_{01}, N_{00} and N_{50} cannot be coloured 1.

i. Let us study the case $c(N_{12}) = 2$. In this case, N_2 is saturated and thus N_{01} and N_{11} must be coloured 3.

- In case N_{34} is coloured 1, the vertices N_{22}, N_{23} and N_{33} must be coloured 3 as V_0 and N_2 are saturated. Consequently, N_{23} is also saturated. It implies that the vertices N_{122}, N_{222}, N_{223} and N_{233} must be all coloured 1. By Fact 5, we conclude that N_{333} must be coloured 2 and then N_3 is also saturated. Consequently, $c(N_{44}) = c(N_{45}) = 3$, but then N_4 has interference at least 3, a contradiction.
- Thus we conclude that N_{34} is coloured 3, as it cannot be coloured 2 thanks to the interference constraint on N_2. Observe that none of the vertices N_{44} and N_{45} can be coloured 1, as it would imply that V_0 is saturated and that the vertices N_{22}, N_{23} and N_{33} should be all coloured 3, leading to a contradiction to Fact 5. N_{44} and N_{45} can neither be both coloured 2 nor 3, due to interference constraints in N_3 and N_4, respectively.

In case $c(N_{44}) = 2$ and $c(N_{45}) = 3$, observe that among N_{23} and N_{33} we have one vertex coloured 1 and the other is coloured 3. Consequently, V_0 and N_4 are both saturated and N_{55} and N_{50} must be coloured 2. But then $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

In case $c(N_{44}) = 3$ and $c(N_{45}) = 2$, we conclude that N_{33} is coloured 1, thanks to Fact 3, and thus V_0 is saturated; consequently, $c(N_{23}) = 3$ and N_4 is saturated, but then $c(N_{55}) = c(N_{50}) = 2$ and $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. Then consider that N_{12} is coloured 3. We claim that neither N_{22} nor N_{23} can receive colour 2. For otherwise, suppose the case where at least one of these vertices would be coloured 2. As N_2 would be saturated, the vertices N_{01} and N_{11} should be both coloured 3. This would imply that N_{112} and N_{122} should be coloured 1 and 3, respectively, due to Fact 3 and the interference constraint in N_1 and N_2. Consequently, as N_1 and N_{12} would be both saturated, N_{22} and N_{23} should be both coloured 2, a contradiction to Fact 3. Observe that N_{22} and N_{23} cannot be both coloured 1 due to the interference in V_0. Let us study the three remaining cases:

- $c(N_{22}) = 1$ and $c(N_{23}) = 3$. At most one of the vertices N_{33} and N_{34} is coloured 2, due to Fact 3. If exactly one of them is coloured 2 (and thus the other is coloured 3 thanks to the interference in V_0), as N_3 is saturated, N_{44} and N_{45} must be coloured 3. This is a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. Thus, N_{33} and N_{34} are both coloured 3 and it implies that N_{44} and N_{45} are both coloured 2, because of Facts 3 and 5. As N_{45} is saturated, N_{55} and N_{50} are both coloured 3 and we reach a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{22}) = 3$ and $c(N_{23}) = 1$. If N_{33} is coloured 2, we observe that N_3 is saturated and N_{34}, N_{44} and N_{45} must be all coloured 3. This contradicts Fact 3.

  We conclude that $c(N_{33}) = 3$. If N_{34} is coloured 2, N_3 is saturated and N_{44} and N_{45} are both coloured 3. Then, N_4 is saturated and $c(N_{55}) = c(N_{50}) = 2$. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. Then, $c(N_{34}) = 3$ and then N_{44} is coloured 2. If N_{45} is coloured 3, N_4 is saturated and then N_{55} and N_{50} must be both coloured 2. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. So $c(N_{45}) = 2$ and N_5 is saturated. As a consequence, we get $c(N_{55}) = c(N_{50}) = c(N_{00}) = c(N_{01}) = 3$. This is another contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{22}) = 3$ and $c(N_{23}) = 3$. N_{33} cannot be coloured 3 thanks to the interference constraint in N_{23}.
  - If $c(N_{33}) = 2$, then N_3 is saturated. In this case, N_{34}, N_{44} and N_{45} cannot be all coloured 3 (Fact 3). So one of them is coloured 1 and the two others are coloured 3 implying that V_0 and N_4 are saturated and N_{55} and N_{50} are both coloured 2. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - If $c(N_{33}) = 1$, then V_0 is saturated. In case N_{34} is coloured 2, N_3 is also saturated and N_{44} and N_{45} must be both coloured 3. Then N_4 is saturated and N_{55} and N_{50} are both coloured 2. This is a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

    Thus we know that $c(N_{34}) = 3$. In case N_{44} is coloured 3, N_4 is saturated and N_{45}, N_{55} and N_{50} should be all coloured 2. This contradicts Fact 3.

Then $c(N_{44}) = 2$. So N_{44} is coloured 2 and we know that N_{23} is saturated. Then, among N_{233}, N_{333} and N_{334} there is exactly one vertex coloured 2, due to Fact 3 and to the interference in N_3. As N_3 is saturated, we conclude that $c(N_{45}) = 3$. But N_4 is saturated, N_{55} and N_{50} must be coloured 2 and we find a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.

(d) Now, we study the case $c(N_2) = c(N_3) = 2$ and $c(N_4) = c(N_5) = 3$. Observe that colours 2 and 3 are symmetric under these hypothesis. In order to use this symmetry, let us consider the possible colourings of N_{23} and N_{45} (up to the symmetries):

i. In case $c(N_{23}) = 2$ and $c(N_{45}) = 3$, observe that N_{34} is necessarily coloured 1, thanks to Fact 3. Consequently, V_0 is saturated, N_{33} is coloured 3 and N_{44} is coloured 2. It implies that N_3 and N_4 are also saturated and that N_{334} and N_{344} are both coloured 1. As N_{34} is also saturated, N_{233} and N_{333} are coloured 3. Moreover, N_{22} is also coloured 3 as V_0 and N_3 are saturated. This is a contradiction as $I_{N_{33}}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. Now consider that $c(N_{23}) = 2$ and $c(N_{45}) = 2$. Since N_3 is saturated and Fact 3 holds, among N_{34} and N_{44} we have one vertex coloured 1 and the other is coloured 3. So V_0 is saturated, N_{33} is coloured 3 and N_4 is then saturated. Consequently, N_{334} and N_{344} are coloured 1 and N_{55} and N_{50} are coloured 2. N_{444} can neither be coloured 3 as N_4 is saturated, nor 1 as $I_{N_{344}}(\mathfrak{T}^2, w_2, c) \geq 3$. So $c(N_{45})$ is saturated and N_{445} and N_{455} are both coloured 1. This is a contradiction as either $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$ or $I_{N_{44}}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. Let us study the case $c(N_{23}) = 2$ and $c(N_{45}) = 1$. So, $c(N_{33}) = c(N_{34}) = 3$ and $c(N_{44}) = 2$. As N_3, N_4 and N_{34} are saturated, N_{233}, N_{333}, N_{334} and N_{344} are coloured 1. As N_3 is saturated, $c(N_{12}) = c(N_{22}) = 3$. N_4 and N_{34} saturated imply that N_{233}, N_{333}, N_{334} and N_{344} are coloured 1. So, by Fact 5, $c(N_{233}) = 3$ and N_{22} is saturated. Consequently, $c(N_{11}) = 2$ and N_2 is saturated. Therefore, $c(N_{112}) = c(N_{122}) = 1$, but we have a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.

iv. We now deal with the case $c(N_{23}) = 1$ and $c(N_{45}) = 2$. Observe that N_{33} cannot be coloured 2, because in this case V_0 and N_3 are saturated and we would have a contradiction to Fact 3 as N_{34} and N_{44} should be both coloured 3. Consequently, N_{33} is coloured 3. In case N_{34} is coloured 3, N_4 is saturated and then N_{45}, N_{55} and N_{50} are coloured 2. This is a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$. So $c(N_{34}) = 2$ and N_3 is saturated. As a consequence, N_{44} is coloured 3 and N_4 is also saturated and the vertices N_{55} and N_{50} must be coloured 2. It implies that N_{45} is saturated and $c(N_{344}) = c(N_{444}) = c(N_{445}) = c(N_{455}) = 1$. As N_3 and N_4 are saturated, N_{334} should be also coloured 1, but this contradicts Fact 5.

v. We now deal with the last subcase in which $c(N_{23}) = 3$ and $c(N_{45}) = 2$ (Recall that colours 2 and 3 are once more symmetric).

- If $c(N_{33}) = 2$, N_3 is saturated. Then N_{34} and N_{44} cannot receive colour 2, cannot be both coloured 1 (Fact 4 with V_0) and cannot be both coloured 3 (Fact 4 with N_4).

– In case $c(N_{34}) = 1$ and $c(N_{44}) = 3$, N_4 and V_0 are saturated. Consequently, $c(N_{334}) = c(N_{344}) = 1$ and N_{34} is also saturated. Thus, $c(N_{12}) = c(N_{22}) = c(N_{233}) = c(N_{333}) = 3$. This is a contradiction to Fact 5.

– So $c(N_{34}) = 3$ and $c(N_{44}) = 1$. One more V_0, N_3 and N_4 are saturated. It implies that $c(N_{12}) = c(N_{22}) = 3$ and then N_{23} is also saturated. Consequently, the vertices N_{233}, N_{333}, N_{334} and N_{344} must be all coloured 1. This contradicts Fact 5.

As $c(N_{33}) \neq 2$, by symmetry, we conclude that $c(N_{44}) \neq 3$. We use this information in the following subcase.

- If $c(N_{33}) = 3$, observe that N_{34} cannot be coloured 3, thanks to Fact 5. Recall that N_{44} is either coloured 1 or 2, by symmetry. Moreover, N_{34} and N_{44} cannot be both coloured 2 due to Fact 5.

– In case $c(N_{34}) = 2$ and $c(N_{44}) = 1$, V_0 and N_3 are saturated. This implies that $c(N_{12}) = c(N_{22}) = 3$. This is a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– So $c(N_{34}) = 1$ and $c(N_{44}) = 2$. N_{55} and N_{50} cannot be both coloured 2, otherwise $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$. So one is coloured 3 and N_4 is saturated. Similarly, N_{12} and N_{22} cannot be both coloured 3, otherwise $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. Thus, one of them is coloured 2 and N_3 is saturated. Then, $c(N_{334}) = c(N_{344}) = 1$ and N_{34} is saturated. Since N_3 is also saturated, we have that $c(N_{233}) = c(N_{333}) = 3$, but then $I_{N_{33}}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

As N_{33} cannot be coloured 3, again by symmetry we conclude that N_{44} cannot be coloured 2. Thus, we have a contradiction to Fact 4 in V_0 as $c(N_{33}) = c(N_{44}) = 1$.

(e) Let us consider now that $c(N_2) = c(N_4) = 2$ and $c(N_3) = c(N_5) = 3$. By Facts 3 and 4, there is at most one vertex in Γ^2 coloured 1. By symmetry, we consider w.l.o.g. that this vertex is in $\{N_{22}, N_{23}, N_{33}, N_{34}\}$. So we know that N_{44}, N_{45} and N_{55} are not coloured 1.

i. $c(N_{34}) = 1$ (and then V_0 is saturated).

- $c(N_{44}) = c(N_{45}) = 2$. In this case, N_4 is saturated. So, $c(N_{23}) = c(N_{33}) = c(N_{55}) = c(N_{50}) = 3$ and N_3 and N_5 are saturated. We then reach a contradiction because $c(N_{334}) = c(N_{344}) = c(N_{445}) = 1$ and then $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{44}) = c(N_{45}) = 3$. So N_{45} is saturated and $c(N_{55}) = c(N_{50}) = 2$. Observe that N_{23} and N_{33} cannot be both coloured 3, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$. If both N_{23} and N_{33} are coloured 2, then N_4 is also saturated and then N_{334}, N_{444}, N_{445} and N_{455} are all coloured 1, contradicting Fact 5. So among N_{23} and N_{33} we have one vertex coloured 2 and the other is coloured 3 and, consequently, N_3 is saturated. So N_{12} and N_{22} are coloured 1 and we have a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.
- Either $c(N_{44}) = 2$ and $c(N_{45}) = 3$, or $c(N_{44}) = 3$ and $c(N_{45}) = 2$. In this case, N_{23} and N_{33} cannot be both coloured 3, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$. Similarly, N_{55} and N_{50} cannot be both coloured 3, otherwise $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. At most two among N_{23}, N_{33}, N_{55} and N_{50} are coloured 2, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. Consequently, one vertex among N_{23} and N_{33} is coloured

2 and the other is coloured 3, the same happens for vertices N_{55} and N_{50} and, then, N_4 is saturated. N_{12} and N_{22} cannot be both coloured 2, otherwise $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of them is coloured 1 and N_3 is saturated, implying that $c(N_{334}) = c(N_{344}) = 1$ and N_{34} is saturated.

If $c(N_{45}) = 3$, then N_5 is saturated and $c(N_{445}) = 1$, but then $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$. If $c(N_{45}) = 2$, we have that $c(N_{44}) = 3$. N_{444} and N_{445} cannot be both coloured 3, otherwise $I_{N_{44}}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of them is coloured 3 and again $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. $c(N_{34}) = 2$. Recall that N_{44}, N_{45} and N_{55} are not coloured 1. Observe that, by Fact 3, at most one of N_{44} and N_{45} is coloured 2. If one of these vertices is coloured 2, N_4 is saturated and N_{55} and N_{50} must be both coloured 1. It implies a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. Consequently, N_{44} and N_{45} are both coloured 3 and N_{45} is saturated. So N_{55} and N_{50} are coloured 2 and N_4 is also saturated implying that $c(N_{344}) = c(N_{444}) = c(N_{445}) = c(N_{455}) = 1$. Since N_{444} is saturated, N_{334} must be coloured 3 and then N_{23} and N_{33} cannot receive colour 3, otherwise $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$. We obtain a contradiction because N_{23} and N_{33} are both coloured 1 and $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. $c(N_{34}) = 3$. Observe that N_{44} and N_{45} cannot be both coloured 3, due to Fact 5.

- $c(N_{44}) = c(N_{45}) = 2$. In this case, N_4 is saturated and then N_{55} and N_{50} must be coloured 3. This is a contradiction because $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{44}) = 2$ and $c(N_{45}) = 3$. Due to the interference in N_5, we have that $c(N_{55}) = c(N_{50}) = 2$ and then N_4 is saturated. However, the vertices N_{23} and N_{33} cannot receive colour 3, due to the interference in N_3, and so they are both coloured 1 and we have a contradiction as $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.
- $c(N_{44}) = 3$ and $c(N_{45}) = 2$. In this case, N_{34} is saturated. If N_{23} and N_{33} are both coloured 2, N_4 is saturated and N_{55} and N_{50} must be coloured 3 and we get $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. So among N_{23} and N_{33} we have one vertex coloured 1 and the other is coloured 2.
  N_{55} and N_{50} can neither be both coloured 3, otherwise $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$, nor both coloured 2, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. So one is coloured 2, the other 3 and N_4 and N_5 are saturated. We then get a contradiction to Fact 5 because $c(N_{334}) = c(N_{344}) = c(N_{444}) = c(N_{445}) = c(N_{455}) = 1$.

(f) Now consider that $c(N_2) = c(N_5) = 2$ and $c(N_3) = c(N_4) = 3$. As in Case 1e, we consider w.l.o.g. that N_{44}, N_{45} and N_{55} are not coloured 1. Observe that N_{44} and N_{45} cannot be both coloured 3, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

i. Consider first that $c(N_{44}) = c(N_{45}) = 2$. Consequently, $c(N_{55}) = c(N_{50}) = 3$ due to the interference constraints in N_{45} and N_5. If N_{00} is coloured 3, N_{50} is saturated and then N_{01} must be coloured 2. As a consequence, N_5 is also saturated and N_{550} and N_{500} must be both coloured 1. This is a contradiction as $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$. So N_{00} is coloured 2 and N_5 is saturated. Thus, $c(N_{01}) = 3$ and N_{550} and N_{500} cannot receive colour 2 (interference in N_5) or 3 (interference in N_{50}). So, $c(N_{550}) = c(N_{500}) = 1$, but them $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. Either $c(N_{44}) = 2$ and $c(N_{45}) = 3$, or $c(N_{44}) = 3$ and $c(N_{45}) = 2$. In this case, observe that N_{55} and N_{50} can neither be both coloured 2 (interference in N_5) nor 3 (interference in N_4). So one is coloured 2, the other is coloured 3 and N_4 is saturated.

- If $c(N_{44}) = 3$ and $c(N_{45}) = 2$, then N_5 is also saturated and N_{34} must be coloured 1. Consequently, V_0 is saturated and $c(N_{23}) = c(N_{33}) = 2$ and $c(N_{00}) = c(N_{01}) = 3$. Due to the interference in N_2, N_{12} and N_{22} must be coloured 3 and then, by Fact 5, N_{11} must be coloured 2. So N_2 is also saturated and then, due to the interference in N_{12}, N_{112} and N_{122} must be both coloured 1. This is a contradiction because $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.
- So $c(N_{44}) = 2$ and $c(N_{45}) = 3$. Observe that N_{33} and N_{34} cannot be both coloured 2, otherwise $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of them is coloured 1 and the other is coloured 2. Thus, V_0 is saturated and N_{23} must be coloured 2. If $c(N_{33}) = 1$ and $c(N_{34}) = 2$, N_{34} is saturated and then $c(N_{334}) = c(N_{344}) = c(N_{444}) = c(N_{445}) = 1$, contradicting Fact 5. So $c(N_{33}) = 2$ and $c(N_{34}) = 1$. Due to the interference in N_2, we have that N_{12} and N_{22} are coloured 3 and then N_3 is also saturated. Then, N_{334} must be coloured 1 due to the interference in N_3 and N_{33}. If N_{344} is coloured 2, N_{33} is saturated and we have a contradiction to Fact 5 because $c(N_{223}) = c(N_{233}) = c(N_{333}) = 1$. So we get $c(N_{344}) = 1$ and N_{34} saturated. This is a contradiction because N_{333} must be coloured 2 and then $I_{N_{33}}(\mathfrak{T}^2, w_2, c) \geq 3$.

2. **Subcase** $c(N_2) = 1$.

W.l.o.g., let $c(N_1) = 2$. We deal with the subcases according to the colouring of N_3, N_4 and N_5: they are all coloured 2 (Case 2a), two of them are coloured 2 (Cases 2b and 2c), only one of them is coloured 2 (Cases 2d and 2e) or they are all coloured 3 (Case 2f).

(a) Consider first the subcase $c(N_3) = c(N_4) = c(N_5) = 2$. In this case, N_4 is saturated and all the vertices N_{23}, N_{33}, N_{34}, N_{44}, N_{45}, N_{55} and N_{50} cannot be coloured 2. Since at most one vertex in Γ^2 is coloured 1, this vertex cannot belong to the set $\{N_{23}, N_{33}, N_{55}, N_{50}\}$ as it would imply a contradiction to Facts 5 in colour 3. So all the vertices in this set are coloured 3, exactly one vertex among N_{34}, N_{44} and N_{45} is coloured 1 and V_0 is saturated. By symmetry, we can consider that N_{45} is coloured 3.

If N_{01} is coloured 2, N_1 is also saturated and all the vertices N_{11}, N_{12} and N_{22} must be coloured 3. This is a contradiction to Fact 5. So $c(N_{01}) = 3$.

In order to avoid a P_5 of vertices coloured 3, N_{00} must be coloured 2. Then, N_{11} and N_{12} must be coloured 3, due to the interference constraint in N_1. Thanks to Fact 5, N_{22} must be coloured 2 and so N_1 and N_3 are saturated. The vertices N_{112} and N_{233} cannot be coloured 3 as we would be in Case 1, then they are both coloured 1 and N_2 is also saturated. Consequently, N_{122} must be coloured 3 and we reach a contradiction as $I_{N_{12}}(\mathfrak{T}^2, w_2, c) \geq 3$.

(b) Let us now suppose that $c(N_3) = c(N_4) = 2$ and $c(N_5) = 3$. We show that there is no feasible colour to N_{44} by examining the three possible cases:

i. Suppose first that N_{44} is coloured 2. So N_4 is saturated and then, if $c(N_{55}) = 3$, as either N_{45} or N_{50} must be coloured 3, we are in Case 1. Thus, N_{55} is coloured 1, V_0 is saturated and N_{23}, N_{33}, N_{34}, N_{45} and N_{50} are coloured 3. Consequently, N_5 is saturated and so N_{00} and N_{01} are coloured 2. Thus, N_1 is saturated and N_{12} and N_{22} must be coloured 3, contradicting Fact 5.

ii. Now consider that $c(N_{44}) = 1$. Thus, V_0 is saturated and N_{34} is coloured 3, otherwise we would be in Case 1.

- Suppose that at one of the vertices N_{23} or N_{33} is coloured 2. Then, N_3 is saturated and the vertices N_{12}, N_{22} and N_{45} must be all coloured 3. So N_{55} is coloured 2, as we are no longer in Case 1, and it implies that N_4 is saturated. As a consequence, N_{50} is coloured 3 and N_5 is also saturated. Thus, N_{00} and N_{01} must be coloured 2, N_1 is saturated and N_{11} is coloured 3. Observe that N_{112} and N_{122} are both coloured 1, otherwise we are in Case 1. So N_2 is also saturated and no feasible colour remains to colour N_{223}.
- So N_{23} and N_{33} are both coloured 3.
  - If N_{22} is coloured 3, N_{12} is coloured 2 (Fact 5), N_{11} is coloured 3 (as we are not in Case 1) and N_{01} is also coloured 3 (interference in V_0 and N_1).

    If $c(N_{00}) = 3$, N_{01} is saturated and then N_{50} is coloured 2. It implies that N_1 is saturated and N_{001} and N_{011} must be both coloured 1. Consequently, N_0 is saturated and N_{000} and N_{500} are both coloured 2. Thus, N_{50} is also saturated and the vertices N_{45} and N_{55} should be both coloured 3. But then we are in Case 1.

    So N_{00} is coloured 2 and N_1 is saturated. Consequently, N_{50} is coloured 3 and N_{55} must be coloured 2 as we are no longer in Case 1. But then no feasible colour remains to colour N_{45}.
  - Thus, we have $c(N_{22}) = 2$. If N_{12} is coloured 2, N_1 is saturated and we have a contradiction to Fact 5, because all the vertices N_{50}, N_{00}, N_{01} and N_{11} must be coloured 3. So, we conclude that $c(N_{12}) = 3$.

    If N_{01} or N_{11} are coloured 2, N_1 is saturated and N_{50} and N_{00} must be coloured 3. In this case, N_{45} and N_{55} cannot receive colour 3, due to the interference in N_5. So they are both coloured 2 and we reach a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

    Consequently, N_{01} and N_{11} are both coloured 3. Observe that N_{45} is also coloured 3, otherwise N_4 is saturated, N_{50} and N_{00} are coloured 3 and we are in Case 1. Consequently, N_{55} and N_{50} are coloured 2, as we are no longer in Case 1 and we cannot violate the interference constraint in N_5. Moreover, N_{00} is also coloured 2, otherwise $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$. But then we have a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. We conclude that N_{44} must be coloured 3. Recall that N_{34} cannot be coloured 2 as we would be in Case 1.

- Consider first the case in which $c(N_{34}) = 1$ and thus V_0 is saturated. If N_{45} is coloured 2, N_4 is saturated and N_{50} and N_{00} should be both coloured 3. But then we are in Case 1. So N_{45} is coloured 3 and N_{55} must be coloured 2. Observe that N_{23} and N_{33} cannot be both coloured 2, due to Fact 3. In case one of these vertices is coloured 2 and the other is coloured 3, observe that

N_3 and N_4 are saturated. Consequently, N_{50} is coloured 3 and N_{45} and N_5 are also saturated. We then reach a contradiction to Fact 5 as all the vertices N_{344}, N_{444}, N_{445} and N_{455} must be coloured 1. So we conclude that N_{23} and N_{33} must be both coloured 3.

If N_{50} is coloured 3, N_5 is saturated. Then, N_{00} and N_{01} must be coloured 2, then N_1 is saturated and we reach a contradiction to Fact 5 as N_{11}, N_{12} and N_{22} must be all coloured 3. So N_{50} is coloured 2 and N_4 is saturated. Consequently, N_{344} and N_{444} are both coloured 1, due to the interference constraints in N_4 and N_{44}. Thus, N_{34} is also saturated and N_{445} must be coloured 3. But then we are in Case 1.

- We deduce that $c(N_{34}) = 3$. We now study the possible colourings of N_{45}.

– If $c(N_{45}) = 2$, N_4 is saturated. The interference constraints in V_0 and N_5 lead us to the conclusion that among N_{55} and N_{50} we have one vertex coloured 1 and the other is coloured 3. Consequently, V_0 is saturated and N_{23} and N_{33} are both coloured 3. This is a contradiction as $I_{N_{34}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– Now consider that $c(N_{45}) = 1$ (V_0 is saturated). The vertices N_{23} and N_{33} cannot be both coloured 2, due to Fact 3. They cannot also be both coloured 3, because of the interference constraint in N_{34}. So among N_{23} and N_{33} we have one vertex coloured 2 and the other is coloured 3 and N_3 is saturated. The vertices N_{55} and N_{50} can neither be both coloured 2, because of the interference in N_4, nor 3, as we are not in Case 1. So one of them is coloured 2 and the other is coloured 3. Thus, N_4 is also saturated.

Similarly, we can conclude that among N_{444} and N_{445} we have one vertex coloured 1 and the other is coloured 3 (recall that these vertices cannot receive colour 2 as N_4 is saturated). Consequently, N_{44} is saturated and the vertices N_{344} and N_{455} must be coloured 1. This is a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– So we have $c(N_{45}) = 3$. Consequently, N_{33}, N_{55} and N_{50} cannot receive colour 3. We thus conclude that two of these vertices are coloured 2 and the other is coloured 1, by considering the interference in V_0 and N_4. We then obtain that N_{334}, N_{344}, N_{444}, N_{445} and N_{455} are all coloured 1. This contradicts Fact 5.

(c) We now treat the case $c(N_3) = c(N_5) = 2$ and $c(N_4) = 3$. Let us consider the possible colours of N_{23}.

i. Suppose first that N_{23} is coloured 1. In this case, V_0 and N_2 are saturated.

- In case N_{33} is coloured 2, N_{34} must be coloured 3 and N_{44} must be coloured 2, otherwise we would be in Case 1. So N_3 is also saturated and N_{45} must be coloured 3. Since N_2 and N_3 are both saturated, N_{12}, N_{22}, N_{223} and N_{233} must be all coloured 3 and then N_{22} is saturated. It implies that N_{11}, N_{112}, N_{122} are coloured 2 and then we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.
- We conclude that N_{33} must be coloured 3. Observe that N_{44} and N_{45} cannot be both coloured 3, as we are no longer in Case 1. Thus, at least one of these vertices is coloured 2. If N_{34} is coloured 2, N_3 is saturated. Then, the vertices N_{12}, N_{22}, N_{223} and N_{233} must be all coloured 3. This contradicts

Fact 5. Consequently, N_{34} is coloured 3 and N_{44} must be coloured 2, otherwise we would be in Case 1. Observe that N_{45} cannot be coloured 2, because, otherwise N_5 will be saturated, $c(N_{55}) = c(N_{50}) = c(N_{00}) = c(N_{01}) = 3$ and $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$. So N_{45} is coloured 3, N_4 is saturated and N_{55} and N_{50} are both coloured 2. However, we are in Case 1 with N_5.

ii. Now consider that $c(N_{23}) = 2$. Observe that neither N_{33} nor N_{34} are coloured 2 due to the interference in N_3.

- Suppose first that $c(N_{33}) = 1$. It implies that V_0 is saturated and that N_{34} is coloured 3. Consequently, N_{44} must be coloured 2, otherwise we are in Case 1, and then N_3 is saturated. So, N_{12}, N_{22} and N_{45} are coloured 3. Observe that among N_{55} and N_{50}, we must have one vertex coloured 2 and the other must be coloured 3 (due to Fact 5 and to the hypothesis that we are not in Case 1). So N_4 is also saturated and it implies that N_{334} and N_{344} are coloured 1. We conclude that N_{33} is saturated and that the vertices N_{223}, N_{233} and N_{333} should be all coloured 3. This contradicts Fact 5.
- Now consider the case in which $c(N_{33}) = 3$ and $c(N_{34}) = 1$. So V_0 is saturated and we can see that N_{44} and N_{45} can neither be both coloured 2 (interference in N_3) nor 3 (Case 1 with N_4). Thus, one is coloured 2 and the other is coloured 3. Consequently, N_3 is saturated and N_{12} and N_{22} are both coloured 3. Furthermore, both N_{334} and N_{344} cannot be coloured 1 (Case 1 with N_{34}). One of them at least is coloured 3. Then N_{55} and N_{50} can neither be both coloured 2 (Case 1 with N_5) nor 3 (otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So among N_{55} and N_{50} we have one vertex coloured 2 and the other is coloured 3. We conclude that N_5 is saturated, N_{00} and N_{01} are coloured 3 and, due to Fact 5, N_{11} is coloured 2. It implies that N_1 is saturated and N_{122} must be coloured 1 (it cannot be coloured 3, otherwise we would be in Case 1 with N_{12}). So N_2 is also saturated and N_{223} and N_{233} must be both coloured 3. This contradicts Fact 5.
- We obtain that N_{33} and N_{34} are both coloured 3. Consequently, N_{44} cannot be coloured 3 (Fact 3 with N_{34}).
  - Suppose first that N_{44} is coloured 1. If $c(N_{45}) = 3$, N_4 is saturated and we are in Case 1 with N_5 instead of V_0, because N_{55} and N_{50} must be both coloured 2. So N_{45} is coloured 2 and it implies that N_{55} and N_{50} must be both coloured 3, due to the interference constraint in V_0 and N_5. Thus, N_4 is saturated. Since N_3 is also saturated, we get that N_{334} and N_{344} are both coloured 1. The vertices N_{444} and N_{445} can neither receive colour 1, due to the interference in N_{44}, nor colour 3, since N_4 is saturated. Thus, they are both coloured 2. But then we have a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - So we get that $c(N_{44}) = 2$ and then N_3 is saturated. Neither N_{12}, nor N_{22} can be coloured 1, otherwise N_2 would also be saturated and it would imply that N_{223} and N_{233} should be coloured 3, leading to a contradiction to Fact 5. So we get that $c(N_{12}) = c(N_{22}) = 3$. Consequently, $c(N_{233}) = c(N_{333}) = c(N_{334}) = c(N_{344}) = 1$, due to interference constraints in N_3, N_{33} and N_{34}. So $c(N_{223}) = 3$ and N_{33} is also saturated. It implies that N_{22} is saturated and then N_{11} can either be coloured 1 or 2. In case it is

coloured 1, N_2 is saturated, N_{112}, N_{122} and N_{222} must be coloured 2 and we have a contradiction as $I_{N_{122}}(\mathfrak{T}^2, w_2, c) \geq 3$. If N_{11} is coloured 2, N_1 is saturated and then N_{112} and N_{122} must be coloured 1. However, we get that $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.

iii. We conclude that N_{23} is coloured 3.

- Suppose first that $c(N_{33}) = 1$. Consequently, V_0 is saturated.
  - Let us first consider the subcase in which N_{34} is coloured 2. Then, N_{44} and N_{45} can neither be both coloured 2, due to the interference in N_3, nor 3, since we are no longer in Case 1. So among N_{44} and N_{45} we have one vertex coloured 2 and the other is coloured 3. It implies that N_3 is saturated. Due to the interference in V_0 and N_5, we conclude that $c(N_{55}) = c(N_{50}) = 3$. So N_4 is saturated implying N_{334} and N_{344} must be both coloured 1. But then N_{33} is also saturated, N_{22} and N_{223} are be both coloured 3 and we are in Case 1 with vertex N_{23}.
  - We conclude that $c(N_{34}) = 3$. Since we are no longer in Case 1, we get that $c(N_{44}) = 2$. N_{45} and N_{55} can neither be both coloured 2 (Fact 3 with N_{45}), nor 3 (interference in N_4). So one of these vertices is coloured 2 and the other is coloured 3, implying that N_5 is saturated and then that $c(N_{50}) = c(N_{00}) = c(N_{01}) = 3$. However, we get a contradiction as neither N_{45} is coloured 3, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, nor N_{55} is coloured 3, otherwise $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.
- Let us consider now the case $c(N_{33}) = 2$. Observe that N_{34} cannot be also coloured 2, due to the interference constraint in N_3.
  - In case $c(N_{34}) = 1$, we have that V_0 is saturated and then N_{44} and N_{45} can neither be both coloured 2 (interference in N_3) nor 3 (Case 1 with N_4). So among N_{44} and N_{45} we find one vertex coloured 2 and the other is necessarily coloured 3. Consequently, N_3 is saturated, N_{12} and N_{22} are coloured 3 and then N_{223} must be coloured 1. So N_2 is also saturated and N_{233} must be coloured 3. This is a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - We conclude that N_{34} must be coloured 3. Consequently, N_{44} cannot be coloured 3, as we are not in Case 1. Let us check the possible colourings of N_{44}.

    If N_{44} is coloured 1, V_0 is saturated. Then, if N_{45} is coloured 3, N_4 is saturated and N_{55} and N_{50} are forced to be coloured 2. But then we are in Case 1 with N_5. So N_{45} is coloured 2, N_3 is saturated and the vertices N_{55} and N_{50} must be coloured 3, due to the interference in N_5. As a consequence, N_4 is also saturated and the vertices N_{334} and N_{344} must be coloured 1. As we are no longer in Case 1, N_{444} must be coloured 2. Due to the interference constraints in N_4 and N_{45}, we get that N_{445} and N_{455} must be both coloured 1. This is a contradiction to Fact 5.

    So N_{44} must be coloured 2 and then N_3 is saturated. Observe that exactly one of the vertices N_{45}, N_{55} and N_{50} must be coloured 1, otherwise N_{45} must be coloured 3 (interference in N_3) and N_{55} and N_{50} must be coloured 2 (interference in N_4) and we are in Case 1. Then, as N_3 and V_0 are saturated we have $c(N_{12}) = c(N_{22}) = 3$, implying that N_{23} is saturated and so $c(N_{223}) = c(N_{233}) = c(N_{333}) = c(N_{334}) = 1$. However, we have that $I_{N_{233}}(\mathfrak{T}^2, w_2, c) \geq 3$.

- So we have that $c(N_{33}) = 3$. Let us now check the possible colourings of $c(N_{34})$.

– First consider that $c(N_{34}) = 1$. Observe that V_0 is saturated.

∗ If $c(N_{44}) = 3$, we get that $c(N_{45}) = 2$ and, consequently, $c(N_{55}) = c(N_{50}) = 3$ (otherwise, $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$). However, observe that $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

∗ So $c(N_{44}) = 2$. If $c(N_{45}) = 2$, N_{45} and N_5 are both saturated implying that N_{55}, N_{50}, N_{00} and N_{01} must be all coloured 3. But then we have a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.
So N_{45} is coloured 3. N_{55} and N_{50} can neither be both coloured 2 (otherwise, Case 1 with N_5), nor 3 (otherwise, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So one of these vertices is coloured 2 and the other is coloured 3. Thus, N_4 and N_5 are saturated and then N_{00} and N_{01} must be coloured 3 and the vertices N_{445} and N_{455} must be coloured 1. In case $c(N_{50}) = 3$, N_{50} is saturated and the vertices N_{555}, N_{550} and N_{500} must be coloured 1, contradicting Fact 5. So, we get that $c(N_{55}) = 3$ and $c(N_{50}) = 2$. Observe that N_{555} and N_{550} can neither receive colour 2 (since N_5 is saturated) nor 3 (otherwise, $I_{N_{55}}(\mathfrak{T}^2, w_2, c) \geq 3$). Thus, they are both coloured 1 and, consequently, N_{500} is coloured 3. It implies that N_{00} is saturated and then we get that N_{11} must be coloured 2. As a consequence, N_1 is saturated and N_{12} and N_{22} are both coloured 3. However, we get that $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– Now consider that $c(N_{34}) = 2$. Let us check the possible colourings of N_{44}.

∗ First suppose that $c(N_{44}) = 1$. If N_{45} is coloured 2, then N_3 is saturated and we have that N_{12} and N_{22} are coloured 3. This is a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. So N_{45} is coloured 3. The vertices N_{55} and N_{50} can neither be both coloured 2 (otherwise, Case 1 with N_5) nor 3 (otherwise, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So one is coloured 2 and the other is coloured 3. As a consequence, N_4 and N_5 are both saturated implying that N_{445} and N_{455} are coloured 1 and then that N_{444} is coloured 2. But then N_{334} and N_{344} must be both coloured 1 (interference in N_{34}) and we have a contradiction to Fact 5.

∗ Now let $c(N_{44}) = 2$. Observe that N_3 and N_{34} are saturated and that N_{12} and N_{22} cannot be both coloured 3, otherwise $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. So among N_{12} and N_{22} we have one vertex coloured 1 and the other is coloured 3. It implies that V_0 is saturated. Observe also that the vertices N_{45}, N_{55} and N_{50} cannot receive colour 2 due to the interference constraint in N_5. Then, we have a contradiction as N_{45}, N_{55} and N_{50} are all coloured 3 and we get $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

∗ We conclude that $c(N_{44}) = 3$. If $c(N_{45}) = 1$, V_0 is saturated. In this case, N_{55} and N_{50} can neither be both coloured 2 (otherwise, Case 1 with N_5) nor 3 (otherwise, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$). So one of these vertices is coloured 2, the other is coloured 3 and we get that N_4 and N_5 are saturated. Thus, N_{445} and N_{455} must be coloured 1 and we are in Case 1 for N_{45}.

N_{45} cannot be coloured 3 as we are no longer in Case 1, so its colour is 2 and N_3 and N_5 are both saturated. N_{55} and N_{50} cannot be both coloured 3, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of these vertices is coloured 1 implying that V_0 is saturated. Consequently, N_{12} and N_{22} are both coloured 3 and we get a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– So we conclude that N_{33} and N_{34} are both coloured 3 and both saturated. If the vertices N_{12}, N_{44} and N_{45} are not coloured 1, they must be all coloured 2 and we have that N_3 is saturated and so $c(N_{223}) = c(N_{233}) = c(N_{333}) = c(N_{334}) = c(N_{344}) = 1$, contradicting Fact 5. So one of these vertices is coloured 1 and V_0 is saturated. In case N_{12} is coloured 1, N_{44} and N_{45} must be coloured 2 and N_{45} is saturated. Consequently, N_{55} and N_{50} are coloured 3 and we have a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.
Then, either N_{44} or N_{45} is coloured 1 (the other being coloured 2) and N_{12} is coloured 2. If N_{44} is coloured 1, then N_{55} and N_{50} are not coloured 2, otherwise $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$. So they are both coloured 3, but then $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.
So we have that N_{44} is coloured 2 and N_{45} is coloured 1. Consequently, N_{55} and N_{50} can neither be both coloured 2 (interference in N_5) nor 3 (interference in N_4). So one is coloured 2 and the other is coloured 3 implying that N_4 and N_5 are saturated. Therefore, $c(N_{445}) = c(N_{455}) = 1$ and we are in Case 1.

(d) We now study the case $c(N_3) = 2$ and $c(N_4) = c(N_5) = 3$. Observe that N_{45} cannot be coloured 3, otherwise we are in Case 1.

i. First consider that $c(N_{45}) = 1$ (V_0 is saturated). If N_{55} is coloured 3, N_{50} must be coloured 2 and we are in Case 2b with central vertex N_5. So N_{55} is coloured 2.

- In case N_{44} is coloured 3, N_{34} must be coloured 2 and then N_{33} must be coloured 3, because we are not in Case 1. Thus, N_4 is saturated, N_{23} and N_{50} must be coloured 2 and N_3 is also saturated. Consequently, N_{334} and N_{344} are both coloured 1.
If N_{00} is coloured 2, N_{50} is saturated and then N_{455} must be coloured 1, N_{45} is also saturated and N_{01} is coloured 3. Since N_{555} and N_{550} must be both coloured 3, we reach a contradiction as $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$.
So we conclude that $c(N_{00}) = 3$. Recall that N_3 is saturated and thus, N_{12} and N_{22} must be both coloured 3. N_{01} and N_{11} can neither be both coloured 2 (otherwise, Case 1) nor 3 (Fact 5). So one is coloured 2 and the other is coloured 3. Thus, N_1 is saturated and N_{122} must be coloured 1, since we are not in Case 1. The vertices N_{223} and N_{233} cannot receive colour 2 as N_3 is saturated, cannot be both coloured 3, thanks to Fact 5, and cannot be both coloured 1, due to the interference in N_2. So one of these vertices is coloured 1 and the other is coloured 3; N_2 is saturated and N_{112} must be coloured 3. Consequently, N_{11} is coloured 2 and N_{01} is coloured 3. We then observe that N_{01} and N_5 are saturated and that N_{001} and N_{011} must be both coloured 1. It leads to a contradiction as $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$.
- We conclude that N_{44} is coloured 2.

– Suppose first that $c(N_{34}) = 2$. In this case, N_{23} and N_{33} are both coloured 3 due to the interference in N_3. Observe that N_{12} and N_{22} can neither be both coloured 2 (interference in N_3) nor 3 (interference in N_{23}). So one is coloured 2 and the other is coloured 3. It implies that N_3 is saturated. Thus, N_{223} and N_{233} are both coloured 1, due to the interference in N_{23}. So N_2 is also saturated. The vertices N_{333} and N_{334} can neither be both coloured 1 (otherwise, $I_{N_{233}}(\mathfrak{T}^2, w_2, c) \geq 3$) nor 3 (Fact 3). So one of them is coloured 1 and the other is coloured 3. As a consequence, N_{23} is saturated, N_{22} is coloured 2 and N_{12} is coloured 3. But then N_{122} and N_{222} must be coloured 2 and we have a contradiction as $I_{N_{22}}(\mathfrak{T}^2, w_2, c) \geq 3$.

– We obtain that $c(N_{34}) = 3$. N_{23} and N_{33} can neither be both coloured 2 (otherwise, Case 1 with N_3) nor 3 (Fact 5). So one of them is coloured 2 and the other is coloured 3. It implies that N_4 is saturated and N_{55} and N_{50} must be coloured 2.

  If $c(N_{00}) = 2$, N_{50} is saturated and thus N_{01} must be coloured 3. Observe that N_{550} and N_{500} can neither be both coloured 1 (interference in N_0) nor 3 (interference in N_5). So one of these vertices is coloured 1 and the other is coloured 3. It implies that N_0 and N_3 are both saturated and thus that N_{000} and N_{001} must be both coloured 3. Then, N_{11} and N_{011} cannot receive colour 1 (N_0 is saturated) neither 3 (otherwise, $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$). So they are both coloured 2 and we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$.

  We conclude that N_{00} must be coloured 3. If N_{01} is coloured 3, N_5 is saturated. In this case, N_{550} and N_{500} can neither be both coloured 1 (interference in N_0) nor 2 (Fact 3). So one of them is coloured 1 and the other is coloured 2 and, as a consequence, N_0 and N_{50} are saturated. Thus, N_{000} and N_{001} must be both coloured 3 and we reach a contradiction to Fact 3.

  So we have that N_{01} must be coloured 2 and $c(N_{11}) = c(N_{12}) = 3$, otherwise $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$. In this case, N_{550} and N_{500} cannot receive colour 2 (interference in N_{50}). They can neither be both coloured 1 (interference in N_0) nor 3 (interference in N_5). Thus, one of these vertices is coloured 1, the other is coloured 3 and N_0 and N_5 are saturated. It implies that one of the vertices N_{000} or N_{001} must be coloured 2 and the other is coloured 3, because they can neither be both coloured 2 (interference in N_{01}) nor 3 (interference in N_{00}). But then, N_{00} is saturated and it implies that $c(N_{011}) = 2$. This leads to a contradiction as $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$.

ii. We then conclude that $c(N_{45}) = 2$. Let us study the possible colourings of N_{44}.

- Suppose now that $c(N_{44}) = 3$. Observe that N_{34} cannot be coloured 3, by Fact 3. If $c(N_{34}) = 2$, then we are in Case 2b with N_4.

  So N_{34} is coloured 1 and V_0 is saturated. Observe that N_{23} and N_{33} can neither be both coloured 2 (otherwise, Case 1 with N_3) nor 3 (interference in N_4). So one of them is coloured 2, the other is coloured 3 and N_4 is saturated. It implies that N_{55} and N_{50} must be coloured 2 and, due to the interference in N_{45}, that N_{445} must be coloured 1. Moreover, N_{334} and N_{344} can neither be both coloured 1 (interference in N_{34}) nor 2 (interference in N_3). Thus, one of them is coloured 1 and the other is coloured 2. As a consequence, N_3 and N_{45} are saturated. We obtain that N_{233} and N_{333} are both coloured 3. So N_{33} cannot be coloured 3, as we are not in Case 1 and then $c(N_{23}) = 3$ and $c(N_{33}) = 2$. Recall that N_3 is saturated and thus N_{12} and N_{22} must be both coloured 3. This is a contradiction to Fact 5.

- Suppose now that $c(N_{44}) = 1$ (and thus that V_0 is saturated).
  - If $c(N_{34}) = 3$, then N_{23} and N_{33} can neither be both coloured 2 (interference in N_3) nor 3 (Fact 5). So one of them is coloured 2 and the other is coloured 3, implying that N_4 is saturated. Consequently, N_{55} and N_{50} must be coloured 2 and then that N_{445} and N_{455} must be both coloured 1 (otherwise, $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$). Thus, N_{344} and N_{444} are both coloured 2, due to the interference in N_{44}. However, we get that $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - We conclude that N_{34} is coloured 2 and thus that N_{23} and N_{33} must be both coloured 3, due to the interference in N_3.
    * If $c(N_{22}) = 3$, then N_{23} is saturated and $c(N_{12}) = 2$, implying that N_3 is also saturated. So N_{223} and N_{233} are both coloured 1 and N_2 is saturated. Consequently, N_{122} and N_{222} are both coloured 2 and we have a contradiction as $I_{N_{12}}(\mathfrak{T}^2, w_2, c) \geq 3$.
    * We obtain that $c(N_{22}) = 2$, and then N_3 is saturated and N_{12} must be coloured 3. Consequently, N_{223} and N_{233} must be both coloured 1 (interference in N_{45}) and N_2 is also saturated. Since N_{122} and N_{222} cannot be both coloured 2 as we are not in Case 1, we conclude that at least one of these vertices is coloured 3 and that N_{23} is saturated. But then we get that $c(N_{333}) = c(N_{334}) = 1$ and we have a contradiction as $I_{N_{233}}(\mathfrak{T}^2, w_2, c) \geq 3$.
- So we have that N_{44} must be coloured 2. Let us now check the possible colourings of N_{34}.
  - In case $c(N_{34}) = 2$, N_3, N_{34} and N_{44} are all saturated. One of N_{12}, N_{22}, N_{23} and N_{33} must be coloured 1, otherwise they are all coloured 3 and we have $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$. So V_0 is also saturated and then N_{55} must be coloured 3. Thus, N_{50} is coloured 2, by Fact 3, and N_{45} is also saturated.

    If both N_{23} and N_{33} are coloured 3, N_4 is saturated and then we have a contradiction to Fact 5, because N_{334}, N_{344}, N_{444}, N_{445} and N_{455} should be all coloured 1.

    So among N_{23} and N_{33} we have one vertex coloured 1, the other is coloured 3 and then N_{12} and N_{22} must be coloured 3.

    If N_{23} is coloured 1 (and then N_{33} is coloured 3), we have that N_2 is saturated and then N_{223} and N_{233} must be coloured 3. But then we have a contradiction to Fact 5.

    So N_{33} must be coloured 1 (and then N_{23} is coloured 3). By Fact 3, we have that N_{223} is coloured 1 and then N_2 is also saturated. Consequently, N_{233} must be coloured 3 and we have a contradiction as $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - Suppose now that $c(N_{34}) = 1$ (so V_0 is saturated).
    * If $c(N_{33}) = 2$, N_3 is also saturated and then N_{12}, N_{22} and N_{23} must be all coloured 3. However, N_{223} and N_{233} must be coloured 1, due to interference

constraints in N_{22}, N_{23} and N_3, which is a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.

* So $c(N_{33}) = 3$. Let us check the possible colourings of N_{23}.

  If N_{23} is coloured 2, N_3 is saturated and then N_{12} and N_{22} are both coloured 3. N_{223} and N_{233} can neither be both coloured 1 (interference in N_2) nor 3 (Fact 5). So one of them is coloured 1, the other is coloured 3 and N_2 is saturated. If N_{223} is coloured 3 (and then $c(N_{233}) = 1$), N_{22} is also saturated. In this case, the vertices N_{11}, N_{112}, N_{122} and N_{222} must be all coloured 2, contradicting Fact 5.

  So we conclude that N_{223} is coloured 1 and N_{233} is coloured 3. Consequently, N_{333} and N_{334} must be coloured 1 (interference in N_3 and N_{33}) and then N_{34} is saturated. Thus, N_{344} and N_{445} must be coloured 3. It implies that N_4 is saturated and then N_{55} and N_{50} are both coloured 2. This is a contradiction as $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.

  We conclude that N_{23} is coloured 3. If N_{22} is also coloured 3, N_{23} is saturated and then N_{12} is coloured 2. N_{223} and N_{233} can neither be both coloured 1 (interference in N_2) nor 2 (interference in N_3). So one of them is coloured 1, the other is coloured 2 and N_2 and N_3 are both saturated. It implies that N_{334} and N_{344} are both coloured 1, N_{34} is saturated and then N_{344} must be coloured 3. But then N_4 is saturated, N_{445} must be coloured 2 and we are in Case 1.

  So N_{22} must be coloured 2. If $c(N_{12}) = 2$, then N_3 is saturated. N_{223} and N_{233} can neither be both coloured 1 (interference in N_2), nor 3 (Fact 3). Thus, one is coloured 1, the other is coloured 3 and N_2 and N_{23} are both saturated. Consequently, N_{12}, N_{122} and N_{222} must be all coloured 2, contradicting Fact 3. Therefore, $c(N_{12}) = 3$, but then N_{223} and N_{233} cannot be coloured 3 (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$). So they are coloured 1 and $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

– We conclude that $c(N_{34}) = 3$. Let us study the possible colourings of N_{55}.

* Suppose first that $c(N_{55}) = 3$. So N_4 and N_5 are saturated. N_{23} and N_{33} can neither be both coloured 1 (interference in V_0) nor 2 (we are not in Case 1). So one of them is coloured 1, the other is coloured 2 and V_0 and N_3 are also saturated. It implies that N_{334} and N_{344} must be both coloured 1 and that N_{50} and N_{00} must be coloured 2. Observe then that N_{445} and N_{455} cannot receive colour 2 (interference in N_{45}) and 3 (N_4 is saturated). So they are both coloured 1 and, by Fact 5, N_{444} must be coloured 2. Since N_{45} is also saturated, N_{555} and N_{550} must be coloured 1. But then we have a contradiction because N_{500} cannot receive colour 1 (Fact 5), 2 (we are not in Case 1) or 3 (N_5 is saturated).

* Now consider that $c(N_{55}) = 2$. Observe that N_{45} is saturated. If N_{50} is coloured 3, N_4 and N_5 are also saturated and we have a contradiction to Fact 5, because N_{344}, N_{444}, N_{445}, N_{455}, N_{555} and N_{550} are all coloured 1.

  So N_{50} is coloured 1 and V_0 and N_0 are saturated. N_{23} and N_{33} can neither be both coloured 2 (we are not in Case 1) nor 3 (Fact 5). So one is coloured 2, the other is coloured 3 and we have that N_3 and N_4 are both saturated.

This leads to a contradiction to Fact 5, because N_{334}, N_{344}, N_{444}, N_{445} and N_{455} are all coloured 1.

∗ We then conclude that N_{55} must be coloured 1 (and V_0 is saturated). Again N_{23} and N_{33} can neither be both coloured 2 nor 3. One of them is coloured 2 and the other is coloured 3 implying that N_3 and N_4 are both saturated. As a consequence, N_{334} and N_{344} are coloured 1 and N_{50} is coloured 2. Thus, N_{445} and N_{455} are both coloured 1, due to the interference in N_4 and N_{45}. So N_{444} must be coloured 2 and N_{45} is saturated. Consequently, N_{555} and N_{550} must be both coloured 3 due to the interference in N_{45} and N_{55}. We obtain that N_5 is also saturated and then that N_{00} and N_{01} are both coloured 2 and both saturated and N_1 is also saturated. So N_{23} is coloured 3 and N_{33} is coloured 2. Furthermore, N_{500} is coloured 1 and N_0 is saturated. But then N_{011}, N_{11}, N_{12} and N_{22} are coloured 3, contradicting Fact 5.

(e) Let us now consider the case $c(N_4) = 2$ and $c(N_3) = c(N_5) = 3$. We study now the subcases concerning to the colour of N_{45}.

i. First consider that $c(N_{45}) = 1$. Recall that V_0 is saturated.

- In case N_{44} is coloured 2, N_{34} is coloured 3 and N_{33} is coloured 2, as we are no longer in Case 1. N_{55} and N_{50} can neither be both coloured 2, otherwise $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, nor 3, otherwise we would be in Case 1 with N_5. So one of these vertices is coloured 2, the other is coloured 3 and N_4 is saturated. But then N_{23} is coloured 3 and we are in Case 2d with central vertex N_3.
- We conclude that N_{44} is coloured 3.

– If N_{34} is coloured 3, N_{34} is saturated and $c(N_{23}) = c(N_{33}) = 2$. Then, N_{22} is coloured 3, otherwise $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$, implying that N_3 is saturated. So $c(N_{12}) = 2$ and N_{23} is also saturated. Thus, N_{223} and N_{233} are both coloured 1, N_2 is saturated, which implies that N_{122} and N_{222} must be coloured 3. Then, we are in Case 1.

– So N_{34} must be coloured 2. In case N_{33} is also coloured 2, then we are in one of the cases from 2a to 2d with central vertex N_{34}. Thus, N_{33} must be coloured 3 implying that N_{23} is coloured 2. N_{12} and N_{22} can neither be both coloured 2 (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$) nor 3 (otherwise, $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$). So one is coloured 2, the other is coloured 3, N_3 is saturated and $c(N_{223}) = c(N_{233}) = 1$ (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$).
Thus, N_2 is also saturated. N_{55} and N_{50} cannot be both coloured 3, as we are not in Case 1. Consequently, (exactly) one of these vertices is coloured 2 and N_4 is saturated. It implies that N_{334} and N_{344} are coloured 1 and then N_{333} must be coloured 2. Thus, N_{23} is saturated and N_{22} must be coloured 3 (otherwise, $I_{N_{23}}(\mathfrak{T}^2, w_2, c) \geq 3$). However, N_{122} and N_{222} must be coloured 3 and we are in Case 1 with vertex N_{22}.

By symmetry, we conclude that $c(N_{34}) \neq 1$.

ii. Suppose now that N_{45} is coloured 2. Observe that N_{44} cannot be coloured 2 as we are no longer in Case 1.

- Consider first the case $c(N_{44}) = 1$ (V_0 is saturated). If N_{34} is coloured 2, N_4 is saturated, N_{23} and N_{33} are both coloured 3 and we are in Case 1 with N_3. So N_{34} is coloured 3 and N_{33} must be coloured 2 (otherwise, Case 1 with N_3). N_{55} and N_{50} cannot be both coloured 3 (otherwise, Case 1 with N_5). So (exactly) one is coloured 2, N_4 is saturated and N_{23} must be coloured 3. However, we are in Case 2d with central vertex N_3.
- We conclude that N_{44} must be coloured 3. Recall that $c(N_{34}) \neq 1$. In case N_{34} is coloured 2, we are in Case 2c with N_4 instead of V_0. So N_{34} is coloured 3 and it is saturated. So $c(N_{23}) \neq 3$, $c(N_{33}) \neq 3$, among N_{23}, N_{33}, N_{55} and N_{50} at most one vertex is coloured 1 (interference in V_0) and at most two are coloured 2 (interference in N_4). Moreover, at most one of the vertices N_{55} and N_{50} is coloured 3, otherwise we are in Case 1 with N_5. So, exactly one of the vertices N_{55} and N_{50} is coloured 3 and N_5 is saturated; exactly two of the vertices N_{23}, N_{33}, N_{55} and N_{50} are coloured 2 and N_4 is saturated. But then we find a contradiction to Fact 5 as N_{334}, N_{344}, N_{444}, N_{445} and N_{455} are all coloured 1.

iii. We then conclude that $c(N_{45}) = 3$ and by symmetry that $c(N_{34}) = 3$. N_{44} cannot be coloured 3 by Fact 5.

- Suppose first that N_{44} is coloured 1 (V_0 is saturated). Consequently, N_{23}, N_{33}, N_{55} and N_{50} must be coloured 2 due to the interference constraints in N_3 and N_5. So N_4 is saturated and N_{334}, N_{344}, N_{445} and N_{455} must be coloured 1 due to the interference in N_{34} and N_{45}. This is a contradiction to Fact 5.
- We obtain that N_{44} must be coloured 2. Among N_{23}, N_{33}, N_{55} and N_{50} at most one vertex is coloured 1 (interference in V_0) and none of them is coloured 3 (interference in N_3 and N_5). So at least 3 of them are coloured 2 and we get a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$.

(f) Now consider that $c(N_3) = c(N_4) = c(N_5) = 3$. By Fact 3, we know that N_{34}, N_{44} and N_{45} are not coloured 3. These vertices are also not coloured 1, otherwise we would be in one of the cases from 2a to 2e with vertex N_4 replacing of V_0. So N_{34}, N_{44} and N_{45} are all coloured 2. Let us check the possible colourings of N_{55}.

i. First consider that N_{55} is coloured 3. If N_{50} is coloured 2, then we are in Case 2d with N_5 instead of V_0. So N_{50} is coloured 1 and V_0 are saturated. However, we obtain that N_{23} and N_{33} are both coloured 2 and we have a contradiction to Fact 5.

ii. Suppose now that $c(N_{55}) = 2$. Observe that N_{50} cannot be coloured 2, by Fact 5.
If $c(N_{50}) = 1$, V_0 is saturated and as N_{44} is saturated we conclude that N_{33} is coloured 3. But then N_3 and N_4 are saturated and we have a contradiction to Fact 5, because all the vertices N_{334}, N_{344}, N_{444}, N_{445} and N_{455} must be coloured 1.
So N_{50} is coloured 3 and N_4, N_5, N_{44} and N_{45} are saturated. Consequently, we find a contradiction to Fact 5 as N_{344}, N_{444}, N_{445}, N_{455} and N_{555} are all coloured 1.

iii. We conclude that $c(N_{55}) = 1$ and V_0 is saturated. N_{23} and N_{33} can neither be both coloured 2, nor 3, due to Facts 5 and 3, respectively. If N_{23} is coloured 3 and N_{33} is coloured 2, we have that N_4, N_{34} and N_{44} are saturated. Thus, N_{334}, N_{344}, N_{444}, N_{445} and N_{455} must be all coloured 1, contradicting Fact 5. Consequently, $c(N_{23}) = 2$ and $c(N_{33}) = 3$. But then we are in Case 2d with vertex N_3 replacing V_0.

3. **Subcase** $c(N_3) = 1$.

Observe that the vertices N_{01}, N_{23}, N_{34} and N_{50} cannot be coloured 1, otherwise we would be in Case 2. Up to symmetries, we study the possible colourings of N_1, N_2, N_4 and N_5: four of the same colour (Case 3a), three of the same colour (Case 3b) or two of the same colour (Cases 3c and 3d).

(a) Let us consider first the case $c(N_1) = c(N_2) = c(N_4) = c(N_5) = 2$. In this case, N_{01}, N_{23}, N_{34} and N_{50} must be coloured 3, due to interference constraints in N_1, N_2, N_4 and N_5, respectively. By symmetry, we consider that if there exists a vertex coloured 1 in Γ^2, then it is in the set $\{N_{33}, N_{44}, N_{45}, N_{55}\}$. Thus, the vertices N_{11} and N_{12} must be coloured 3 and we are in Case 2 with respect to N_{11}.

(b) Now let $c(N_1) = c(N_2) = c(N_4) = 2$ and $c(N_5) = 3$. Observe that the vertices N_{11}, N_{12} and N_{22} cannot be coloured 2, otherwise we would be in the previous Cases 1 or 2. If these vertices are all coloured 3, N_{01} and N_{23} cannot receive colour 3 as we would be in Case 2. So N_{01} and N_{23} must be both coloured 2 and we reach a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$. So one of the vertices N_{11}, N_{12} and N_{22} is coloured 1 and V_0 is saturated.

i. If $c(N_{11}) = 1$, N_{12} and N_{22} must be coloured 3. So N_{23} is coloured 2 (it cannot be coloured 3 as we would be in Case 2) and then N_2 is saturated. Consequently, N_{33} and N_{34} are coloured 3. Observe that N_{44} and N_{45} can neither be both coloured 2 (Fact 4 with N_4) nor 3 (Fact 5). So N_4 is saturated and N_{55} and N_{50} are both coloured 3. Then we find a contradiction as we are in Case 1 with vertex N_5.

ii. In case N_{22} is coloured 1 and $c(N_{11}) = c(N_{12}) = 3$, we have that N_{01} is coloured 2. So N_1 is saturated, N_{00} and N_{50} must be coloured 3 and we are in Case 2 with N_{50}.

iii. So we have that $c(N_{12}) = 1$ and $c(N_{11}) = c(N_{22}) = 3$. If $c(N_{23}) = 2$, we have that N_2 is saturated, N_{33} and N_{34} must be coloured 3 and among the vertices N_{44} and N_{45} we have one vertex coloured 2 and the other is coloured 3. Consequently, N_{55} and N_{50} must be coloured 3 and we are in Case 1. So $c(N_{23}) = 3$.

Observe that among N_{34}, N_{44} and N_{45} we have at most one vertex coloured 2, otherwise we would be in one of the Cases 1 or 2. Similarly, at most one of the vertices N_{45}, N_{55} and N_{50} is coloured 3. In case there is a vertex coloured 2 among N_{34} and N_{44}, due to two vertices coloured 2 in the set $\{N_{45}, N_{55}, N_{50}\}$, we have a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. Observe that we cannot have all the vertices N_{34}, N_{44} and N_{45} coloured 3 as we would be in Case 2. So,

N_{34} and N_{44} are coloured 3 and N_{45} is coloured 2. Since that there is a vertex in N_{55} and N_{50} coloured 2, we conclude that N_4 is saturated and then N_{33} is coloured 3. This is a contradiction to Fact 5.

(c) We now study the case $c(N_1) = c(N_2) = 2$ and $c(N_4) = c(N_5) = 3$. By symmetry, we consider that the vertices N_{00}, N_{01}, N_{11}, N_{12}, N_{22} and N_{23} are not coloured 1. Then, the vertices N_{11}, N_{12} and N_{22} must coloured 3, otherwise we would be in Cases 1 or 2. By the same reason, N_{01} and N_{12} must be coloured 2. As N_1 is saturated, N_{00} is coloured 3. Consequently, we can neither colour N_{50} with colours 1 or 3, because we would be in Case 2, nor colour it with colour 2, due to the interference in N_1.

(d) Let us consider now that $c(N_1) = 2$, $c(N_2) = 3$ and that among N_4 and N_5 we have one vertex coloured 2 and the other is coloured 3. By symmetry, we can once more consider that the vertices N_{00}, N_{01}, N_{11}, N_{12}, N_{22} and N_{23} are not coloured 1.

i. In case N_{12} is coloured 3, all the vertices N_{11}, N_{22} and N_{23} must be coloured 2, otherwise we would be in Cases 1 or 2. So N_1 is saturated and N_{00} and N_{01} must be coloured 3. Then, as in Case 3c no feasible colour remains to colour N_{50}.

ii. Thus N_{12} is coloured 2. It implies that $c(N_{01}) = c(N_{11}) = c(N_{22}) = 3$, otherwise we would be in Cases 1 or 2. Consequently, N_2 is saturated, N_{23} and N_{34} are coloured 2, and thus N_{33} must be coloured 1. So N_3 is also saturated and N_{223} and N_{233} must be coloured 2. Then we are in Case 1 with N_{23}.

Case: V_0 **has exactly one neighbour coloured 1**.

We also consider that no vertex v has two neighbours with its own colour, otherwise we can consider that v is V_0 and we are in the previous case. This fact is extensively used in this proof and many times it is omitted. W.l.o.g, let N_0 be the only neighbour of V_0 coloured 1 and let $c(N_1) = 2$.

1. Suppose first that $c(N_2) = 2$. Consequently, $c(N_3) = 3$, otherwise N_2 would have two neighbours coloured 2. We have three cases to analyse:

(a) In case $c(N_4) = c(N_5) = 2$, we claim that $c(N_{01}) = c(N_{50}) = 3$. In fact, if not, one of the vertices N_0, N_1 or N_5 would have two neighbours with their colours. By the same reason, we conclude $N_{00} = 2$. At this point, observe that N_1 and N_5 are saturated, thanks to the set $\{N_1, N_2, N_4, N_5, N_{00}\}$. Consequently, the vertices N_{11} and N_{12} cannot receive colour 2 and they cannot be both coloured 3 as N_{11} would have two neighbours with its colour. Similarly, we can conclude that at least one vertex of N_{22} and N_{33} is coloured 1 and also one of N_{34} and N_{44} and one of N_{45} and N_{55}. This is a contradiction because $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

(b) Suppose now that $c(N_4) = 2$ and $c(N_5) = 3$. Observe that $c(N_{01}) = 3$. By the hypothesis that no vertex has two neighbours with the same colour, we conclude that among the vertices N_{11} and N_{12} at least one of them is coloured

1, none of them can receive colour 2 and they cannot be both coloured 3. The same is valid for the vertices N_{22} and N_{23}. Observe also that these four vertices cannot be all coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$. Then consider that three of these vertices are coloured 1. Thus, since V_0 is saturated, we must be able colour the remaining vertices of Γ^2 with colours 2 and 3. If we consider that $c(N_{33}) = 2$, then all the other colours of vertices in Γ^2 are fixed by the hypothesis that each vertex has no two neighbours with its colour. One may check that, in this case, $c(N_{44}) = c(N_{55}) = c(N_{50}) = 2$. Thus, $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction. In case we colour N_{33} with colour 3, one can check that there is no feasible colour for N_{45}. Consequently, we conclude that among N_{11} and N_{12} there is one vertex coloured 1 and the other is coloured 3; and the same holds for vertices N_{22} and N_{23}.

We now show by contradiction that no colour is feasible to N_{55}.

i. First suppose that $N_{55} = 1$. Thus, we already know that V_0 is saturated and we can no longer use colour 1 to colour vertices in Γ^2. If we suppose that $c(N_{45}) = 2$, we observe that we cannot colour the vertices N_{34} and N_{44} with colours 2 and 3. Thus, let $c(N_{45}) = 3$. In this case, $c(N_{50}) = c(N_{44}) = 2$, $c(N_{34}) = 3$ and $c(N_{33}) = 2$. We observe that $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

ii. Suppose now that $c(N_{55}) = 2$. Observe that N_{45} cannot be coloured 2. Suppose then that $c(N_{45}) = 1$. Again V_0 is saturated and we cannot have colour 1 in the remaining vertices of Γ^2. If $c(N_{44}) = 2$, then $c(N_{33}) = 2$ and $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction. Thus, let $c(N_{44}) = 3$. In this case $c(N_{34}) = 2$ and $c(N_{33}) = 3$. Consequently, N_3 and N_4 are saturated. It implies that $c(N_{334}) = c(N_{344}) = 1$. As a consequence, $c(N_{444}) = 3$, $c(N_{445}) = 1$ and, since N_4 is saturated, no colour is feasible to colour N_{455}.

   We must consider then the case in which $c(N_{45}) = 3$. As a consequence we have $c(N_{50}) = 2$. Since $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 2$, we conclude that $c(N_{44}) = 1$, $c(N_{34}) = 3$ and $c(N_{33}) = 2$. We obtain that N_3 and N_4 are saturated. Consequently, $c(N_{334}) = c(N_{344}) = 1$, but then N_{444} has two neighbours coloured 1, a contradiction.

iii. The last subcase to consider is the one in which $c(N_{55}) = 3$. Observe that it implies $c(N_{50}) = 2$ and that N_{45} cannot be coloured 3. In case $c(N_{45}) = 1$, V_0 is saturated and then N_{44} cannot be coloured 1. Suppose first that $c(N_{44}) = 2$. Observe that N_4 is saturated and that $c(N_{34}) = 3$. Consequently, no feasible colour remains to colour N_{33}. Then consider that $c(N_{44}) = 3$. Consequently, $c(N_{34}) = 2$ and N_4 and N_5 are saturated. This is a contradiction as the vertices N_{445} and N_{455} should be both coloured 1, as they are at distance two from N_4 and N_5, but then N_{45} would have two neighbours with the same colour. Thus, $c(N_{45}) = 2$ and N_4 is saturated. If N_{44} is coloured 1, then N_{33} and N_{34} should be both coloured 3, a contradiction. Consequently, $c(N_{44}) = 3$. In this case, we get $c(N_{33}) = 3$, $c(N_{34}) = 1$ and N_3 is saturated. However, N_{334} and N_{344} should be both coloured 1, a contradiction since $c(N_{34}) = 1$.

(c) Now suppose that $c(N_4) = 3$ and $c(N_5) = 2$. First observe that $c(N_{01}) = 3$ and $c(N_{23}) = 1$, thanks to the hypothesis that no vertex has two neighbours with the same colour. By the same hypothesis, we can conclude that N_{11} and N_{12} cannot receive colour 2 and at most one of them is coloured 3. By the same reasoning, we can conclude that at least one of the vertices N_{44} and N_{45} is coloured 1. Thus, V_0 is saturated and no other vertex at distance two from V_0 can receive colour 1. Consequently, by using this information combined with the hypothesis that no vertex has two neighbours with its colour we conclude that $c(N_{33}) = c(N_{34}) = 2$. Thus, we conclude that $c(N_{44}) = 1$ and $c(N_{45}) = 2$. Since $c(N_{45}) = c(N_5) = 2$, we obtain that $c(N_{55}) = c(N_{50}) = 3$. This implies that $c(N_{00}) = 2$. However, $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 3$, thanks to the vertices N_1, N_2, N_{00}, N_{34} and N_{45}.

(d) Finally, if $c(N_4) = c(N_5) = 3$, then N_4 has two neighbours with its own colour and we are in the previous case.

2. Suppose then that $c(N_2) = 3$. We consider the possible colourings of N_3, N_4 and N_5:

(a) First, it is not possible to have $c(N_3) = c(N_4) = c(N_5) = 2$ as N_4 would have two neighbours with its colour.

(b) Then, consider the case in which $c(N_3) = c(N_4) = 2$ and $c(N_5) = 3$. Once more we know that N_{50}, N_{00} and N_{01} cannot be coloured 1, otherwise N_0 would have two neighbours with its own colour. Similarly, none of the vertices N_{23}, N_{33}, N_{34}, N_{44} and N_{45} can receive colour 2, otherwise N_3 or N_4 would have two neighbours coloured 2. We prove now that no colour is feasible for N_{55}.

i. First, consider that $c(N_{55}) = 1$.

- Suppose also that $c(N_{45}) = 1$. Consequently, we get $c(N_{44}) = 3$, otherwise N_{45} has two neighbours with colour 1. In case N_{34} is coloured 1, V_0 is saturated and we reach a contradiction, because $c(N_{23}) = c(N_{33}) = 3$ and N_{23} would have two neighbours coloured 3. Thus, suppose that $c(N_{34}) = 3$. It implies that $c(N_{33}) = 1$ and $c(N_{23}) = 3$. As a consequence, $c(N_{12}) = c(N_{22}) = 2$, because V_0 is saturated and $c(N_{23}) = 3$. We then get a contradiction since N_{12} has two neighbours coloured 2.
- We conclude then that N_{45} is coloured 3. Since $c(N_5) = 3$, we obtain that $c(N_{44}) = 1$. In case $c(N_{34}) = 1$, we have that V_0 is saturated and both N_{23} and N_{33} should be coloured 3. This would be a contradiction as N_{23} would have two neighbours coloured 3. Consequently, $c(N_{34}) = 3$. If N_{33} is coloured 1, we have $c(N_{23}) = 3$. Once more $c(N_{12}) = c(N_{22}) = 2$ and we have a contradiction as N_{12} has two neighbours coloured 2. So $c(N_{33}) = 3$ and, consequently, $c(N_{23}) = 1$. Since V_0 is saturated and no vertex has two neighbours with its own colour, either we have $c(N_{11}) = c(N_{22}) = 2$ and $c(N_{12}) = 3$ or we have $c(N_{11}) = c(N_{22}) = 3$ and $c(N_{12}) = 2$. In the first case, we have a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$ and in the latter case we also have a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$ (recall that $c(N_{50}) = 2$ and in the set $\{N_{00}, N_{01}\}$ we have one vertex coloured 2 and the other coloured 3).

ii. Suppose then that $c(N_{55}) = 2$. We distinguish three cases.

- $c(N_{44}) = c(N_{45}) = 1$, we have that $c(N_{34}) = 3$. In case $c(N_{33}) = 1$, we have that $c(N_{23}) = 3$ and V_0 is saturated. This is a contradiction as N_{12} and N_{22} have no feasible colouring. Then consider the case $c(N_{33}) = 3$. Observe that $c(N_{344}) = 2$, otherwise N_{34} or N_{44} have two neighbours with their colour. Consequently, N_4 is saturated and all the vertices N_{444}, N_{445} and N_{455} should be coloured 3, as they all have two adjacent neighbours coloured 1 and they are all at distance two from N_4. This is a contradiction as N_{455} would have two neighbours with its own colour.
- $c(N_{45}) = 1$ and $c(N_{44}) = 3$. Suppose that $c(N_{33}) = c(N_{34}) = 1$. Thus, V_0 is saturated and $c(N_{23}) = 3$. Once more we get a contradiction as N_{12} and N_{22} should be both coloured 2. Thus, consider now that $c(N_{33}) = 3$ and $c(N_{34}) = 1$. Observe that $c(N_{23}) = 1$ and V_0 is saturated. If $c(N_{22}) = 2$, we get that $c(N_{12}) = 3$ and $c(N_{11}) = 2$. Since at least one of the vertices N_{50} and N_{00} must be coloured 2, we reach a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$. In case $c(N_{22}) = 3$, we get that $c(N_{12}) = 2$ and $c(N_{11}) = 3$. Since N_2 is saturated, we conclude that $c(N_{01}) = 2$. Once more we obtain a contradiction as $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$. Let us now study the case $c(N_{33}) = 1$ and $c(N_{34}) = 3$. In case $c(N_{50}) = 2$, N_4 is saturated and we obtain a contradiction as all the vertices N_{334}, N_{344} and N_{444} should be coloured 1. Thus, consider that $c(N_{50}) = 3$. In this case, N_5 is saturated and we get a contradiction as N_{00} and N_{01} should be both coloured 2. Since we do not have the case $c(N_{33}) = 3$ and $c(N_{34}) = 3$ as N_{34} would have two neighbours with its colour, we conclude that $c(N_{45}) = 3$.
- So $c(N_{45}) = 3$, then we get that $c(N_{44}) = 1$ (otherwise N_{45} has two neighbours of the same colour), $c(N_{50}) = 2$ and $c(N_{00}) = 3$. In this case, we easily obtain a contradiction as N_4 is saturated and the vertices N_{445} and N_{455} have no feasible colouring.

iii. We conclude that $c(N_{55}) = 3$. As a consequence, we get $c(N_{45}) = 1$ and $c(N_{50}) = 2$. If $c(N_{44}) = 1$, then $c(N_{455}) = 2$ and N_4 is saturated. But then all the vertices N_{34}, N_{344}, N_{444} and N_{445} should be coloured 3. This would be a contradiction. Consequently, $c(N_{44}) = 3$. In this case, in the set $\{N_{00}, N_{01}\}$ there is exactly one vertex coloured 2 and the other is coloured 3, thanks to the interference constraint in vertex N_5 and to the hypothesis that no vertex has two neighbours with its own colour. Similarly, we can conclude that in the set $\{N_{445}, N_{455}\}$ there is exactly one vertex coloured 1 and the other is coloured 2. Since N_5 is saturated, we get $c(N_{34}) = 1$. So, N_{45} is saturated and both vertices N_{555} and N_{550} should be coloured 2. This would be a contradiction as N_{550} would have two neighbours coloured 2.

(c) Now let $c(N_3) = c(N_5) = 2$ and $c(N_4) = 3$. We show now that no colour is feasible to N_{55}.

i. Suppose first that $c(N_{55}) = 1$.

- First consider that $c(N_{45}) = 1$. Then N_{44} cannot be coloured 1 because we would have $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$.
  - Then, suppose that N_{44} is coloured 2 and N_{34} is coloured 1. Since V_0 is saturated, all the remaining vertices in Γ^2 are not coloured 1. In case N_{33}

is coloured 2, we have that N_3 is saturated and thus $c(N_{22}) = c(N_{23}) = 3$. This is a contradiction to the hypothesis that no vertex has two neighbours with its colour as $c(N_2) = 3$. In case N_{33} is coloured 3, we have that $c(N_{23}) = 2$, then $c(N_{22}) = 3$ and $c(N_{12}) = 2$. But then, $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

- Consequently, if N_{44} is coloured 2, N_{34} must be coloured 3 (observe it cannot be coloured 2 as it would have two neighbours N_3 and N_{44} coloured 2). If N_{50} is coloured 2, N_5 is saturated and the vertices N_{445}, N_{455} and N_{555} should be all coloured 3 (as N_{45} and N_{55} are both coloured 1). This is a contradiction as N_{455} has two neighbours with its own colour. Consequently, we have $c(N_{50}) = 3$. Observe that among the vertices N_{445} and N_{455} at least one of them is coloured 3. Thus, N_4 is saturated and in the set $\{N_{23}, N_{33}\}$ we have exactly one vertex coloured 1 (due to the interference constraint in V_0) and the other is coloured 2. Since V_0 and N_3 are saturated, the vertices N_{12} and N_{22} should be both coloured 3. This is a contradiction as $c(N_2) = 3$.
- Thus, $c(N_{44}) = 3$ and N_{34} can be either coloured 1 or 2. If $c(N_{34}) = 1$, we get that V_0 is saturated. If N_{33} is coloured 2, N_{23} is necessarily coloured 3 and N_{12} and N_{22} should be both coloured 2. This is a contradiction as N_{12} would have two neighbours coloured 2. Thus N_{33} is coloured 3. It implies that $c(N_{23}) = 2$, then $c(N_{22}) = 3$, $c(N_{12}) = 2$ and $c(N_{01}) = c(N_{11}) = 3$. This is a contradiction as $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 3$.
- We conclude that $c(N_{44}) = 3$ and $c(N_{34}) = 2$. Observe that $c(N_{344}) = 1$ and $c(N_{445}) = 2$, thanks to the hypothesis that no vertex has two neighbours with its colour. Since we get that N_{45} is saturated, we have $c(N_{444}) = 2$ and, consequently, $c(N_{455}) = 3$. Observe now that N_{34} and N_4 are saturated (because among N_{33} and N_{334} we have exactly one vertex coloured 1 and the other is coloured 3). As a consequence, $c(N_{23}) = 1$ and $c(N_{50}) = 2$. At this point the colours of the remaining vertices in Γ^2 are fixed as V_0 is saturated. We have $c(N_{00}) = c(N_{01}) = c(N_{12}) = 3$ and $c(N_{11}) = c(N_{22}) = 2$. Thus we observe that $I_{N_1}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

• Then, consider now that N_{45} is coloured 2. It implies that $c(N_{50}) = 3$ and that among N_{00} and N_{01} we have exactly one vertex coloured 2 and the other is coloured 3. Consequently, N_5 is saturated and among N_{555} and N_{550} we have exactly one vertex coloured 1 and one vertex coloured 3. In case N_{44} is coloured 1, N_{55} is saturated. Thus, N_{445}, N_{455} and N_{500} are all coloured 3. This is a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$. If $c(N_{44}) = 3$, we obtain that $c(N_{445}) = 1$ and that $c(N_{455}) = 3$. Consequently, N_{55} is saturated and $c(N_{500}) = 3$. Once more we have a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$.

• Suppose then that $c(N_{45}) = 3$.

- If $c(N_{50}) = 2$, we have that $c(N_{00}) = 3$. Since $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 2$ and $c(N_4) = c(N_{45}) = 3$, we conclude that among N_{34}, N_{44} and N_{445} we have exactly two vertices coloured 1 and the other is coloured 2. Consequently, we get N_5 is saturated and thus $c(N_{01}) = 3$. This implies that $c(N_{500}) = 1$ and then N_{55} is saturated. Thus, we get a contradiction as we have no feasible colouring for the vertices N_{455} and N_{555}.

– So $c(N_{50}) = 3$. If $c(N_{34}) = c(N_{44}) = 1$, we observe that V_0 is saturated and that among N_{23} and N_{33} we have exactly one vertex coloured 2 and one coloured 3. Consequently, N_4 is saturated and we reach a contradiction as no colouring is feasible to the vertices N_{334}, N_{344} and N_{444}.

In case N_{44} is coloured 1, then N_{34} is coloured 2, we observe that among N_{23} and N_{33} we have one vertex coloured 1 and the other is coloured 3. As a consequence, we get that V_0 and N_4 are saturated. Since $c(N_{334}) = 1$, no colouring is feasible for the vertex N_{344}. If N_{44} is coloured 2 (and so N_{34} is coloured 1), observe that N_5 is saturated, since there is a vertex coloured 2 and another coloured 3 in the set $\{N_{00}, N_{01}\}$ and we also find a vertex coloured 1 and another coloured 2 among vertices N_{445} and N_{455}. Consequently, the vertices N_{555} and N_{550} receive colours 1 and 3 (in some order). Thus, N_{55} is saturated and then $c(N_{500}) = 3$. This is a contradiction as $I_{N_{50}}(\mathfrak{T}^2, w_2, c) \geq 3$. Since no other colouring is feasible for N_{34} and N_{44} as we cannot assign them the colour 3, we conclude that the colour of N_{55} cannot be 1.

ii. Let us consider now the case $c(N_{55}) = 2$. It implies that $c(N_{50}) = 3$ and, consequently, the vertices N_{00} and N_{01} receive colours 2 and 3 in some order. Thus, N_5 is saturated. In case N_{44} and N_{45} are both coloured 1, the vertices N_{34}, N_{445} and N_{455} must be all coloured 3. This is a contradiction as $I_{N_4}(\mathfrak{T}^2, w_2, c) \geq 3$. In case $c(N_{44}) = 1$ and $c(N_{45}) = 3$, no colouring is feasible to the vertices N_{445} and N_{455}. If $c(N_{44}) = 3$ and $c(N_{45}) = 1$, observe that $c(N_{34}) = c(N_{445}) = 1$ and that one vertex among N_{555} and N_{550} is coloured 1. Thus, $I_{N_{45}}(\mathfrak{T}^2, w_2, c) \geq 3$, a contradiction.

iii. We then conclude that the only possible colour for N_{55} is the colour 3. Recall N_{50} cannot be coloured 1 as N_0 would have two neighbours with its own colour.

- Let us first consider the case in which $c(N_{50}) = 2$. As a consequence, we obtain $c(N_{00}) = 3$ and $c(N_{45}) = 1$.

– If $c(N_{01}) = 2$, we can easily check that N_1 and N_5 are saturated. Observe also that N_0 is saturated as N_0 has a neighbour, the vertex V_0, coloured 1 and 3 other vertices at distance two also coloured 1 which are N_{45}, one vertex in the set $\{N_{11}, N_{12}\}$ and another in the set $\{N_{550}, N_{500}\}$. Consequently, we reach a contradiction as N_{001} and N_{011} should be both coloured 3, but then N_{001} would have two neighbours with colour 3.

– Thus, $c(N_{01}) = 3$ in this case. It implies that $c(N_{500}) = 1$ and that the colour 3 does not appear in the vertices N_{000}, N_{001} and N_{011}. These three vertices can also not be all coloured 1 or 2, as N_{001} would have two neighbours of the same colour. We cannot have two of these vertices coloured 1 as we would have $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 3$. Consequently, in the set $\{N_{000}, N_{001}, N_{011}\}$ we have one vertex coloured 1 and two vertices coloured 2. This implies that N_0 and N_1 are saturated. We then reach a contradiction as no feasible colour remains to assign to N_{11}.

- Then, we conclude that N_{50} must be coloured 3 and then we get $c(N_{00}) = 2$ and $c(N_{01}) = 3$. Observe that if $c(N_{45}) = 2$, we have a contradiction as N_5 is saturated and all the vertices N_{555}, N_{550} and N_{500} should be coloured 1. Thus we have that $c(N_{45}) = 1$. Observe that the vertices N_{11} and N_{12} cannot be both coloured the same, as we would either violate the interference constraint in N_0 (recall that there is one vertex coloured 1 in the set $\{N_{550}, N_{500}\}$) or we would have a vertex with two neighbours of the same colour. In case N_{11} and N_{12} are coloured 1 and 2, in any order, observe that since N_0 and N_1 are saturated, no colouring is feasible for the vertices N_{001} and N_{011}. We also have no feasible colouring for these vertices in case N_{12} is coloured 1 (and then N_0 is saturated) or 2 (in this case N_1 is saturated) and the vertex N_{11} is coloured 3.

Thus, $c(N_{12}) = 3$ and suppose first that $c(N_{11}) = 1$. Since N_0 is saturated, the vertices N_{000}, N_{001} and N_{011} can be just coloured 2 or 3. In case $c(N_{000}) = 2$, we obtain that $c(N_{001}) = 3$ and $c(N_{011}) = 2$. We reach a contradiction as $I_{N_{00}}(\mathfrak{T}^2, w_2, c) \geq 3$ (observe that one vertex among N_{550} and N_{500} is coloured 2). If $c(N_{000}) = 3$, we have that $c(N_{001}) = 2$ and $c(N_{011}) = 3$. Then, we also find a contradiction as $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 3$.

Consequently, $c(N_{11}) = 2$ and N_1 is saturated. In this case, no colouring is feasible for the vertices N_{122}, N_{22} and N_{23} and we complete the proof of this case as no colour is feasible for the vertex N_{55}.

(d) In case we have $c(N_3) = 2$ and $c(N_4) = c(N_5) = 3$, we are in a symmetric case to 1b.

(e) If $c(N_3) = 3$ and $c(N_4) = c(N_5) = 2$, we obtain a symmetric case to 1c.

(f) The case $c(N_3) = c(N_5) = 3$ and $c(N_4) = 2$ is symmetric to 2a.

(g) Finally, it is not possible to have $c(N_3) = c(N_4) = 3$ as N_3 would have two neighbours, N_2 and N_4, with its own colour.

Case: V_0 **has no neighbour coloured 1**.

Now we consider that no vertex has a neighbour with its own colour, otherwise we are in the previous case. W.l.o.g, we may conclude that $c(N_0) = c(N_2) = c(N_4) = 2$ and $c(N_1) = c(N_3) = c(N_5) = 3$. Thus, we obtain $c(N_{01}) = c(N_{12}) = c(N_{23}) = c(N_{34}) = c(N_{45}) = c(N_{50}) = 1$. This is a contradiction as $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 3$.

Now we present the colouring providing the corresponding upper bound.

For a weighted 3-improper 3-colouring of $(\mathfrak{T}^2, w_2)$ set, for $0 \leq j \leq 2$, $E_j = \{(j, 0) + a(3f_1) + b(f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 2$, assign the colour $j + 1$ to all the vertices in E_j. See Figure 7(e).

Now we prove that $(\mathfrak{T}^2, w_2)$ does not admit a weighted 4.5-improper 2-colouring. Again, by contradiction, suppose that there exists a weighted 4.5-improper 2-colouring c of $(\mathfrak{T}^2, w_2)$ with the interference function w_2. A vertex can have at most four neighbours of the same colour as it. We analyse some cases:

1. There exists a vertex V_0 with four of its neighbours coloured with its own colour, say 1. Therefore among the vertices of Γ^2 at most one is coloured 1. Consider

the two neighbours of V_0 coloured 2. First, consider the case in which they are adjacent and let them be N_0 and N_1. In Γ^2, N_0 has three neighbours and four vertices at distance two; since at most one being of colour 1, these vertices produce in N_0 an interference equal to 4 and as N_1 is also of colour 2, then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. In case the two neighbours of V_0 coloured 2 are non adjacent, let them be N_i and N_j. At least one of them, say N_i has its three neighbours in Γ^2 coloured 2 and it has also at least three vertices at distance two in Γ^2 coloured 2; taking into account that N_j is coloured 2 and at distance two from N_i, we get $I_{N_i}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

2. No vertex has four neighbours with its colour and there exists at least one vertex V_0 coloured 1 that has three neighbours of the same colour 1.

(a) The three other neighbours of V_0 coloured 2 are consecutive and let them be N_0, N_1 and N_2. N_{34}, N_{44} and N_{45} are all coloured 2, otherwise N_4 would have four neighbours coloured 1 and we would be in Case 1. At most one of N_{01}, N_{11} and N_{12} has colour 2, otherwise N_1 would have four neighbours coloured 2 and we would be again in Case 1.

i. N_{11} is coloured 2. Then $c(N_{01}) = c(N_{12}) = 1$. As already $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4$, there is at most another vertex in Γ^2 coloured 1. So either the three vertices N_{22}, N_{23} and N_{33} or the three vertices N_{00}, N_{50} and N_{55} are all coloured 2 and then $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 5$ or $I_{N_5}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

ii. N_{01} is coloured 2 (the case N_{12} is symmetric). Then, $c(N_{11}) = c(N_{12}) = 1$. One of N_{00} and N_{50} is of colour 1 otherwise, N_0 has four neighbours of colour 2. But then $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$ so all the other vertices of Γ^2 are coloured 2. Therefore, $I_{N_2}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

iii. N_{01}, N_{11} and N_{12} all have colour 1. In that case $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$. Therefore all the other vertices of Γ^2 are coloured 2 and $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$. So the other vertices at distance two of N_0 are coloured 1 and then $I_{N_{01}}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(b) Among the three vertices of colour 2, only two are consecutive. W.l.o.g., let the three vertices of colour 2 be N_0, N_1 and N_3. At least one vertex of N_{50}, N_{00}, N_{01} is coloured 1, otherwise N_0 has four neighbours of the same colour as it and we would be in the previous case. Similarly at least one vertex of N_{01}, N_{11}, N_{12} is coloured 1, otherwise N_1 has four neighbours with its colour and we would be in the previous case. At least one vertex of N_{23}, N_{33}, N_{34} is coloured 1, otherwise N_3 has three consecutive neighbours of the same colour as it and we are in the previous case. Suppose N_{01} is coloured 2, then $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 4.5$ and exactly one of N_{50}, N_{00} and one of N_{11}, N_{12} is coloured 1 and N_{45}, N_{55} are coloured 2, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$. Then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. So, $c(N_{01}) = 1$. If both N_{50}, N_{00} are coloured 2, then $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$ with three neighbours coloured 2 and at least four vertices at distance two coloured 2, namely N_3 and three vertices among N_{45}, N_{55}, N_{11}, N_{12} (at most one vertex of these could be of colour 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$). So, one of N_{50}, N_{00} is coloured 1 and all the other vertices in $\{N_{11}, N_{12}, N_{22}, N_{44}, N_{45}, N_{55}\}$ are coloured 2 implying that $I_{N_3}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(c) No two vertices of colour 2 are consecutive. W.l.o.g, let these vertices be N_0, N_2, N_4. The three neighbours of N_0 (resp. N_1, N_2) in Γ^2 that are not neighbours of V_0 cannot be all coloured 2, otherwise we are in Case (a). So exactly one neighbour of N_0, N_1, N_2 in Γ^2 is coloured 1, otherwise $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$. Furthermore all the other vertices of Γ^2 are coloured 2. Then, if $c(N_{12}) = c(N_{45}) = 2$, we conclude that $I_{N_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction. Consequently, w.l.o.g., suppose that $c(N_{12}) = 1$. In this case, N_{23} has at least three neighbours coloured 2 and we are in some previous case.

3. No vertex has three neighbours coloured with its own colour, but there exists at least one vertex, say V_0, of colour 1 that has two neighbours coloured 1.

(a) These two neighbours are consecutive say N_0 and N_1. The neighbours of N_3 and N_4 in Γ^2 are all coloured 1, otherwise they would have at least three neighbours with the same colour. Similarly, at least one of N_{12} and N_{22} is coloured 1, otherwise N_2 would have at least three neighbours also coloured 2. Then, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(b) These two neighbours are of the form N_i and N_{i+2}, for some $0 \leq i \leq 3$. W.l.o.g., let these neighbours be N_0 and N_2. Thus, the three neighbours of N_4 in Γ^2, N_{34}, N_{44} and N_{45} are coloured 1 and at least one vertex of N_{23} and N_{33} (resp. N_{55} and N_{50}) is coloured 1. Moreover, at least one vertex of N_{01}, N_{11} and N_{12} must be coloured 1, otherwise N_1 would have three neighbours with its colour. Consequently, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

(c) These two neighbours are of the form N_i and N_{i+3}, for some $0 \leq i \leq 2$. W.l.o.g., let these neighbours be N_0 and N_3. Again, at least three vertices among N_{01}, N_{11}, N_{12}, N_{22} and N_{23} and at least three other vertices among N_{34}, N_{44}, N_{45}, N_{55} and N_{50} are coloured 1. Consequently, $I_{V_0}(\mathfrak{T}^2, w_2, c) \geq 5$, a contradiction.

4. No vertex has two neighbours of the same colour. Suppose V_0 is coloured 1 and has only one neighbour N_0 coloured 1. Then, its other five neighbours are coloured 2 and N_2 has two neighbours of the colour 2, a contradiction.

A weighted 5-improper 2-colouring of $(\mathfrak{T}^2, w_2)$ is obtained as follows: for $0 \leq j \leq 1$, let $F_j = \{(j, 0) + a(2f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$ and $F'_j = \{(j-1, 1) + a(2f_1) + b(f_1 + 2f_2) \mid \forall a, b \in \mathbb{Z}\}$. Then, for $0 \leq j \leq 1$, assign the colour $j+1$ to all the vertices in F_j and in F'_j. See Figure 7(f).

Since each vertex has six neighbours and twelve vertices at distance two in $\mathfrak{T}$, there is no weighted t-improper 1-colouring of $(\mathfrak{T}^2, w_2)$, for any $t < 12$. Obviously, there is a weighted 12-improper 1-colouring of $\mathfrak{T}^2$. $\square$

4. Integer Linear Programming Formulations, Algorithms and Results

In this section, we look at how to solve the Weighted Improper colouring and Threshold Improper colouring for general instances inspired by the practical motivation. We present integer linear programming models for both problems. These models can be solved exactly for small sized instances using solvers like CPLEX[1]. For larger instances, the solvers can take a prohibitive time to provide exact solutions. It is usually possible to obtain a sub-optimal solution stopping the solver after a limited time. If the time is too short, the quality of the solution may be unsatisfactory. Thus, we introduce two algorithmic approaches to find good solutions for Threshold Improper colouring in a short time: a simple polynomial-time greedy heuristic and an exact Branch-and-Bound algorithm. We compare the three methods on different sets of instances, among them Poisson-Voronoi tessellations as they are good models of antenna networks [5, 13, 14].

4.1. Integer Linear Programming Models

Given an edge-weighted graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+^*$, and a positive real threshold t, we model Weighted Improper Colouring by using two kinds of binary variables. Variable x_{ip} indicates if vertex i is coloured p and variable c_p indicates if colour p is used, for every $1 \leq i \leq n$ and $1 \leq p \leq l$, where l is an upper bound for the number of colours needed in an optimal weighted t-improper colouring of G. l can be trivially chosen of value n, but a better value may be given by the results of Section 2. The model follows:

$$
\begin{array}{rcl}
\min & \sum_{p=1}^{l} c_p & \\
\text{subject to} & & \\
& \sum_{ij \in E \text{ and } j \neq i} w(i,j) x_{jp} \leq t + M(1 - x_{ip}) & \forall i \in V, 1 \leq p \leq l \\
& c_p \geq x_{ip} & \forall i \in V, 1 \leq p \leq l \\
& \sum_{p=1}^{l} x_{ip} = 1 & \forall i \in V \\
& x_{ip} \in \{0,1\} & \forall i \in V, 1 \leq p \leq l \\
& c_p \in \{0,1\} & 1 \leq p \leq l
\end{array}
$$

where M is a large integer. For instance, it is sufficient to choose $M > \sum_{uv \in E} w(u,v)$.

For Threshold Improper Colouring, given an edge-weighted graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+^*$, and a number of possible colours $k \in \mathbb{N}^*$, the model we consider is:

[1] http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/

$$
\begin{array}{lll}
\min & t & \\
\text{subject to} & & \\
& \sum_{ij \in E \text{ and } j \neq i} w(i,j)x_{jp} \leq t + M(1 - x_{ip}) & \forall i \in V, 1 \leq p \leq l \\
& \sum_{p=1}^{k} x_{ip} = 1 & \forall i \in V \\
& x_{ip} \in \{0, 1\} & \forall i \in V, 1 \leq p \leq l
\end{array}
$$

We give directly these models to the ILP solver CPLEX without using any preprocessing or any other technique to speed the search for an optimal solution.

4.2. Algorithmic approach

In this section, we show a Branch-and-Bound algorithm and a randomised greedy heuristic to tackle THRESHOLD IMPROPER COLOURING. Both are based on common procedures to determine the order in which vertices are coloured and colours are tried for a single vertex. Although, the Branch-and-Bound needs an ordering of the vertices to be coloured as input while the heuristic colours the vertices at the same time the order is being processed.

4.2.1. Order of vertices and colours

The order in which the vertices are chosen to be coloured follows essentially the same idea as the DSATUR algorithm, created by Daniel Brélaz [6].

Consider a graph $G = (V, E, w)$, $w : E \to \mathbb{R}_+^*$ and a partial colouring $c : U \to \{1, \ldots, k\}$, where $U \subseteq V$. We say that vertex v is *coloured* if $v \in U$, otherwise it is *uncoloured*. We define the *total potential interference* in vertex v to be:

$$
I_{c,v}^{tot} = \sum_{\{u \in V \mid uv \in E \text{ and } v \notin U\}} w(u, v),
$$

which is the sum of interferences for all colours induced in v by all its already coloured neighbours.

The idea for both algorithms is to first colour vertices with highest total potential interference. Whenever more than one vertex has the highest total potential interference, one of them is chosen at random. At the beginning, when all vertices have $I_{c,v}^{tot} = 0$, one of the highest weighted degree is chosen instead.

Consider the following steps:

1. Colour a random vertex with maximal sum of incoming weights.
2. Colour a random vertex with maximal total potential interference.
3. If all vertices all coloured, stop. Otherwise, repeat step 2.

Note that the total potential interference does not depend on the actual colours assigned to the vertices. Thus, in order to decide which is the next vertex to be coloured, both algorithms, Branch-and-Bound and heuristic, use these three steps. However, the Branch-and-Bound algorithm needs an order to colour the vertices as input. So, we decide which order to give to the Branch-and-Bound algorithm as input by running these three steps and using a single colour.

The procedure above specifies the order of vertices. For the order of colours to try, we define the *potential interference* in vertex v for colour x as:

$$I_{c,v,x} = \sum_{\{u \in V | uv \in E \text{ and } c(v)=x\}} w(u,v)$$

Anytime one of our algorithms colours a vertex, it tries the colours in order of increasing potential interference.

4.2.2. Branch-and-Bound Algorithm

Having an ordering procedure for both vertices and colours, we construct a simple Branch-and-Bound algorithm using them. The order of vertices to colour is fixed before running the algorithm, following the procedure in Section 4.2.1. Then, the ordered vertices are coloured by a recursive function that tries all the possible colours for each vertex as far as no interference constraint is violated. The order in which the colours are tried is also presented in the previous section. Our algorithm outputs all the feasible colourings it finds and, as all the possible colours are tried, the one using the minimum number of colours is an optimal one.

Here you have a pseudo code for the algorithm:

Algorithm 1: Branch&Bound

input : edge-weighted graph (G, w), number of colours k, partial colouring c, upper bound t and corresponding colouring $\tilde{c}$, order in which vertices should be coloured O

output: new upper bound t' and corresponding colouring $\tilde{c}$'

if $\max_{v \in V} I_v(G, w, c) \geq t$ **then**
    return t and $\tilde{c}$
if *all vertices are coloured in c* **then**
    return ($\max_{v \in V} I_v(G, w, c)$ and c)
v = next vertex uncoloured in c according to O
for $x \in$ *possible colours in order of increasing* $I_{c,v,x}$ **do**
    (t and $\tilde{c}$) = Branch&Bound(G, k, $c \cap (v \leftarrow x)$, t, $\tilde{c}$, O)
return t and $\tilde{c}$

Where by $c \cap (v \leftarrow x)$ we mean a partial colouring where colour of vertex v (which was uncoloured in c) is set to x, and colours of all other vertices are as in c. The algorithm is first called with all vertices uncoloured and $t = \infty$.

This algorithm displays a problematic behaviour. Imagine the partial colouring of the first few vertices yields good results locally, but implies a suboptimal interference at a more distant part of the graph. As the solution search takes exponential time in number of vertices, it is easy to envision that the time required to change the colouring of first vertices can be prohibitively long.

4.2.3. Greedy Heuristic

Here we propose a randomised greedy heuristic that, repeated multiple, but not exponentially many times, finds similar solutions to the above Branch-and-Bound without the mentioned problem. On the other hand, there are some solutions that are impossible to find with it, no matter the number of tries. An example of such an unobtainable solution is the optimal colouring of infinite square grid with 2 colours.

Algorithm 2: Levelling Heuristic

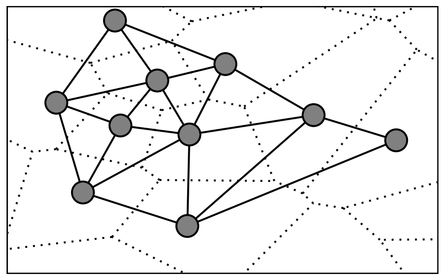
input : edge-weighted graph (G, w), number of colours k, upper bound t
output: **failed** or a colouring c

$c(v) = \emptyset \quad \forall v \in V$
for $i \in \{1, \ldots, |V|\}$ **do**
  $v =$ next, in order of increasing $I^{tot}_{c,v}$, vertex uncoloured in c
  for $x \in$ *possible colours in order of increasing* $I_{c,v,x}$ **do**
    if *colouring* v *with* x *does not cause* $\max_{v \in V} I_v(G, w, c) \geq t$ **then**
      $c(v) = x$
      break the inner loop
  if $c(v) = \emptyset$ **then**
    return failed
return c

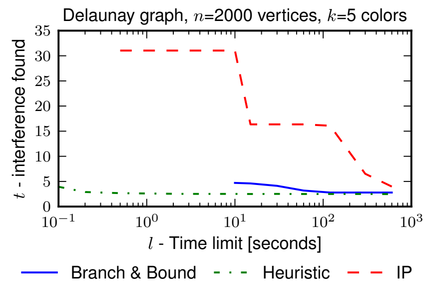
Note that there is substantial randomness in this algorithm. The first vertex is the one of the ones with highest weighted degree. In the extreme case of regular graphs, this already means any vertex at random. If we use the simple interference function defined in Section 3, then the second vertex is a random neighbour of the first vertex. Any time there are multiple vertices with maximum total potential interference, we choose one at random. Similarly, the choice of colours is also random in case of equal potential interference.

Above algorithm is first called with $t = \infty$. Whenever it returns a colouring, we set $t = \max_{v \in V} I_v(G, w, c)$ for further iterations. It is repeated for a given number of times, or until a time limit is reached. In all instances in the following sections the program is constrained by a time limit. This means that the algorithm is called for an unknown, but probably big number of times (e.g. for a 6-regular grid of 1024 vertices the program performs on average over 500 runs of the algorithm per second).

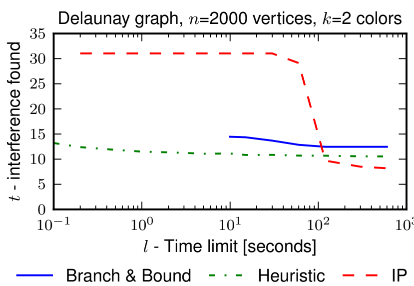
As a *randomised greedy colouring* heuristic, it has to be ran multiple times to achieve satisfactory results. This is not a practical issue due to low computational cost of each run. The local immutable colouring decision is taken in time $O(k\Delta)$. Then, after each such decision, the interference has to be propagated, which takes linear time in the vertex degree. This gives a computational complexity bound $O(kn\Delta)$-time.



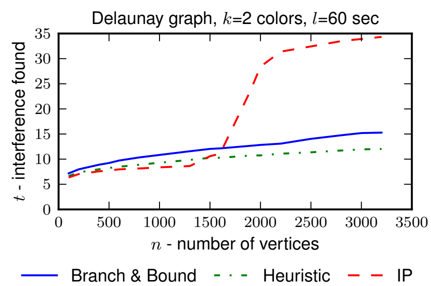
(a) Example Delaunay graph, dotted lines delimit corresponding Voronoi diagram cells



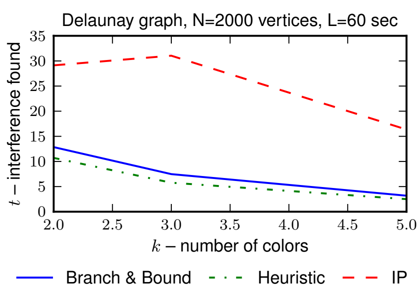
(b) Over time



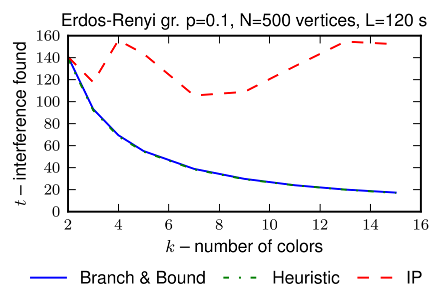
(c) Over time



(d) Over size



(e) Over colours



(f) Over colours

Figure 9: Results comparison for Levelling heuristic, Branch-and-Bound algorithm and Integer Linear Programming Formulation.

4.3. Validation

In this section we validate our algorithmic approaches at THRESHOLD IMPROPER COLOURING, by examining performance of their implementations. Tests cover a wide range of parameters, mostly on Delaunay graphs (see section 4.3.2).

4.3.1. Implementation

The ILP model is constructed out of the input graph and given directly to the CPLEX ILP solver. Branch-and-Bound algorithm is implemented in a straightforward way in the Python programming language. The greedy heuristic has a highly optimised implementation in the Cython programming language[2].

In results displayed below, all programs are run simultaneously on the same quad-core enterprise-grade CPU. Both the Branch-and-Bound and greedy heuristic are limited to a single core. CPLEX is allowed to both the remaining cores.

4.3.2. Graphs

We consider random Delaunay graphs (dual of Voronoi diagram). This kind of graphs is an intuitive approximation of a network of irregular cells. To obtain a graph in this class, take a set of random points uniformly distributed over a square. These represent the vertices of the graph. To obtain the edges, compute a Delaunay triangulation. This can be done e.g. with Fortune's algorithm described in [12] in $O(n \log n)$ time.

See Figure 9(a) for a depiction of a fragment of such graph. Vertices are arranged according to the positions of original random points. Dotted lines delimit corresponding Voronoi diagram cells. Only edges between vertices visible on the illustration are displayed.

Note that, to follow the model of the physical motivation, we are dealing with very sparse graphs. The average degree in Delaunay graph G converges to six (this results follows from the observation that G is planar and triangulated, thus $|E(G)| = 3|V(G)| - 6$ by Euler's formula). To get an idea about the proposed algorithms' performance in denser graphs, we also run some tests on Erdös-Rényi graphs with expected degree equal to 50.

The interference model we consider in all experiments is the one described in Section 3: adjacent nodes interfere by 1 and nodes at distance two interfere by 1/2.

4.3.3. Results

Figure 9 shows a performance comparison of the above-mentioned algorithms. For all the plots, each data point represents an average over a number (between 24 and 100) of different graphs. The experiment procedure is as follows. For each graph size considered in an experiment, a number of graphs is generated. Each of those graphs is transformed into a set of instances, one for each desired number of allowed colours. All the programs are run on each instance, once for each desired value of time limit. Finally, a data point is created with results and all the parameters, averaged over the number of graphs.

Figures 9(b) and 9(c) plot how results for a problem instance get enhanced with increasing time limits. Plot 9(d) shows how well all the programmes scale with increasing graph sizes. Plots 9(e) and 9(f) show decreasing interference along increasing the number of colours allowed.

One immediate observation about both the heuristic and Branch-and-Bound algorithm is that they provide good solutions in relatively short time. On the

[2]This is the faster implementation envisioned in [3].

other hand, with limited time, they fail to improve up to optimal results, especially with a low number of allowed colours. An example near-optimal solution found in around three minutes was not improved by Branch-and-Bound in over six days.

The heuristic, is able to provide good results in sub-second times and scales better with increasing graph sizes than the Branch-and-Bound. It is also not prone to spending a lot time exploring a sub-optimal branch of a decision tree. Still, in many cases it is unable to obtain optimal results and displays a worse end result than an integer linear program, given enough time.

Solving the ILP does not scale with increasing graph sizes as well as our simple algorithms. Furthermore, Figure 9(e) reveals one problem specific to ILP. When increasing the number of allowed colours, obtaining small interferences gets easier. But this introduces additional constraints in the formulation, thus increasing the complexity for a solver.

Proposed algorithms also work well for denser graphs. Figure 9(f) plots interferences for different numbers of colours allowed found by the programs for an Erdös-Rényi graph with $n = 500$ and $p = 0.1$. This gives us an average degree equal to 50. Both Branch-and-Bound and heuristic programs achieve acceptable, and nearly identical, results. But the large number of constraints makes the integer linear programming formulation very inefficient.

5. Conclusion, Open Problems and Future Directions

In this paper, we introduced and studied a new colouring problem, Weighted Improper Colouring. This problem is motivated by the design of telecommunication antenna networks in which the interference between two vertices depends on different factors and can take various values. For each vertex, the sum of the interferences it receives should be less than a given threshold value.

We first give general bounds on the weighted-improper chromatic number. We then study the particular case of infinite paths, trees and grids: square, hexagonal and triangular. For these graphs, we provide their weighted-improper chromatic number for all possible values of t. Finally, we propose a heuristic and a Branch-and-Bound algorithm to find good solutions of the problem. We compare their results with the one of an integer linear programming formulation on cell-like networks, Poisson-Voronoi tessellations.

Many problems remain to be solved:

- The study of the grid graphs, we considered a specific function where vertices at distance one interfere by 1 and vertices at distance two by 1/2. Other weight functions should be considered. e.g. $1/d^2$ or $1/(2^{d-1})$, where d is the distance between vertices.
- Other families of graphs could be considered, for example hypercubes.
- We showed that the Threshold Improper Colouring problem can be transformed into a problem with only two possible weights on the edges 1 and ∞, that is a mix of proper and improper colouring. This simplify

the nature of the graph interferences but at the cost of an important increase of instance sizes. We want to further study this. In particular, let $G = (V, E, w)$ be an edge-weighted graph where the weights are all equal to 1 or M. Let G_M be the subgraph of G induced by the edges of weight M; is it true that if $\Delta(G_M) << \Delta(G)$, then $\chi_t(G, w) \leq \chi_t(G) \leq \left\lceil \frac{\Delta(G,w)+1}{t+1} \right\rceil$? A similar result for $L(p, 1)$-labelling [15] suggests it could be true.

Note that the problem can also be solved *algorithmically* for other classes of graphs and for other functions of interference. We started looking in this direction in [4]. The problem can be expressed as a linear program and then be solved exactly using solvers such as CPLEX or Glpk[3] for small instances of graphs. For larger instances, we propose a heuristic algorithm inspired by DSATUR [6] but adapted to the specifics of our colouring problem. We used it to derive colouring with few colours for Poisson-Voronoi tessellations as they are good models of antenna networks [5, 13, 14]. We plan to further investigate the algorithmic side of our colouring problem.

References

[1] Aardal, K., van Hoesel, S., Koster, A., Mannino, C., Sassano, A., 2007. Models and solution techniques for frequency assignment problems. Annals of Operations Research 153 (1), 79–129.

[2] Alouf, S., Altman, E., Galtier, J., Lalande, J., Touati, C., 2005. Quasi-optimal bandwidth allocation for multi-spot MFTDMA satellites. In: INFOCOM 2005. 24th Annual Joint Conference of the IEEE Computer and Communications Societies. Proceedings IEEE. Vol. 1. IEEE, pp. 560–571.

[3] Araujo, J., Bermond, J.-C., Giroire, F., Havet, F., Mazauric, D., Modrzejewski, R., 2011. Weighted improper colouring. In: Iliopoulos, C., Smyth, W. (Eds.), Combinatorial Algorithms. Vol. 7056 of Lecture Notes in Computer Science. Springer Berlin / Heidelberg, pp. 1–18.

[4] Araujo, J., Bermond, J.-C., Giroire, F., Havet, F., Mazauric, D., Modrzejewski, R., 2011. Weighted Improper Colouring. Research Report RR-7590, INRIA.

[5] Baccelli, F., Klein, M., Lebourges, M., Zuyev, S., 1997. Stochastic geometry and architecture of communication networks. Telecom. Systems 7 (1), 209–227.

[6] Brélaz, D., 1979. New methods to color the vertices of a graph. Communications of the ACM 22 (4), 251–256.

[3] http://www.gnu.org/software/glpk/

[7] Brooks, R. L., 1941. On colouring the nodes of a network. Mathematical Proceedings of the Cambridge Philosophical Society 37 (02), 194–197.

[8] Correa, R., Havet, F., Sereni, J.-S., 2009. About a Brooks-type theorem for improper colouring. Australasian Journal of Combinatorics 43, 219–230.

[9] Cowen, L., Goddard, W., Jesurum, C., 1995. Defective coloring revisited. J. Graph Theory 24, 205–219.

[10] Cowen, L. J., Cowen, R. H., Woodall, D. R., 1986. Defective colorings of graphs in surfaces: Partitions into subgraphs of bounded valency. Journal of Graph Theory 10 (2), 187–195.

[11] Fischetti, M., Lepschy, C., Minerva, G., Romanin-Jacur, G., Toto, E., 2000. Frequency assignment in mobile radio systems using branch-and-cut techniques. European Journal of Operational Research 123 (2), 241–255.

[12] Fortune, S., 1987. A sweepline algorithm for voronoi diagrams. Algorithmica 2 (1), 153–174.

[13] Gupta, P., Kumar, P., 2000. The capacity of wireless networks. Information Theory, IEEE Transactions on 46 (2), 388–404.

[14] Haenggi, M., Andrews, J., Baccelli, F., Dousse, O., Franceschetti, M., 2009. Stochastic geometry and random graphs for the analysis and design of wireless networks. Selected Areas in Communications, IEEE Journal on 27 (7), 1029–1046.

[15] Havet, F., Reed, B., Sereni, J.-S., 2008. L(2,1)-labelling of graphs. In: Proceedings of the nineteenth annual ACM-SIAM symposium on Discrete algorithms. SODA '08. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, pp. 621–630.

[16] Karp, R., 1972. Reducibility among combinatorial problems. In: Miller, R., Thatcher, J. (Eds.), Complexity of Computer Computations. Plenum Press, pp. 85–103.

[17] Lovász, L., 1966. On decompositions of graphs. Studia Sci. Math. Hungar. 1, 273–238.

[18] Mannino, C., Sassano, A., 2003. An enumerative algorithm for the frequency assignment problem. Discrete Applied Mathematics 129 (1), 155–169.

[19] Woodall, D. R., 1990. Improper colorings of graphs. In: Nelson, R., Wilson, R. J. (Eds.), Pitman Res. Notes Math. Ser. Vol. 218. Longman Scientific and Technical, pp. 45–63.

[20] Yeh, R. K., 2006. A survey on labeling graphs with a condition at distance two. Discrete Mathematics 306 (12), 1217 – 1231.