

Weighted Improper Colouring

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Abstract. In this paper, we study a new colouring problem up to our best knowledge inspired by the imperative of practical networks. In real-life wireless networks, nodes interfere with one another with various intensities depending on numerous parameters: distance between them, the geographical topography, obstacles, etc. We model this with a noise matrix. The interference perceived by a node then is the sum of all the noise of the nodes emitting on the same frequency. The problem is then to determine the minimum number of colours (or frequencies) needed to colour the whole graph so that the interference does not exceed a given threshold. We provide several general results, such as bounds on this number of colours (e.g. a Brook's like theorem). We then study the practical case of square of infinite grids which corresponds to operators' network and a noise decreasing with the distance. We provide the chromatic number of the square, triangular and hexagonal grids for all possible admissible interference levels. Finally, we model the problem using linear programming, propose and test a heuristic and an exact branch&bound algorithms on random cell-like graphs, namely the Poisson Voronoi tessellations.

1 Introduction

Given a graph $G = (V, E)$, a k -colouring of G is a function $c : V \rightarrow \{1, \dots, k\}$. The colouring c is *proper* if $(u, v) \in E$ implies $c(u) \neq c(v)$. The *chromatic number* of a given graph G is the minimum integer k such that G admits a proper k -colouring, denoted by $\chi(G)$. The goal of the VERTEX COLOURING problem is to determine $\chi(G)$, for a given graph G . It is well-known that determining the chromatic number $\chi(G)$ of a given graph G is NP-hard.

A k -colouring c is *l -improper* if, for every vertex $u \in V$, $|\{v \in N(u) \mid c(v) = c(u)\}| \leq l$. Given a positive integer l , the *l -improper chromatic number* of a given graph G is the minimum integer k such that G has an l -improper k -colouring, which is denoted by $\chi_l(G)$. For a given graph G and an integer l , the goal of the IMPROPER COLOURING problem is to determine $\chi_l(G)$ [5]. Given a graph G and an integer l , the problem of computing the l -improper chromatic number $\chi_l(G)$ is also NP-hard. Indeed in [6], the authors proved that the problem of deciding if there is a l -improper k -colouring of a given graph G is NP-complete for all pairs (l, k) of integers with $l \geq 2$ and $k \geq 1$. In [5], the authors proved the NP-completeness for particular classes of graphs. For instance, given a graph G with

maximum degree at most 8, the problem of deciding if there is a 1-improper 4-colouring of G is NP-complete. Given a planar graph G with maximum degree 4, the problem of deciding if there is a 1-improper 2-colouring of G is NP-complete. A third NP-complete problem is the problem of deciding if there is a 2-improper 2-colouring of a given planar graph G with maximum degree 6.

In this work we define and study a new, up to our best knowledge, variation of the improper colouring problem for edge-weighted graphs.

Given an edge-weighted graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+$, and a threshold $t \in \mathbb{R}_+$, we say that c is a *weighted t -improper k -colouring* of G if c is a k -colouring of the vertices of G in such a way that, for each vertex $u \in V$, the following constraint is satisfied:

$$\sum_{\{v \in N(u) \mid c(v) = c(u)\}} w(u, v) \leq t.$$

Given a threshold $t \in \mathbb{R}_+$, the minimum integer k such that the graph G admits a weighted t -improper k -colouring is the *weighted t -improper chromatic number* of G , denoted by $\chi_t^w(G)$. Given an edge-weighted graph $G = (V, E, w)$ and a threshold $t \in \mathbb{R}_+$, determining $\chi_t^w(G)$ is the goal of the WEIGHTED IMPROPER COLOURING problem, that we define. The WEIGHTED IMPROPER COLOURING problem is clearly NP-complete even for the three particular instances described in [5]. Indeed given a graph G , we get the NP-completeness by choosing $\forall e \in E$, $w(e) = 1$, and $t = l$.

On the other hand, we also define the THRESHOLD IMPROPER COLOURING problem which is, for a given edge-weighted graph $G = (V, E, w)$ and a positive integer k , to determine the minimum real t such that G admits a weighted t -improper k -colouring that is called *minimum k -threshold* of G , denoted by $\omega_k^w(G)$. The THRESHOLD IMPROPER COLOURING problem is also NP-complete even for the previous particular cases by choosing $\forall e \in E$, $w(e) = 1$. However, if the graph G has bounded tree-width, then we present a polynomial-time algorithm to decide if $\omega_k^w(G) \leq t$.

Due to lack of space, the proofs of the results are omitted and can be found in the appendix or in the corresponding research report³.

Related Work. The motivation of these problems is the FREQUENCY ASSIGNMENT PROBLEM (FAP). FAP has several variations that were already studied in the literature (see [1] for a survey). In most of these variations, the main constraint to be satisfied is that if two vertices (mobile phones, antennas, spots, etc.) are close, then the difference between the frequencies that are assigned to them must be greater than some function that usually depends on their distance.

There is a strong relationship between most of these variations and the $L(p_1, \dots, p_d)$ -labelling. In this problem, the goal is to find a colouring of the vertices of a given graph G in such a way that the difference between the colours assigned to vertices at distance i must be at least p_i , for every $i = 1, \dots, d$.

³ The report can be found at the url http://www-sop.inria.fr/members/Julio-Cesar.Silva_Araujo/wic/

For some other variations, for each non-satisfied interference constraint a penalty must be paid. In particular, the goal of the MINIMUM INTERFERENCE ASSIGNMENT PROBLEM (MI-FAP) is to minimise the total penalties that must be paid, when the number of frequencies to be assigned is given. This problem can also be studied for only *co-channel interferences*, in which the penalties are applied only if the two vertices have the same frequency. Although, MI-FAP under these constraints does not correspond to WEIGHTED IMPROPER COLOURING, because we consider the co-channel interference, i.e. penalties, just between each vertex and its neighbourhood.

The two closest related works we found in the literature are [11] and [7]. However, they both apply penalties over co-channel interference, but also to the *adjacent channel interference*, i.e. when the colours of adjacent vertices differ by one unit. Moreover, their results are not similar to ours. In [11], they propose an enumerative algorithm for the problem, while in [7] a Branch-and-Cut method is proposed and applied over some instances.

Particular Practical Cases. Our initial motivation to look at these problems was the design of satellite antennas for multi-spot MFTDMA satellites [2]. In this technology, satellites transmit signals to areas on the ground called spots. These spots form a grid like structure which is modeled in the reference by an hexagonal-cell graph. To each spot is assigned a radio channel or colour. Spots are interfering with one another and a spot can use a colour only if the interference level does not exceed a given threshold t . The level of interference between two spots depends on their distance. The authors of [2] introduce a factor of mitigation γ and the interferences of remote spots are reduced by a factor $1 - \gamma$. When the interference level is too low, the nodes are considered to not interfere anymore. Considering such types of interference where nodes at distance at most i affect one another leads to the study of the i -th power of the graph modelling the network and a case of special interest is the power of grid graphs.

Results. In this work, we study both parameters $\chi_t^w(G)$ and $\omega_k^w(G)$. We present some bounds for general graphs, in particular Theorems 1 and 2. We then study some particular cases of the edge-weights function w modelling infinite square, hexagonal and triangular grids. As stated, these grids are often used to model networks of antennas (see for example [4, 12]). For these graphs, we provide $\chi_t^w(G)$ for all possible values of the threshold t , for a particular w , Theorems 6, 7 and 8. We then propose heuristics to solve the problem for general graphs. We compare them to an integer program formulation on random cell-like graphs, namely Voronoi diagrams of random points of the plan, classically used in the literature to model telecommunication networks [3,8,9]. For properties of Poisson Voronoi tessellation, see for example [13].

2 General Results

2.1 Bounds

Upper bound for Weighted Improper Colouring Lovász [10] proved a Brooks'-like theorem for the IMPROPER COLOURING problem. Namely, it holds that $\chi_l(G) \leq \lceil \frac{\Delta(G)+1}{l+1} \rceil$ for a graph G of maximum degree $\Delta(G)$.

In what follows, we show that this result nicely extends for WEIGHTED IMPROPER COLOURING. Nicely means here that an edge only contributes its weight to the value of the bound.

Given an edge-weighted graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+^*$, and $u \in V$, let $d_w(u) = \sum_{v \in N(u)} w(u, v)$. Denote by $\Delta_w(G) = \max_{u \in V} d_w(u)$. Given a k -colouring $c : V \rightarrow \{1, \dots, k\}$ of G , we denote $d_w^i(u) = \sum_{\{v \in N(u) \mid c(v)=i\}} w(u, v)$, for every vertex $u \in V$ and color $i = 1, \dots, k$. Finally, we note $gcd(w)$ the greatest common divisor of the weights of w . We use here the generalization of the gcd to non integer numbers (e.g. in \mathbb{R}) where a number x is said to divide a number y if the fraction y/x is an integer. The important property of the $gcd(w)$ we use here is that it is the smallest possible difference between two interferences, that is, for two vertices u and v , if $d_w^i(u) > d_w^j(v)$ then $d_w^i(u) \geq d_w^j(v) + gcd(w)$.

Theorem 1. *Given an edge-weighted graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+^*$, and a threshold $t \in \mathbb{R}_+^*$, then the following inequality holds, for any real $\epsilon > 0$:*

$$\chi_t^w(G) \leq \lceil \frac{\Delta_w(G) + \epsilon}{t + gcd(w)} \rceil.$$

Proof. We say that a k -colouring c of G is *well-balanced* if c satisfies the following property:

Property 1. For all vertex $v \in V$, if it is coloured i , then $d_w^i(v) \leq d_w^j(v)$, for every $j = 1, \dots, k$.

First, we prove that for any $k \geq 2$, there exists a well-balanced k -colouring of G . To prove this fact one may just colour G arbitrarily with k colours and then repeat the following procedure: if there exists a vertex v coloured i and a colour j such that $d_w^i(v) > d_w^j(v)$, then recolour v with colour j . Observe that this procedure neither increase (we just move a vertex from one colour to another) nor decrease (a vertex without neighbour on its colour is never moved) the number of colours with this process. Moreover, the weight of the edges between vertices coloured i and j has just increased. All the other edges between colours have not changed. Thus, each time that a vertex is recoloured, the sum of the weights of the edges between the colours increase by a minimal amount of $gcd(w)$. Consequently, this procedure finishes and produces a well-balanced k -colouring of G in finite time (as we know that the sum of the weights cannot exceed $|E| \max_{e \in E} w(e)$).

Observe that in any well-balanced k -colouring c of a graph G , the following holds:

$$d_w(v) = \sum_{u \in N(v)} w(u, v) \geq k d_w^i(v). \quad (1)$$

Let $k^* = \lceil \frac{\Delta_w(G) + \epsilon}{t + \gcd(w)} \rceil \geq 2$ and c^* be a well-balanced k^* -colouring. We claim that c^* is a weighted t -improper k^* -colouring of G .

By contradiction, suppose that there is a vertex v in G such that $c^*(v) = i$ and that $d_w^i(v) > t$. Since c^* is well-balanced, $d_w^j(v) > t$, for all $j = 1, \dots, k^*$. By the definition of $\gcd(w)$, it leads to $d_w^j(v) \geq t + \gcd(w)$ for all $j = 1, \dots, k^*$. Combining this inequality with Inequality (1), we obtain:

$$\Delta_w(G) \geq d_w(v) \geq k^*(t + \gcd(w)),$$

giving

$$\Delta_w(G) \geq d_w(v) \geq \Delta_w(G) + \epsilon,$$

a contradiction. The result follows. \square

Note that when all weights are one, we obtain the bound for the improper colouring derived in [10]. As a matter of fact, in this case $\gcd(w) = 1$ and $\Delta_w(G) = \Delta$ is an integer. Thus, there exist two integers a, b such that $\Delta = a(t + 1) + b$, with $b \leq k - 1$. For an $\epsilon < 1$, we have $\lceil \frac{\Delta + \epsilon}{t + 1} \rceil = a + 1 = \lceil \frac{\Delta + 1}{t + 1} \rceil$. Note also that the bound is tight for complete graphs with same weight on each edge.

Open Problem: characterise all graphs for which the bound is tight.

Upper bound for Threshold Improper colouring. Let $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+^*$, be an edge-weighted graph and k be a positive integer. $V' = \{u \in V, d(u) \geq k\}$ represents the set of nodes with degree at least k . $G' = (V', E', w')$ is the graph obtained after removing all nodes of $V \setminus V'$.

Lemma 1. $\omega_k^w(G) = \omega_k^w(G')$

For the rest of the section, we only consider edge-weighted graphs with minimum degree at least k . For each $v \in V$, let $E_{min}^{k-1}(v)$ be the set of $d(v) - (k - 1)$ least weighted edges incident to v .

Theorem 2. Let $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+^*$, be an edge-weighted graph and k be a positive integer. Then, $\omega_k^w(G) \leq \max_{v \in V} w(E_{min}^{k-1}(v))$, where $w(E_{min}^{k-1}(v)) = \sum_{e \in E_{min}^{k-1}(v)} w(e)$.

2.2 Transformation

In this section, we prove that the THRESHOLD IMPROPER COLOURING problem can be transformed into a problem mixing proper and improper colouring. More precisely, given an edge-weighted graph $G = (V, E, w)$ such that $\forall e \in E$, $w(e) \in \mathbb{N} \cup \{\infty\}$, and a positive integer k , the problem of computing $\omega_k^w(G)$ can be transformed into the problem of computing $\omega_k^w(G^* = (V^*, E^*, w^*))$ such that for any $e \in E^*$, $w^*(e) \in \{1, \infty\}$. We describe in Appendix A.1 how to construct the graph G^* proving also that the desired property on the weights is always verified. We prove that the two problems are equivalent, that is

Theorem 3. *Let $G = (V, E, w)$ be an edge-weighted graph such that $\forall e \in E$, $w(e) \in \mathbb{N} \cup \{\infty\}$, and a positive integer k . Let $G^* = (V^*, E^*, w^*) = G^{s+1}$ be the edge-weighted graph constructed from G verifying that $\forall e \in E^*$, $w^*(e) \in \{1, \infty\}$. Then $\omega_k^w(G) = \omega_k^w(G^*)$.*

After the transformation, the problem remains to find the minimum l such that a (non-weighted) l -improper k -colouring of G^* exists with the constraint that some subgraphs of G^* must admit a proper colouring. The equivalence of the two problems have been proved for integers weights but it is possible to adapt the transformation taking into consideration rational weights.

Open Problem: For such graphs with only two possible weight values 1 and ∞ , we think that for a high density of edges with weight 1, the problem may be equivalent to the improper colouring problem for the graph after removing the edges of weight ∞ .

3 Squares of Particular Graphs

As mentioned in the introduction, the general problem is motivated by networks of antennas that resemble grids [2]. In this network, the noise generated by an antenna undergoes an attenuation with the distance it travels. We model this specific case with squares of infinite grid. Given a graph G , the square of G , denoted by G^2 , is the graph obtained from G by adding all edges (u, v) such that u and v are at distance 2 in G . We study a particular case of the function of weights. Given a graph $G = (V, E)$ and its square $G^2 = (V, E^2)$, we study from now on the function $w^* : E \rightarrow \{1, 0.5\}$ such that $w^*(e) = 0.5$ if, and only if, $e \in E^2 \setminus E$. We choose this function as an example of signal attenuation.

We show an example of a graph on 4 vertices and its square in Figure 3. Vertices at distance one have an interference of one and vertices at distance two have interference of 0.5.

For any $t \in \mathbb{R}_+$, we provide the weighted t -improper chromatic number of infinite square grids in Theorem 6, hexagonal grids in Theorem 8, and triangular grids in Theorem 7. For all values of the threshold t , we provide constructions using the minimum number of colours, along with proofs of their optimality. For the triangular grids, the proofs involved too many subcases to be readable because of the higher degree of this grid. We thus ran the integer program defined in Section 4⁴. It would be nice to obtain concise combinatorial proofs of these bounds. As a preliminary we study the cases of infinite paths and trees.

3.1 Infinite paths and trees

In this section, we characterise the weighted t -improper chromatic number of the square of an infinite path, for all positive real t . Moreover, we present a simple upper bound for $\chi_t^w(T^2)$, for a given tree T .

⁴ The code can be found at the url http://www-sop.inria.fr/members/Julio-Cesar.Silva_Araujo/wic/

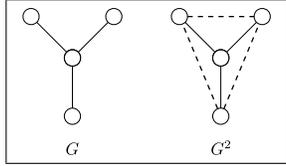
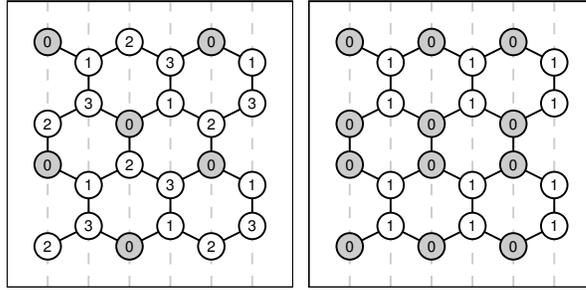


Fig. 1. A graph and its square. Solid (dashed) lines: interference one (one half).



(a) $t = 0, k = 4$ (b) $t = 2, k = 2$

Fig. 2. Optimal constructions for the hexagonal grid.

Theorem 4. Let $P = (V, E)$ be an infinite path. Then,

	<i>if</i>	$0 \leq t < 1$	$1 \leq t < 3$	$3 \leq t$
$\chi_t^w(G^2) =$		3	2	1

Theorem 5. Let $T = (V, E)$ be a tree. Then, $\lceil \frac{\Delta(T)-|t|}{2t+1} \rceil + 1 \leq \chi_t^w(T^2) \leq \lceil \frac{\Delta(T)-1}{2t+1} \rceil + 2$.

3.2 Grids

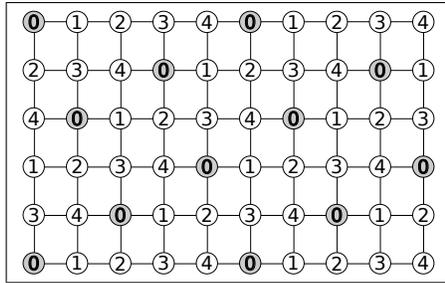
In this section, we show the optimal values of $\chi_t^w(G^2)$, whenever G is an infinite square grid, or triangular grid or a hexagonal one, for all the possible values of t . We provide examples of optimal constructions in Figure 2, 3 and 4.

Square Grid. The square grid is the graph in which the vertices are all integer linear combinations $ae_1 + be_2$ of the two vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$, for any $a, b \in \mathbb{Z}$. Each vertex (a, b) has four neighbours: its left neighbour $(a - 1, b)$, its right neighbour $(a + 1, b)$, its top neighbour $(a, b + 1)$ and its down neighbour $(a, b - 1)$.

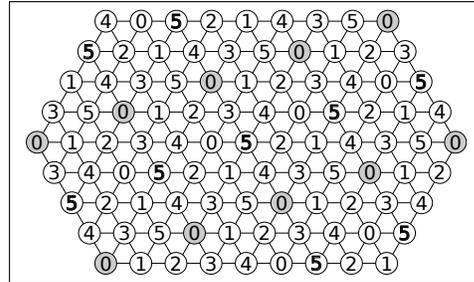
Theorem 6. If G is an infinite square grid, then

	<i>if</i>	$0 \leq t < 0.5$	$0.5 \leq t < 1$	$1 \leq t < 3$	$3 \leq t < 8$	$8 \leq t$
$\chi_t^w(G^2) =$		5	4	3	2	1

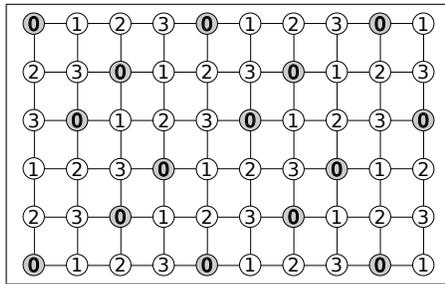
Triangular Grid. The triangular grid is graph whose vertices are all the integer linear combinations $ae_1 + be_2$ of the two vectors $e_1 = (1, 0)$ and $e_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus we may identify the vertices with 2 the ordered pairs (a, b) of integers. Each vertex $v = (a, b)$ has six neighbours: its left neighbour $(a - 1, b)$, its right neighbour $(a + 1, b)$, its leftup neighbour $(a - 1, b + 1)$, its rightup neighbour $(a, b + 1)$, its leftdown neighbour $(a, b - 1)$ and its rightdown neighbour $(a + 1, b - 1)$.



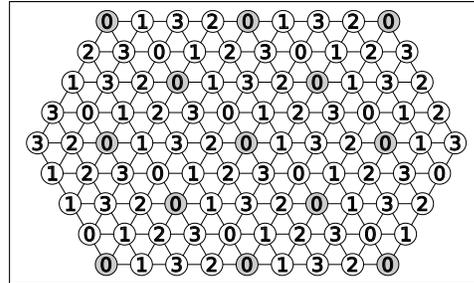
(a) Weighted 0-improper 5-colouring of G^2 , for square grid G .



(a) Weighted 0.5-improper 6-colouring of G^2 , for triangular grid G .



(b) Weighted 0.5-improper 4-colouring of G^2 , for square grid G .



(b) Weighted 1.5-improper 4-colouring of G^2 , for triangular grid G .

Fig. 3. Constructions for the square grid.

Fig. 4. Constructions for the triangular grid.

Theorem 7. *If G is an infinite triangular grid, then*

<i>if</i>	$t = 0$	$t = 0.5$	$t = 1$	$1.5 \leq t < 3$	$3 \leq t < 5$	$5 \leq t < 12$	$12 \leq t$
$\chi_t^w(G^2) =$	7	6	5	4	3	2	1

Hexagonal Grid. The hexagonal grid graph is the graph whose vertex set is the set of pairs of integers (a, b) , $a, b \in \mathbb{Z}$ and each vertex (a, b) has 3 neighbours: $(a, b - 1)$, $(a, b + 1)$ and $(a - 1, b)$, if b is even, or $(a + 1, b)$, otherwise.

Theorem 8. *If G is an infinite hexagonal grid, then*

<i>if</i>	$0 \leq t < 1$	$1 \leq t < 2$	$2 \leq t < 6$	$6 \leq t$
$\chi_t^w(G^2) =$	4	3	2	1

4 Algorithms, Integer Programs and Results

In this section we introduce several algorithmic approaches to the problems introduced in this paper. We start with a simple greedy heuristic for THRESHOLD

IMPROPER COLOURING. Then we introduce a branch and bound algorithm inspired by it. Finally we present linear programming models for both problems coped with in this paper.

Levelling Heuristic. We developed a simple heuristic aiming at obtaining a leveled distribution of interference over the nodes. Thus it attempts at solving the THRESHOLD IMPROPER COLOURING problem. The principal idea is to colour each node to *minimise the local interference*. This is achieved by tracking potential interference — the interference induced by already coloured vertices on the ones yet to colour. Every time a node is selected to be coloured, it is selected among the nodes with the greatest sum of potential interferences in all colours. Then colours are considered in increasing order of potential interference. Initially it was the only decision criterion. Performance is greatly enhanced by supplying an *interference target* $-t_t$. Then the local colouring decision is taken with the additional constraint, that *interference in no other node is increased past t_t* . For first run t_t is set to a very big number and updated to $T - \varepsilon$ whenever solution with interference T is found. The pseudo-code of the heuristic and its detailed explanation can be found in Appendix C.

As a *randomised greedy colouring* heuristic, it has to be run multiple times to achieve good results. Still it is efficient enough to be practical. The local, immutable colouring decision is taken in time $O(k)$. Then, after each such decision, the interference has to be propagated, which takes time linear to the node degree.

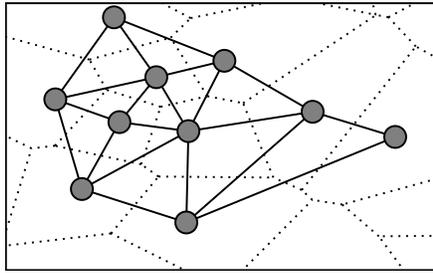
Branch and Bound Algorithm. We also implemented a simple branch and bound algorithm inspired by the above heuristic. It chooses colours for vertices in the same order as the heuristic. It also follows the heuristic order for colours tried. Despite its naive implementation in the Python programming language, it is capable to produce good colourings in a short time. As a branch and bound algorithm it is also guaranteed to find the optimal solution in a finite, alas exponential, time.

Integer Programming Models. It is not difficult to model both problems WEIGHTED IMPROPER COLOURING and THRESHOLD IMPROPER COLOURING by integer programs. The formulations are given in Appendix B. In the following, we compare the solutions proposed by CPLEX for these models with the ones produced by the described Branch-and-Bound algorithm and the Levelling heuristic.

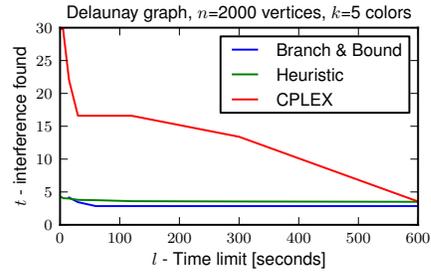
4.1 Results

Figure 5 shows a performance comparison of algorithms mentioned above. The comparison is based on Delaunay graph (dual of Voronoi diagram) for a set of random points. This kind of graph is a natural approximation of a network of irregular cells. We consider a weighted version of the squared graph, as described in Section 3.

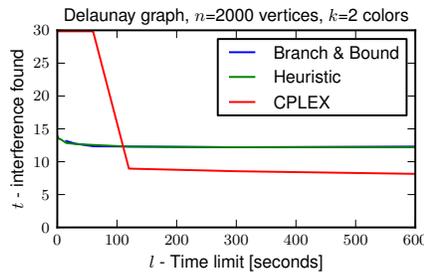
Figure 5(c) shows how the solutions get optimised over time. Ten random graphs were generated. Then each program was run with different time limits. Best interference averaged over the ten graphs is plotted against the time limits.



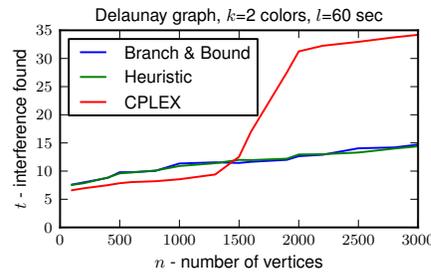
(a) Example Delaunay graph, dotted lines delimit corresponding Voronoi diagram cells



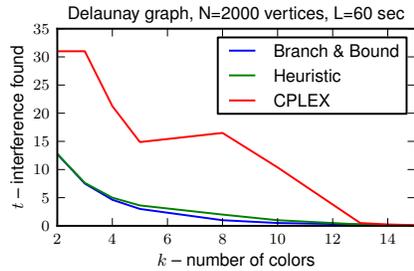
(b) Over time



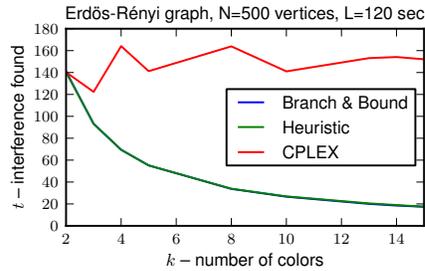
(c) Over time



(d) Over size



(e) Over colours



(f) Over colours

Fig. 5. Results comparison for Levelling heuristic, branch and bound algorithm and linear program run with CPLEX.

Similarly, plot 5(e) represents interference when changing numbers of colours used. Finally Figure 5(d) plots average results for ten graphs for each graph size.

One immediate observation about both the heuristic and branch&bound algorithm is that they deliver fast. Despite their naive implementation in a slow programming language they tend to find near-optimal results in matter of seconds even for very large graphs. On the other hand, they fail to improve up to optimal results, especially with a low number of colours allowed. Although it is easy to envision an implementation faster by orders of magnitude, this

may still give little improvement — once a near-optimal solution is found, the branch&bound algorithm tends to get stuck.

CPLEX mitigates this problem with its branch&cut algorithm and a very good implementation. However, it can not take advantage of any specialised knowledge of the problem, only the basic linear program representation. Thus it takes much more time to produce first good results. Despite taking advantage of multi-core processing, it does not scale with increasing graph sizes as well as our simple algorithms. Furthermore, Figure 5(e) reveals one problem specific to linear programming. When increasing the number of colours allowed, obtaining small interferences gets easier. But this introduces additional constraints in the linear program, thus increasing the complexity for a solver.

To sum up, which solution to choose depends on your needs. If its vital to obtain optimal results, the graph is small or you can wait a very long time for a colouring, you should go with linear program. For big graphs or short deadlines the other solutions are preferable, but come at a cost of probably slightly suboptimal interferences.

Above observations are valid only for the very particular case of the simple interference function and very sparse graphs. The average degree in Delaunay graph converges to 6. Proposed algorithms also work quite well for denser graphs. Figure 5(f) plots interferences for different numbers of colors allowed found by the programs for an Erdős-Rényi graph with $n=500$ and $p=0.1$. This gives us an average degree of 50. Both branch&bound and heuristic programs achieve acceptable, and nearly identical, results. But the sheer number of constraints renders the linear program nearly unusable.

5 Conclusion, Open Problems and Future Directions

In this paper, we introduced and studied a new colouring problem, WEIGHTED IMPROPER COLOURING. Given a graph G , a weight function $w : E \rightarrow \mathbb{R}_+^*$ and a threshold value $t \in \mathbb{R}_+$, the problem is to determine the minimum number of colours needed to colour G in such a way that, for every vertex $u \in V$,

$$\sum_{v \in N(u) | c(v)=c(u)} w(u, v) \leq t.$$

This problem is motivated by the design of telecommunication antenna network in which the interferences between two nodes depends on different factors and can take various values. For each node, the sum of the interference it receives should be less than a given threshold value.

We first give general bounds on the chromatic number. We then study the particular case of square, triangular and hexagonal grids. For these graphs, we provide their chromatic number for all possible values of t . Finally, we propose a heuristic and a branch&bound algorithm to find good solutions of the problem. We compare their results with the one of an integer program on cell-like networks, Poisson Voronoi tessellations.

Open problems and future directions. A large number of problems remain:

- We plan to determine for which graphs other than complete graphs with a uniform weight function the Brook’s like bound is tight.
- For the study of the grid graphs, we considered a specific function where node at distance one interfere by 1 and nodes at distance 2 by 1/2. Other weight functions should be considered. e.g. $1/d^2$, where d is the distance between nodes.
- Other families of graphs could be considered, for example Hypercubes.
- We are working on the study of specific weight functions with only two kinds of weights 1 and ∞ . Note that we proved in Section 2.2 that all instances of the THRESHOLD IMPROPER COLOURING can be reduced to an instance with only these two weights. The problem then boils down to a combination of a proper and a classic improper colouring for these graphs. We think that for a high density of edges with weight 1, the problem may be equivalent to considering only the improper colouring.

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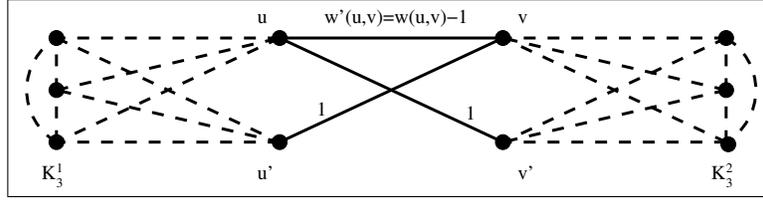


Fig. 6. Construction of G^{i+1} from G^i using edge (u, v) with $k = 4$. Dashed edges represent edges with infinite weights.

A Proofs of Section General Results

Proof of Lemma 1:

Proof. \Leftarrow If there is a weighted t -improper k -colouring of G' , then it is easy to get a weighted t -improper k -colouring of G choosing, for each node $u \in V \setminus V'$, a colour different than colours of its neighbours. It is always possible because $d(u) \leq k - 1$.

\Rightarrow If there is a weighted t -improper k -colouring of G , then there is a weighted t -improper k -colouring of G' choosing $\forall v \in V'$, $c_{G'}(v) = c_G(v)$. \square

Proof of Theorem 2:

Proof. Let $G_{min}^{k-1} = G[E \setminus \{\bigcup_{v \in V} E_{min}^{k-1}(v)\}]$. Observe that the maximum degree of a vertex in $G_{min}^{k-1} \leq k - 1$. Consequently, G_{min}^{k-1} admits a k -proper colouring c of its vertices.

Observe that the maximum interference of a vertex v in G when G is coloured by the colouring c is exactly $\max_{v \in V} w(E_{min}^{k-1}(v))$ and the result follows. \square

A.1 Transformation

In this section, we prove that the THRESHOLD IMPROPER COLOURING problem can be transformed into a problem mixing proper and improper colouring. More precisely, given an edge-weighted graph $G = (V, E, w)$ such that $\forall e \in E$, $w(e) \in \mathbb{N} \cup \{\infty\}$, and a positive integer k , the problem of computing $\omega_k^w(G)$ can be transformed into the problem of computing $\omega_k^w(G^* = (V^*, E^*, w^*))$ such that for any $e \in E^*$, $w^*(e) \in \{1, \infty\}$. We first describe how to construct the graph G^* proving also that the desired property on the weights is always verified. Then, we prove that the two problems are equivalent.

Let $G = (V, E, w)$ be an edge-weighted graph such that $\forall e \in E$, $w(e) \in \mathbb{N} \cup \{\infty\}$, and let k be a positive integer. We will construct the graphs G^{i+1} from G^i , for $i = 0 \dots s$ with $G^0 = G$. We prove in Lemma 2 that there exists a finite integer s such that $G^{s+1} = (V^{s+1}, E^{s+1}, w^{s+1})$ verifying that $\forall e \in E^{s+1}$, $w^{s+1}(e) \in$

$\{1, \infty\}$. We set $G^* = G^{s+1}$. Then we prove in Theorem 9 that $\omega_k^w(G) = \omega_k^{w^*}(G^*)$. We first describe the construction of the graph G^i , $0 \leq i \leq s+1$.

Consider the graph $G^i = (V^i, E^i, w^i)$, $0 \leq i \leq s$. Let us define $w_{max}^i(G^i) = \max_{e \in E^i} \{w^i(e) \mid w^i(e) \neq \infty\}$. If $\forall e \in E^i, w^i(e) = \infty$, then it is done. If there is at least one edge $e \in E^i$ such that $w^i(e) \neq \infty$ and $w_{max}^i = 1$, then it is done. Otherwise, there exists $(u, v) \in E^i$ such that $2 \leq w^i(u, v) < \infty$.

Let K_{k-1}^1 and K_{k-1}^2 be two complete graphs such that $\forall e \in E(K_{k-1}^1) \cup E(K_{k-1}^2), w^{i+1}(e) = \infty$. We construct the graph $G^{i+1} = (V^{i+1}, E^{i+1}, w^{i+1})$ with $V^{i+1} = V^i \cup V(K_{k-1}^1) \cup V(K_{k-1}^2) \cup \{u', v'\}$ and $E^{i+1} = E^i \cup F$. $\forall x \in V(K_{k-1}^1), \forall y \in \{u, u'\}, (x, y) \in F$ with $w^{i+1}(x, y) = \infty$. $\forall x \in V(K_{k-1}^2), \forall y \in \{v, v'\}, (x, y) \in F$ with $w^{i+1}(x, y) = \infty$. $(u, v') \in F$ with $w^{i+1}(u, v') = 1$, and $(u', v) \in F$ with $w^{i+1}(u', v) = 1$. Finally $\forall e \in V^i \setminus \{(u, v)\}, w^{i+1}(e) = w^i(e)$ and $w^{i+1}(u, v) = w^i(u, v) - 1$. Figure 6 shows the transformation of edge $(u, v) \in E^i$ to get G^{i+1} with $k = 4$.

Lemma 2. *There exists a finite integer s such that $G^{s+1} = (V^{s+1}, E^{s+1}, w^{s+1})$ is such that $\forall e \in E^{s+1}, w^{s+1}(e) \in \{1, \infty\}$.*

Proof. Given $G^i = (V^i, E^i, w^i)$, we define the function f as follows: $f(G^i) = \sum_{\{(x,y) \in E^i \mid w^i(x,y) \neq \infty\}} (w^i(x,y) - 1)$. We will prove that if $f(G^i) > 0$, then $f(G^{i+1}) = f(G^i) - 1$. Consider $i \geq 1$. We assume that $f(G^i) > 0$, that is there exists an edge $(u^i, v^i) \in E^i$ such that $w^i(u^i, v^i) \geq 2$. This edge will be used to transform the graph (Figure 6). We have $\forall e \in E^i \setminus \{(u^i, v^i)\}, w^{i+1}(e) = w^i(e)$. Furthermore $\forall e \in E^{i+1} \setminus E^i, w^{i+1}(e) \in \{1, \infty\}$, and so these edges do not contribute to $f(G^{i+1})$. Finally $w^{i+1}(u, v) = w^i(u, v) - 1$, and so we get $f(G^{i+1}) = f(G^i) - 1$. Since $f(G^0)$ is a finite integer, there exists a finite integer s such that $f(G^{s+1}) = 0$. It means that G^{s+1} is such that $\forall e \in E^{s+1}, w^{s+1}(e) \in \{1, \infty\}$ by definition of the function f . □

Theorem 9. *Let $G = (V, E, w)$ be an edge-weighted graph such that $\forall e \in E, w(e) \in \mathbb{N} \cup \{\infty\}$, and a positive integer k . Let $G^* = (V^*, E^*, w^*) = G^{s+1}$ be the edge-weighted graph constructed from G verifying that $\forall e \in E^*, w^*(e) \in \{1, \infty\}$. Then $\omega_k^w(G) = \omega_k^{w^*}(G^*)$.*

Proof. Consider the graphs $G^i = (V^i, E^i, w^i)$ and $G^{i+1} = (V^{i+1}, E^{i+1}, w^{i+1})$, $0 \leq i \leq s$. $(u, v) \in E^i$ denotes the edge used to get G^{i+1} in the transformation. We will prove that, $\forall i \in \{0, \dots, s\}, \omega_k^w(G^i) = \omega_k^{w^*}(G^{i+1})$.

\Rightarrow Suppose that there exists a weighted t-improper k-colouring of G^i . $\forall x \in V^i, c_{G^i}(x)$ denotes the colour of x and $d_{G^i}(x) = \sum_{\{y \in N(x) \mid c_{G^i}(y) = c_{G^i}(x)\}} w^i(x, y)$. By assumption $d_{G^i}(x) \leq t, \forall x \in V^i$. We now assign colours to nodes of G^{i+1} . First $\forall x \in V^i, c_{G^{i+1}}(x) = c_{G^i}(x)$ (recall that $V^i \subset V^{i+1}$). We assign to the nodes of $K_{k-1}^1, k-1$ different colours, each different than $c_{G^{i+1}}(u)$. Formally $\forall x, y \in V(K_{k-1}^1), x \neq y$, then $c_{G^{i+1}}(x) \neq c_{G^{i+1}}(y), c_{G^{i+1}}(x) \neq c_{G^{i+1}}(u)$, and $c_{G^{i+1}}(y) \neq c_{G^{i+1}}(u)$. Furthermore, $c_{G^{i+1}}(u') = c_{G^{i+1}}(u)$. Then we assign to the nodes of $K_{k-1}^2, k-1$ different colours, each different than $c_{G^{i+1}}(v)$. Formally $\forall x, y \in V(K_{k-1}^2), x \neq y$, then $c_{G^{i+1}}(x) \neq c_{G^{i+1}}(y), c_{G^{i+1}}(x) \neq$

$c_{G^{i+1}}(v)$, and $c_{G^{i+1}}(y) \neq c_{G^{i+1}}(v)$. Furthermore, $c_{G^{i+1}}(v') = c_{G^{i+1}}(v)$. By construction, we have $d_{G^{i+1}}(u') \leq d_{G^{i+1}}(u)$ and $d_{G^{i+1}}(v') \leq d_{G^{i+1}}(v)$. Furthermore $\forall x \in V(K_{C-1}^1) \cup V(K_{C-1}^2)$, $d_{G^{i+1}}(x) = 0$. Finally $d_{G^{i+1}}(u) = d_G^i(u)$ because if $c_{G^{i+1}}(u) = c_{G^{i+1}}(v)$, then $c_{G^{i+1}}(u) = c_{G^{i+1}}(v')$, and the two adjacent edges have weights $w^i(u, v) - 1$ and 1 , respectively (other contributions to this penalty do not change). Similarly $d_{G^{i+1}}(v) = d_G^i(v)$. Thus there exists a weighted t-improper k-colouring of G^{i+1} .

\Leftarrow Suppose that there exists a weighted t-improper k-colouring of G^{i+1} . $\forall x \in V^{i+1}$, $c_{G^{i+1}}(x)$ denotes the colour of x . $\forall x \in V^{i+1}$, we set $d_{G^{i+1}}(x) = \sum_{\{y \in N(x) | c_{G^{i+1}}(y) = c_{G^{i+1}}(x)\}} w^i(x, y)$. By assumption $d_{G^{i+1}}(x) \leq t$, $\forall x \in V^{i+1}$. By construction $d_{G^{i+1}}(u') \leq d_{G^{i+1}}(u)$ and $\forall x \in V(K_{k-1}^1) \cup V(K_{k-1}^2)$, $d_{G^{i+1}}(x) = 0$ because of the infinite weights at adjacent edges. We now assign to nodes of G^i the same colours than in G^{i+1} . Formally, $\forall x \in V^i$, $c_G^i(x) = c_{G^{i+1}}(x)$. By assumption we get that $\forall x \in V^i \setminus \{u, v\}$, $d_G^i(x) \leq t$. Finally $d_G^i(u) = d_{G^{i+1}}(u)$ because if edge (u, v) of weight $w^i(u, v) - 1$ participates to $d_{G^{i+1}}(u)$, then edge (u, v') participates also to $d_{G^{i+1}}(u)$ ($w^{i+1}(u, v') = 1$); then (u, v) participates to $d_G^i(u)$ with weight $w^i(u, v)$ (other contributions do not change). Similarly $d_G^i(v) = d_{G^{i+1}}(v)$. Thus there exists a weight t-improper k-colouring of G . \square

In the worst case, the number of nodes of G^* is $m(w_{max} - 1)(n + 2k)$ and the number of edges of G^* is $m + m(w_{max} - 1)k(k + 1)$ with $n = |V|$ and $m = |E|$. The worst case is when each edge $e \in E$ is such that $w(e) = w_{max}$ even if in that case we can do a simpler transformation.

In conclusion, this construction permits to transform the THRESHOLD IMPROPER COLOURING problem into a problem mixing proper and improper colouring. Indeed the problem remains to find the minimum l such that a (non-weighted) l -improper k -colouring of G^* exists with the constraint that some subgraphs of G^* must admit a proper colouring. The equivalence of the two problems have been proved to integers weights but it is possible to adapt the transformation taking into consideration rational weights.

B Integer Programming Models

It is not difficult to model both problems WEIGHTED IMPROPER COLOURING and THRESHOLD IMPROPER COLOURING by integer programming formulations.

Given an edge-graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+^*$, and a positive real threshold t , we model WEIGHTED IMPROPER COLOURING by using two kinds of variables. Variables x_{ip} indicate if vertex i is coloured p and variables c^p indicate if colour p is used, for every $1 \leq i \leq n$ and $1 \leq p \leq l$, where l is an upper bound for the number of colours needed in an optimal weighted t -improper colouring of G . The model follows:

$$\begin{array}{ll}
\min & \sum_p c^p \\
\text{subject to} & \\
& \sum_{j \neq i} w(i, j) x_{jp} \leq t + M(1 - x_{ip}) \quad (\forall i \in V, \forall p \in \{1, \dots, l\}) \\
& c^p \geq x_{ip} \quad (\forall i \in V, \forall p \in \{1, \dots, l\}) \\
& \sum_p x_{ip} = 1 \quad (\forall i \in V) \\
& x_{ip} \in \{0, 1\} \quad (\forall i \in V, \forall p \in \{1, \dots, l\}) \\
& c^p \in \{0, 1\} \quad (\forall p \in \{1, \dots, l\})
\end{array}$$

where M is a large integer. For instance, one may choose $M \geq \sum_{(u,v) \in E} w(u, v)$. This model corresponds to a simpler case of the one presented in [1], Section 3.5.

For THRESHOLD IMPROPER COLOURING, given an edge-weighted graph $G = (V, E, w)$, $w : E \rightarrow \mathbb{R}_+^*$, and a positive integer k , we do not need the variables c^p to model it, since the maximum number of colours k is fixed. Consequently, the model we consider is:

$$\begin{array}{ll}
\min & t \\
\text{subject to} & \\
& \sum_{j \neq i} w(i, j) x_{jp} \leq t + M(1 - x_{ip}) \quad (\forall i \in V, \forall p \in \{1, \dots, k\}) \\
& \sum_p x_{ip} = 1 \quad (\forall i \in V) \\
& x_{ip} \in \{0, 1\} \quad (\forall i \in V, \forall p \in \{1, \dots, k\})
\end{array}$$

C More on the Levelling Heuristic

The pseudo-code of the heuristic we developed is shown in Algorithm 1. Two important auxiliary arrays are used: I and I' . $I_{v,c}$ stores what is the potential interference for color c in node v . I'_v stores the sum of all potential interferences in node v . They are initialized to zero in lines 3-6 and are updated whenever a vertex is coloured in lines 16-18. I' is the base for the order in which vertices are coloured — in lines 10-11 we choose a random vertex v among the ones with greatest I'_v . Then, in lines 12-13, we try each possible colour c in order given by non-decreasing values of $I_{v,c}$. The condition in line 14 checks if colouring v with colour c would induce threshold greater than t_t in any node. If not, v is coloured with c , I and I' get updated and we exit the loop at line 13.

Algorithm 1: Levelling — randomised greedy heuristic

```
1  $t_t \leftarrow \infty$ 
2 foreach  $try \in \{0, 1, \dots, max\_tries\}$  do
3   foreach  $v \in V$  do
4     foreach  $k \in \{1, 2, \dots, K\}$  do
5        $I_{v,c} \leftarrow 0$ 
6      $I'_v \leftarrow 0$ 
7    $T \leftarrow V$ 
8    $possible \leftarrow \mathbf{true}$ 
9   while  $T \neq \emptyset \wedge possible$  do
10     $T' \leftarrow \{x \in T : I'_x = \max I'\}$ 
11     $v \leftarrow$  random element from  $T'$ 
12     $C \leftarrow (1, 2, \dots, k)$  sorted to give  $I_{v,i} \leq I_{v,i+1}$ 
13    foreach  $c \in C$  do
14      if  $v$  CAN BE COLOURED  $c$  then
15        colour  $v$  with colour  $c$ 
16        foreach  $w \in N(v)$  do
17           $I_{w,c} \leftarrow I_{w,c} + f(v, w)$ 
18           $I'_w \leftarrow I'_w + f(v, w)$ 
19        break
20    if  $n$  was coloured then
21       $T \leftarrow T \setminus v$ 
22    else
23       $possible \leftarrow \mathbf{false}$ 
24  if  $possible$  then
25     $t_t \leftarrow t_t - \varepsilon$ 
```
