THÈSE DE DOCTORAT

SPÉCIALITÉ : MATHÉMATIQUES

Long time behavior of a mean-field model of interacting spiking neurons

Comportement en temps long d'un modèle champ moyen de neurones à décharge en interactions

Quentin Cormier

INRIA

Présentée en vue de l'obtention du grade de DOCTEUR

Dirigée par : Etienne Tanré et Romain Veltz

Soutenue le : 15 janvier 2021

Jury

M François Delarue	Professeur	Université Côte d'Azur	Examinateur
M Pierre-Emmanuel Jabin	Professeur	Pennsylvania State University	Examinateur
Mme Eva Löcherbach	Professeur	Université Paris 1	Rapporteur
M Stéphane Mischler	Professeur	Université Paris Dauphine	Rapporteur
M Denis Talay	Directeur de recherche	Inria Sophia-Antipolis	Membre invité
M Etienne Tanré	Chargé de recherche	Inria Sophia-Antipolis	Directeur de thèse
M Romain Veltz	Chargé de recherche	Inria Sophia-Antipolis	Directeur de thèse

Abstract

We study the long time behavior of a McKean-Vlasov stochastic differential equation (SDE), driven by a Poisson measure. In neuroscience, this SDE models the dynamics of the membrane potential of a typical neuron in a large network. The model can be derived by considering a finite network of generalized Integrate-And-Fire neurons and by taking the limit where the number of neurons goes to infinity. Hence the McKean-Vlasov SDE is a mean-field model of spiking neurons.

We study existence and uniqueness of the solution this McKean-Vlasov SDE and describe its invariant probability measures. For small enough interaction parameter J, we prove uniqueness and global stability of the invariant measure. For J arbitrary large however, the invariant measures may not be unique. We give a sufficient condition ensuring the local stability of such a given invariant probability measure. Our criterion involves the location of the zeros of an explicit holomorphic function associated to the considered stationary solution. When all the zeros have negative real part, we prove that stability holds. We then give sufficient general conditions ensuring the existence of periodic solutions through a Hopf bifurcation: at some critical interaction parameter J_0, the invariant probability losses its stability and periodic solutions appear for J close to J_0. To obtain these results, we combine probabilistic and deterministic methods. In particular, a key tool in this analysis is a nonlinear Volterra Integral equation satisfied by the synaptic current.

Finally, we illustrate these results with examples which are tractable analytically. Additionally, we give numerical methods to approximate the solution of the mean-field equation and to predict numerically the bifurcations.

Keywords: McKean-Vlasov stochastic processes; Long time behavior; Mean-field interaction; Volterra integral equation; Hopf bifurcation

Résumé

Nous étudions le comportement en temps long d'une équation différentielle stochastique (EDS) de type McKean-Vlasov, dirigée par une mesure de Poisson. En neurosciences, cette EDS modélise la dynamique du potentiel de membrane d'un neurone typique dans un grand réseau. Le modèle peut-être obtenu en considérant un réseau fini de neurones de type Intègre-Et-Tire généralisé et en prenant la limite où le nombre de neurones tend vers l'infini. Cette EDS est donc un modèle champ moyen de neurones à décharge.

Nous étudions l'existence et l'unicité de la solution de cette EDS McKean-Vlasov et nous donnons ses mesures de probabilité invariantes. Si le paramètre d'interaction J est suffisamment petit, nous prouvons l'unicité et la stabilité globale de la mesure invariante. Pour un J quelconque cependant, il peut y avoir plusieurs mesures de probabilité invariantes. Nous donnons une condition suffisante assurant la stabilité locale d'une telle mesure invariante. Notre critère fait intervenir les zéros d'une fonction holomorphe associée à la solution stationnaire considérée. Lorsque tous les zéros sont de partie réelle négative, nous prouvons la stabilité. Nous donnons finalement des conditions générales suffisantes assurant l'existence de solutions périodiques par le biais d'une bifurcation de Hopf : pour un certain paramètre d'interaction critique J_0, la probabilité invariante perd sa stabilité et des solutions périodiques apparaissent pour J suffisamment proche de J_0. Pour obtenir ces résultats, nous combinons des méthodes probabilistes et déterministes. En particulier, dans cette analyse, un outil clé est l'équation intégrale de Volterra non linéaire satisfaite par le courant synaptique.

Enfin, nous illustrons ces résultats par des exemples que l'on peut traiter de manière analytique. En outre, nous donnons des méthodes numériques pour approximer la solution de l'équation champ moyen et pour prédire numériquement les bifurcations.

Mots clefs: Processus stochastique de McKean-Vlasov; Comportement en temps long; Equation Champ moyen; Equation intégrale de Volterra; Bifurcation de Hopf

Remerciements

Avant tout, je souhaite remercier ma fabuleuse partenaire, Sacha. Merci infiniment pour tous les sacrifices que tu as fait pour moi. J'ai été souvent très occupé par ce travail. Merci pour ta compréhension et ton soutient inébranlable ! Je suis très heureux avec toi.

Je remercie très chaleureusement mes deux directeurs de thèse, Etienne Tanré et Romain Veltz. Vous formez un duo très complémentaire et je suis très chanceux de vous avoir eu comme encadrants ! Romain, merci beaucoup pour tout le temps que tu m'as accordé dans ton bureau à m'apprendre des choses déterministes ! Merci également pour tes nombreux et judicieux conseils de lecture, j'ai beaucoup appris grâce à toi. Merci de m'avoir plus d'une fois sauvé alors que j'étais complètement bloqué et merci pour ton enthousiasme débordant sur les simulations numériques (#Julia). Etienne, merci infiniment pour ta disponibilité et pour tous tes géniaux conseils (en maths bien sûr, mais pas uniquement). Merci pour ta patience que j'ai pu (involontairement, évidemment !) mettre à l'épreuve et pour toute ton implication dans ce travail d'encadrement. Merci de m'avoir lancé et motivé sur ces belles questions. Merci de m'avoir toujours poussé à faire plus et mieux ! J'étais entre de très bonnes mains.

Je suis très fier et très honoré de la composition mon jury. François Delarue et Pierre-Emmanuel Jabin, je vous remercie d'avoir accepté d'en faire partie, d'avoir passé du temps à examiner ma thèse et de m'avoir posé de très intéressantes questions qui me sont autant de pistes pour la suite. Stéphane Mischler et Eva Löcherbach, merci infiniment d'avoir rapporté ma thèse. Stéphane, un grand merci pour tout le temps que vous y avez passé et pour vos judicieux pointeurs bibliographiques. Eva, merci encore pour votre rapport très détaillé qui atteste d'une lecture très approfondie de ma thèse. Je suis très touché et je vous remercie également pour toutes vos questions qui me motivent à continuer l'étude de ce modèle. Denis, merci d'avoir accepté d'être membre invité de cette soutenance. Je te remercie pour ta disponibilité tout au long de ma thèse : c'était génial de pouvoir te poser toutes mes questions sur les processus stochastiques. Je suis impatient de découvrir ton futur livre !

Nicolas Fournier, je vous remercie pour toutes nos discussions, pour m'avoir écouté toute une matinée et pour tous vos conseils ! Benjamin Aymard, merci beaucoup pour ton aide avec les schémas numériques, ça m'a été très utile. Par ailleurs, je tiens à remercier l'anonyme auteur du rapport de l'article publié à “Stochastic Processes and their Applications”. Son impressionnant rapport m'a été très précieux : d'une part il m'a permis d'améliorer considérablement l'article (et donc une partie de ce manuscrit), et d'autre part, ses judicieux conseils ont largement contribué à améliorer mon style de rédaction.

Mes années à Sophia-Antipolis ont été très agréables grâce aux géniales personnes que j'ai eu la chance de rencontrer à l'INRIA. Je pense en particulier à Radu, Milica, Pascal, Émilie, Lorenzo, Benjamin, Yuri et bien d'autres. Merci pour toutes nos discussions et pour tous les super moments que nous avons passés ensemble ! Merci également à toi Damien pour nos nombreuses discussions, je te souhaite le meilleur pour la fin de ta thèse. Mes derniers mots sont pour ma famille, en particulier pour mes parents. Merci pour votre soutien tout au long de ma (longue !) scolarité.

Contents

Main notations

$A * B$ Defined on page 19
$$A * B(t,s) = \int_s^t A(t,u)B(u,s)du.$$

δ_x The Dirac mass at x, for all $A \in \mathcal{B}(\mathbb{R}_+)$, $\delta_x(A) = \mathbb{1}_A(x)$.

$\lfloor x \rfloor$ Floor function of x.

$\mathbb{1}$ The indicator function: $\mathbb{1}_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise.} \end{cases}$

$\widehat{g}(z)$ $= \int_0^\infty e^{-zt} g(t)dt$, the Laplace transform of g, defined for $z \in \mathbb{C}$ on page 51.

$\Im(z)$ The imaginary part of $z \in \mathbb{C}$.

$\Re(z)$ The real part of $z \in \mathbb{C}$.

$\langle \nu, g \rangle$ $= \int_{\mathbb{R}_+} g(x)\nu(dx)$.

$\nu(g)$ $= \int_{\mathbb{R}_+} g(x)\nu(dx)$.

b The deterministic drift $b : \mathbb{R}_+ \to \mathbb{R}$.

$\mathcal{B}_\alpha$ For $\alpha > 0$, the operator defined on page 79 by $\mathcal{B}_\alpha \phi = -J(\alpha)\langle \phi, f \rangle \partial_x \nu_\alpha^\infty$.

$B_\eta^T(d)$ $= \{a \in C_T^0, \quad \sup_{t\in[0,T]} |a_t - d_t| < \eta\}$, the open balls of C_T^0 defined on page 128.

$B_\lambda^\infty(h,\rho)$ $= \{c \in L_\lambda^\infty, \ ||c - h||_\lambda^\infty < \rho\}$, the open balls of L_λ^∞ defined on page 84.

$\mathcal{B}(\mathbb{R}_+)$ The Borel sets of $\mathbb{R}_+$.

$\mathcal{B}(\mathbb{R}_+;\mathbb{R})$ The space of measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$.

$c_{\boldsymbol{a},\tau}$ $= \tau H_{\boldsymbol{a},\tau}^{2\pi}(\pi_{\boldsymbol{a},\tau})$, for $\boldsymbol{a} \in C_{2\pi}^0$ and $\tau > 0$. Defined on page 138.

C_T^0 The space of continuous T-periodic functions. Defined on page 126.

$C_T^{0,0}$ $= \{y \in C_T^0, \ \int_0^T y(s)ds = 0\}$, defined on page 126.

$D_h F(h) \cdot c$ The Fréchet derivative of F at the point h, evaluated at c.

f The rate function $f : \mathbb{R}_+ \to \mathbb{R}_+$.

$H_{\boldsymbol{a}}^\nu(t,s)$ $= \mathbb{P}(\tau_s^{\boldsymbol{a},\nu} > t)$, the survival function defined on page 10.

$H_{\boldsymbol{a}}^{x}(t,s)$ $= H_{\boldsymbol{a}}^{\delta_x}(t,s)$, when the initial condition is the Dirac measure at x, defined on page 19.

$H_{\boldsymbol{a}}(t,s)$ $= H_{\boldsymbol{a}}^{\delta_0}(t,s)$, when the initial condition is the Dirac measure at 0.

$H_{\alpha}^{\nu}(t)$ $= H_{\alpha}^{\nu}(t,0)$, for constant current ($\boldsymbol{a} \equiv \alpha$).

$H_{\boldsymbol{a},\tau}(t,s)$ $= H_{\boldsymbol{d}}(\tau t, \tau s)$, with $d(t) = a(\frac{t}{\tau})$ defined for $\boldsymbol{a} \in C^0_{2\pi}$ and $\tau > 0$ on page 138.

$H_{\boldsymbol{a},\tau}^{2\pi}(t,s)$ $= \sum_{k\geq 0} H_{\boldsymbol{a},\tau}(t, s-2\pi k)$, defined on page 138.

$H_{\boldsymbol{a},\tau}^{2\pi}(y)(t)$ $= \int_0^{2\pi} H_{\boldsymbol{a},\tau}^{2\pi}(t,s)y(s)ds$, defined for $y \in L^1([0,2\pi])$ on page 132.

$\bar{H}_{\boldsymbol{h}}^{\alpha}$ $= H_{\alpha+\boldsymbol{h}} - H_{\alpha}$, defined on page 100.

$J(\alpha)$ $= \frac{\alpha}{\gamma(\alpha)}$.

$K_{\boldsymbol{a}}^{\nu}(t,s)$ $= -\frac{d}{dt}\mathbb{P}(\tau_s^{\boldsymbol{a},\nu} > t)$, the density of the first jump defined on page 10.

$K_{\boldsymbol{a}}^{x}(t,s)$ $= K_{\boldsymbol{a}}^{\delta_x}(t,s)$, when the initial condition is the Dirac measure at x, defined on page 19.

$K_{\boldsymbol{a}}(t,s)$ $= K_{\boldsymbol{a}}^{\delta_0}(t,s)$, when the initial condition is the Dirac measure at 0.

$K_{\alpha}^{\nu}(t)$ $= K_{\alpha}^{\nu}(t,0)$, for constant current ($\boldsymbol{a} \equiv \alpha$).

$K_{\boldsymbol{a},\tau}(t,s)$ $= \tau K_{\boldsymbol{d}}(\tau t, \tau s)$, with $d(t) = a(\frac{t}{\tau})$, defined for $\boldsymbol{a} \in C^0_{2\pi}$ and $\tau > 0$ on page 138.

$K_{\boldsymbol{a},\tau}^{2\pi}(t,s)$ $= \sum_{k\geq 0} K_{\boldsymbol{a},\tau}(t, s-2\pi k)$. Defined on page 138.

$K_{\boldsymbol{a},\tau}^{2\pi}(y)(t)$ $= \int_0^{2\pi} K_{\boldsymbol{a},\tau}^{2\pi}(t,s)y(s)ds$, defined for $y \in L^1([0,2\pi])$ on page 132.

$\bar{K}_{\boldsymbol{h}}^{\alpha}$ $= K_{\alpha+\boldsymbol{h}} - K_{\alpha}$, defined on page 100.

L^1_{λ} $= \{h \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}),\ ||h||^1_{\lambda} < \infty\}$, defined for $\lambda \geq 0$ on page 51. Equipped with the norm

$$||h||^1_{\lambda} = \int_{\mathbb{R}_+} |h(s)|e^{\lambda s}ds.$$

L^{∞}_{λ} $= \{h \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}),\ ||h||^{\infty}_{\lambda} < \infty\}$, defined for $\lambda \geq 0$ on page 74. Equipped with the norm

$$||h||^{\infty}_{\lambda} = \operatorname*{ess\,sup}_{t\geq 0}|h_t|e^{\lambda t}.$$

$\mathcal{L}(E;F)$ Given two Banach spaces E and F, $\mathcal{L}(E;F)$ is the set of linear bounded operators from E to F.

$\mathcal{L}^*_{\alpha}$ For $\alpha > 0$, the generator of the Markov process corresponding to an isolated neuron subject to the current α. Defined on page 79 by

$$\mathcal{L}^*_{\alpha}\phi = -\partial_x\left[(b+\alpha)\phi\right] - f\phi + \langle \phi, f\rangle \delta_0.$$

$\mathcal{M}(f^2)$ $= \{\nu \in \mathcal{P}(\mathbb{R}_+),\ \int_{\mathbb{R}_+} f^2(x)\nu(dx) < +\infty\}$, defined on page 73.

N A Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with Lebesgue intensity.

$N(L)$ $= \{x,\ L(x) = 0\}$, the kernel of the linear operator L.

$\mathcal{P}(\mathbb{R}_+)$ The set of probability measures on $\mathbb{R}_+$.

$r^\nu_{\boldsymbol{a}}(t,s)$ $= \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s})$, the instantaneous jump rate of $Y^{\boldsymbol{a},\nu}_{t,s}$, defined on page 10.

$r^x_{\boldsymbol{a}}(t,s)$ $= r^{\delta_x}_{\boldsymbol{a}}(t,s)$, when the initial condition is the Dirac measure at x, defined on page 19.

$r_{\boldsymbol{a}}(t,s)$ $= r^{\delta_0}_{\boldsymbol{a}}(t,s)$, when the initial condition is the Dirac measure at 0.

$r^\nu_\alpha(t)$ $= r^\nu_\alpha(t,0)$, for constant current ($\boldsymbol{a} \equiv \alpha$).

$R(L)$ $= \{y,\ \exists x:\ L(x) = y\}$, the range of the linear operator L.

$\mathcal{V}^1_\lambda$ $= \{\kappa \in \mathcal{B}(\Delta;\mathbb{R}),\ ||\kappa||^1_\lambda < \infty\}$, defined for $\lambda \geq 0$ on page 58. Equipped with the norm

$$||\kappa||^1_\lambda = \operatorname*{ess\,sup}_{t\geq 0} \int_{\mathbb{R}_+} |\kappa(t,s)| e^{\lambda(t-s)} ds.$$

X_t The solution of the McKean-Vlasov SDE (1.2). Defined on page 3. That is (X_t) solves

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t \int_{\mathbb{R}_+} X_{u-} \mathbb{1}_{\{z \leq f(X_{u-})\}} \mathbf{N}(du,dz).$$

$Y^{\boldsymbol{a},\nu}_{t,s}$ The solution of the linear non-homogeneous SDE (1.9) driven by the current $\boldsymbol{a}$, starting at time s with law ν. Defined on page 10. It solves

$$\forall t \geq s, \quad Y^{\boldsymbol{a},\nu}_{t,s} = Y^{\boldsymbol{a},\nu}_{s,s} + \int_s^t b(Y^{\boldsymbol{a},\nu}_{u,s})du + \int_s^t a_u du - \int_s^t \int_{\mathbb{R}_+} Y^{\boldsymbol{a},\nu}_{u-,s} \mathbb{1}_{\{z \leq f(Y^{\boldsymbol{a},\nu}_{u-,s})\}} \mathbf{N}(du,dz).$$

Δ $= \{(t,s) \in \mathbb{R}^2, t \geq s\}$. Defined on page 25.

$\gamma(\alpha)$ $= \nu^\infty_\alpha(f)$, the jump rate under the invariant measure ν^∞_α.

ν^∞_α The invariant measure of $(Y^{\alpha,\nu}_{t,0})_{t\geq 0}$. Defined on page 40 by

$$\nu^\infty_\alpha(dx) = \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}dy\right) \mathbb{1}_{[0,\sigma_\alpha)}(x)dx.$$

$\Phi(\nu,h)$ $= J(\alpha) r^\nu_{\alpha+h} - (\alpha+h)$, defined for $\nu \in \mathcal{M}(f^2)$ and $h \in L^\infty_\lambda$ on page 97.

$\varphi^{\boldsymbol{a}}_{t,s}(x)$ The deterministic flow, defined on page 18. It is the solution of

$$\forall t \geq s, \quad \frac{d}{dt}\varphi^{\boldsymbol{a}}_{t,s}(x) = b(\varphi^{\boldsymbol{a}}_{t,s}(x)) + a_t$$
$$\varphi^{\boldsymbol{a}}_{s,s}(x) = x.$$

$\varphi_t^{\alpha}(x)$ $= \varphi_{t,0}^{\boldsymbol{a}}(x)$ for constant current ($\boldsymbol{a} \equiv \alpha$).

$\pi_{\boldsymbol{a},\tau}$ The invariant measure of the Markov Chain on $[0, 2\pi]$ with probability density kernel transition given by $K_{\boldsymbol{a},\tau}^{2\pi}$.

$\Psi_\alpha(t)$ $= \int_0^\infty \frac{d}{dx} H_\alpha^x(t) \nu_\alpha^\infty(dx)$, defined on page 89.

$\rho_{\boldsymbol{a},\tau}$ $= \frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}$, defined on page 138.

σ_α $= \inf\{x \geq 0,\ b(x) + \alpha = 0\}$, defined on page 41.

$\sigma_{\boldsymbol{a}}(t)$ $= \lim_{u\to-\infty} \varphi_{t,u}^{\boldsymbol{a}}(0)$, defined on page 34.

$\tau_s^{\boldsymbol{a},\nu}$ $= \inf\{t \geq s : Y_{t,s}^{\boldsymbol{a},\nu} \neq Y_{t-,s}^{\boldsymbol{a},\nu}\}$, the time of the first jump of $Y^{\boldsymbol{a},\nu}$ after s. Defined on page 10.

$\Theta_\alpha(t)$ $= \int_0^\infty \frac{d}{dx} r_\alpha^x(t) \nu_\alpha^\infty(dx)$, defined on page 77.

$\Xi_\alpha(t)$ $= \int_0^\infty \frac{d}{dx} K_\alpha^x(t) \nu_\alpha^\infty(dx)$, defined on page 91.

$\xi_\alpha(t)$ $= r_\alpha(t) - \gamma(\alpha)$, defined on page 53.

$\xi_\alpha^x(t)$ $= r_\alpha^x(t) - \gamma(\alpha)$.

Chapter 1

Introduction

1.1 The model

1.1.1 The particle system

We study a model of network of neurons. Let $b : \mathbb{R}_+ \to \mathbb{R}$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ be two deterministic smooth functions. For each $N \in \mathbb{N}$, we consider a Piecewise-Deterministic Markov Process (PDMP) $\mathbf{X}^N_t = (X^{1,N}_t, \cdots, X^{N,N}_t) \in \mathbb{R}^N_+$. For $i \in \{1, \cdots, N\}$, $X^{i,N}_t$ models the membrane potential of the neuron i in the network. It emits spikes at random times. The spiking rate of neuron i at time t is $f(X^{i,N}_t)$: it only depends on the potential of neuron i. When the neuron i emits a spike, say at time τ, its potential is reset ($X^{i,N}_{\tau+} = 0$) and the potential of the other neurons (say neuron j) increases by an amount $J^N_{i\to j}$, where the connection strength $J^N_{i\to j} \geq 0$ is fixed:

$$\forall j \neq i, \quad X^{j,N}_{\tau_+} = X^{j,N}_{\tau_-} + J^N_{i\to j}.$$

The *weight matrix* $(J^N_{i\to j})_{i,j\in\{1,\cdots,N\}^2}$ is assumed to be deterministic and constant in time. Between two spikes, the potentials of each neuron evolve according to the one dimensional ODE

$$\frac{d}{dt} X^{i,N}_t = b(X^{i,N}_t).$$

The drift function b models the *subthreshold* dynamics: it describes how the membrane potentials evolve between the jumps. We assume that $b(0) \geq 0$ such that the dynamics stay on $\mathbb{R}_+$. The rate function f models the intensity of the jumps of the neurons.

This process is indeed a PDMP. In particular, it is a Markov process (see [Dav84]). Equivalently, the model can be described using a system of SDEs driven by Poisson measures. Let $(\mathbf{N}^i(du, dz))_{i=1,\cdots,N}$ be a family of N independent Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$. Let $(X^{i,N}_0)_{i=1,\cdots,N}$ be a family of N random variables on $\mathbb{R}_+$, independent of the Poisson measures. Then $(X^{i,N})$ is a *càdlàg* process solution of the system of SDEs:

$$\begin{aligned}\forall i = 1, \cdots, N, \quad X^{i,N}_t =& X^{i,N}_0 + \int_0^t b(X^{i,N}_u)du + \sum_{j\neq i} J^N_{j\to i} \int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(X^{j,N}_{u-})\}} \mathbf{N}^j(du, dz) \\ &- \int_0^t \int_{\mathbb{R}_+} X^{i,N}_{u-} \mathbb{1}_{\{z\leq f(X^{i,N}_{u-})\}} \mathbf{N}^i(du, dz).\end{aligned} \tag{1.1}$$

We plot in Figure 1.1 a typical trajectory of $(X_t^{i,N})$ with $N = 2$ neurons.

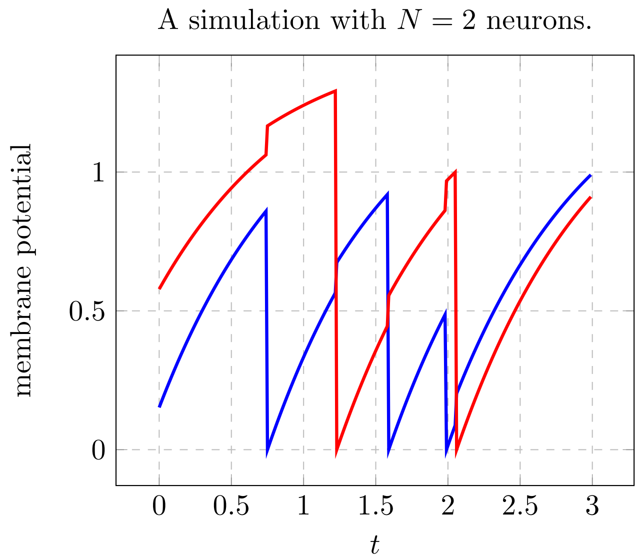


Figure 1.1: A simulation of the particle system with $N = 2$ neurons, $b(x) = 3/2-x$, $f(x) = x^2$ and $J^2_{1\to 2} = J^2_{2\to 1} = 0.2$. When one of the neuron is spiking its potential is reset to zero and the other one receives a kick of size 0.2.

To summarize, the model is parametrized by the rate function f, the drift function b, the weight matrix $(J^N_{i\to j})_{i,j\in\{1,\cdots,N\}^2}$ and the law of the initial conditions. We are interested in the limit of large network ($N \to \infty$). We now describe the limit equation associated to (1.1).

1.1.2 The McKean-Vlasov SDE

We now assume that the initial conditions $(X_0^{i,N})_{i\in\{1,\cdots,N\}}$ are independent and identically distributed ($i.i.d.$) with probability law ν. We furthermore assume that the weights between the neurons are all equal:

$$\forall i, j \in \{1, \cdots, N\}, \quad J^N_{i\to j} = \frac{J}{N}.$$

The deterministic constant $J \geq 0$ models the strength of the interactions. Under these additional assumptions, the particles are indistinguishable from one to the other and the particle system is exchangeable: $i, j \in \{1, \cdots, N\}$ with $i < j$, we have

$$\mathcal{L}((X_t^{1,N}, \cdots, X_t^{N,N})_{t\geq 0}) = \\ \mathcal{L}((X_t^{1,N}, \cdots, X_t^{i-1,N}, X_t^{j,N}, X_t^{i+1,N}, \cdots, X_t^{j-1,N}, X_t^{i,N}, X_t^{j+1,N}, \cdots, X_t^{N,N})_{t\geq 0}).$$

The scaling $J^N_{i\to j} = \frac{J}{N}$ corresponds to the scaling of the law of large numbers. Note that

$$\mathbb{E}\,\frac{J}{N}\sum_{j\neq i}\int_0^t\int_{\mathbb{R}_+}\mathbb{1}_{\{z\leq f(X^{j,N}_{u-})\}}\mathbf{N}^j(du,dz) = \frac{J}{N}\sum_{j\neq i}\int_0^t\mathbb{E}\,f(X^{j,N}_{u-})du$$
$$= \frac{N-1}{N}J\int_0^t\mathbb{E}\,f(X^{1,N}_{u-})du.$$

Furthermore, for any fixed deterministic time $u\geq 0$, the process $X^{i,N}$ jumps at time u with probability zero. Thus, we deduce that $\mathbb{E}\,f(X^{1,N}_{u-}) = \mathbb{E}\,f(X^{1,N}_u)$.

As the number of neurons N goes to infinity, one expects *propagation of chaos* to hold: as N goes to infinity, any pair of neurons of the network (say $X^{1,N}_t$ and $X^{2,N}_t$) becomes more and more independent and each neuron (say $(X^{1,N}_t)$) converges in law to the solution of the following McKean-Vlasov SDE:

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t\mathbb{E}\,f(X_u)du - \int_0^t\int_{\mathbb{R}_+}X_{u-}\mathbb{1}_{\{z\leq f(X_{u-})\}}\mathbf{N}(du,dz). \tag{1.2}$$

In this equation, $\mathbf{N}$ is a Poisson measure on $\mathbb{R}^2_+$ with intensity being the Lebesgue measure $dudz$, the initial condition X_0 has law ν and is independent of the Poisson measure. This SDE is nonlinear in the sense of McKean-Vlasov, because of the interaction term $\mathbb{E}\,f(X_u)$ which depends on the law of the solution X. Informally, Equation (1.2) can be understood in the following way:

$$\text{Between the jumps, } (X_t) \text{ solves the ODE} \quad \dot{X}_t = b(X_t) + J\,\mathbb{E}\,f(X_t)$$
$$\text{and } (X_t) \text{ jumps to zero at a rate } f(X_t).$$

Eq. (1.2) is the main object of this thesis: we study the well-posedness as well as the qualitative properties of the solution, in particular its long time behavior. This model, described here with a probabilistic formalism, can also be understood via the following Partial Derivative Equation (*PDE*).

1.1.3 The nonlinear Fokker-Planck equation

Let $\nu(t,dx) := \mathcal{L}(X_t)$ be the law of X_t. It solves the following nonlinear Fokker-Planck equation, in the sense of measures:

$$\partial_t\nu(t,dx) + \partial_x\left[(b(x)+J\tilde{r}^\nu(t))\nu(t,dx)\right] + f(x)\nu(t,dx) = \tilde{r}^\nu(t)\delta_0 \tag{1.3}$$
$$\nu(0,dx) = \mathcal{L}(X_0), \quad \tilde{r}^\nu(t) = \int_{\mathbb{R}_+}f(x)\nu(t,dx).$$

Here δ_0 is the Dirac measure in 0. Note that $\tilde{r}^\nu(t) = \mathbb{E}\,f(X_t)$. If furthermore $\mathcal{L}(X_t)$ has a density for all t, that is $\mathcal{L}(X_t) = \nu(t,x)dx$ then $\nu(t,x)$ solves the following strong form of the Fokker-Planck equation (1.3)

$$\partial_t\nu(t,x) + \partial_x\left[(b(x)+J\tilde{r}^\nu(t))\nu(t,x)\right] + f(x)\nu(t,x) = 0, \tag{1.4}$$
$$\nu(0,x)dx = \mathcal{L}(X_0), \quad \tilde{r}^\nu(t) = \int_{\mathbb{R}_+}f(x)\nu(t,x)dx,$$

with the boundary condition:

$$\forall t > 0, \quad (b(0) + J\tilde{r}^\nu(t))\,\nu(t,0) = \tilde{r}^\nu(t). \tag{1.5}$$

Many questions arising in this thesis can be formulated equivalently using the nonlinear SDE (1.2) or using the nonlinear PDE (1.3). We shall come back in Chapter 2 (see Section 2.7) on the derivation of the PDE (1.3) and discuss conditions such that $\mathcal{L}(X_t)$ admits densities solving (1.4) and (1.5).

1.1.4 Two changes of parameters

The model, described by the mean-field equation (1.2), is parametrized by the drift b, the jump rate f, the interaction parameter J and the initial condition X_0 with law ν.

A change of time

Let $\tau > 0$. Consider (X_t) a solution of (1.2) and let for all $t, x \geq 0$

$$V_t := X_{\tau t}, \quad \tilde{b}(x) := \tau b(x), \quad \tilde{f}(x) := \tau f(x).$$

Let $\tilde{\mathbf{N}} := \mathbf{N} \circ g^{-1}$ be the push-forward measure of $\mathbf{N}$ by the function

$$g(t,z) := (\tau t, z/\tau).$$

Note that $\tilde{\mathbf{N}}(du, dz)$ is again a Poisson measure of intensity $dudz$. We have

$$\begin{aligned} V_t &= X_0 + \int_0^{\tau t} b(X_u)du + J\int_0^{\tau t} \mathbb{E}\, f(X_u)du - \int_0^{\tau t}\int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{\tau z \leq \tau f(X_{u-})\}}\mathbf{N}(du,dz) \\ &= X_0 + \int_0^t \tilde{b}(V_s)ds + J\int_0^t \mathbb{E}\,\tilde{f}(V_s)ds - \int_0^t\int_{\mathbb{R}_+} V_{s-}\mathbb{1}_{\{w\leq \tilde{f}(V_{s-})\}}\tilde{\mathbf{N}}(ds,dw). \end{aligned}$$

We have made the change of variables $u = \tau s$ in the first two integrals and the change of variables $(u,z) = g(s,w)$ in the third integral. So (V_t) is a (weak) solution of (1.2) with $V_0 = X_0$, $\tilde{J} = J$, $\tilde{b}$, $\tilde{f}$ and $\tilde{\mathbf{N}}$. So accelerate the time by a factor τ is equivalent to scale b and f by τ, while keeping J fixed. We shall use this fact in Chapter 5 (see Section 5.3.4).

Reduction to $J = 1$.

Consider (X_t) a solution of (1.2) and let for all $t, x \geq 0$:

$$V_t := \frac{X_t}{J}, \quad \tilde{b}(x) := \frac{1}{J}b(Jx), \quad \text{and} \quad \tilde{f}(x) := f(Jx).$$

Then (V_t) solves (1.2) with $V_0 = \frac{X_0}{J}$, $\tilde{b}$, $\tilde{f}$ and $\tilde{J} = 1$. So without loss of generality we can reduce (1.2) to $J = 1$. However, we will never use this reduction in this work. We prefer to fix b, f (and the initial condition ν) and to let J varies. Indeed, J has a clear interpretation: it models the intensity of the interactions between two neurons (see (1.1)). For instance in Chapter 3, we study the dynamics for J small enough (b and f being fixed) and in Chapter 5, we study the Hopf bifurcations as J varies.

1.2 Biological background and motivations

With 10^{11} neurons in the human brain, neurons are the basic unit that makes perception, learning, feeling and behavior possible. Despite the important variability between neurons (there are more that 10^3 different types of neurons), the nerve cells all share a common structure (a soma, an axon and the synapses). Neurons "communicate" via action potentials (or *spikes*). Those are highly stereotyped signals where the membrane potential of the spiking neuron rapidly raises and falls [MKJSSH13, Ch. 2].

Models of a single neuron

The first mathematical description of a single neuron is the integrate-and-fire model of Lapicque (1907). In his seminal paper [Lap07], Lapicque models the membrane potential of a neuron by the scalar ODE $\tau\dot{V} = -V(t) + RI$, until the *spiking time*, which is defined by the time at which the membrane potential $V(t)$ reaches a fixed deterministic threshold. After the spike, the membrane potential is reset to a resting value.

A more sophisticated and more precise model is later proposed by Hodgkin–Huxley [HH52]. It models jointly the membrane potential together with the voltage gated ions channels of the neuron. Altogether the system is described by a $4D$ nonlinear ODE. A key feature is that the spikes are intrinsic to the dynamics: no threshold is required. In addition to its complexity, one drawback of Hodgkin–Huxley model is that the spikes are not well-defined: while we observe that the Hodgkin–Huxley dynamics exhibits important variations of the membrane potential (that we interpret as spikes) the precise timing of these action potentials in not well defined in this continuous model. In contrary, in Integrate-and-Fire models, the spiking times are well defined, and so it is easier to define the mean number of spikes per unit of time, which is an important quantity when modeling network of neurons.

Spike trains (that is the times of the successive spikes) of real neurons, particularly *in vivo*, are noisy. This variability cannot be correctly reproduced by the deterministic models of Lapicque and Hodgkin–Huxley. There are two classical ways to introduce noise in the Integrate-and-fire models. The first approach is to add a diffusion term in the dynamics (typically an additive Brownian motion). The second approach is to abandon the notion of a fixed deterministic threshold and to replace it with a "soft" threshold. In this case, a neuron fires with a probability which depends on its membrane potential. This is known in the physics literature as "escape rate" models, "noisy outputs" or "generalized integrate-and-fire" model. We refer to [Bri88; Ger95], as well as [GKNP14, Ch. 9] and the references therein. In [DGLP15], the authors proposed to use Poisson measures (see (1.1)) to describe this model in rigorous mathematical terms. Using motoneurons, in [JBHD11] the authors validated such escape rate model with real data and suggested $f(x) \approx \exp(a_0 + a_1 x)$ for some constant a_0 and a_1 fitted experimentally.

Neurons in a network

A spike of a neuron propagates along the axon of the neuron and reaches its synapses. This triggers a chemical reaction that induces a communication with the post-synaptic neurons. The mechanism is the following: vesicles, stocked in the pre-synaptic neuron, merge with the membrane. This releases neurotransmitters which are recaptured by the post synaptic neuron: this in turn increases the membrane potential of the post-synaptic neuron. We refer to [MKJSSH13, Ch. 8] for more details. The jumps mechanism of (1.1), introduced in [DGLP15], is a caricature of such chemical synapse transmissions: a spike of the pre-synaptic neuron induces an instantaneous jump in the post-synaptic neurons.

To support our mean-field assumption, we need large enough networks of neurons, such that the local interactions are correctly captured by the mean-field current. Moreover such network have to be small enough such that the population is homogeneous (the neurons are similar in shape) and such that the population is sufficiently connected. These assumptions are typically fulfilled within cortical columns (such as for instance V1). Those are structures of a diameter 50μm to 1mm with 10^2 to 10^5 neurons. We refer to [WTGWSLM04; SHMHSWHN06; SDG17] for a discussion about this scenario.

To describe the activity of a population of neurons, one often uses rate models, that describe the mean activity of the neurons of the considered population. For instance, in the Wilson-Cowan model [WC72] the authors consider $E(t)$ the proportion of neurons that are firing at time t and assume that this quantity solves the following ODE:

$$\tau \dot{E}(t) = -E(t) + [1 - \xi E(t)]\, F(JE(t)).$$

In this equation, ξ, J are constants (J models the connectivity of the population) and $F : \mathbb{R} \to \mathbb{R}$ is a nonlinear function. While being useful to make predictions, the main drawback of these methods is that the evolution equation is a coarse grain description (mesoscopic scale), which is not derived from the dynamics of the underlying neurons.

Interest of the model for the neuroscience

As opposed to rate models, (1.2) works simultaneously at the microscopic and at the mesoscopic scales. Consider (X_t) a solution of (1.2). On one hand, (X_t) models the membrane potential of a "typical" integrate-and-fire neuron with escape rate f, under the influence of the other neurons through the mean-field interactions. On the other hand, $\mathbb{E}\, f(X_t)$ is the mean activity of the population. This ability to bridge the gap between these two scales is particularly interesting in view of recent experimental advances. For instance, using micro electrode arrays, it is possible to measure simultaneously the membrane potential of neighborhood neurons [ODBBF15].

We shall see that the model can exhibit spontaneous oscillations (see Chapter 5 and 6): this could provide a better understanding of some features of neural oscillations. For instance, the model can help to make predictions about the frequencies of the spontaneous oscillations, knowing the parameters of a typical neuron. Similarly the model exhibits bi-stability (see

Chapter 6). Such feature (coexistence of two stable states) is sometime alleged to explain the process of decision making [MKJSSH13, Appendix F.]. When bi-stability disappears, this leads to a cusp bifurcation where some small modification of a parameter can give brutal changes of the dynamics (see Chapter 6).

1.3 Previous works and results

Equation (1.1) and its mean-field version (1.2) is a close variant of the model introduced by [DGLP15]. In their work, the limit equation writes

$$X_t = X_0 - \lambda \int_0^t (X_u - \mathbb{E}\, X_u)\, du + \int_0^t \mathbb{E}\, f(X_u) du - \int_0^t \int_{\mathbb{R}_+} X_{u-} \mathbb{1}_{\{z \le f(X_{u-})\}} \mathbf{N}(du, dz). \quad (1.6)$$

Here the constant λ is non-negative and the term $-\lambda(X_u - \mathbb{E}\, X_u)$ accounts for a callback to the mean value of the membrane potential at time t, $\mathbb{E}\, X_t$. This callback models electrical synapses. The only difference between (1.2) and (1.6) is this callback to the mean value. In (1.2), we have replaced it with a deterministic drift $b(X_t)$. The authors proved existence and uniqueness of a weak solution of the limit equation (1.6) as well as the convergence of the corresponding particle system towards this solution (*propagation of chaos*), under the additional assumption that the initial condition ν is compactly supported: there exists $A_0 > 0$ such that $\mathbb{P}(X_0 \in [0, A_0]) = 1$.

In [FL16], the authors were able to remove this assumption of a compact support of the initial condition. They proved path-wise uniqueness of (1.6) and study the *propagation of chaos* under very few assumptions on the initial datum. They also obtained results on the long time behavior of the solution of (1.6). They proved that (1.6) admits two invariant measures: the Dirac mass at 0, δ_0 and a non-trivial one. They moreover gave sufficient conditions ensuring that the Dirac mass δ_0 is not stable: if $\mathbb{P}(X_0 = 0) < 1$, then X_t does not converge in law to 0. When $\lambda = 0$, they proved that the non-trivial invariant measure is globally attractive. Note that $\lambda = 0$ corresponds to $b \equiv 0$ in (1.2). To do so, they used the strong form of the Fokker-Planck equation (1.4). They relied on the fact that the boundary condition is constant: when $b \equiv 0$, (1.5) become:

$$\nu(t, 0) = \frac{1}{J}.$$

Denote by ν_∞ the non-trivial invariant probability measure. They proved that

$$\forall t \ge 0, \quad \frac{d}{dt} ||\nu(t, \cdot) - \nu_\infty(\cdot)||_{L^1(\mathbb{R}_+)} \le 0,$$

where $\nu(t, \cdot)$ denotes the density of X_t. In particular, they assumed that the initial condition X_0 has a density which itself satisfies the boundary condition. They obtained the convergence of $\nu(t, \cdot)$ to ν_∞ in L^1 norm.

Still for the case $b \equiv 0$, in [DV21] the authors gave another proof of the stability of the non-trivial invariant probability measure. They obtained local stability results (that is they assume that the initial condition is close enough to the invariant probability measure) with explicit rate of convergence. They *linearized* the Fokker-Planck equation (1.4) around the invariant measure and studied the spectrum of this linear mapping. They showed a spectral gap giving the exponential rate of convergence.

In [DV17], the authors explored numerically the behavior of the solution of (1.6) for $\lambda > 0$. They showed that periodic solutions appeared for $f(x) = x^p$ with p and λ large enough. They suggested that the transition towards the oscillations occurred via a Hopf bifurcation, where the invariant measure lost its stability.

The nonlinear Fokker-Planck equation (1.3), or its strong version (1.4), belongs to the family of nonlinear transport equations with a boundary condition. Such PDE have been studied in the context of population dynamics (see for instance [GM74; Prü83; Web85; Per07]). In [GM74], the authors studied the following transport equation

$$\begin{aligned}
&\partial_t \rho(t,x) + \partial_x \rho(t,x) + \lambda(x, P(t))\rho(t,x) = 0 \\
&P(t) = \int_0^\infty \rho(t,x)dx \\
&\rho(t,0) = \int_0^\infty \beta(x, P(t))\rho(t,x)dx.
\end{aligned}$$

Here, $\rho(t,x)dx$ is the proportion of the population with age x. The quantity $\lambda(x, P(t))$ is the death rate, while $\int_0^\infty \beta(x, P(t))\rho(t,x)dx$ is the birth rate. The authors characterized the stationary solutions of this PDE and found a criterion assuring the local stability of the stationary solutions. They derived a Volterra integral equation and used it to obtain the stability criteria. More recently, [PPS10; PPS13; MW18; MQW18] have re-explored these models for neuroscience applications (see [CCDR15; Che17b] for a probabilistic interpretation of some of these PDEs using Hawkes processes). In [MW18] (see also [Gab18]), the authors studied such PDE in the sense of measures. In our setting, that means to use (1.3) (and not the strong form (1.4)). They considered solutions in the space of bounded Radon measures

$$M^1(\mathbb{R}_+) = \{g \in (\mathcal{C}(\mathbb{R}_+))', \text{ Supp } g \in \mathbb{R}_+\}$$

and used recent tools developed for the semigroups on this Banach space (see [MS16]). They obtained results on the long time behavior for weak-enough interactions. To do so, they processed by perturbing the case for which there is no interaction. Our PDE (1.4) differs from theirs in the sense that we have a nonlinear transport term (theirs is constant and equal to one) and our boundary condition is more complex: in particular the boundary condition (1.5) is nonlinear with respect to the solution $\nu(t, dx)$.

A celebrated variant of (1.2) is the following "standard" integrate-and-fire model with a fixed deterministic threshold [CCP11b; CPSS15; DIRT15a; DIRT15b]. In this case, the limit equation writes (see [DIRT15a])

$$X_t = X_0 + \int_0^t b(X_u)du + J\,\mathbb{E}\,M_t + W_t - M_t, \tag{1.7}$$

where $(W_t)_{t\geq 0}$ is a Brownian motion, $M_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k)$, $\tau_0 = 0$ and

$$\tau_k = \inf\{t > \tau_{k-1},\ X_{t-} \geq 1\}.$$

That is, the τ_k are the successive spiking times, corresponding to the times where X_t reaches the value 1, which is the value of the threshold. The random variable M_t is the number of spikes of (X_t) up to time t. Note that such fixed deterministic threshold corresponds (informally) to our case to the choice $f(x) = +\infty \mathbb{1}_{[1,\infty)}(x)$, and that between the jump, a diffusion W_t is added to the dynamics. In [CCP11b] the authors show that the limit equation (1.7) may have blow-up, that is $\frac{d}{dt}\mathbb{E}\, M_t$ becomes infinite in finite time. This result and its proof can easily be adapted to (1.2) in the case where f *explodes* at a finite location and that b is lower bounded. Consider for instance

$$f(x) = \frac{1}{1-x}, \quad \text{and} \quad b(x) = 1. \tag{1.8}$$

However, if f is regular enough and does not grow too fast to infinity, such blow-up phenomena disappears and the limit equation is well-posed for all times. Indeed, the blow-up appearing in (1.7) is reminiscent of a "cascade" of spikes, which is better understood at the level of the particle systems. Assume all the neurons have their potential close to the threshold (for all $i \in \{1, \cdots, N\}$, $X_t^{i,N} \geq 1 - \frac{J}{N}$ at some time t), then a spike of one of the N neurons triggers an instantaneous spikes of all the other neurons! This translates to a blow-up at the level of the limit equation. However, if f is regular enough, such cascade of spikes cannot happen, thanks the Poisson structure of (1.2).

In [DO16] (see also [HKL18; HP19]), the authors studied the long time behavior of the finite particle system (1.1). They assumed that $b(x) = -\alpha x$ for some constant $\alpha \geq 0$, that f is globally Lipschitz with $f(0) = 0$ and that f is differentiable at 0. When $\alpha > 0$, they proved almost sure extinction in finite time of the particle system. So $(\delta_0)^{\otimes N}$ is the unique, globally attractive, invariant measure. For $\alpha = 0$, they proved the existence of a globally attractive non-trivial invariant probability measure. To do so, they proved that the process is Harris recurrent, by exhibiting an explicit regeneration scheme. Finally, we mention the recent work [LM20]. The authors consider $b(x) = -\alpha x$ for some constant $\alpha > 0$ and $f(x) = \min(x, 1)$. So, the result of [DO16] applies: the finite particle system extincts in finite time almost surely. For J large enough, the authors proved that the nonlinear equation (1.2) admits a non-trivial invariant measure, globally attractive, using a coupling argument. Moreover, they show that in this specific situation, the finite particle system is *metastable*: it spends a long time close to the solution of the nonlinear equation (1.2), before finally extincts.

1.4 Contributions

1.4.1 The Volterra integral equation: Chapter 2

The main difficulty of (1.2) (or its PDE version (1.3)) is to control the nonlinear interaction $t \mapsto J\,\mathbb{E}\, f(X_t)$. In particular, there is no simple autonomous equation for this quantity. To overcome this difficulty, we introduce a "linearized" version of (1.2), for which we can derive a closed integral equation of the jump rate.

Fix $s \geq 0$ and let $\boldsymbol{a} : [s, \infty) \to \mathbb{R}_+$ be a non-negative deterministic function, that we call the *external current*. It replaces the interaction $J\mathbb{E} f(X_u)$ in (1.2). We assume that $\boldsymbol{a} : [s, \infty) \to$

$\mathbb{R}_+$ is a Borel-measurable locally integrable function ($\forall t \geq s$, $\int_s^t a_u du < \infty$). Consider the linear non-homogeneous SDE

$$\forall t \geq s, \quad Y_{t,s}^{\boldsymbol{a},\nu} = Y_{s,s}^{\boldsymbol{a},\nu} + \int_s^t b(Y_{u,s}^{\boldsymbol{a},\nu})du + \int_s^t a_u du - \int_s^t \int_{\mathbb{R}_+} Y_{u-,s}^{\boldsymbol{a},\nu} \mathbb{1}_{\{z \leq f(Y_{u-,s}^{\boldsymbol{a},\nu})\}} \mathbf{N}(du, dz), \tag{1.9}$$

where $\mathcal{L}(Y_{s,s}^{\boldsymbol{a},\nu}) = \nu$. Under quite general assumptions on b and f, this SDE has a path-wise unique solution (see Lemma 2.13). We denote the jump rate of this SDE by:

$$\forall t \geq s, \quad r_{\boldsymbol{a}}^{\nu}(t,s) := \mathbb{E}\, f(Y_{t,s}^{\boldsymbol{a},\nu}). \tag{1.10}$$

Moreover, taking $s = 0$ and $Y_{0,0}^{\boldsymbol{a},\nu} = X_0$, it holds that $(Y_{t,0}^{\boldsymbol{a},\nu})_{t \geq 0}$ is a solution to (1.2) if it satisfies the closure condition

$$\forall t \geq 0, \quad a_t = J r_{\boldsymbol{a}}^{\nu}(t, 0). \tag{1.11}$$

Conversely, any solution to (1.2) is a solution to (1.9) with $a_t = J\, \mathbb{E}\, f(X_t)$.

Consider $\tau_s^{\boldsymbol{a},\nu}$ the time of the first jump of $Y^{\boldsymbol{a},\nu}$ after s

$$\tau_s^{\boldsymbol{a},\nu} := \inf\{t \geq s : Y_{t,s}^{\boldsymbol{a},\nu} \neq Y_{t-,s}^{\boldsymbol{a},\nu}\}. \tag{1.12}$$

We introduce the survival function $H_{\boldsymbol{a}}^{\nu}(t,s)$ and the density of the first jump $K_{\boldsymbol{a}}^{\nu}(t,s)$ to be

$$H_{\boldsymbol{a}}^{\nu}(t,s) := \mathbb{P}(\tau_s^{\boldsymbol{a},\nu} > t), \quad K_{\boldsymbol{a}}^{\nu}(t,s) := -\frac{d}{dt}\mathbb{P}(\tau_s^{\boldsymbol{a},\nu} > t). \tag{1.13}$$

We prove in Chapter 2 that the function $r_{\boldsymbol{a}}^{\nu}$ satisfies the Volterra integral equation

$$\forall t \geq s, \quad r_{\boldsymbol{a}}^{\nu}(t,s) = K_{\boldsymbol{a}}^{\nu}(t,s) + \int_s^t K_{\boldsymbol{a}}^{\delta_0}(t,u) r_{\boldsymbol{a}}^{\nu}(u,s) du, \tag{1.14}$$

Consequently (1.11) and (1.14) give a third formulation of this mean-field model. We use this Volterra integral equation to obtain a new proof of existence and path-wise uniqueness of the solution of the nonlinear equation (1.2). As in [FL16], we do not require the initial condition to be compactly supported. Moreover, we give sufficient condition ensuring that the jump rate $\mathbb{E}\, f(X_t)$ is uniformly bounded in time:

$$\sup_{t \geq 0} \mathbb{E}\, f(X_t) < \infty.$$

1.4.2 Long time behavior for weak enough interactions: Chapter 3

Our first main result is Theorem 3.7. We study the long time behavior of the solution of the nonlinear SDE (1.2) under the assumption that the interactions are small. We prove that there exists a constant $J^* > 0$, only depending on b and f, such that for all $J \in [0, J^*]$, the SDE (1.2) has a unique invariant probability measure which is globally attractive. Moreover, the rate of convergence to this invariant measure is exponential.

The outline of the proof is the following. We first study the case $J = 0$ (no interaction). Consider $\alpha > 0$ and let $(Y_{t,0}^{\alpha,\nu})$ be the solution of (1.9). That is $(Y_{t,0}^{\alpha,\nu})_{t\geq 0}$ corresponds to a single neuron (no interaction) subject to a constant current $\boldsymbol{a} \equiv \alpha$. We prove that $(Y_{t,0}^{\alpha,\nu})$ has a unique invariant measure given by

$$\nu_\alpha^\infty(dx) := \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) \mathbb{1}_{[0,\sigma_\alpha)}(x)$$

In this equation, the upper bound σ_α of the support of the invariant measure, is given by

$$\sigma_\alpha := \inf\{x \geq 0,\ b(x)+\alpha = 0\}.$$

Because we assume that $b(0) \geq 0$, it holds that $\sigma_\alpha \in \mathbb{R}_+^* \cup \{+\infty\}$. The normalization constant $\gamma(\alpha) > 0$ is such that ν_α^∞ is a probability measure. Moreover, it holds that

$$\nu_\alpha^\infty(f) = \int_{\mathbb{R}_+} f(x)\nu_\alpha^\infty(dx) = \gamma(\alpha).$$

In other words, $\gamma(\alpha)$ is the jump rate under the invariant measure ν_α^∞. For constant current $\boldsymbol{a} \equiv \alpha$, the Volterra integral equation (1.14) is of convolution type. So, techniques using the Laplace transform are available. We prove that $(Y_{t,0}^{\alpha,\nu})$ converges in law to ν_α^∞ at an exponential rate. More precisely, we prove that there exists a constant $\lambda_\alpha^* > 0$ (only depending on b, f and α and explicitly determined by the Laplace transform of the survival function $H_\alpha^{\delta_0}(t,0)$) such that for all $\lambda < \lambda_\alpha^*$, it holds that

$$\sup_{t\geq 0} |r_\alpha^\nu(t,0) - \gamma(\alpha)|e^{\lambda t} < \infty.$$

Second, we remark that the invariant measures of (1.2) are

$$\{\nu_\alpha^\infty,\ \alpha = J\gamma(\alpha)\}.$$

For J small enough, we prove existence and uniqueness of the invariant measure of (1.2): there is a unique $\alpha^* > 0$ such that $\alpha^* = J\gamma(\alpha^*)$.

Third, we extend the convergence result to non-constant current. We consider $\lambda < \lambda_{\alpha^*}^*$ and $\boldsymbol{a}$ a deterministic function such that

$$|a_t - \alpha^*| \leq Ce^{-\lambda t},$$

for some constant $C > 0$. We then prove that

$$|r_{\boldsymbol{a}}^\nu(t,0) - \gamma(\alpha^*)| \leq D(C)e^{-\lambda t},$$

for some new constant $D(C)$ related to C. To do so, we use a perturbation argument involving the Volterra integral equation (1.14).

Finally a fixed point argument ends the proof: we consider J small enough such that $JD(C) \leq C$. It follows that the Picard iterations

$$a_{n+1}(t) := Jr_{\boldsymbol{a}_n}^\nu(t,0), \quad a_0 := \alpha^*$$

satisfy

$$\forall n \in \mathbb{N}, \quad |a_n(t) - \alpha_*| \le Ce^{-\lambda t}.$$

We then let n go to infinity, and use that these Picard iterations converge to the nonlinear current $J\,\mathbb{E}\,f(X_t)$ to obtain

$$\forall t \ge 0, \quad |J\,\mathbb{E}\,f(X_t) - \alpha_*| \le Ce^{-\lambda t}.$$

1.4.3 Local stability of an invariant probability measure: Chapter 4

We study the stability of the invariant measures of (1.2) for J arbitrary large. Consider ν_α^∞ an invariant probability measure of the nonlinear equation (1.2). The constant $\alpha \ge 0$ satisfies $\alpha = J\gamma(\alpha)$. We give a sufficient condition ensuring the local stability of ν_α^∞. If this criteria of stability is met, then starting from any initial condition ν close enough to the invariant measure ν_α^∞, the law of X_t, solution of (1.2), converges to ν_α^∞. The convergence holds at an exponential rate. The criteria involves the location of the roots of an explicit holomorphic function. Consider for all $t \ge 0$

$$r_\alpha^x(t) := r_\alpha^{\delta_x}(t, 0) = \mathbb{E}\,f(Y_{t,0}^{\alpha,\delta_x})$$

the jump rate at time t of an isolated neuron subject to a constant current α and starting at time 0 with the initial potential x. Let:

$$\forall t \ge 0, \quad \Theta_\alpha(t) := \int_0^\infty \frac{d}{dx} r_\alpha^x(t) \nu_\alpha^\infty(dx).$$

We prove that for all $\lambda < \lambda_\alpha^*$, $t \mapsto \Theta_\alpha(t)e^{\lambda t} \in L^1(\mathbb{R}_+)$. So the Laplace transform of Θ_α is a holomorphic function on $\{z \in \mathbb{C},\ \Re(z) > -\lambda_\alpha^*\}$. Theorem 4.13 states that if

$$\sup\{\Re(z) > -\lambda_\alpha^*,\ J\widehat{\Theta}_\alpha(z) = 1\} < 0,$$

then the invariant measure ν_α^∞ is locally stable. If moreover it holds that

$$\forall x \ge 0, \quad f(x) + b'(x) \ge 0,$$

we prove in Theorem 4.14 that this last stability criteria is automatically satisfied. The proof of Theorem 4.13 relies on the implicit function theorem. Given $\lambda \in (0, \lambda_\alpha^*)$, consider the following weighed $L^\infty(\mathbb{R}_+)$ space:

$$L_\lambda^\infty := \{h \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}),\ ||h||_\lambda^\infty < \infty\}, \quad \text{with} \quad ||h||_\lambda^\infty := \operatorname*{ess\,sup}_{t \ge 0} |h_t| e^{\lambda t}.$$

Given an initial condition ν, we define the following function

$$\Phi(\nu, h) := J r_{\alpha+h}^\nu(\cdot, 0) - (\alpha + h),$$

where $r_{\alpha+h}^\nu(t, 0) := \mathbb{E}\,f(Y_{t,0}^{\alpha+h,\nu})$ and $h \in L_\lambda^\infty$. Note that when $\Phi(\nu, h) = 0$, then (1.11) holds with $\boldsymbol{a} \equiv \alpha + h$. Hence such current $\alpha + h$ can be used to define a solution of the nonlinear SDE (1.2).

By carefully inspecting the perturbation argument of Chapter 3, we can prove that the function $L^\infty_\lambda \ni h \mapsto \Phi(\nu, h) \in L^\infty_\lambda$ is $\mathcal{C}^1$ Fréchet differentiable in a neighborhood of $h = 0$. We then apply the implicit function theorem at the point $(\nu^\infty_\alpha, 0)$, for which we have

$$\Phi(\nu^\infty_\alpha, 0) = J r^{\nu^\infty_\alpha}_\alpha - \alpha = J\gamma(\alpha) - \alpha = 0.$$

To do so, the Fréchet derivative of Φ with respect to h at the point $(\nu^\infty_\alpha, 0)$ (denoted $D_h\Phi(\nu^\infty_\alpha, 0)$) needs to be invertible with continuous inverse. We prove that $D_h\Phi(\nu^\infty_\alpha, 0)$ can be represented using a convolution with respect to Θ_α and so the invertible condition can be stated in term of the location of the zeros of the above holomorphic function.

1.4.4 Periodic solutions via Hopf bifurcation: Chapter 5

In this chapter, we give sufficient general conditions on b, f and J to have a Hopf bifurcation: at some critical interaction parameter J_0, an invariant probability measure of the process become instable and periodic solutions appear for J close to J_0. Our conditions can be stated using the same explicit holomorphic function of Chapter 4. We assume that for $J = J_0$, there exists an invariant measure $\nu^\infty_{\alpha_0}$ of (1.2) (and so the constant $\alpha_0 > 0$ satisfies $\alpha_0 = J_0\gamma(\alpha_0)$) such that

$$\exists \tau_0 > 0, \quad J_0 \widehat{\Theta}_{\alpha_0}(i/\tau_0) = 1 \quad \text{with} \quad i^2 = -1.$$

That is, the criterion of stability studied in Chapter 4 is violated in α_0 with a pair of purely imaginary zeros $\pm i/\tau_0$. The main result, Theorem 5.9, gives sufficient conditions ensuring the existence of a family of periodic solutions of (1.2). Those periodic solutions are strong solutions of the Fokker-Planck equation (1.4). The proof of the result relies on two main arguments. First, we study an isolated neuron subject to a periodic current. That is, given $\boldsymbol{a}$ a T-periodic non-negative continuous function, we study the long time behavior of $(Y^{\boldsymbol{a},\nu}_{t,0})_{t\geq 0}$, the solution of (1.9). Because of the forcing periodic input, the law of $Y^{\boldsymbol{a},\nu}_{t,0}$ asymptotically oscillates as t goes to infinity. To characterize the T-periodic limit, we introduce the following discrete time Markov Chain. We consider $(\tau_i)_{i\geq 1}$ the times of the successive jumps of $Y^{\boldsymbol{a},\nu}_{t,0}$. We then let

$$\phi_i := \tau_i - \left\lfloor \frac{\tau_i}{T} \right\rfloor T$$

be the *phase* of the i-th jump. Then, $(\phi_i)_{i\geq 1}$ is a Markov Chain on $[0, T]$. We prove that this Markov Chain has a unique invariant measure, denoted $\pi_{\boldsymbol{a}}$. Finally, this invariant measure $\pi_{\boldsymbol{a}}$ is used to construct the limit periodic law of $Y^{\boldsymbol{a},\nu}_{t,0}$. We then go back to the non-linear equation (1.2). We parameterize the periodic solutions as the zeros of a non-linear function. Two difficulties have to be managed here. First, the period T of the solution is also unknown: we use the scaling argument described in Section 1.1.4 to only consider 2π-periodic functions. Second, the periodic solutions are a priori to be found in a infinite dimensional Banach space. We use the Lyapunov–Schmidt method to reduce the problem to a space of dimension two. Finally, we solve this finite dimensional problem by the mean of the implicit function theorem.

1.4.5 Examples and numerical simulations: Chapter 6

In this last chapter, we illustrate the theoretical results with explicit examples and numerical simulations. We first study in details the following case where

$$\forall x \geq 0, \quad b(x) = m - x \quad \text{ and } f(x) = x^2,$$

for some constant $m \geq 0$. Depending on the value of m and J, the nonlinear SDE (1.2) may have 1,2 or 3 invariant probability measures. We study analytically the case $m = 0$ as well as m large. We then study the following example:

$$\forall x \geq 0, \quad b(x) = m - x \quad \text{ and } \quad f(x) := \begin{cases} 0 & \text{for } 0 \leq x < 1, \\ 1/\beta & \text{for } x \geq 1, \end{cases}$$

for some constants $m > 1$ and $\beta > 0$. We will see that for β small enough the model exhibits (many) Hopf bifurcations and the spectral conditions of Chapter 5 can be analytically verified. Finally, we describe and compare two numerical methods: a Monte Carlo Euler scheme to simulate the particle system (1.1) and a finite volume scheme to approximate the solution of the Fokker-Planck equation (1.3). In addition, we give an algorithm to determine numerically the stability of the invariant probability measure of (1.2). We rely on the results of Chapter 4: the stability is given by the location of the zeros of an explicit holomorphic functions. We approximate those zeros by computing the eigenvalues of an explicit large matrix. This matrix is derived from the finite volume scheme.

Chapter 2

Well-posedness of the mean-field equation

We study existence and path-wise uniqueness of the McKean-Vlasov SDE (1.2) and describe some qualitative properties of the solution. We give sufficient conditions ensuring that the mean number of jumps per unit of time (the *jump rate*) is uniformly bounded in time and we study the regularity of the marginals of the solution (existence of a densities). Our proof for the existence and uniqueness is based on the Volterra integral equation (1.14) and on a fixed point argument. Some of the material of this chapter is taken from the first part of the published paper [CTV20a].

2.1 Introduction

We consider the McKean-Vlasov equation (1.2)

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t \int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{z\le f(X_{u-})\}}\mathbf{N}(du,dz),$$

where the initial condition X_0 is independent of the Poisson measure $\mathbf{N}$, of intensity $dudz$. We give conditions on $b : \mathbb{R}_+ \to \mathbb{R}$, $f : \mathbb{R}_+ \to \mathbb{R}_+$ and on $\mathcal{L}(X_0)$, the law of the initial condition X_0, such that this nonlinear SDE admits a path-wise unique solution.

Results on the existence of a solution to (1.2), in a slightly different context (in particular, with $b(x) = -\kappa x$ for $\kappa \ge 0$), have been obtained in [DGLP15]. One difficulty is that the function f is not assumed to be globally Lipschitz. In [DGLP15], the authors assumed that the initial condition ν is compactly supported. This property is preserved at any time $t > 0$. So, the behavior of the solution in the case of a locally Lipschitz continuous rate function f is similar to the case with a function f globally Lipschitz continuous. When the initial condition is not compactly supported, the situation is more delicate. In [FL16], the authors proved existence and path-wise uniqueness of the solution to (1.2) (again in a slightly different setting where $b(x) = -\kappa x$). To do so, they first proved path-wise uniqueness of (1.2), using an ad-hoc distance. They defined

$$\forall x \ge 0, \quad H(x) := f(x) + \arctan(x)$$

and proved that if $(X_t), (\tilde{X}_t)$ are two solutions of (1.2) driven by the same Poisson measure, then it holds that for all $T > 0$, there exists a constant C_T such that

$$\sup_{t \leq T} \mathbb{E} \left| H(X_t) - H(\tilde{X}_t) \right| \leq C_T \, \mathbb{E} \left| H(X_0) - H(\tilde{X}_0) \right|.$$

Moreover, they proved the tightness of the particle system (1.1) which yields existence of a solution by considering a converging sub-sequence. This method also gives the propagation of chaos. In this chapter, we give another proof for the existence and uniqueness of the solution of (1.2). As in [FL16], we do not require the initial condition ν to be compactly supported nor f to be Lipschitz. To proceed, we first derive a *closed* equation for the jump rate: the Volterra integral equation (1.14). We give two different derivations of this equation (see Proposition 2.19 and Proposition 2.24). We use this equation together with a fixed point argument to obtain the result. We find this method more robust. Indeed, it is not obvious to adapt the proof of [FL16] to our setting, where b is arbitrary. Moreover, we obtain a global in time bound of the jump rate.

Remark 2.1. *Note that the "global" existence/uniqueness results obtained for this model differs from those of the "standard" Integrate-and-Fire model with a fixed deterministic threshold* (1.7)*. In [CCP11a], the authors proved that a blow-up phenomenon appears when the law of the initial condition is close enough to the threshold ϑ: at the blow-up time, the jump rate of the solution diverges to infinity. Here, under our assumptions, the situation is completely different: the jump rate is uniformly bounded in time (see Theorem 2.8).*

2.2 Notations and main results

Let us introduce some notations and definitions. We denote by $\mathcal{P}(\mathbb{R}_+)$ the set of probability measures on $\mathbb{R}_+$ and by $\mathbf{N}(du, dz)$ a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dudz$. For $s \geq 0$ and $\nu \in \mathcal{P}(\mathbb{R}_+)$, let $Y^{\boldsymbol{a},\nu}_{s,s}$ be a ν-distributed random variable independent of $\mathbf{N}$. We consider the canonical filtration $(\mathcal{F}^s_t)_{t \geq s}$ associated to the Poisson measure $\mathbf{N}$ and to the initial condition $Y^{\boldsymbol{a},\nu}_{s,s}$, that is the completion of

$$\sigma\{Y^{\boldsymbol{a},\nu}_{s,s}, \mathbf{N}([s,r] \times A) : \; s \leq r \leq t, \; A \in \mathcal{B}(\mathbb{R}_+)\}.$$

Definition 2.2. *Let $s \geq 0$ and $\boldsymbol{a} : [s, \infty) \to \mathbb{R}_+$ be a measurable locally integrable function ($\forall t \geq s, \; \int_s^t a_u du < \infty$).*

- *A process $(Y^{\boldsymbol{a},\nu}_{t,s})_{t \geq s}$ is said to be a solution of the non-homogeneous linear equation* (1.9) *with a current $\boldsymbol{a}$ if $(Y^{\boldsymbol{a},\nu}_{t,s})_{t \geq s}$ is $(\mathcal{F}^s_t)_{t \geq s}$-adapted, càdlàg, a.s. $\forall t \geq s, \; \int_s^t f(Y^{\boldsymbol{a},\nu}_{u,s}) du < \infty$ and* (1.9) *holds a.s.*
- *A $(\mathcal{F}^0_t)_{t \geq 0}$-adapted càdlàg process $(X_t)_{t \geq 0}$ is said to solve the nonlinear SDE* (1.2) *if $t \mapsto \mathbb{E} f(X_t)$ is measurable locally integrable and if $(X_t)_{t \geq 0}$ is a solution of* (1.9) *with $s = 0$, $Y^{\boldsymbol{a},\nu}_{0,0} = X_0$ and $\forall t \geq 0, \; a_t = J \, \mathbb{E} f(X_t)$.*

For any measurable function g, we write $\nu(g) := \int_0^\infty g(x) \nu(dx)$ whenever this integral makes sense.

Assumption 2.3. *We assume that $b : \mathbb{R}_+ \to \mathbb{R}$ is a globally Lipschitz function with $b(0) \geq 0$.*

Remark 2.4. *The assumption $b(0) \geq 0$ ensures that the solution of (1.2) stays in $\mathbb{R}_+$ (and is required if one wishes the associated particle system (1.1) to be well-defined on $(\mathbb{R}_+)^N$, where N is the number of particles).*

Assumption 2.5. *We assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{C}^1$ strictly increasing function with $f(0) = 0$ and there exists a constant C_f such that:*

2.5*(a) for all* $x, y \geq 0,\ f(x+y) \leq C_f(1 + f(x) + f(y))$.

2.5*(b) for all* $A > 0$, $\sup_{x\geq 0} Af'(x) - f(x) < \infty$.

2.5*(c) for all* $x \geq 0,\ |b(x)| \leq C_f(1 + f(x))$.

Remark 2.6. *These assumptions ensure that $f(x) > 0$ for all $x > 0$. Using that f is non-decreasing and 2.5(a), one has*

$$\forall A > 0, \exists C_A > 0, \quad \forall x \geq 0, \quad f(Ax) \leq C_A(1 + f(x)). \tag{2.1}$$

Moreover, 2.5(a) also implies that f grows at most at a polynomial rate: there exists constants C_0 and $p > 0$ such that

$$\forall x \geq 0, \quad f(x) \leq C_0(1 + x^p). \tag{2.2}$$

Indeed, this follows from

$$\forall n \in \mathbb{N}, \quad f(2^{n+1}) \leq C_f(1 + 2f(2^n)),$$

and the fact that f is non-decreasing. We can choose $p := 1 + \log_2 C_f$.

Assumption 2.7. *The law of the initial condition $\nu \in \mathcal{P}(\mathbb{R}_+)$ satisfies $\nu(f^2) < \infty$.*

Let us give our main result.

Theorem 2.8. *Let $J \geq 0$. Under Assumptions 2.3, 2.5 and 2.7, the nonlinear SDE (1.2) has a path-wise unique solution $(X_t)_{t\geq 0}$ in the sense of Definition 2.2. Furthermore the function $t \mapsto \mathbb{E}\, f(X_t)$ is $\mathcal{C}^1$ and there is a finite constant $\bar{r} > 0$ (only depending on b, f and J) such that*

$$\sup_{t\geq 0} \mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)) \quad \text{and} \quad \limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}.$$

2.3 Technical notations and lemmas

Between its random jumps, the SDE (1.9) is reduced to a non-homogeneous ODE. Let us introduce its flow $\varphi^{\boldsymbol{a}}_{t,s}(x)$, which by definition is the solution of

$$\begin{aligned} \forall t \geq s, \quad \frac{d}{dt}\varphi^{\boldsymbol{a}}_{t,s}(x) &= b(\varphi^{\boldsymbol{a}}_{t,s}(x)) + a_t \\ \varphi^{\boldsymbol{a}}_{s,s}(x) &= x. \end{aligned} \tag{2.3}$$

Given $s \in \mathbb{R}$, we consider $L^\infty_{\text{loc}}([s,\infty);\mathbb{R}_+)$ the space of non-negative and locally bounded measurable functions on $[s,\infty)$. The following standard results on the ODE (2.3) will be useful all along:

Lemma 2.9. *Let $\boldsymbol{a} \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$. Assume b satisfies Assumption 2.3. Then:*

2.9(a) *For all $x \geq 0$ and $s \geq 0$, the integral equation*

$$\varphi_t = x + \int_s^t b(\varphi_u)du + \int_s^t a_u du,$$

has a unique continuous in time solution, that we denote $\varphi^{\boldsymbol{a}}_{t,s}(x)$. This is the flow associated to the drift b and to the external current $\boldsymbol{a}$.

2.9(b) *Given $\boldsymbol{a}$ and $\boldsymbol{d}$ in $L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, the flow satisfies the following comparison principle:*

$$[\forall t \geq 0,\ a_t \geq d_t] \implies [\forall x \geq y \geq 0,\ \forall t \geq s \geq 0,\ \varphi^{\boldsymbol{a}}_{t,s}(x) \geq \varphi^{\boldsymbol{d}}_{t,s}(y)].$$

Furthermore, denote by L the Lipschitz constant of b, we have

$$\left|\varphi^{\boldsymbol{a}}_{t,s}(x) - \varphi^{\boldsymbol{d}}_{t,s}(x)\right| \leq e^{L(t-s)} \int_s^t |a_u - d_u| du.$$

2.9(c) *The function $(t,s) \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is continuous on $\{(t,s) : 0 \leq s \leq t < \infty\}$.*

We have explicit expressions for $H^\nu_{\boldsymbol{a}}(t,s)$ and $K^\nu_{\boldsymbol{a}}(t,s)$, defined by (1.13)

$$\forall t \geq s, \quad H^\nu_{\boldsymbol{a}}(t,s) = \int_0^\infty \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx). \tag{2.4}$$

$$K^\nu_{\boldsymbol{a}}(t,s) = \int_0^\infty f(\varphi^{\boldsymbol{a}}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx). \tag{2.5}$$

Notation 2.10. *Given two "kernels" α and β, it is convenient to follow the notation of [GLS90] and define:*

$$\forall t \geq s, \quad (\alpha * \beta)(t,s) := \int_s^t \alpha(t,u)\beta(u,s)du. \tag{2.6}$$

Notation 2.11. *To shorten notations, we write for all $x \geq 0$:*

$$r^x_{\boldsymbol{a}}(t,s) := r^{\delta_x}_{\boldsymbol{a}}(t,s), \quad H^x_{\boldsymbol{a}}(t,s) := H^{\delta_x}_{\boldsymbol{a}}, \quad K^x_{\boldsymbol{a}}(t,s) := K^{\delta_x}_{\boldsymbol{a}},$$

where δ_x is the Dirac measure in x. When $x = 0$, we omit the 0 and simply write $r_{\boldsymbol{a}}$. We use the same conventions for H and K.

With these two notations, the Volterra equation (1.14) becomes

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}}. \tag{2.7}$$

From the definition (1.13), one can check directly the following relation

$$1 * K^\nu_{\boldsymbol{a}} = 1 - H^\nu_{\boldsymbol{a}}. \tag{2.8}$$

Notation 2.12. *When the input current $\boldsymbol{a}$ is constant and equal to α ($\boldsymbol{a} \equiv \alpha$), (1.9) is homogeneous and we write*

$$\begin{aligned} \forall t \geq 0, \quad Y_t^{\alpha,\nu} &:= Y_{t,0}^{\boldsymbol{a},\nu}, \\ r_\alpha^\nu(t) &:= r_{\boldsymbol{a}}^\nu(t,0), \\ K_\alpha^\nu(t) &:= K_{\boldsymbol{a}}^\nu(t,0), \\ H_\alpha^\nu(t) &:= H_{\boldsymbol{a}}^\nu(t,0), \\ \varphi_t^\alpha(x) &:= \varphi_{t,0}^{\boldsymbol{a}}(x). \end{aligned}$$

Note that in this homogeneous situation, the operation $*$ corresponds to the classical convolution operation. In particular this operation is commutative in the homogeneous setting and (2.7) is a convolution Volterra equation.

2.4 Derivation of the Volterra integral equation

2.4.1 Well-posedness of the non-homogeneous linear equation (1.9)

Fix $s \geq 0$ and let $\boldsymbol{a} \in L^\infty_{\text{loc}}([s,\infty);\mathbb{R}_+)$. We consider the non-homogeneous linear SDE (1.9).

Lemma 2.13. *Assume b satisfies Assumption 2.3. Let $f \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)$ and $\nu \in \mathcal{P}(\mathbb{R}_+)$. Then the SDE (1.9) has a path-wise unique solution on $[s,\infty)$ in the sense of Definition 2.2.*

Proof. We give a direct proof by considering the jumps of $Y_{t,s}^{\boldsymbol{a},\nu}$ and by solving the equation between the jumps.

- **Step 1**: we assume that f is bounded. There exists a constant $K < \infty$ such that

  $$\sup_{x \geq 0} f(x) \leq K.$$

  In this case, the solution of (1.9) can be constructed in the following way. Define by induction

  $$\tau_0 := \inf\{t \geq s : \int_s^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(\varphi_{u,s}^{\boldsymbol{a}}(Y_{s,s}^{\boldsymbol{a},\nu}))\}} \mathbf{N}(du,dz) > 0\},$$

  $$\forall n \geq 0, \quad \tau_{n+1} := \inf\{t \geq \tau_n : \int_{\tau_n}^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z \leq f(\varphi_{u,\tau_n}^{\boldsymbol{a}}(0))\}} \mathbf{N}(du,dz) > 0\}.$$

  Using that $f \leq K$, it follows that $a.s.$ $\lim_{n\to\infty} \tau_n = +\infty$. We set

  $$Y_{t,s}^{\boldsymbol{a},\nu} = \varphi_{t,s}^{\boldsymbol{a}}(Y_{s,s}^{\boldsymbol{a},\nu}) \mathbb{1}_{t \in [s,\tau_0)} + \sum_{n \geq 0} \varphi_{t,\tau_n}^{\boldsymbol{a}}(0) \mathbb{1}_{t \in [\tau_n,\tau_{n+1})},$$

  we can directly verify that $t \mapsto Y_{t,s}^{\boldsymbol{a},\nu}$ is almost surely a solution of (1.9).

Path-wise uniqueness of equation (1.9) follows immediately from Lemma 2.9(a): two solutions have to be equal almost surely before the first jump, from which we deduce that the two solutions have to jump at the same time. By induction on the number of jumps, the two trajectories are almost surely equal.

- **Step 2**: We come back to the general case where f is not assumed to be bounded and we adapt the strategy of [FL16, proof of Prop. 2]. We use Step 1 with $f^K(x) := f(\min(x, K))$ for some $K > 0$. Let us denote $Y^{\boldsymbol{a},\nu,K}_{t,s}$ the solution of (1.9) where f has been replaced by f^K. The boundedness of f^K implies the path-wise uniqueness of $Y^{\boldsymbol{a},\nu,K}_{t,s}$. We introduce $\zeta_K := \inf\{t \geq s : \, Y^{\boldsymbol{a},\nu,K}_{t,s} \geq K\}$, it holds that $Y^{\boldsymbol{a},\nu,K}_{t,s} = Y^{\boldsymbol{a},\nu,K+1}_{t,s}$ for all $t \in [s, \zeta_K]$ and all $K \in \mathbb{N}$. Moreover, $\zeta_K < \zeta_{K+1}$. We define $\zeta := \sup_K \zeta_K$ and deduce the existence and uniqueness of a solution $t \mapsto Y^{\boldsymbol{a},\nu}_{t,s}$ of (1.9) on $[0, \zeta($ such that $\limsup_{t\to\zeta} Y^{\boldsymbol{a},\nu}_{t,s} = \infty$ on the event $\{\zeta < \infty\}$. But any solution of (1.9) satisfies for all $t \geq s$, $Y^{\boldsymbol{a},\nu}_{t,s} \leq \varphi^{\boldsymbol{a}}_{t,s}(Y^{\boldsymbol{a},\nu}_{s,s}) < \infty$ a.s. and so it holds that $\zeta = +\infty$ a.s.

□

Lemma 2.14. *Let b, f and ν satisfying Assumptions 2.3, 2.5 and 2.7. Let $s \geq 0$ and $\boldsymbol{a} \in L^\infty_{loc}([s,\infty); \mathbb{R}_+)$. Consider the solution $(Y^{\boldsymbol{a},\nu}_{t,s})_{t\geq s}$ of (1.9). The functions $t \mapsto \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s})$, $t \mapsto \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s})$, $t \mapsto \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s})|b(Y^{\boldsymbol{a},\nu}_{t,s})|$ and $t \mapsto \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{t,s})$ are locally bounded on $[s,\infty)$. Moreover, $t \mapsto \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}) =: r^\nu_{\boldsymbol{a}}(t,s)$ is continuous on $[s,\infty)$.*

Proof. Consider the interval $[s, T]$ for some $T > 0$. Let $A := \operatorname{ess\,sup}_{t\in[s,T]} a_t$. Denote by L the Lipschitz constant of b. Denote by C_T any constant only depending on b, f, A and T, that may change from line to line. It is clear that

$$\forall t \in [s,T], a.s. \quad Y^{\boldsymbol{a},\nu}_{t,s} \leq Y^{\boldsymbol{a},\nu}_{s,s} + \int_s^t [b(Y^{\boldsymbol{a},\nu}_{u,s}) + a_u]du \leq Y^{\boldsymbol{a},\nu}_{s,s} + L\int_s^t Y^{\boldsymbol{a},\nu}_{u,s} du + T(b(0) + A).$$

By Grönwall's inequality, there exists a constant C_T such that

$$a.s. \quad Y^{\boldsymbol{a},\nu}_{t,s} \leq C_T(1 + Y^{\boldsymbol{a},\nu}_{s,s})$$

Using (2.1), we have

$$a.s. \quad f(Y^{\boldsymbol{a},\nu}_{t,s}) \leq C_T(1 + f(Y^{\boldsymbol{a},\nu}_{s,s})).$$

Because there exists a constant C_f such that $f \leq C_f(1 + f^2)$, we deduce that there is another constant C_T such that

$$a.s. \quad f^2(Y^{\boldsymbol{a},\nu}_{t,s}) \leq C_T(1 + f^2(Y^{\boldsymbol{a},\nu}_{s,s})).$$

Using Assumption 2.7, we deduce that $t \mapsto \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{t,s})$ is bounded on $[s,T]$. This implies that $t \mapsto \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s})$ and $t \mapsto f'(Y^{\boldsymbol{a},\nu}_{t,s})$ are also bounded on $[s,T]$. Assumptions 2.5(b) and 2.5(c) give the existence of C_0 such that

$$\forall x \geq 0, \quad f'(x)|b(x)| \leq C_0(1 + f^2(x)),$$

and so $t \mapsto \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s})|b(Y^{\boldsymbol{a},\nu}_{t,s})|$ is also bounded on $[s,T]$. We now apply the Itô formula (see for instance Theorem 32 of [Pro05, Chap. II]) to $Y^{\boldsymbol{a},\nu}_{t,s}$. It gives for any $\epsilon > 0$

$$f(Y^{\boldsymbol{a},\nu}_{t+\epsilon,s}) = f(Y^{\boldsymbol{a},\nu}_{t,s}) + \int_t^{t+\epsilon} f'(Y^{\boldsymbol{a},\nu}_{u,s})[b(Y^{\boldsymbol{a},\nu}_{u,s}) + a_u]du$$
$$- \int_t^{t+\epsilon}\int_0^\infty f(Y^{\boldsymbol{a},\nu}_{u-,s})\mathbb{1}_{\{z\leq f(Y^{\boldsymbol{a},\nu}_{u-,s})\}}\mathbf{N}(du,dz).$$

Taking the expectation, it follows that

$$\mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t+\epsilon,s}) - \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}) = \int_t^{t+\epsilon} \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{u,s})[b(Y^{\boldsymbol{a},\nu}_{u,s}) + a_u]du - \int_t^{t+\epsilon} \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{u,s})du,$$

from which we deduce that $t \mapsto \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s})$ is locally Lipschitz and consequently continuous. □

2.4.2 Study of the marginals of the solution of (1.9): first derivation

Let $s \geq 0$ and $\boldsymbol{a} \in L^\infty_{\text{loc}}([s,\infty);\mathbb{R}_+)$ be fixed. Consider $(Y^{\boldsymbol{a},\nu}_{t,s})_{t\geq s}$ the path-wise unique solution of equation (1.9) driven by the current $\boldsymbol{a}$. Following [FL16], we define:

$$\tau_{s,t} := \sup\{u \in [s,t] : Y^{\boldsymbol{a},\nu}_{u,s} \neq Y^{\boldsymbol{a},\nu}_{u-,s}\},$$

the time of the last jump before t, with the convention that $\tau_{s,t} = s$ if there is no jump during $[s,t]$. It follows directly from (1.9) that:

$$\forall t \geq s,\ a.s.\quad Y^{\boldsymbol{a},\nu}_{t,s} = \varphi^{\boldsymbol{a}}_{t,s}(Y^{\boldsymbol{a},\nu}_{s,s})\mathbb{1}_{\{\tau_{s,t}=s\}} + \varphi^{\boldsymbol{a}}_{t,\tau_{s,t}}(0)\mathbb{1}_{\{\tau_{s,t}>s\}}.$$

We also define:

$$\forall t \geq s,\quad J_t := \int_s^t\int_0^\infty \mathbb{1}_{\{z\leq f(Y^{\boldsymbol{a},\nu}_{u-,s})\}}\mathbf{N}(du,dz),$$

the number of jumps between s and t. Because s is fixed in this analysis, we write J_t (and not $J_{t,s}$) to simplify the notations.

Lemma 2.15. *For all $t \geq u \geq s$, we have*

$$\mathbb{P}(J_t = J_u|\mathcal{F}_u) = H^{Y^{\boldsymbol{a},\nu}_{u,s}}_{\boldsymbol{a}}(t,u)\ a.s.$$

where $H^x_{\boldsymbol{a}}$ is given by (1.13) (with $\nu = \delta_x$).

Proof. We have $\{J_t = J_u\} = \{\int_u^t\int_0^\infty \mathbb{1}_{\{z\leq f(Y^{\boldsymbol{a},\nu}_{\theta-,s})\}}\mathbf{N}(d\theta,dz) = 0\}$. Moreover, $\mathcal{F}_u$ and $\sigma\{\mathbf{N}([u,\theta]\times A) : \theta \in [u,t], A \in \mathcal{B}(\mathbb{R}_+)\}$ are independent. It follows from the Markov property satisfied by $(Y^{\boldsymbol{a},\nu}_{t,s})_{t\geq s}$ that

$$a.s.\quad \mathbb{P}(J_t = J_u|\mathcal{F}_u) = g(Y^{\boldsymbol{a},\nu}_{u,s})$$

where: $g(x) := \mathbb{P}(\int_u^t\int_0^\infty \mathbb{1}_{\{z\leq f(\varphi^{\boldsymbol{a}}_{\theta,u}(x))\}}\mathbf{N}(d\theta,dz) = 0) = H^x_{\boldsymbol{a}}(t,u)$. □

Lemma 2.16 (See also [FL16], Proposition 25)**.** *Grant Assumptions 2.3, 2.5 and 2.7. Let $s \geq 0$ and $\boldsymbol{a} \in L^\infty_{loc}([s,\infty);\mathbb{R}_+)$. For all $t > s$, the law of $\tau_{s,t}$ is given by:*

$$\mathcal{L}(\tau_{s,t})(du) = H^\nu_{\boldsymbol{a}}(t,s)\delta_s(du) + r^\nu_{\boldsymbol{a}}(u,s)H_{\boldsymbol{a}}(t,u)\mathbb{1}_{\{s<u<t\}}du.$$

Proof. First, from Lemma 2.15, it follows that:

$$\mathbb{P}(\tau_{s,t} = s) = \mathbb{P}(J_t = 0) = \mathbb{E}(H_{\boldsymbol{a}}^{Y^{\boldsymbol{a},\nu}_{s,s}}(t,s)) = H^\nu_{\boldsymbol{a}}(t,s).$$

Let now $u \in (s,t]$ and $h > 0$ be such that: $s < u-h < u \leq t$. We have

$$\begin{aligned}\mathbb{P}(\tau_{s,t} \in (u-h,u]) &= \mathbb{P}(J_u > J_{u-h}, J_t = J_u)\\ &= \mathbb{E}(\mathbb{1}_{\{J_u > J_{u-h}\}}\mathbb{P}(J_t = J_u|\mathcal{F}_u))\\ &= \mathbb{E}(\mathbb{1}_{\{J_u > J_{u-h}\}}H_{\boldsymbol{a}}^{Y^{\boldsymbol{a},\nu}_{u,s}}(t,u)).\end{aligned}$$

Let $A := \operatorname{ess\,sup}_{u\in[s,t]} a_u$. On the event $\{J_u > J_{u-h}\}$, the process jumps at least once on $(u-h,u]$ and so, by Lemma 2.9(b), we have $Y^{\boldsymbol{a},\nu}_{u,s} \in [0, \varphi^{\boldsymbol{a}}_{u,u-h}(0)] \subset [0, \varphi^A_h(0)]$. It follows that

$$|\mathbb{P}(\tau_{s,t} \in (u-h,u]) - \mathbb{E}(\mathbb{1}_{\{J_u > J_{u-h}\}}H_{\boldsymbol{a}}(t,u))| \leq \sup_{x\in[0,\varphi^A_h(0)]} |H^x_{\boldsymbol{a}}(t,u) - H_{\boldsymbol{a}}(t,u)|\,\mathbb{P}(J_u > J_{u-h}).$$

From Lemma 2.18 below, we have

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(J_u > J_{u-h}) = r^\nu_{\boldsymbol{a}}(u,s).$$

Using Lemma 2.9(c), $x \mapsto H^x_{\boldsymbol{a}}(t,u)$ is continuous at $x = 0$. The continuity of $h \mapsto \varphi^A_h(0)$ at $h = 0$ yields

$$\lim_{h\downarrow 0}\frac{1}{h}|\mathbb{P}(\tau_{s,t} \in (u-h,u]) - \mathbb{E}(\mathbb{1}_{\{J_u > J_{u-h}\}}H_{\boldsymbol{a}}(t,u))| = 0.$$

Combining the two results, we obtain the stated formula:

$$\lim_{h\downarrow 0}\frac{1}{h}\mathbb{P}(\tau_{s,t} \in (u-h,u]) = r^\nu_{\boldsymbol{a}}(u,s)H_{\boldsymbol{a}}(t,u).$$

This proves the result. □

Lemma 2.17. *Let b, f satisfying Assumptions 2.3 and 2.5. Let $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\nu(f) < \infty$. Fix $s \geq 0$, and consider $\boldsymbol{a} \in L^\infty_{loc}([s,\infty);\mathbb{R}_+)$. We have*

$$\lim_{\delta\to 0}\int_{\mathbb{R}_+}\sup_{0\leq\theta\leq\delta}\left[f(\varphi^{\boldsymbol{a}}_{\theta+s,s}(x)) - f(x)\right]\nu(dx) = 0.$$

Proof. It suffices to apply the Dominated Convergence Theorem, using in particular Assumption 2.5(a). □

Lemma 2.18 (See also [FL16], Lemma 23)**.** *For all $u \in (s,t]$ we have:*

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{P}(J_u > J_{u-h}) = r^\nu_{\boldsymbol{a}}(u,s).$$

Proof. Again let $A := \operatorname{ess\,sup}_{u \in [s,t]} a_u < \infty$. We have:

$$\begin{aligned}
&|hr^\nu_{\boldsymbol{a}}(u,s) - \mathbb{P}(J_u > J_{u-h})| \\
&\quad \leq |hr^\nu_{\boldsymbol{a}}(u,s) - hr^\nu_{\boldsymbol{a}}(u-h,s)| + \left| hr^\nu_{\boldsymbol{a}}(u-h,s) - \mathbb{E}\int_{u-h}^u f(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s}))d\theta \right| \\
&\qquad + \left| \mathbb{E}\int_{u-h}^u f(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s}))d\theta - \left[1 - \mathbb{E}\exp\left(-\int_{u-h}^u f(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s}))d\theta\right)\right]\right| \\
&\quad =: \Delta^1_h + \Delta^2_h + \Delta^3_h.
\end{aligned}$$

From the continuity of $u \mapsto r^\nu_{\boldsymbol{a}}(u,s)$ (Lemma 2.14) it follows that $\lim_{h\downarrow 0} \frac{\Delta^1_h}{h} = 0$. Moreover,

$$\Delta^2_h = \left| \int_{u-h}^u \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{u-h,s})d\theta - \mathbb{E}\int_{u-h}^u f(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s}))d\theta \right|.$$

Using Lemma 2.17 (with $\nu = \mathcal{L}(Y^{\boldsymbol{a},\nu}_{u-h,s})$ and $s = u-h$), we deduce that $\Delta^2_h = \mathcal{O}(h)$ as $h \to 0$. Finally, using that $\forall x \geq 0,\ |x - (1-e^{-x})| \leq x^2$ we have

$$\Delta^3_h \leq \mathbb{E}\left(\int_{u-h}^u f(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s}))d\theta\right)^2.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\Delta^3_h \leq h\,\mathbb{E}\int_{u-h}^u f^2(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s}))d\theta.$$

Because b is Lipschitz, we can find a constant C^A_t such that

$$\forall x \geq 0, \quad \varphi^{\boldsymbol{a}}_{\theta,u-h}(x) \leq C^A_t(1+x).$$

So, using Assumption 2.5(a), we can find another constant C^A_T such that

$$f^2(\varphi^{\boldsymbol{a}}_{\theta,u-h}(Y^{\boldsymbol{a},\nu}_{u-h,s})) \leq C^A_T(1 + f^2(Y^{\boldsymbol{a},\nu}_{u-h,s})).$$

Because $t \mapsto \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{t,s})$ is locally bounded (see Lemma 2.14), there exists a constant C_t with such that

$$\Delta^3_h \leq C_t h^2.$$

This shows that $\lim_{h\downarrow 0} \frac{\Delta^3_h}{h} = 0$. Combining the three results ends the proof. □

Proposition 2.19 (See also [FL16], Theorem 12)**.** *Grant Assumptions 2.3, 2.5 and 2.7 . Let $s \geq 0$ and $\boldsymbol{a} \in L^\infty_{loc}([s,\infty);\mathbb{R}_+)$. Let $Y^{\boldsymbol{a},\nu}_{t,s}$ be the solution of equation (1.9), starting from*

$\mathcal{L}(Y^{\boldsymbol{a},\nu}_{s,s}) = \nu$. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable function such that $\mathbb{E}\,|\phi(Y^{\boldsymbol{a},\nu}_{t,s})| < \infty$. It holds that

$$\mathbb{E}\,\phi(Y^{\boldsymbol{a},\nu}_{t,s}) = \int_s^t \phi(\varphi^{\boldsymbol{a}}_{t,u}(0))H_{\boldsymbol{a}}(t,u)r^{\nu}_{\boldsymbol{a}}(u,s)du + \int_0^\infty \phi(\varphi^{\boldsymbol{a}}_{t,s}(x))H^x_{\boldsymbol{a}}(t,s)\nu(dx). \tag{2.9}$$

In particular, $r^{\nu}_{\boldsymbol{a}}(t,s) = \mathbb{E}\,f(Y^{\boldsymbol{a},\nu}_{t,s})$ solves the Volterra equation (2.7)

$$r^{\nu}_{\boldsymbol{a}} = K^{\nu}_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^{\nu}_{\boldsymbol{a}}.$$

Proof. We have, for all $t \geq s$

$$\begin{aligned}\mathbb{E}\,\phi(Y^{\boldsymbol{a},\nu}_{t,s}) &= \mathbb{E}\,\phi(Y^{\boldsymbol{a},\nu}_{t,s})\mathbb{1}_{\{\tau_{s,t}=s\}} + \mathbb{E}\,\phi(Y^{\boldsymbol{a},\nu}_{t,s})\mathbb{1}_{\{\tau_{s,t}>s\}}\\ &= \mathbb{E}\,\phi(\varphi^{\boldsymbol{a}}_{t,s}(Y^{\boldsymbol{a},\nu}_{s,s}))\mathbb{1}_{\{\tau_{s,t}=s\}} + \mathbb{E}\,\phi(\varphi^{\boldsymbol{a}}_{t,\tau_{s,t}}(0))\mathbb{1}_{\{\tau_{s,t}>s\}}\\ &=: A^1_t + A^2_t.\end{aligned}$$

Using Lemma 2.15, it follows that

$$\begin{aligned}A^1_t &= \mathbb{E}[\phi(\varphi^{\boldsymbol{a}}_{t,s}(Y^{\boldsymbol{a},\nu}_{s,s}))\,\mathbb{P}(J_t = J_s|\mathcal{F}_s)] = \mathbb{E}[\phi(\varphi^{\boldsymbol{a}}_{t,s}(Y^{\boldsymbol{a},\nu}_{s,s}))H^{Y^{\boldsymbol{a},\nu}_{s,s}}_{\boldsymbol{a}}(t,s)]\\ &= \int_0^\infty \phi(\varphi^{\boldsymbol{a}}_{t,s}(x))H^x_{\boldsymbol{a}}(t,s)\nu(dx).\end{aligned}$$

Moreover, using Lemma 2.16, we have $A^2_t = \int_s^t \phi(\varphi^{\boldsymbol{a}}_{t,u}(0))r^{\nu}_{\boldsymbol{a}}(u,s)H_{\boldsymbol{a}}(t,u)du$. Taking $\phi = f$ we obtain the Volterra equation (2.7). □

Note that using Lemma 2.16, $\int_s^t \mathcal{L}(\tau_{s,t})(du) = 1$ gives

$$H^{\nu}_{\boldsymbol{a}} + H_{\boldsymbol{a}} * r^{\nu}_{\boldsymbol{a}} = 1. \tag{2.10}$$

We prove in the next Lemma that (2.7) admits a unique solution. Define:

$$\Delta := \{(t,s) \in \mathbb{R}^2,\ t \geq s\}. \tag{2.11}$$

Lemma 2.20. *Consider $k, h \in \mathcal{C}(\Delta;\mathbb{R})$. Let $s \in \mathbb{R}$, the Volterra integral equation*

$$\forall t \geq s, \quad x(t,s) = h(t,s) + \int_s^t k(t,u)x(u,s)du,$$

has a unique continuous solution $t \mapsto x(t,s)$.

Proof. Fix $T > s$. It is enough to prove the existence and uniqueness result on $[s,T]$. We consider the Banach space $(\mathcal{C}([s,T];\mathbb{R}), ||\cdot||_{\infty,T})$ and define on this space the following operator: $\Gamma : x \mapsto h + k * x$. Let $M^T_s = \sup_{u \leq [s,T]} |k(u)| < \infty$. The operator $\Gamma : \mathcal{C}([s,T];\mathbb{R}) \to \mathcal{C}([s,T];\mathbb{R})$ is such that for all $n \in \mathbb{N}$, the iteration Γ^n is an affine operator with linear part

$\Gamma_0^n : x \mapsto k^{*(n)} * x$. To prove that Γ^n is contracting for n large enough, it is equivalent to prove that Γ_0^n is contracting for n large enough. By induction, it is easily shown that

$$\forall x \in \mathcal{C}([s,T];\mathbb{R}), \forall n \in \mathbb{N}, \quad ||\Gamma_0^n(x)||_{\infty,t} := \sup_{u\in[s,t]} |(\Gamma_0^n(x))(u)| \leq \frac{||x||_{\infty,T}(M_s^T(t-s))^n}{n!}.$$

Consequently for all $x \in \mathcal{C}([s,T];\mathbb{R})$ and $n \in \mathbb{N}$, $||\Gamma_0^n(x)||_{\infty,T} \leq \frac{(M_s^T(T-s))^n}{n!}||x||_{\infty,T}$ and so Γ_0^n is contracting for n large enough. We deduce that the operator Γ^n is also contracting and has a unique fixed point in $\mathcal{C}([s,T];\mathbb{R})$. It is also a fixed point of Γ. This ends the proof. $\square$

Corollary 2.21. *Grant Assumptions 2.3, 2.5 and 2.7. Let $s \geq 0$ be fixed and $\boldsymbol{a} \in L^\infty_{loc}([s,\infty);\mathbb{R}_+)$. Then equation (2.7) has a unique continuous solution $t \mapsto r^\nu_{\boldsymbol{a}}(t,s)$ on $[s,\infty)$.*

Proof. By Lemma 2.20, it suffices to prove that $(t,s) \mapsto K^\nu_{\boldsymbol{a}}(t,s)$ and $(t,s) \mapsto K_{\boldsymbol{a}}(t,s)$ are continuous from Δ to $\mathbb{R}$, which is a consequence of the explicit expression (2.5) and the Dominated Convergence Theorem. $\square$

We shall need the following well-known result on Volterra equation:

Lemma 2.22. *Consider $k, w \in \mathcal{C}(\Delta;\mathbb{R})$ two kernels. The Volterra equation $x = w + k * x$ has a unique solution given by $x = w + r * w$, where $r : \Delta \to \mathbb{R}$ is the "resolvent" of k, i.e. the unique solution of*

$$r = k + k * r.$$

Proof. It is clear from the proof of Lemma 2.20 that both Volterra equations have a unique solution. Moreover, we have: $w + k * (w + r * w) = w + k * w + (r - k) * w = w + r * w$. By uniqueness, we deduce that $x = w + r * w$. $\square$

Remark 2.23. *In view of (1.14) (with $\nu = \delta_0$), $r_{\boldsymbol{a}}$ is the resolvent of $K_{\boldsymbol{a}}$. So (1.14) and Lemma 2.22 yields*

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + r_{\boldsymbol{a}} * K^\nu_{\boldsymbol{a}}. \tag{2.12}$$

2.4.3 A second derivation

We now give a second proof for (2.12). This proof is more direct and use the time of the first jump of the process: it makes explicit the renewal structure of (1.2).

Proposition 2.24. *Consider b, f and ν such that Assumption 2.3, 2.5 and 2.7 hold. Let $s \geq 0$ and $\boldsymbol{a} \in L^\infty_{loc}([s,\infty);\mathbb{R}_+)$. Consider $(Y^{\boldsymbol{a},\nu}_{t,s})$ the solution of (1.9). Then (2.7) holds*

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}},$$

where $K^\nu_{\boldsymbol{a}}, K_{\boldsymbol{a}}$ and $r^\nu_{\boldsymbol{a}}$ are defined by (1.13).

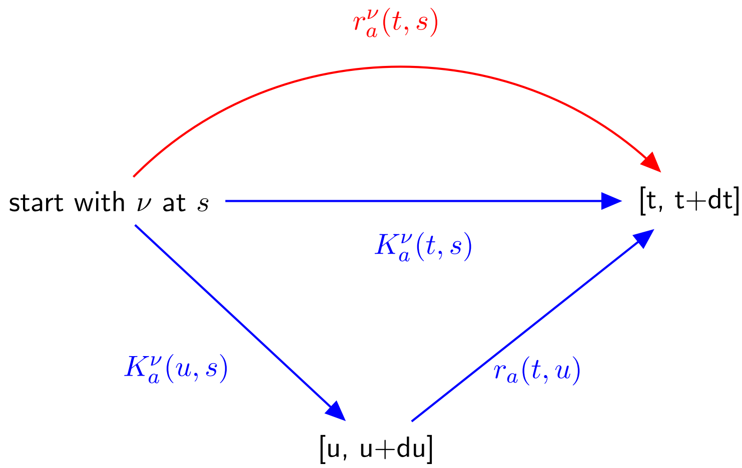


Figure 2.1: The probability that $Y^{\boldsymbol{a},\nu}_{\cdot,s}$ spikes between t and $t+dt$ is, by definition, approximately equals to $r^\nu_{\boldsymbol{a}}(t,s)dt$. Either such spike is the first (and so there are no spikes between s and t), which occurs with probability $K^\nu_{\boldsymbol{a}}(t,s)dt$. Either there is a first spike, say between u and $u+du$ for some $u\in(s,t)$. This happens with probability $K^\nu_{\boldsymbol{a}}(u,s)du$ and the process is reset to 0 at this time u, such that the required probability is $K^\nu_{\boldsymbol{a}}(u,s)r_{\boldsymbol{a}}(t,u)dudt$. Integrating over all $u\in(s,t)$, we find the Volterra integral equation (2.12). We refer to the proof of Proposition 2.24 for more details.

Proof. Let $t\geq s$. We have

$$r^\nu_{\boldsymbol{a}}(t,s)=\mathbb{E}\,f(Y^{\boldsymbol{a},\nu}_{t,s})=\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbb{P}(Y^{\boldsymbol{a},\nu}_{\cdot,s}\text{ has at least one jump between } t \text{ and } t+\delta).$$

Let $\tau^{\nu,\boldsymbol{a}}_s$ be defined by (1.12), the first spiking time of $Y^{\boldsymbol{a},\nu}_{u,s}$ after s. The law of $\tau^{\nu,\boldsymbol{a}}_s$ is $K^\nu_{\boldsymbol{a}}(u,s)du$. We have

$$r^\nu_{\boldsymbol{a}}(t,s)=\mathbb{E}\,f(Y^{\boldsymbol{a},\nu}_{t,s})=\mathbb{E}\,f(Y^{\boldsymbol{a},\nu}_{t,s})\mathbb{1}_{\{\tau^{\nu,\boldsymbol{a}}_s\geq t\}}+\mathbb{E}\,f(Y^{\boldsymbol{a},\nu}_{t,s})\mathbb{1}_{\{\tau^{\nu,\boldsymbol{a}}_s\in(s,t)\}}.$$

The first term is equal to $\lim_{\delta\downarrow 0}\frac{1}{\delta}\mathbb{P}(Y^{\boldsymbol{a},\nu}_{\cdot,s}$ has its first jump between t and $t+\delta)=K^\nu_{\boldsymbol{a}}(t,s)$. Using the strong Markov property at the stopping time $\tau^{\nu,\boldsymbol{a}}_s$ and exploiting the fact that $(Y^{\boldsymbol{a},\nu}_{\cdot,s})$ is reset to 0 at this time, we find that the second term is equal to

$$\mathbb{E}\,f(Y^{\boldsymbol{a},\nu}_{t,s})\mathbb{1}_{\{\tau^{\nu,\boldsymbol{a}}_s\in(s,t)\}}=\int_s^t r_{\boldsymbol{a}}(t,u)K^\nu_{\boldsymbol{a}}(u,s)du.$$

So, we deduce that

$$r^\nu_{\boldsymbol{a}}(t,s)=K^\nu_{\boldsymbol{a}}(t,s)+\int_s^t r_{\boldsymbol{a}}(t,u)K^\nu_{\boldsymbol{a}}(u,s)du.$$

Using the notation (2.6), this is (2.12). So by Lemma 2.22 and Remark 2.23, we obtain (2.7). It ends the proof. $\square$

2.5 A priori estimates on the jump rates

Lemma 2.25. *Grant Assumptions 2.3, 2.5 and 2.7 . Let $s \geq 0$ and $\boldsymbol{a} \in L^\infty_{loc}([s,\infty);\mathbb{R}_+)$. Let $Y^{\boldsymbol{a},\nu}_{t,s}$ be the solution of equation (1.9), starting from $\mathcal{L}(Y^{\boldsymbol{a},\nu}_{s,s}) = \nu$. Then the functions $t \mapsto \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s})$, $t \mapsto \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s})b(Y^{\boldsymbol{a},\nu}_{t,s})$ and $t \mapsto \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{t,s})$ are continuous on $[s,\infty)$.*

Proof. The proof relies on Proposition 2.19. Consider the interval $[s,T]$ for some fixed $T > s \geq 0$ and let $A := \operatorname{ess\,sup}_{t\in[s,T]} a_t$. Let $\phi \in \{f', f'b, f^2\}$. By Lemma 2.9(c), the function $(t,u) \mapsto \phi(\varphi^{\boldsymbol{a}}_{t,u}(0))H_{\boldsymbol{a}}(t,u)r^\nu_{\boldsymbol{a}}(u,s)$ is uniformly continuous on $\{(t,u) : s \leq u \leq t \leq T\}$. Consequently

$$t \mapsto \int_s^t \phi(\varphi^{\boldsymbol{a}}_{t,u}(0))H_{\boldsymbol{a}}(t,u)r^\nu_{\boldsymbol{a}}(u,s)du \text{ is continuous on } [s,T].$$

The continuity of $t \mapsto \int_0^\infty \phi(\varphi^{\boldsymbol{a}}_{t,s}(x))H^x_{\boldsymbol{a}}(t,s)\nu(dx)$ follows from the Dominated Convergence Theorem. For instance, for $\phi \equiv f'$, there exists a constant C_T such that

$$\forall t \in [s,T], \forall x \geq 0, \quad f'(\varphi^{\boldsymbol{a}}_{t,s}(x)) \overset{2.5(b)}{\leq} C_f(1+f(\varphi^{\boldsymbol{a}}_{t,s}(x))) \leq C_f(1+f(\varphi^{A}_{t-s}(x))) \leq C_T(1+f(x)),$$

from which the result follows easily using Assumption 2.7. Similar estimates hold for $\phi(x) := f'(x)b(x)$ (using Assumption 2.5(c)) and for $\phi(x) := f^2(x)$. ☐

We now give a uniform in time a priori estimate of the jump rate of (1.2).

Proposition 2.26. *Grant Assumptions 2.3, 2.5 and 2.7. Consider $(X_t)_{t\geq 0}$ a solution of the nonlinear equation (1.2) in the sense of Definition 2.2. Then $t \mapsto \mathbb{E}\, f(X_t) \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ and there is a finite constant $\bar{r} > 0$ (only depending on b, f and J) such that*

$$\sup_{t\geq 0} \mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)) \quad \text{and} \quad \limsup_{t\to\infty} \mathbb{E}\, f(X_t) \leq \bar{r}.$$

Moreover, $\bar{r}$ can be chosen to be an increasing function of J.

Proof. By applying the same arguments as in the proof of Lemma 2.14 it is clear that the functions

$$t \mapsto \mathbb{E}\, f(X_t), t \mapsto \mathbb{E}\, f'(X_t), t \mapsto \mathbb{E}\, f^2(X_t) \text{ and } t \mapsto \mathbb{E}\, |b(X_t)| f'(X_t)$$

are locally bounded. Applying the Itô formula and taking expectations yields

$$\mathbb{E}\, f(X_t) = \mathbb{E}\, f(X_0) + \int_0^t \mathbb{E}\, f'(X_u)b(X_u)du + J\int_0^t \mathbb{E}\, f'(X_u)\,\mathbb{E}\, f(X_u)du - \int_0^t \mathbb{E}\, f^2(X_u)du. \tag{2.13}$$

We deduce that $t \mapsto \mathbb{E}\, f(X_t)$ is continuous. Define for all $t \geq 0$, $a_t := J\,\mathbb{E}\, f(X_t)$. From Lemma 2.13, it is clear that:

$$a.s. \quad \forall t \geq 0, \quad X_t = Y^{\boldsymbol{a},\nu}_{t,0},$$

where $(Y^{\boldsymbol{a},\nu}_{t,0})_{t\geq 0}$ is the solution of (1.9) driven by $\boldsymbol{a}$. In particular, Lemma 2.25 applies and the functions $t \mapsto \mathbb{E}\, f'(X_t)$, $t \mapsto \mathbb{E}\, f^2(X_t)$ and $t \mapsto \mathbb{E}\, f'(X_t)b(X_t)$ are continuous. From equation (2.13), we deduce that $t \mapsto \mathbb{E}\, f(X_t) \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ and

$$\frac{d}{dt}\,\mathbb{E}\, f(X_t) = \mathbb{E}\, f'(X_t)b(X_t) + J\,\mathbb{E}\, f'(X_t)\,\mathbb{E}\, f(X_t) - \mathbb{E}\, f^2(X_t).$$

We have using the Cauchy-Schwarz inequality:

1. $$\begin{aligned} J\,\mathbb{E}\, f'(X_t)\,\mathbb{E}\, f(X_t) - \frac{1}{4}\,\mathbb{E}\, f^2(X_t) &\leq J\,\mathbb{E}\, f'(X_t)\,\mathbb{E}\, f(X_t) - \frac{1}{4}\,\mathbb{E}^2\, f(X_t) \\ &= \mathbb{E}\, f(X_t)\left[J\,\mathbb{E}\, f'(X_t) - \frac{1}{8}\,\mathbb{E}\, f(X_t) - \frac{1}{8}\,\mathbb{E}\, f(X_t)\right] \\ &\leq 2\beta^2, \end{aligned}$$
   where $\beta := \sup_{x\geq 0} Jf'(x) - \frac{1}{8}f(x) < \infty$ (by Assumption 2.5(b)). We used $\sup_{y\geq 0} y(\beta - \frac{1}{8}y) \leq 2\beta^2$ to obtain the last inequality. Note that β is a non-decreasing function of J.

2. Moreover we have, using 2.5(b) $C_0 := \sup_{x\geq 0} f'(x)b(x) - \frac{1}{4}f^2(x) \overset{2.5(c)}{\leq} \sup_{x\geq 0} C_f f'(x)(1 + f(x)) - \frac{1}{4}f^2(x) < \infty$.

Combining the points 1 and 2 and gives

$$\frac{d}{dt}\,\mathbb{E}\, f(X_t) \leq \frac{1}{2}[(2C_0 + 4\beta^2) - \mathbb{E}^2\, f(X_t)]. \tag{2.14}$$

We define: $\bar{r} := \sqrt{2C_0 + 4\beta^2}$ and deduce from the Lemmas 2.27 and 2.28 below that

$$\sup_{t\geq 0}\mathbb{E}\, f(X_t) \leq \max(\bar{r}, \mathbb{E}\, f(X_0)), \quad \text{and} \quad \limsup_{t\to\infty}\mathbb{E}\, f(X_t) \leq \bar{r}.$$

□

Lemma 2.27. *Let $x_0, M \in \mathbb{R}_+$. The following ODE:*

$$\dot{x} = \frac{1}{2}(M^2 - x^2), \quad x(0) = x_0$$

has a unique solution given by:

$$x(t) = \begin{cases} M\frac{e^{Mt}-A}{e^{Mt}+A} & \text{if } x_0 < M \\ M & \text{if } x_0 = M \\ M\frac{e^{Mt}+A}{e^{Mt}-A} & \text{if } x_0 > M \end{cases},$$

for some constant $A > 0$ determined by the initial condition x_0.

Proof. A direct computation show that the given formula is indeed a solution. Uniqueness of the solution is a consequence of the following Lemma: □

Lemma 2.28. *Let $y \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R}_+)$ satisfying*

$$\dot{y} \leq \frac{1}{2}(M^2 - y^2), \quad y(0) = x_0.$$

Then it holds for all $t \geq 0$, $y(t) \leq x(t)$, where x is the solution of the ODE of Lemma 2.27.

Proof. Let $z(t) := y(t) - x(t)$. We have $z(0) = 0$. Assume there exists $u > 0$ such that $z(u) > 0$. Define:

$$s := \sup\{t \in [0,u] : z(t) = 0\}.$$

It follows from the continuity of z that $z(s) = 0$. Because $z(u) > 0$ we have $0 \leq s < u$. Furthermore, z is non-negative on $[s,u]$, so for all $t \in [s,u]$ we have

$$\dot{z} \leq \frac{1}{2}(x^2 - y^2) = -\frac{1}{2}z \cdot (x+y) \leq 0.$$

We used here the fact that both x and y are non-negative functions. To conclude, we get the following contradiction:

$$0 = z(s) \geq z(u) > 0.$$

□

2.6 Existence and uniqueness: proof of Theorem 2.8

We first prove existence and uniqueness on $[0,T]$, for some (small) $T > 0$ to specify. We then iterate this construction on $[T,2T], [2T,3T]$ and so on.

Let b, f and ν satisfying Assumptions 2.3, 2.5 and 2.7. Let $\bar{r}$ be given by Proposition 2.26. Let $\bar{a} := J\max(\bar{r}, \nu(f))$. For $A, T > 0$, we consider the set

$$\mathcal{C}_A^T := \{\boldsymbol{a} \in \mathcal{C}([0,T];\mathbb{R}_+), \quad \sup_{t\in[0,T]} a_t \leq A\}.$$

Lemma 2.29. *Let $A > 0$ and $\boldsymbol{a} \in \mathcal{C}_A^T$. Consider $Y_{t,0}^{\boldsymbol{a},\nu}$ the solution of (1.9). There exists a constant C_0 (only depending on b, f and J) such that*

$$\forall t \geq 0, \quad Jr_{\boldsymbol{a}}^{\nu}(t,0) \leq \bar{a} + C_0 t(1 + A + A^2).$$

Proof. The function $t \mapsto r_{\boldsymbol{a}}^{\nu}(t,0) = \mathbb{E} f(Y_{t,0}^{\boldsymbol{a},\nu})$ is $\mathcal{C}^1$ and, using similar arguments to those in Proposition 2.26, there exists some constant C_0 (which may change from line to line) such that

$$\begin{aligned}
\frac{d}{dt}\mathbb{E} f(Y_{t,0}^{\boldsymbol{a},\nu}) &= \mathbb{E} f'(Y_{t,0}^{\boldsymbol{a},\nu})[b(Y_{t,0}^{\boldsymbol{a},\nu}) + a_t] - \mathbb{E} f^2(Y_{t,0}^{\boldsymbol{a},\nu}) \\
&\overset{2.5(c)}{\leq} \left[C_0\,\mathbb{E} f'(Y_{t,0}^{\boldsymbol{a},\nu})(1 + f(Y_{t,0}^{\boldsymbol{a},\nu})) - \frac{1}{2}\mathbb{E} f^2(Y_{t,0}^{\boldsymbol{a},\nu})\right] + \left[A\,\mathbb{E} f'(Y_{t,0}^{\boldsymbol{a},\nu}) - \frac{1}{2}\mathbb{E}^2 f(Y_{t,0}^{\boldsymbol{a},\nu})\right] \\
&\overset{2.5(b)}{\leq} C_0 + AC_0 + \left[A\,\mathbb{E} f(Y_{t,0}^{\boldsymbol{a},\nu}) - \frac{1}{2}\mathbb{E}^2 f(Y_{t,0}^{\boldsymbol{a},\nu})\right] \\
&\leq C_0(1 + A + A^2).
\end{aligned}$$

We used the inequality $\sup_{x\geq 0} Ax - x^2/2 \leq A^2/2$. It ends the proof. $\square$

We choose $A := \bar{a} + 1$ and
$$T := \frac{1}{C_0(1 + A + A^2)},$$
such that $\sup_{t\in[0,T]} Jr^\nu_{\boldsymbol{a}}(t,0) \leq A$. Consider the following application:
$$\begin{array}{cccc} G & : & \mathcal{C}^T_A & \to & \mathcal{C}^T_A \\ & & \boldsymbol{a} & \mapsto & Jr^\nu_{\boldsymbol{a}}(\cdot, 0). \end{array} \tag{2.15}$$
The space $\mathcal{C}^T_A$, equipped with the supremum norm $||\boldsymbol{a}||_{\infty,T} := \sup_{t\in[0,T]} |a_t|$, is complete. We now prove that the application G is contracting. Let $\boldsymbol{a}, \boldsymbol{d} \in \mathcal{C}^T_A$. Given $t, s \in [0, T]$ with $t \geq s$, we denote by $r^\nu_{\boldsymbol{a}}(t,s)$ and $r^\nu_{\boldsymbol{d}}(t,s)$ their corresponding jump rate. Both $r^\nu_{\boldsymbol{a}}$ and $r^\nu_{\boldsymbol{d}}$ satisfy the Volterra equation (2.7). It follows that the difference $\Delta_0 := r^\nu_{\boldsymbol{a}} - r^\nu_{\boldsymbol{d}}$ solves
$$\begin{aligned} \Delta_0 &= K^\nu_{\boldsymbol{a}} - K^\nu_{\boldsymbol{d}} + K_{\boldsymbol{a}} * (r^\nu_{\boldsymbol{a}} - r^\nu_{\boldsymbol{d}}) + (K_{\boldsymbol{a}} - K_{\boldsymbol{d}}) * r^\nu_{\boldsymbol{d}} \\ &= W_0 + K_{\boldsymbol{a}} * \Delta_0 \end{aligned}$$
with
$$W_0 := K^\nu_{\boldsymbol{a}} - K^\nu_{\boldsymbol{d}} + (K_{\boldsymbol{a}} - K_{\boldsymbol{d}}) * r^\nu_{\boldsymbol{d}} \tag{2.16}$$
Consequently, Δ_0 solves the following non-homogeneous Volterra equation with kernel $K_{\boldsymbol{a}}$
$$\Delta_0 = W_0 + K_{\boldsymbol{a}} * \Delta_0. \tag{2.17}$$
Using Lemma 2.22 and Remark 2.23, we obtain
$$\Delta_0 = W_0 + r_{\boldsymbol{a}} * W_0. \tag{2.18}$$

Lemma 2.30. *There exists a constant M^A_T only depending on T, f, b and A such that, for all $\boldsymbol{a}, \boldsymbol{d} \in \mathcal{C}^T_A$*
$$\forall 0 \leq s \leq t \leq T,\ \forall x \in \mathbb{R}_+, \quad |K^x_{\boldsymbol{a}} - K^x_{\boldsymbol{d}}|(t,s) \leq M^A_T(1 + f^2(x)) \int_s^t |a_u - d_u| du.$$

Proof. Fix $\boldsymbol{a}$ and $\boldsymbol{d}$ in $\mathcal{C}^T_A$. The constant M^A_T may change from line to line. We have
$$\begin{aligned} |K^x_{\boldsymbol{a}} - K^x_{\boldsymbol{d}}|(t,s) &= \left| f(\varphi^{\boldsymbol{a}}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right) - f(\varphi^{\boldsymbol{d}}_{t,s}(x)) \exp\left(-\int_s^t f(\varphi^{\boldsymbol{d}}_{u,s}(x))du\right)\right| \\ &\leq \left| f(\varphi^{\boldsymbol{a}}_{t,s}(x)) - f(\varphi^{\boldsymbol{d}}_{t,s}(x))\right| \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right) \\ &\quad + f(\varphi^{\boldsymbol{d}}_{t,s}(x)) \left| \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right) - \exp\left(-\int_s^t f(\varphi^{\boldsymbol{d}}_{u,s}(x))du\right)\right| \\ &=: M_1 + M_2. \end{aligned}$$

Assumptions 2.3 and 2.5 together with Lemma 2.9(b) give

$$\begin{aligned} M_1 &\leq |f(\varphi^{\boldsymbol{a}}_{t,s}(x)) - f(\varphi^{\boldsymbol{d}}_{t,s}(x))| \\ &\leq M^A_T(1+f(x))|\varphi^{\boldsymbol{a}}_{t,s}(x) - \varphi^{\boldsymbol{d}}_{t,s}(x)| \\ &\leq M^A_T(1+f(x)) \int_s^t |a_u - d_u| du. \end{aligned}$$

Furthermore, using that $\forall x, y \geq 0$ $|e^{-x} - e^{-y}| \leq |x-y|$, we have

$$\begin{aligned} M_2 &\leq M^A_T(1+f(x)) \int_s^t |f(\varphi^{\boldsymbol{a}}_{u,s}(x)) - f(\varphi^{\boldsymbol{d}}_{u,s}(x))| du \\ &\leq M^A_T(1+f(x))^2 \int_s^t \int_s^u |a_\theta - d_\theta| d\theta du \\ &\leq M^A_T(1+f^2(x)) \int_s^t |a_u - d_u| du. \end{aligned}$$

Combining the two estimates, we get the result. □

Proof of Theorem 2.8. We first prove existence and uniqueness up to time T. Again, we write M^A_T for any constant that may depend on T, A, b, f and J and that can change from line to line. By Assumptions 2.5 and 2.7 and Lemma 2.30 it follows that

$$\forall \boldsymbol{a}, \boldsymbol{d} \in \mathcal{C}^T_A,\ \forall t \in [0,T], \quad |K^\nu_{\boldsymbol{a}} - K^\nu_{\boldsymbol{d}}|(t,0) \leq M^A_T \int_0^t |a_u - d_u| du.$$

Moreover, since $\sup_{t\in[0,T]} r_{\boldsymbol{d}}(t,0) \leq \frac{A}{J}$ we have

$$|(K_{\boldsymbol{a}} - K_{\boldsymbol{d}}) * r_{\boldsymbol{d}}|(t,0) = \left| \int_0^t (K_{\boldsymbol{a}} - K_{\boldsymbol{d}})(t,u) r_{\boldsymbol{d}}(u,0) du \right| \leq M^A_T \int_0^t |a_u - d_u| du.$$

Let W_0 be given by (2.16). We deduce that there is a constant M^A_T such that

$$\forall \boldsymbol{a}, \boldsymbol{d} \in \mathcal{C}^T_A,\ \forall t \in [0,T], \quad |W_0|(t,0) \leq M^A_T \int_0^t |a_u - d_u| \, du.$$

Using (2.18), we deduce that

$$\begin{aligned} |\Delta_0(t,0)| \leq & |W_0|(t,0) + \int_0^t r_{\boldsymbol{a}}(t,u) |W_0|(u,0) du \\ \leq & M^A_T \int_0^t |a_u - d_u| \, du. \end{aligned}$$

We have proved that there is a constant M^A_T such that:

$$\forall \boldsymbol{a}, \boldsymbol{d} \in \mathcal{C}^T_A, \forall t \in [0,T], \quad ||Jr^\nu_{\boldsymbol{a}}(\cdot, 0) - Jr^\nu_{\boldsymbol{d}}(\cdot,0)||_{\infty,t} \leq M^A_T \int_0^t ||a - d||_{\infty,u} du.$$

Using this estimate, we deduce that G has a unique fixed point $\boldsymbol{a}^* \in \mathcal{C}_A^T$. It is then easy to check that $(Y_{t,0}^{\boldsymbol{a}^*,\nu})_{t\in[0,T]}$, driven by the current $\boldsymbol{a}^*$ and with initial condition $Y_{0,0}^{\boldsymbol{a}^*,\nu} = X_0$, defines a solution of (1.2) up to time T. This proves existence of a strong solution to (1.2) on $[0,T]$. Now, if $(X_t)_{t\geq 0}$ is a strong solution of (1.2) in the sense of Definition 2.2, let for all $t \geq 0$, $a_t := J\,\mathbb{E}\,f(X_t)$. The arguments of Lemma 2.14 show that the function $\boldsymbol{a}$ is continuous. Moreover, we have $\sup_{t\geq 0} a_t \leq \max(J\bar{r}, J\,\mathbb{E}\,f(X_0)) \leq A$ and consequently $\boldsymbol{a} \in \mathcal{C}_A^T$. It is clear that $(X_t)_{t\geq 0}$ solves (1.9) with $a_t := J\,\mathbb{E}\,f(X_t)$ and $Y_{0,0}^{\boldsymbol{a},\nu} := X_0$. We deduce that $\boldsymbol{a}$ is the unique fixed point of G: $\forall t \in [0,T]$: $a_t = a_t^*$. Consequently, by Lemma 2.13, we have $a.s.$ $\forall t \in [0,T]$ $X_t = Y_{t,0}^{\boldsymbol{a},\nu}$. This proves path-wise uniqueness on $[0,T]$.

In order to extend this construction after time T, we have to check that the law of X_T satisfies the same assumptions as ν. Indeed, using Proposition 2.26, we have $J\,\mathbb{E}\,f(X_T) \leq \bar{a}$. Thus, we can iterate the construction between T and $2T$, etc. This ends the proof of Theorem 2.8. □

2.7 On weak and strong solutions of the Fokker-Planck equation

We end this Chapter with a discussion on the solutions of the Fokker-Planck equations (1.3) and (1.4). We say that $(\nu(t,dx))_{t\geq 0}$ is a solution of (1.3) if for any $\mathcal{C}^1$ compactly supported test function $g : \mathbb{R}_+ \to \mathbb{R}$, the function $t \mapsto \int_{\mathbb{R}_+} g(x)\nu(t,dx)$ is differentiable with

$$\frac{d}{dt}\int_{\mathbb{R}_+} g(x)\nu(t,dx) = \int_{\mathbb{R}_+} g'(x)\left[b(x) + J\tilde{r}^\nu(t)\right]\nu(t,dx) - \int_{\mathbb{R}} g(x)f(x)\nu(t,dx) + g(0)\tilde{r}^\nu(t),$$

and for all $t \geq 0$, $\tilde{r}^\nu(t) = \int_{\mathbb{R}_+} f(x)\nu(t,dx)$.

Remark 2.31. *Consider $(\nu(t,dx))_{t\geq 0}$ a solution of (1.3). Let*

$$\forall t \geq 0, \quad a_t := J\tilde{r}^\nu(t) = J\int_0^\infty f(x)\nu(t,dx).$$

Then for all $t \geq 0$, $\tilde{r}^\nu(t) = r_{\boldsymbol{a}}^\nu(t,0)$ (and so $a_t = Jr_{\boldsymbol{a}}^\nu(t,0)$).

Proposition 2.32. *Let b, f satisfying Assumptions 2.3 and 2.5. Let $\nu \in \mathcal{P}(\mathbb{R}_+)$ satisfying Assumption 2.7. Consider (X_t) the solution of (1.2) starting from ν. Let*

$$\tilde{r}^\nu(t) := \mathbb{E}\,f(X_t) \quad \text{and} \quad a_t := J\,\mathbb{E}\,f(X_t).$$

Then $\nu(t,dx) := \mathcal{L}(X_t)$ is a solution of (1.3). Assume moreover that b is $\mathcal{C}^1$ and that the initial condition $\nu(0,\cdot)$ has a density. Then for any $t \geq 0$, the law of X_t has a density, denoted $\nu(t,x)$. It satisfies:

1. *For all $t > 0$, the function $x \mapsto \nu(t,x)$ is continuous at $x = 0$ and the boundary condition (1.5) holds:*

$$\nu(t,0) = \frac{\tilde{r}^\nu(t)}{b(0) + J\tilde{r}^\nu(t)}.$$

2. *Assume that $x \mapsto \nu(0,x)$ is continuous. Then for all $t > 0$, the following limit exists and*

$$\lim_{\epsilon \downarrow 0} \nu(t, \varphi^{\boldsymbol{a}}_{t,0}(0) + \epsilon) - \nu(t, \varphi^{\boldsymbol{a}}_{t,0}(0) - \epsilon)$$
$$= \exp\left(- \int_0^t (f + b')(\varphi^{\boldsymbol{a}}_{u,0}(0)) du \right) \left[\nu(0,0) - \frac{\tilde{r}^\nu(0)}{b(0) + J\tilde{r}^\nu(0)} \right].$$

If the density of the initial condition satisfies the boundary condition (1.5), then $\nu(t, \cdot)$ is continuous on $\mathbb{R}_+$. Otherwise, $\nu(t,\cdot)$ has a discontinuity at $x = \varphi^{\boldsymbol{a}}_{t,0}(0)$.

3. *Assume that $x \mapsto \nu(0,x)$ is $\mathcal{C}^1(\mathbb{R}_+)$. Then the function $(t,x) \mapsto \nu(t,x)$ is $\mathcal{C}^1$ on the open set*

$$\{(t,x) \in (\mathbb{R}^*_+)^2,\ \varphi^{\boldsymbol{a}}_{t,0}(0) \neq x\}.$$

Furthermore (1.4) holds on this domain.

Proof. Using Itô's formula, we obtain immediately that $\mathcal{L}(X_t)$ solves (1.3):

$$\frac{d}{dt} \mathbb{E}\, g(X_t) = \mathbb{E}\, g'(X_t)\, (b(X_t) + J\tilde{r}^\nu(t)) + \mathbb{E}\, (g(0) - g(X_t))\, f(X_t).$$

Assume now that $\nu = \mathcal{L}(X_0)$ has a density. It holds that $X_t = Y^{\boldsymbol{a},\nu}_{t,0}$, and so by Proposition 2.19, one has

$$\begin{aligned}\mathbb{E}\, g(Y^{\boldsymbol{a},\nu}_{t,0}) &= \int_0^t g(\varphi^{\boldsymbol{a}}_{t,u}(0)) H_{\boldsymbol{a}}(t,u) \tilde{r}^\nu(u) du + \int_0^\infty g(\varphi^{\boldsymbol{a}}_{t,0}(x)) H^x_{\boldsymbol{a}}(t,0) \nu(x) dx \\ &= A^1_t + A^2_t.\end{aligned}$$

Let $t > 0$ be fixed. The function $u \mapsto \varphi^{\boldsymbol{a}}_{t,u}(0)$ is decreasing on $(-\infty, t]$ and:

$$\frac{d}{du} \varphi^{\boldsymbol{a}}_{t,u}(0) = -\left[b(0) + a_u \right] \exp\left(\int_u^t b'(\varphi^{\boldsymbol{a}}_{\theta,u}(0)) d\theta \right).$$

We used that $\boldsymbol{a}$ is $\mathcal{C}^1$ and so $u \mapsto \varphi^{\boldsymbol{a}}_{t,u}(0)$ is also $\mathcal{C}^1$. Let

$$\sigma_{\boldsymbol{a}}(t) := \lim_{u \to -\infty} \varphi^{\boldsymbol{a}}_{t,u}(0), \tag{2.19}$$

such that $u \mapsto \varphi^{\boldsymbol{a}}_{t,u}(0)$ is a bijection from $(-\infty, t]$ to $[0, \sigma_{\boldsymbol{a}}(t))$. We denote by $x \mapsto \beta^{\boldsymbol{a}}_t(x)$ its inverse. The change of variable $x = \varphi^{\boldsymbol{a}}_{t,u}(0)$ yields:

$$A^1_t = \int_0^{\varphi^{\boldsymbol{a}}_{t,0}(0)} g(x) H_{\boldsymbol{a}}(t, \beta^{\boldsymbol{a}}_t(x)) \frac{\tilde{r}^\nu(\beta^{\boldsymbol{a}}_t(x))}{b(0) + a_{\beta^{\boldsymbol{a}}_t(x)}} \exp\left(\int_{\beta^{\boldsymbol{a}}_t(x)}^t b'(\varphi^{\boldsymbol{a}}_{\theta, \beta^{\boldsymbol{a}}_t(x)}(0)) d\theta \right) dx.$$

Moreover, the function $x \mapsto \varphi^{\boldsymbol{a}}_{t,0}(x)$ is non-decreasing and

$$\frac{d}{dx} \varphi^{\boldsymbol{a}}_{t,0}(x) = \exp\left(\int_0^t b'(\varphi^{\boldsymbol{a}}_{u,0}(x)) du \right).$$

So $x \mapsto \varphi^{\boldsymbol{a}}_{t,0}(x)$ is a bijection from $\mathbb{R}_+$ to $[\varphi^{\boldsymbol{a}}_{t,0}(0), +\infty)$. We denote by $y \mapsto \gamma^{\boldsymbol{a}}_t(y)$ its inverse. Then the change of variable $y = \varphi^{\boldsymbol{a}}_{t,0}(x)$ yields

$$A^2_t = \int_{\varphi^{\boldsymbol{a}}_{t,0}(0)}^{\infty} g(y) H_{\boldsymbol{a}}^{\gamma^{\boldsymbol{a}}_t(y)}(t,0)\nu(\gamma^{\boldsymbol{a}}_t(y)) \exp\left(\int_0^t b'(\varphi^{\boldsymbol{a}}_{u,0}(\gamma^{\boldsymbol{a}}_t(y)))du\right)dy.$$

Altogether, we deduce that $\mathcal{L}(X_t)$ has a density $\nu(t,x)$ and

$$\begin{aligned}\nu(t,x) =& H_{\boldsymbol{a}}(t, \beta^{\boldsymbol{a}}_t(x)) \frac{\tilde{r}^\nu(\beta^{\boldsymbol{a}}_t(x))}{b(0) + a_{\beta^{\boldsymbol{a}}_t(x)}} \exp\left(\int_{\beta^{\boldsymbol{a}}_t(x)}^t b'(\varphi^{\boldsymbol{a}}_{\theta,\beta^{\boldsymbol{a}}_t(x)}(0))d\theta\right)\mathbb{1}_{[0,\varphi^{\boldsymbol{a}}_{t,0}(0))}(x) \\ &+ H_{\boldsymbol{a}}^{\gamma^{\boldsymbol{a}}_t(x)}(t,0)\nu(\gamma^{\boldsymbol{a}}_t(x)) \exp\left(\int_0^t b'(\varphi^{\boldsymbol{a}}_{u,0}(\gamma^{\boldsymbol{a}}_t(x)))du\right)\mathbb{1}_{[\varphi^{\boldsymbol{a}}_{t,0}(0),\infty)}(x).\end{aligned}$$

In particular, for $x = 0$, we have $\beta^{\boldsymbol{a}}_t(0) = t$ and so for $t > 0$

$$\nu(t,0) = \frac{\tilde{r}^\nu(t)}{b(0) + J\tilde{r}^\nu(t)}.$$

The end of the proof follows easily from the explicit expression of $\nu(t,x)$. In particular, Point 2 follows from the fact that when $x = \varphi^{\boldsymbol{a}}_{t,0}(0)$, we have $\beta^{\boldsymbol{a}}_t(x) = \gamma^{\boldsymbol{a}}_t(x) = 0$. □

2.8 Discussions and perspectives

We gave sufficient conditions ensuring existence and path-wise uniqueness of the solution of the nonlinear SDE (1.2). To do so, we derived a closed equation for the jump rate: the Volterra equation (1.14). Finally, in Section 2.7, we discussed the regularity of the marginals of the solution. In our setting, the function f grows at most at a polynomial rate (see (2.2)) and b is globally Lipschitz. It would be interesting to see if the conclusion of Theorem 2.8 still holds if f grows to infinity faster than such polynomial rate. In particular, does the jump rate stay uniformly bounded in time for such f? In the limit scenario where f converges to infinity at a finite point $x_0 > 0$, the same blow-up phenomena than in [CCP11b] may exist (consider for instance b and f given by (1.8)). Assume now b is not globally Lipschitz and that the flow converges to infinity in a finite time (consider for instance $b(x) = x^2$). How f should behave at infinity to compensate the explosion of the deterministic flow? The question of the right "balance" between the drift b and the rate function f, such that (1.2) has a unique well-behave solution, is left open. We derived an integral equation for the jump rate of the limit equation. It would be particularly interesting to study the fluctuations of the solution of (1.1) around its mean-field limit. Recently, in [ELL19], [HS19], [FST19] and [Che17a] different methods have been proposed to study similar questions. Many extensions of this work are conceivable. One can generalize the model to multi-populations, such as a population of excitatory neurons and a population of inhibitory neurons. That is, consider the following McKean-Vlasov SDE

$$\begin{aligned}
X_t^e &= X_0^e + \int_0^t b_e(X_u^e)du + J_{ee}\int_0^t \mathbb{E}\, f_e(X_u^e)du + J_{ie}\int_0^t \mathbb{E}\, f_i(X_u^i)du \\
&\quad - \int_0^t \int_{\mathbb{R}_+} X_{u-}^e \mathbb{1}_{\{z \leq f_e(X_{u-}^e)\}} \mathbf{N}_e(du, dz). \qquad (2.20) \\
X_t^i &= X_0^i + \int_0^t b_i(X_u^i)du + J_{ei}\int_0^t \mathbb{E}\, f_e(X_u^e)du + J_{ii}\int_0^t \mathbb{E}\, f_i(X_u^i)du \\
&\quad - \int_0^t \int_{\mathbb{R}_+} X_{u-}^i \mathbb{1}_{\{z \leq f_i(X_{u-}^i)\}} \mathbf{N}_i(du, dz).
\end{aligned}$$

In this equation, X_t^e (respectively X_t^i) models the membrane potential of a typical excitatory (respectively inhibitory) neuron. They jump to zero with rates $f_e(X_t^e)$ and $f_i(X_t^i)$. The constants J_{ee} and J_{ei} are non-negative (they model excitatory synapses) while the constants J_{ii} and J_{ie} are non-positive (to account for the inhibitory synapses). The initial conditions X_0^e and X_0^i and the Poisson measures $\mathbf{N}^e$ and $\mathbf{N}^i$ are independent. The dynamics lives now on $\mathbb{R}$ (not only $\mathbb{R}_+$ as before). A further generalization is to consider spatially structured populations [DOR15] : in this asymptotic scenario, the number of populations goes to infinity. The behavior of such system, including the existence of traveling waves, is completely open. We finally mention a last extension where a diffusion is added to the dynamics:

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du + \sigma B_t - \int_0^t \int_{\mathbb{R}_+} X_{u-} \mathbb{1}_{\{z \leq f(X_{u-})\}} \mathbf{N}(du, dz).$$

Here $\sigma > 0$ and (B_t) is a standard Brownian motion independent of the Poisson measure $\mathbf{N}$ and of the initial condition X_0. Assume f is a smooth approximation of $+\infty \mathbb{1}_{[1,\infty)}(x)$. Then this last equation can be seen as a regularized version of (1.7). It would be interesting to study the distance between this smooth regularization version and the solution of (1.7).

Chapter 3

Long time behavior with weak interactions

We study the long time behavior of the solution of the mean-field equation (1.2) in the setting of weak enough interactions. We prove that for a J small enough, the solution converges to the unique (in this case) invariant probability measure. To this aim, we first replace temporary the interaction part of the equation by a deterministic external quantity (called the external current). For constant current, we obtain the convergence to the invariant probability measure. Using a perturbation method, we extend this result to more general external currents. Finally, we prove the result for the nonlinear McKean-Vlasov equation.
This Chapter is based on the second part of the article [CTV20a].

3.1 Introduction

Consider (X_t) the solution of the nonlinear SDE (1.2)

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t \int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{z\leq f(X_{u-})\}}\mathbf{N}(du, dz),$$

where $\mathcal{L}(X_0) := \nu$ and where $\mathbf{N}$ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity measure $dudz$, independent of X_0.

Under Assumptions 2.3, 2.5 and 2.7, this SDE has a path-wise unique solution and the associated jump rate $t \mapsto \mathbb{E}\, f(X_t)$ is uniformly bounded in time (see Theorem 2.8).

We study here the long time behavior of (X_t), under the assumption that J is sufficiently small (*weakly connecting regime*).

A possible approach is to study the nonlinear Fokker-Planck equation (1.4). Such nonlinear transport equations with boundary conditions have been studied in the context of population dynamics (see for instance [GM74; Prü83; Web85; Per07]). The PDE (1.4) differs from theirs in the sense that we have a nonlinear transport term (theirs is constant and equal to one) and our boundary condition is more complex.

The long time behavior of the PDE (1.4) has been studied in [FL16] and in [DV21] in the case where $b \equiv 0$. In this situation, one can simplify the PDE (1.4) with a simpler boundary condition

$$\nu(t,0) = \frac{1}{J}.$$

The authors proved that if the density of the initial condition satisfies this boundary condition and regularity assumptions, then $\nu(t,\cdot)$ converges to the density of the invariant probability measure as t goes to infinity. The convergence holds in L^1 or in stronger norms (see [DV21]). For $b \not\equiv 0$, the boundary condition is more delicate and their methods cannot be easily applied.

Actually the long time behavior of the solution to (1.2) may be remarkably intricate. Depending on the choice of f, b and J, equation (1.2) may have multiple invariant probability measures. Even if the invariant probability measure is unique, it is not necessarily a stable one and oscillations may appear (see Chapter 6 for explicit examples). In [DV17], the authors have numerically illustrated this phenomenon in a setting close to ours.

Our main result describes the long time behavior of the solution to (1.2) in the weakly connected regime (Theorem 3.7). If the connection strength J is small enough, we prove that (1.2) has a unique invariant probability measure which is globally stable. We give the explicit expression of this non-trivial invariant distribution and starting from any initial condition X_0, we prove the convergence in law of X_t to it, exponentially fast, as t goes to infinity. We argue that this result is very general: it does not depend on the explicit shape of the functions f or b. For stronger connection strengths J, such a result cannot hold true in general as (1.2) may have multiple invariant probability measures.

Note that we prove convergence in law, which is weaker than convergence in L^1 norm. On the other hand, we require very few on the initial condition, in particular, we do not assume the existence of a density for the initial condition in Theorem 3.7.

The proof of Theorem 3.7 is organized as follows. Given $\nu \in \mathcal{P}(\mathbb{R}_+)$ and $\boldsymbol{a} \in \mathcal{C}(\mathbb{R};\mathbb{R}_+)$, consider $(Y^{\boldsymbol{a},\nu}_{t,0})$ the solution of the non-homogeneous linear equation (1.9) starting at time 0 with law ν. First, we give in Proposition 3.17 the long time behavior of $(Y^{\alpha,\nu}_{t,0})$ the solution of (1.9) with a constant current ($\boldsymbol{a} \equiv \alpha$). Any solution converges in law to a unique invariant probability measure ν^∞_α (Proposition 3.9). In that case, the Volterra equation (1.14) is of convolution type and it is possible to study finely its solution using Laplace transform techniques. Second, we prove, for small J, the uniqueness of a constant current α^* such that

$$\forall t \geq 0, \quad \alpha^* = J\mathbb{E}f(Y^{\alpha^*,\nu^\infty_{\alpha^*}}_{t,0}).$$

Third, we extend the previous convergence result to non-constant currents $\boldsymbol{a}$ satisfying

$$|a_t - \alpha^*| \leq Ce^{-\lambda t}, \tag{3.1}$$

for some $\lambda > 0$ sufficiently small. Using a perturbation method, we prove that

$$Y^{\boldsymbol{a},\nu}_{t,0} \xrightarrow[t\to\infty]{\mathcal{L}} \nu^\infty_{\alpha^*}.$$

Fourth, in Theorem 3.7, we give the long time behavior of the solution to the nonlinear equation (1.2) for small J. Here, we use a fixed point argument.

The layout of the chapter is as follows. Our main results are given in Section 3.2. In Section 3.3, we characterize the invariant probability measures of (1.2). In Section 3.5 we study the long time behavior of the solution to (1.9) with a constant current α. In Section 3.6, we introduce the perturbation method. Finally Section 3.7 is devoted to the proof of our main result (Theorem 3.7).

3.2 Assumptions and main results

Recall that in between its random jumps, the SDE (1.9) is reduced to a non-homogeneous ODE. We denote by $\varphi^{\boldsymbol{a}}_{t,s}(x)$ its flow, which solves (2.3). If $\boldsymbol{a} \equiv \alpha$, we write $\varphi^{\alpha}_{t}(x) = \varphi^{\boldsymbol{a}}_{t,0}(x)$.

Assumption 3.1. *Assume that $b : \mathbb{R}_+ \to \mathbb{R}$ is a Lipschitz function with $b(0) > 0$ and that b is bounded from above:*

$$\exists C_b \geq 0, \forall x \geq 0, \quad b(x) \leq C_b. \tag{3.2}$$

Assume moreover that there is a positive constant C_φ such that for all $\boldsymbol{a}$, $\boldsymbol{d} \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}_+)$

$$\forall x \geq 0,\ \forall t \geq s, \quad \left|\varphi^{\boldsymbol{a}}_{t,s}(x) - \varphi^{\boldsymbol{d}}_{t,s}(x)\right| \leq C_\varphi \int_s^t |a_u - d_u|\, du. \tag{3.3}$$

Assumption 3.2. *We assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{C}^1$ strictly increasing function with $f(0) = 0$ and there exists a constant C_f such that*

3.2(a) for all $x, y \geq 0,\ f(x + y) \leq C_f(1 + f(x) + f(y))$.

3.2(b) for all $\theta \geq 0,\ \sup_{x\geq 0}\{\theta f'(x) - f(x)\} < \infty$.
Define $\psi(\theta) := \sup_{x\geq 0}\{\theta f'(x) - \frac{1}{2} f^2(x)\} < \infty$. We also assume that

$$\lim_{\theta\to+\infty} \frac{\psi(\theta)}{\theta^2} = 0.$$

3.2(c) for all $x \geq 0,\ |b(x)| \leq C_f(1 + f(x))$.

Remark 3.3. *Grant Assumptions 3.1 and 3.2. Then Assumptions 2.3 and 2.5 hold. So Theorem 2.8 applies: if $\nu \in \mathcal{P}(\mathbb{R}_+)$ is such that $\nu(f^2) < \infty$, (1.2) has a path-wise unique solution $(X_t)_{t\geq 0}$ and $\sup_{t\geq 0} \mathbb{E}\, f(X_t) < \infty$. Let us detail the differences between the two sets of assumptions. The assumption $b(0) > 0$ ensures that δ_0, the Dirac measure at 0, is not an invariant probability measure of (1.2). Assumption (3.3) is a strengthen version of the conclusion of Lemma 2.9(b). The only difference between 3.2 and 2.5 is 3.2(b): it is a strengthen version of 2.5(b) which is useful to obtain some global estimates on the jump rate (see Proposition 3.40).*

Notation 3.4. *Denote for all $\alpha \geq 0$ the probability measure*

$$\nu^\infty_\alpha(dx) := \frac{\gamma(\alpha)}{b(x) + \alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y) + \alpha} dy\right) \mathbb{1}_{\{x\in[0,\sigma_\alpha]\}} dx, \tag{3.4}$$

where $\gamma(\alpha)$ is the normalization

$$\gamma(\alpha) := \left[\int_0^{\sigma_\alpha} \frac{1}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) dx\right]^{-1}. \tag{3.5}$$

The upper bound σ_α of the support of ν_α^∞ is given by

$$\sigma_\alpha := \lim_{t\to\infty} \varphi_t^\alpha(0) \in \mathbb{R}_+^* \cup \{+\infty\}. \tag{3.6}$$

Remark 3.5. *1. This definition of σ_α is coherent with (2.19), because when $\boldsymbol{a}$ is constant and equal to α, it holds that $\varphi_{t,u}^{\boldsymbol{a}}(0) = \varphi_{t-u}^\alpha(0)$.*

2. For all $\alpha \geq 0$, $\gamma(\alpha) = \nu_\alpha^\infty(f)$.

3. We prove in Proposition 3.9 that for any $\alpha \geq 0$, ν_α^∞ is the unique invariant probability measure of (1.9) with $\boldsymbol{a} \equiv \alpha$.

We say that $\nu \in \mathcal{P}(\mathbb{R}_+)$ is an invariant probability measure of (1.2) if $\nu(f^2) < \infty$ and if for all $t \geq 0$, $\mathcal{L}(X_t) = \nu$, where $(X_t)_{t\geq 0}$ is the path-wise unique solution of (1.2), starting with law ν.

Proposition 3.6. *Let b and f satisfying Assumptions 3.1 and 3.2. The probability measure ν_α^∞ is an invariant measure of (1.2) iff*

$$\frac{\alpha}{\gamma(\alpha)} = J. \tag{3.7}$$

Moreover, define $J_m := \sup\{J_0 \geq 0 :\ \forall J \in [0, J_0]$ equation (3.7) has a unique solution$\}$, then $J_m > 0$. Consequently, for all $0 \leq J < J_m$ the nonlinear process (1.2) has a unique invariant probability measure.

We now state our main result: the convergence to the unique invariant probability measure for weak enough interactions.

Theorem 3.7. *Under Assumptions 3.1 and 3.2, there exists strictly positive constants J^* and λ (both only depending on b and f) satisfying*

$$0 < J^* < J_m \quad \text{and} \quad 0 < \lambda < f(\sigma_0),$$

(σ_0 and J_m are defined in Notation 3.4 and Proposition 3.6) with the following properties. For all $J \in [0, J^*]$ and all $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\nu(f^2) < \infty$, there exists a constant D (only depending on b, f, J, λ and $\nu(f)$) such that

$$\forall t \geq 0, \quad |\mathbb{E} f(X_t) - \gamma(\alpha^*)| \leq De^{-\lambda t}.$$

Here, $(X_t)_{t\geq 0}$ is the solution of the nonlinear SDE (1.2) starting with law ν and α^* is the unique solution of (3.7). Moreover, it holds that (X_t) converges in law to $\nu_{\alpha^*}^\infty$ at an exponential rate. If $\phi : \mathbb{R}_+ \to \mathbb{R}$ is a bounded Lipschitz-continuous function, it holds that

$$\exists D' > 0, \forall t \geq 0, \quad |\mathbb{E}\, \phi(X_t) - \nu_{\alpha^*}^\infty(\phi)| \leq D' e^{-\lambda t},$$

where the constant D' only depends on b, f, J, ν, λ and ϕ through its infinite norm and its Lipschitz constant.

Note that in Theorem 3.7, the unique invariant probability measure $\nu_{\alpha^*}^\infty$ is globally stable: for weak enough interactions, starting from any initial condition, the solution converges to $\nu_{\alpha^*}^\infty$.

Examples

Given $p \geq 1$, $\mu > 0$ and $\kappa \leq 0$, define, for all $x \geq 0$

$$f(x) := x^p, \quad \text{and} \quad b(x) := \mu - \kappa x.$$

Then (b, f) satisfies the Assumptions 3.1 and 3.2. In that case, the flow is given by

$$\varphi^{\boldsymbol{a}}_{t,s}(x) = x e^{-\kappa(t-s)} + \frac{\mu}{\kappa}[1 - e^{-\kappa(t-s)}] + \int_s^t e^{-\kappa(t-u)} a_u du.$$

We have for all $x, y \in \mathbb{R}_+$, $f(x+y) \leq 2^{p-1}(f(x)+f(y))$. Moreover $\psi(\theta) = \frac{1}{2}\theta^{\frac{2p}{p+1}}(p-1)^{\frac{p-1}{p+1}}(1+p)$, so Assumption 3.2(b) holds.

Consequently, Theorem 3.7 applies. When $\kappa > 0$, the invariant probability measures are compactly supported and not necessarily unique. Consider for instance $b(x) = \mu - x$, $f(x) = x^2$. If μ is small enough, a numerical study shows that there exists $0 < \alpha_1 < \alpha_2 < \infty$ such that the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is increasing on $[0, \alpha_1)$, decreasing on $[\alpha_1, \alpha_2)$ and finally increasing on $[\alpha_2, \infty)$. Thus, if $J \in (\frac{\alpha_2}{\gamma(\alpha_2)}, \frac{\alpha_1}{\gamma(\alpha_1)})$, the nonlinear equation (1.2) admits exactly 3 non-trivial invariant probability measures. A numerical study shows that only two of the three are locally stable (bi-stability).

Another interesting example is the following. Assume $b(x) = 2 - 2x$ and $f(x) = x^{10}$. Then, a numerical study shows that the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is increasing on $\mathbb{R}_+$ and consequently for all $J \geq 0$, (1.2) admits a unique invariant probability measure. But if $J \in [0.7, 1.05]$ a further numerical analysis shows that the law of the solution of (1.2) asymptotically oscillates, betraying that the invariant probability measure is not locally stable. We refer to Chapter 6 for a detailed study (both numerically and theoretically) of such examples.

In particular, the assumption on the size of J cannot be removed in Theorem 3.7.

3.3 The invariant probability measures: proof of Proposition 3.6

We now study the invariant probability measures of the nonlinear process (1.2). We follow the strategy of [FL16]: we first study the linear process driven by a constant current α and show that it has a unique invariant probability measure. We then use this result to study the invariant probability measures of the nonlinear equation (1.2). Let $\alpha \geq 0$ and $(Y^{\alpha,\nu}_t)_t$ the solution of the following SDE:

$$Y^{\alpha,\nu}_t = Y^{\alpha,\nu}_0 + \int_0^t b(Y^{\alpha,\nu}_u)du + \alpha t - \int_0^t \int_{\mathbb{R}_+} Y^{\alpha,\nu}_{u-} \mathbb{1}_{\{z \leq f(Y^{\alpha,\nu}_{u-})\}} \mathbf{N}(du, dz). \tag{3.8}$$

Remark 3.8. *Eq. (3.8) is (1.9) with for all $t \geq 0$, $a_t = \alpha$ and $s = 0$ (see Notations 2.12). That is, we have $Y^{\alpha,\nu}_t = Y^{\alpha,\nu}_{t,0}$.*

Recall that $\sigma_\alpha = \lim_{t\to\infty} \varphi^\alpha_t(0)$. It holds that

$$\sigma_\alpha := \inf\{x \geq 0,\ b(x) + \alpha = 0\}.$$

Proposition 3.9. *Grant Assumptions 2.3 and 2.5. Let α such that $b(0) + \alpha > 0$. Then the SDE (3.8) has a unique invariant probability measure ν^∞_α given by (3.4):*

$$\nu^\infty_\alpha(dx) := \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) \mathbb{1}_{[0,\sigma_\alpha)}(x) dx,$$

where $\gamma(\alpha)$ is the normalizing factor given by (3.5). Moreover we have $\nu^\infty_\alpha(f) = \gamma(\alpha)$ and $\nu^\infty_\alpha(f^2) < \infty$.

Remark 3.10. *We assume only Assumptions 2.3 and 2.5 (and not 3.1 and 3.2). A proof of this result can be found in [FL16, Prop. 21] with $b(x) := -\kappa x$ and with slightly different assumptions on f. We give here a proof based on different arguments. Note that the general method introduced by [Cos90] to find the stationary measures of a PDMP can be applied here; we use a method introduced in this paper to prove the uniqueness part.*

Proof. We first check that ν^∞_α is indeed an invariant measure of (3.8).
Claim 1 We have $\nu^\infty_\alpha(f^2) < \infty$.
First $b(0) + \alpha > 0$ yields $\sigma_\alpha > 0$. The function $t \mapsto \varphi^\alpha_t(0)$ is a bijection from $\mathbb{R}_+$ to $[0, \sigma_\alpha)$. Consequently, the changes of variable $x = \varphi^\alpha_t(0)$ and $y = \varphi^\alpha_u(0)$ give

$$\begin{aligned}\nu^\infty_\alpha(f^2) &= \gamma(\alpha) \int_0^{\sigma_\alpha} \frac{f^2(x)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) dx \\ &= \gamma(\alpha) \int_0^\infty f^2(\varphi^\alpha_t(0)) \exp\left(-\int_0^t f(\varphi^\alpha_u(0)) du\right) dt.\end{aligned}$$

Using that f is continuous and strictly positive, we have

$$\lim_{t\to\infty} f(\varphi^\alpha_t(0)) = f(\sigma_\alpha) > 0.$$

So, for all $\lambda < f(\sigma_\alpha)$, there exists a constant C_λ such that

$$\forall t \geq 0, \quad \exp\left(-\int_0^t f(\varphi^\alpha_u(0)) du\right) \leq C_\lambda e^{-\lambda t}.$$

If $\sigma_\alpha < \infty$, then $f^2(\varphi^\alpha_t(0)) \leq f^2(\sigma_\alpha) < \infty$. If $\sigma_\alpha = \infty$, using (2.2), there exists some constants C_0, $p > 0$ such that

$$f^2(\varphi^\alpha_t(0)) \leq C_0 e^{2pLt},$$

where L denotes the Lipschitz constant of b. Overall, we deduce that $\nu^\infty_\alpha(f^2) < \infty$.
Claim 2 We have: $K^{\nu^\infty_\alpha}_\alpha(t) = \gamma(\alpha) H_\alpha(t)$.
We recall that $H_\alpha(t) = H^{\delta_0}_\alpha(t, 0)$. We have, for all $t \geq 0$:

$$K^{\nu^\infty_\alpha}_\alpha(t) = \int_0^{\sigma_\alpha} f(\varphi^\alpha_t(x)) \exp\left(-\int_0^t f(\varphi^\alpha_u(x)) du\right) \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) dx. \tag{3.9}$$

The change of variable $y = \varphi_u^\alpha(0)$ yields:

$$K_\alpha^{\nu_\alpha^\infty}(t) = \int_0^{\sigma_\alpha} f(\varphi_t^\alpha(x)) \exp\left(-\int_0^t f(\varphi_u^\alpha(x))du\right) \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^{t(x)} f(\varphi_u^\alpha(0))du\right)dx,$$

where $t(x)$ is the unique $t \geq 0$ such that $\varphi_t^\alpha(0) = x$. We now make the change of variable $x = \varphi_s^\alpha(0)$ and obtain (using the semi-group property satisfied by $\varphi_t^\alpha(0)$):

$$\begin{aligned} K_\alpha^{\nu_\alpha^\infty}(t) &= \gamma(\alpha) \int_0^\infty f(\varphi_t^\alpha(\varphi_s^\alpha(0))) \exp\left(-\int_0^t f(\varphi_u^\alpha(\varphi_s^\alpha(0)))du\right) \exp\left(-\int_0^s f(\varphi_u^\alpha(0))du\right)ds \\ &= \gamma(\alpha) \int_t^\infty f(\varphi_\theta^\alpha(0)) \exp\left(-\int_0^\theta f(\varphi_u^\alpha(0))du\right)d\theta \\ &= \gamma(\alpha)\left[H_\alpha(t) - \lim_{\theta\to\infty} \exp\left(-\int_0^\theta f(\varphi_u^\alpha(0))du\right)\right]. \end{aligned}$$

Claim 2 follows from $\lim_{\theta\to\infty} \exp\left(-\int_0^\theta f(\varphi_u^\alpha(0))du\right) = 0$ (see the proof of Claim 1).

We now consider $(Y_t^{\alpha,\nu_\alpha^\infty})_{t\geq 0}$ the solution of equation (3.8) starting from $\mathcal{L}(Y_0^{\alpha,\nu_\alpha^\infty}) = \nu_\alpha^\infty$. Proposition 2.19 applies, so $r_\alpha^{\nu_\alpha^\infty}(t) = \mathbb{E}\, f(Y_t^{\alpha,\nu_\alpha^\infty})$ is the unique solution of the Volterra equation

$$r_\alpha^{\nu_\alpha^\infty} = K_\alpha^{\nu_\alpha^\infty} + K_\alpha * r_\alpha^{\nu_\alpha^\infty}.$$

Using Claim 2 and the relation (2.8), we verify that the constant function $\gamma(\alpha)$ is also solution:

$$K_\alpha^{\nu_\alpha^\infty} + K_\alpha * \gamma(\alpha) = \gamma(\alpha)H_\alpha + \gamma(\alpha)(1 - H_\alpha) = \gamma(\alpha).$$

By uniqueness (Lemma 2.21), we deduce that $\forall t \geq 0,\ r_\alpha^{\nu_\alpha^\infty}(t) = \gamma(\alpha)$.

Finally, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. Using Proposition 2.19, we have

$$\begin{aligned} \mathbb{E}\,\phi(Y_t^{\alpha,\nu_\alpha^\infty}) &= \gamma(\alpha)\int_0^t \phi(\varphi_{t-u}^\alpha(0))H_\alpha(t-u)du + \int_0^\infty \phi(\varphi_t^\alpha(x))H_\alpha^x(t)\nu_\alpha^\infty(dx) \\ &= \gamma(\alpha)\int_0^t \phi(\varphi_u^\alpha(0))H_\alpha(u)du \\ &\quad + \int_0^{\sigma_\alpha} \phi(\varphi_t^\alpha(x)) \exp\left(-\int_0^t f(\varphi_u^\alpha(x))du\right)\frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}dy\right)dx. \end{aligned}$$

The change of variables $y = \varphi_u^\alpha(0)$ and $x = \varphi_\theta^\alpha(0)$ yields

$$\begin{aligned} \mathbb{E}\,\phi(Y_t^{\alpha,\nu_\alpha^\infty}) &= \gamma(\alpha)\int_0^t \phi(\varphi_u^\alpha(0))H_\alpha(u)du \\ &\quad + \gamma(\alpha)\int_0^\infty \phi(\varphi_t^\alpha(\varphi_\theta^\alpha(0))) \exp\left(-\int_0^t f(\varphi_u^\alpha(\varphi_\theta^\alpha(0)))du\right) \exp\left(-\int_0^\theta f(\varphi_u^\alpha(0))du\right)d\theta \\ &= \gamma(\alpha)\int_0^t \phi(\varphi_u^\alpha(0))H_\alpha(u)du + \gamma(\alpha)\int_t^\infty \phi(\varphi_u^\alpha(0)) \exp\left(-\int_0^u f(\varphi_\theta^\alpha(0))d\theta\right)du \\ &= \gamma(\alpha)\int_0^\infty \phi(\varphi_u^\alpha(0))H_\alpha(u)du \\ &= \nu_\alpha^\infty(\phi). \end{aligned}$$

This proves that $\forall t \geq 0$, $\mathcal{L}(Y_t^{\alpha,\nu_\alpha^\infty}) = \nu_\alpha^\infty$ and consequently ν_α^∞ is an invariant probability measure of (3.8). Moreover, we have

$$\nu_\alpha^\infty(f) = \gamma(\alpha) \int_0^{\sigma_\alpha} \frac{f(x)}{b(x)+\alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha} dy\right) dx = \gamma(\alpha).$$

It remains to prove that the invariant probability measure is unique. Following [Dav84] and [Cos90], we define $\mathcal{B}^{ac}(\mathbb{R}_+)$ the set of bounded function $h : \mathbb{R}_+ \to \mathbb{R}$ such that for all $x \in \mathbb{R}_+$, the function $t \mapsto h(\varphi_t^\alpha(x))$ is absolutely continuous on $\mathbb{R}_+$. For $h \in \mathcal{B}^{ac}(\mathbb{R}_+)$, we define $\mathcal{H}h(x) := \frac{d}{dt} h(\varphi_t^\alpha(x))\big|_{t=0}$.

Claim 3 Let $h \in \mathcal{B}^{ac}(\mathbb{R}_+)$, then for all $x \geq 0$ we have

$$\tfrac{d}{dt} \mathbb{E}\, h(Y_t^{\alpha,\delta_x})\Big|_{t=0} = \mathcal{L}h(x) \quad \text{with} \quad \mathcal{L}h(x) := \mathcal{H}h(x) + (h(0) - h(x))f(x).$$

Let $\tau_1^x = \inf\{t \geq 0 : \ Y_t^{\alpha,\delta_x} \neq Y_{t-}^{\alpha,\delta_x}\}$ and $\tau_2^x = \inf\{t > \tau_1^x : \ Y_t^{\alpha,\delta_x} \neq Y_{t-}^{\alpha,\delta_x}\}$ be the times of the first and second jumps of (Y_t^{α,δ_x}). We have

$$\begin{aligned} \mathbb{E}\, h(Y_t^{\alpha,\delta_x}) &= \mathbb{E}\, h(Y_t^{\alpha,\delta_x}) \mathbb{1}_{\{t<\tau_1^x\}} + \mathbb{E}\, h(Y_t^{\alpha,\delta_x}) \mathbb{1}_{\{\tau_1^x \leq t < \tau_2^x\}} + \mathbb{E}\, h(Y_t^{\alpha,\delta_x}) \mathbb{1}_{\{t \geq \tau_2^x\}} \\ &=: A_t^1 + A_t^2 + A_t^3. \end{aligned}$$

By Lemma 2.15, we have $A_t^1 = h(\varphi_t^\alpha(x))\mathbb{P}(t < \tau_1^x) = h(\varphi_t^\alpha(x))H_\alpha^x(t)$. It follows that $\frac{d}{dt} A_t^1\big|_{t=0} = \mathcal{H}h(x) - h(x)f(x)$. Moreover using that the density of τ_1^x is $s \mapsto K_\alpha^x(s)$ it holds that $A_t^2 = \int_0^t h(\varphi_{t-s}^\alpha(0)) K_\alpha^x(s) H_\alpha^0(t-s) ds$. We deduce that $\frac{d}{dt} A_t^2\big|_{t=0} = h(0)f(x)$. Then, using that h is bounded, we have $A_t^3 \leq ||h||_\infty \int_0^t \int_0^t K_\alpha^x(u) K_\alpha^0(s-u) du ds = \mathcal{O}(t^2)$. This proves Claim 3.

Let g be a bounded measurable function. We follow the method of [Cos90, proof of Th. 3(a)] and define

$$\forall x \geq 0, \quad \lambda_g(x) := \int_0^\infty g(\varphi_u^\alpha(x)) \exp\left(-\int_0^u f(\varphi_\theta^\alpha(x)) d\theta\right) du.$$

Claim 4 The function λ_g belongs to $\mathcal{B}^{ac}(\mathbb{R}_+)$ and satisfies $\mathcal{H}\lambda_g(x) = f(x)\lambda_g(x) - g(x)$.
Using the semi-group property of $\varphi_t^\alpha(x)$ we have

$$\lambda_g(\varphi_t^\alpha(x)) = \exp\left(\int_0^t f(\varphi_\theta^\alpha(x)) d\theta\right) \left[\lambda_g(x) - \int_0^t g(\varphi_u^\alpha(x)) \exp\left(-\int_0^u f(\varphi_\theta^\alpha(x)) d\theta\right) du\right].$$

This proves that λ_g is in $\mathcal{B}^{ac}(\mathbb{R}_+)$ with $\frac{d}{dt}\lambda_g(\varphi_t^\alpha(x)) = f(\varphi_t^\alpha(x))\lambda_g(\varphi_t^\alpha(x)) - g(\varphi_t^\alpha(x))$ and gives the stated formula.

Consider now ν an invariant probability measure with $\nu(f) < \infty$. The Markov property at time $t = 0$ together with Claim 3 shows that $\frac{d}{dt} \mathbb{E}\, \lambda_g(Y_t^{\alpha,\nu})\big|_{t=0} = \frac{d}{dt} \int_0^\infty \mathbb{E}\, \lambda_g(Y_t^{\alpha,\delta_x}) \nu(dx)\Big|_{t=0} = \nu(\mathcal{L}\lambda_g)$. The exchange of the derivative at time $t = 0$ and the integral on $\mathbb{R}_+$ is legitimate thanks to the Dominated Convergence Theorem. Claim 4 and the fact that ν is an invariant measure then show that

$$0 = \frac{d}{dt} \mathbb{E}\, \lambda_g(Y_t^{\alpha,\nu})\Big|_{t=0} = \lambda_g(0)\nu(f) - \nu(g).$$

The same computations can be done with $g \equiv 1$, giving $\lambda_1(0)\nu(f) = 1$. It follows that

$$\nu(g) = \frac{\lambda_g(0)}{\lambda_1(0)} = \int_0^\infty g(x)\nu_\alpha^\infty(dx).$$

We deduce that necessarily $\nu = \nu_\alpha^\infty$. □

Remark 3.11. *The formula $K_\alpha^{\nu_\alpha^\infty}(t) = \gamma(\alpha)H_\alpha(t)$ (proved in Claim 2) can be generalized to non-constant currents and has a simple probabilistic interpretation. We refer to Chapter 4, Lemma 4.50.*

The next lemma characterizes the invariant probability measures of (1.2).

Lemma 3.12. *Under Assumptions 3.1 and 3.2, the invariant probability measures of the nonlinear equation (1.2) are $\{\nu_\alpha^\infty \mid \alpha = J\gamma(\alpha), \alpha \in \mathbb{R}_+\}$.*

Proof. Let ν be an invariant probability measure of (1.2) and $\mathcal{L}(X_0) = \nu$. We have

$$\forall t \geq 0, \quad \mathbb{E}\, f(X_t) = \nu(f) =: p.$$

Let $\alpha := Jp$. The process $(X_t)_{t\geq 0}$ solves (3.8) and ν is an invariant probability measure of equation (3.8). It implies that $\nu = \nu_\alpha^\infty$. Moreover $p = \gamma(\alpha)$ and so necessarily $\frac{\alpha}{\gamma(\alpha)} = J$.

Conversely, let $\alpha \geq 0$ such that $\frac{\alpha}{\gamma(\alpha)} = J$. Let $(Y_t^{\alpha,\nu_\alpha^\infty})$ be the solution of (3.8) with $\mathcal{L}(Y_0^{\alpha,\nu_\alpha^\infty}) = \nu_\alpha^\infty$. We have seen that $\mathbb{E}\, f(Y_t^{\alpha,\nu_\alpha^\infty}) = \gamma(\alpha)$, it follows that $\alpha = J\,\mathbb{E}\, f(Y_t^{\alpha,\nu_\alpha^\infty})$. Consequently $(Y_t^{\alpha,\nu_\alpha^\infty})_{t\geq 0}$ solves (1.2) and ν_α^∞ is one of its invariant probability measure. □

The problem of finding the invariant probability measures of the mean-field equation (1.2) has been reduced to finding the solutions of the scalar equation (3.7). When J is small enough, we can prove that it has a unique solution, which concludes the proof of Proposition 3.6.

Lemma 3.13. *Under Assumptions 3.1 and 3.2, eq. (3.7) has at least one solution $\alpha^* > 0$. Moreover, there is a constant $J_0 > 0$ such that for all $J \in [0, J_0]$ (3.7) has a unique solution.*

Proof. Recall (3.5). By the changes of variable $y = \varphi_u^\alpha(0)$ and $x = \varphi_t^\alpha(0)$, it holds that

$$\gamma(\alpha)^{-1} = \int_0^\infty \exp\left(-\int_0^t f(\varphi_u^\alpha(0))du\right)dt. \tag{3.10}$$

In particular, the function $\alpha \mapsto \gamma(\alpha)$ is non-decreasing. Furthermore, using that $b(x) \leq C_b$, we have

$$\begin{aligned}\frac{\alpha}{\gamma(\alpha)} &\geq \alpha \int_0^\infty \exp\left(-\int_0^t f((\alpha + C_b)u)du\right)dt \\ &\geq \frac{\alpha}{\alpha + C_b}\int_0^\infty \exp\left(-\frac{1}{\alpha + C_b}\int_0^\theta f(z)dz\right)d\theta.\end{aligned}$$

We deduce that $\lim_{\alpha\to+\infty} \alpha\gamma(\alpha)^{-1} = +\infty$. Let $U(\alpha) := \alpha\gamma(\alpha)^{-1}$. One has $U(0) = 0$, $\lim_{\alpha\to+\infty} U(\alpha) = +\infty$ and U is continuous on $\mathbb{R}_+$. It follows that the equation $U(\alpha) = J$ has at least one solution α^*. Moreover, one can show that the function U has a (right) derivative at $\alpha = 0$ and $U'(0) = 1/\gamma(0) > 0$. Consequently, there is $\alpha_0 > 0$ such that U is strictly increasing on $[0, \alpha_0]$. Using $\lim_{\alpha\to+\infty} U(\alpha) = +\infty$, we can find α_1 such that: $\forall \alpha \geq \alpha_1, U(\alpha) \geq 1$. Finally let $J_0 := \min_{\alpha\in[\alpha_0,\alpha_1]} U(\alpha) > 0$. Let $J < J_0$, it is clear that the equation $U(\alpha) = J$ has exactly one solution on $\mathbb{R}_+$. This solution belongs to $[0, \alpha_0]$. □

Remark 3.14. *Here, the Assumption $b(0) > 0$ is crucial: consider for instance $b(x) = -x$ and $f(x) = x^2$. We will see in Chapter 6, Proposition 6.1 that*

$$\lim_{\alpha\downarrow 0} \frac{\alpha}{\gamma(\alpha)} = +\infty.$$

3.4 The convergence of the jump rate implies the convergence in law of the time marginals

In this section, we prove that a good control of the jump rate $t \mapsto \mathbb{E} f(X_t)$ is sufficient to deduce the convergence in law of the solution of (1.2) to the invariant distribution.

Proposition 3.15. *Grant Assumptions 3.1, 3.2. Let $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\nu(f^2) < \infty$. Let $(X_t)_{t\geq 0}$ be the solution of the nonlinear equation (1.2), with $\mathcal{L}(X_0) = \nu$. Assume that there exist constants $\lambda, C > 0$ and $\alpha^* \geq 0$ (that may depend on b, f, ν, and J) such that*

$$\forall t \geq 0, \quad |\mathbb{E} f(X_t) - \gamma(\alpha^*)| \leq Ce^{-\lambda t},$$

and that α^* satisfies equation (3.7): $\frac{\alpha^*}{\gamma(\alpha^*)} = J$. Then

$$X_t \xrightarrow[t\to\infty]{\mathcal{L}} \nu^\infty_{\alpha^*}.$$

Moreover, if $\phi : \mathbb{R}_+ \to \mathbb{R}$ is any bounded Lipschitz-continuous function, it holds that

$$\forall 0 < \lambda' < \min(\lambda, f(\sigma_0)),\ \exists D > 0 \quad s.t. \quad \forall t \geq 0, \quad |\mathbb{E}\,\phi(X_t) - \nu^\infty_{\alpha^*}(\phi)| \leq De^{-\lambda' t},$$

where the constant D only depends on $b, f, J, C, \nu, \lambda'$ and ϕ through its infinite norm and its Lipschitz constant.

We will often use the following observations.

Remark 3.16. *Under Assumptions 3.1 and 3.2, it holds that:*

3.16*(a) for all $\boldsymbol{a} \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$ and all ν in $\mathcal{P}(\mathbb{R}_+)$, it holds that*

$$\forall t \geq s \geq 0, \quad H^\nu_{\boldsymbol{a}}(t, s) \leq H_0(t - s).$$

3.16(b) Using (3.2), the flow grows at most linearly: for all $\alpha > 0$, it holds that

$$\forall t, x \geq 0, \quad \varphi_t^\alpha(x) \leq x + (C_b + \alpha)t.$$

3.16(c) Using that f is continuous and strictly increasing, we have

$$\forall \alpha \geq 0, \quad \lim_{t\to\infty} f(\varphi_t^\alpha(0)) = f(\sigma_\alpha) \geq f(\sigma_0) > 0.$$

Proof of Proposition 3.15. Let $(X_t)_{t\geq 0}$ be the solution of (1.2) and $\phi : \mathbb{R}_+ \to \mathbb{R}$ a bounded Lipschitz-continuous function, with Lipschitz constant l_ϕ. Consider $\lambda' \in (0, \min(\lambda, f(\sigma_0)))$. We denote by D any constant only depending on $b, f, J, C, \nu, \lambda'$, $||\phi||_\infty$ and l_ϕ which shall change from line to line. Define for all $t \geq 0$, $a_t := J\,\mathbb{E}\,f(X_t)$. It holds that $(X_t)_{t\geq 0}$ is a solution of (1.9) with driving current $\boldsymbol{a}$. Denote $r_{\boldsymbol{a}}^\nu(t,0) = \mathbb{E}\,f(X_t)$. By Proposition 2.19, we have

$$\mathbb{E}\,\phi(X_t) = \int_0^t \phi(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)r_{\boldsymbol{a}}^\nu(u,0)du + \int_0^\infty \phi(\varphi_{t,0}^{\boldsymbol{a}}(x))H_{\boldsymbol{a}}^x(t,0)\nu(dx).$$

Using Remarks 3.16(a) and 3.16(c), and $\lambda' < f(\sigma_0)$, we deduce that

$$\forall t \geq 0, \quad \int_0^\infty \phi(\varphi_{t,0}^{\boldsymbol{a}}(x))H_{\boldsymbol{a}}^x(t,0)\nu(dx) \leq De^{-\lambda' t}$$

for some constant D. Moreover, one has, using the change of variable $x = \varphi_v^{\alpha^*}(0)$

$$\begin{aligned}\nu_{\alpha^*}^\infty(\phi) &= \int_0^{\sigma_{\alpha^*}} \phi(x)\nu_{\alpha^*}^\infty(dx) = \int_0^\infty \phi(\varphi_v^{\alpha^*}(0))\gamma(\alpha^*)H_{\alpha^*}(v)dv \\ &= \int_0^t \phi(\varphi_{t,u}^{\alpha^*}(0))H_{\alpha^*}(t,u)\gamma(\alpha^*)du + \int_t^\infty \phi(\varphi_v^{\alpha^*}(0))\gamma(\alpha^*)H_{\alpha^*}(v)dv.\end{aligned}$$

The last equality is obtained with the change of variable $v = t - u$. The second term is controlled by

$$\begin{aligned}\int_t^\infty \phi(\varphi_u^{\alpha^*}(0))\gamma(\alpha^*)H_{\alpha^*}(u)du &\leq ||\phi||_\infty\gamma(\alpha^*)\int_t^\infty \frac{f(\varphi_u^{\alpha^*}(0))}{\inf_{v\geq t} f(\varphi_v^{\alpha^*}(0))}\exp\left(-\int_0^u f(\varphi_\theta^{\alpha^*}(0))d\theta\right)du \\ &= \frac{||\phi||_\infty\gamma(\alpha^*)}{f(\varphi_t^{\alpha^*}(0))}\exp\left(-\int_0^t f(\varphi_\theta^{\alpha^*}(0))d\theta\right) \\ &\leq De^{-\lambda' t},\end{aligned}$$

for some constant D. We used again Remark 3.16(c). It remains to show that

$$\Delta_t := \left|\int_0^t \phi(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)r_{\boldsymbol{a}}^\nu(u,0)du - \int_0^t \phi(\varphi_{t,u}^{\alpha^*}(0))H_{\alpha^*}(t,u)\gamma(\alpha^*)du\right|$$

goes to zero exponentially fast. One has

$$\begin{aligned}
\Delta_t &\leq \int_0^t \left|\phi(\varphi^{\boldsymbol{a}}_{t,u}(0)) - \phi(\varphi^{\alpha^*}_{t,u}(0))\right| H_{\boldsymbol{a}}(t,u) r^\nu_{\boldsymbol{a}}(u,0)du \\
&\quad + \int_0^t |H_{\boldsymbol{a}}(t,u) - H_{\alpha^*}(t,u)|\, |\phi(\varphi^{\alpha^*}_{t,u}(0))| r^\nu_{\boldsymbol{a}}(u,0)du \\
&\quad + \int_0^t H_{\alpha^*}(t,u) |\phi(\varphi^{\alpha^*}_{t,u}(0))|\, |r^\nu_{\boldsymbol{a}}(u,0) - \gamma(\alpha^*)|\, du \\
&=: \Delta^1_t + \Delta^2_t + \Delta^3_t.
\end{aligned}$$

Using that for all $t \geq 0$, $|r^\nu_{\boldsymbol{a}}(t,0) - \gamma(\alpha^*)| \leq Ce^{-\lambda' t}$ ($\lambda' < \lambda$) and Remark 3.16(a), we obtain

$$\begin{aligned}
\Delta^3_t &\leq C||\phi||_\infty \int_0^t H_0(t,u) e^{-\lambda' u} du \\
&= C||\phi||_\infty e^{-\lambda' t} \int_0^t H_0(t-u) e^{\lambda'(t-u)} du \\
&\leq \left[C||\phi||_\infty \int_0^\infty H_0(u) e^{\lambda' u} du \right] e^{-\lambda' t} =: De^{-\lambda' t}.
\end{aligned}$$

The fact that $u \mapsto H_0(u)e^{\lambda' u}$ belongs to $L^1(\mathbb{R}_+)$ follows from $\lambda' < f(\sigma_0)$. By Theorem 2.8, one can find a constant $\bar{r}$ (with $\gamma(\alpha^*) \leq \bar{r}$) such that

$$\forall t \geq 0, \quad \mathbb{E}\, f(X_t) = r^\nu_{\boldsymbol{a}}(t,0) \leq \bar{r}.$$

Moreover, Assumption (3.3) and Remark 3.16(a) give

$$\begin{aligned}
\Delta^1_t &\leq \bar{r} l_\phi \int_0^t |\varphi^{\boldsymbol{a}}_{t,u}(0) - \varphi^{\alpha^*}_{t,u}(0)| H_0(t,u) du \\
&\leq \bar{r} l_\phi C_\varphi \int_0^t \int_u^t |a_\theta - \alpha^*| d\theta H_0(t,u) du.
\end{aligned}$$

Using that $\int_u^t |a_\theta - \alpha^*| d\theta \leq JC \int_u^t e^{-\lambda'\theta} d\theta \leq \frac{JCe^{-\lambda' u}}{\lambda'}$, one has

$$\begin{aligned}
\Delta^1_t &\leq \frac{\bar{r} l_\phi C_\varphi JC}{\lambda'} e^{-\lambda' t} \int_0^t e^{\lambda'(t-u)} H_0(t-u) du \\
&\leq \left[\frac{\bar{r} l_\phi C_\varphi JC}{\lambda'} \int_0^\infty H_0(u) e^{\lambda' u} du \right] e^{-\lambda' t} =: De^{-\lambda' t}.
\end{aligned}$$

Finally, using the inequality $|e^{-A} - e^{-B}| \leq e^{-\min(A,B)}|A - B|$ together with Remark 3.16(a) we obtain

$$\Delta^2_t \leq ||\phi||_\infty \bar{r} \int_0^t H_0(t-u) \int_u^t \left| f(\varphi^{\boldsymbol{a}}_{\theta,u}(0)) - f(\varphi^{\alpha^*}_{\theta,u}(0)) \right| d\theta du.$$

We set $\bar{\alpha} := J\bar{r}$. From $\boldsymbol{a} \leq \bar{\alpha}$, Remark 3.16(b) yields

$$\varphi^{\boldsymbol{a}}_{\theta,u}(0) \leq (\bar{\alpha} + C_b)(\theta - u).$$

So using Assumption 2.5(b) and (2.2), we find that there exists $C_0, p > 0$ such that

$$f'(\varphi^{\boldsymbol{a}}_{\theta,u}(0)) \leq C_0(1 + (\theta - u)^p).$$

So, using Assumption 2.3, we find that

$$\left| f(\varphi^{\boldsymbol{a}}_{\theta,u}(0)) - f(\varphi^{\alpha^*}_{\theta,u}(0)) \right| \leq C_0(1 + (\theta - u)^p) \int_u^\theta \left| \varphi^{\boldsymbol{a}}_{v,u}(0) - \varphi^{\alpha^*}_{v,u}(0) \right| dv.$$

Overall, we find that there exists a constant D such that

$$\int_u^t \left| f(\varphi^{\boldsymbol{a}}_{\theta,u}(0)) - f(\varphi^{\alpha^*}_{\theta,u}(0)) \right| d\theta \leq D \left(1 + (t-u)^{p+2}\right) e^{-\lambda' u}.$$

So

$$\begin{aligned} \Delta_t^2 &\leq D||\phi||_\infty \bar{r} e^{-\lambda' t} \int_0^t H_0(t-u) \left(1 + (t-u)^{p+2}\right) e^{\lambda'(t-u)} du. \\ &\leq \left[D||\phi||_\infty \bar{r} \int_0^\infty H_0(u) \left(1 + u^{p+2}\right) e^{\lambda' u} du \right] e^{-\lambda' t}. \end{aligned}$$

Again, the integral is finite because $\lambda' < f(\sigma_0)$. Combining the three estimates, we have proved the result. □

3.5 Long time behavior with constant drift

The goal of this section is to study the rate of convergence to the invariant probability measure when $J = 0$ (no interaction). We use Laplace transform techniques to characterize the convergence. We state here the main result of the section.

Proposition 3.17. *Grant Assumptions 3.1 and 3.2. Let $\alpha \geq 0$. One can find a constant $\lambda^*_\alpha \in (0, f(\sigma_\alpha)]$ (only depending on b, f and α) such that for any $0 < \lambda < \lambda^*_\alpha$, there exists a constant D (only depending on f, b, α and λ) such that for all $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\nu(f^2) < \infty$:*

$$\forall t \geq 0, \quad |\mathbb{E}\, f(Y^{\alpha,\nu}_t) - \gamma(\alpha)| \leq D e^{-\lambda t} \int_0^\infty [1 + f(x)] |\nu - \nu^\infty_\alpha|(dx), \tag{3.11}$$

Moreover, one has

$$Y^{\alpha,\nu}_t \xrightarrow[t \to \infty]{\mathcal{L}} \nu^\infty_\alpha.$$

Remark 3.18. *In the above result, λ^*_α is explicitly known in terms of f, b and α (see its expression (3.14)) and is optimal (see Remark 3.27). Note also that (3.11) states explicitly the dependence on the initial distribution ν through its distance to the invariant measure ν^∞_α.*

Let us first mention that it suffices to prove (3.11). Indeed, we then apply Proposition 3.15 with $\tilde{b}(x) = b(x) + \alpha$ and $J = 0$. With this choice, the process $(Y^{\alpha,\nu}_t)_{t \geq 0}$ is the solution of (1.2) and 0 solves (3.7). This gives the convergence in law of $Y^{\alpha,\nu}_t$ to ν^∞_α.

The proof of (3.11) is divided in two main steps. In Section 3.5.1, we first consider $\nu = \delta_0$, and prove that for all $\lambda < \lambda^*_\alpha$, the function $t \mapsto |r_\alpha(t) - \gamma(\alpha)| e^{\lambda t}$ belongs to $L^1(\mathbb{R}_+)$. Then, in Section 3.5.2, we extend the result to arbitrary initial condition and obtain the uniform estimate (3.11).

3.5.1 Convergence starting from $\nu = \delta_0$

Study of the Volterra equation

In the case where $\boldsymbol{a}$ is constant and equal to α, the Volterra equation (1.14) is a linear homogeneous convolution Volterra equation. If moreover the initial condition ν is δ_0, the jump rate $r_\alpha(t) := \mathbb{E}\, f(Y_t^{\alpha,\delta_0})$ satisfies

$$r_\alpha = K_\alpha + K_\alpha * r_\alpha. \tag{3.12}$$

For such equations, it is very natural to use Laplace transform techniques as convolutions become scalar products with this transformation. Furthermore, the "kernel" K_α is non-negative. Volterra equation with positive kernels have been studied in the context of Renewal theory. The main reference on this question is a paper of Feller [Fel41]. We refer to [Fel41, Th. 4] for this method. However, in our case the rate of convergence is exponential. In order to achieve the optimal rate of convergence, we use general methods from the Volterra integral equation theory, and especially the Whole-line Paley-Wiener Theorem.

Definition 3.19 (Laplace transform)**.** *Let $g : \mathbb{R}_+ \mapsto \mathbb{R}$ be a measurable function. The Laplace transform of g is the following function*

$$\widehat{g}(z) := \int_0^\infty e^{-zt} g(t)dt,$$

defined for all $z \in \mathbb{C}$ for which the integral exists.

Note that the Laplace transforms of H_α and K_α are well defined for all $z \in \mathbb{C}$ with $\Re(z) > -f(\sigma_\alpha)$. This follows from $\forall \lambda < f(\sigma_\alpha)$, $\sup_{t\geq 0} H_\alpha(t) e^{\lambda t} < \infty$. The same holds for K_α. Integrating by parts the Laplace transform of K_α shows that

$$\forall z \in \mathbb{C},\ \Re(z) > -f(\sigma_\alpha) \implies \widehat{K_\alpha}(z) = 1 - z\widehat{H_\alpha}(z). \tag{3.13}$$

It is also useful to introduce the following Banach space

Definition 3.20. *For any $\lambda \in \mathbb{R}$, let $L^1_\lambda := \{h \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}) : ||h||^1_\lambda < \infty\}$ the space of Borel-measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$, equipped with the norm*

$$||h||^1_\lambda := \int_{\mathbb{R}_+} |h(s)| e^{\lambda s} ds.$$

The long time behavior of r_α is related to the location of the poles of $\widehat{r}_\alpha$. Equation (3.12) gives

$$\forall \Re(z) > 0, \quad \widehat{r}_\alpha(z) = \frac{\widehat{K}_\alpha(z)}{1 - \widehat{K}_\alpha(z)}.$$

This suggests to study the location of the zeros of $1 - \widehat{K}_\alpha(z) = z\widehat{H}_\alpha(z)$.

On the zeros of $\widehat{H}_\alpha$

Along this section, we grant Assumptions 2.3 and 2.5 and consider α such that $b(0) + \alpha > 0$.

Lemma 3.21. *For all $z \in \mathbb{C}$ with $\Re(z) \geq 0$, it holds that $\widehat{H}_\alpha(z) \neq 0$.*

Proof. First, K_α being non-negative, we have

$$|\widehat{K}_\alpha(z)| \leq \int_0^\infty |e^{-tz}| K_\alpha(t)dt < \int_0^\infty K_\alpha(t)dt = 1 \text{ if } \Re(z) > 0.$$

It yields $\Re(z) > 0 \implies \widehat{H}_\alpha(z) \neq 0$.

Second, H_α being a real-valued function, $\widehat{H}_\alpha(z) = 0$ iff $\widehat{H}_\alpha(\bar{z}) = 0$, so it is sufficient to locate the zeros of $\widehat{H}_\alpha$ in the region $\Im(z) \geq 0$. Following [Fel41, proof of Th.4, (b)], if $z = iy$, $y > 0$ then

$$iy\widehat{H}_\alpha(iy) = 1 - \widehat{K}_\alpha(iy) = \int_0^\infty (1 - \cos(yt))K_\alpha(t)dt + i\int_0^\infty \sin(yt)K_\alpha(t)dt.$$

Consequently, $\widehat{K}_\alpha(iy) = 1$ for some $y > 0$ would imply that for Lebesgue almost every $t \geq 0$, $(1 - \cos(yt))K_\alpha(t) = 0$, that is, a.e. $K_\alpha(t) = 0$. It obviously contradicts the assumption $f(x) > 0$ for $x > 0$. It follows that $\forall y > 0$, $\widehat{H}_\alpha(iy) \neq 0$. Finally for $z = 0$, we have $\widehat{H}_\alpha(0) = \int_0^\infty H_\alpha(t)dt \neq 0$. □

Lemma 3.22. *The zeros of $\widehat{H}_\alpha$ are isolated.*

Proof. The function $\widehat{H}_\alpha$ is holomorphic on $\Re(z) > -f(\sigma_\alpha)$. Thus, its zeros are isolated. □

Lemma 3.23. *For all $z \in \mathbb{C}$ with $\Re(z) > -f(\sigma_\alpha)$, it holds that*

$$|\widehat{K}_\alpha(z)| \leq \frac{\phi_\alpha(\Re(z))}{|\Im(z)|},$$

where for all $x \in \mathbb{R}$, $\phi_\alpha(x) := ||K'_{\alpha,x}||_1$ and $K_{\alpha,x}(t) := e^{-xt}K_\alpha(t)$, $K'_{\alpha,x}(t) := \frac{d}{dt}K_{\alpha,x}(t)$.

Consequently, the zeros of $\widehat{H}_\alpha$ are within a "cone":

$$\forall z \in \mathbb{C},\ \Re(z) > -f(\sigma_\alpha), \quad \widehat{H}_\alpha(z) = 0 \implies |\Im(z)| \leq \phi_\alpha(\Re(z)).$$

Proof. Let $z = x + iy$, $y > 0, x > -f(\sigma_\alpha)$. We have

$$\widehat{K}_\alpha(z) = \int_0^\infty e^{-zt}K_\alpha(t)dt = \int_0^\infty e^{-iyt}K_{\alpha,x}(t)dt = \int_0^\infty \frac{e^{-iyt}}{iy}K'_{\alpha,x}(t)dt.$$

The last equality follows by an integration by part (recall that $K_{\alpha,x}(0) = 0$ because $f(0) = 0$). It yields

$$|\widehat{K}_\alpha(z)| \leq \frac{||K'_{\alpha,x}||_1}{|y|}.$$

We deduce that for $|y| > ||K'_{\alpha,x}||_1$, we have $\widehat{K}_\alpha(z) \neq 1$ and also $\widehat{H}_\alpha(z) \neq 0$. □

Consequently, from Lemmas 3.21, 3.22 and 3.23, we can define the abscissa of the "first" zero of $\widehat{H}_\alpha$:

$$\lambda^*_\alpha := -\sup\{\Re(z) |\ \Re(z) > -f(\sigma_\alpha),\ \widehat{H}_\alpha(z) = 0\}, \tag{3.14}$$

with the convention that $\lambda^*_\alpha = f(\sigma_\alpha)$ if the set of zeros is empty. We have proved that

$$0 < \lambda^*_\alpha \leq f(\sigma_\alpha) \leq \infty.$$

The parameter λ^*_α is key here as it gives the speed of convergence to the invariant probability measure. It only depends on α, b and f.

Convergence with optimal rate

We use the following Whole Line Paley-Wiener Theorem, which is one of the most important ingredients of the convolution Volterra integral equations theory.

Theorem 3.24 (Whole-line Paley-Wiener)**.** *Let $k \in L^1(\mathbb{R};\mathbb{R})$. There exists a function $x \in L^1(\mathbb{R};\mathbb{R})$ satisfying the equation*

$$\forall t \geq 0, \quad x(t) = k(t) + \int_{\mathbb{R}} k(t-u)x(u)du$$

if and only if

$$\forall y \in \mathbb{R}, \quad \widehat{k}(iy) := \int_{\mathbb{R}} e^{-iyt}k(t)dt \neq 1.$$

Note that here $\widehat{k}(iy)$ is actually the Fourier transform of k evaluated at $y \in \mathbb{R}$.

Proof. See [GLS90, Th. 4.3, Chap. 2]. We prove later, in details, an extension of this theorem (see Proposition 3.38). □

Recall that the constant $\lambda^*_\alpha > 0$ is defined by (3.14). The main result of this section is:

Proposition 3.25. *Assume b and f satisfy Assumptions 2.3 and 2.5. Let $\alpha > -b(0)$. Define*

$$\forall t \geq 0, \quad \xi_\alpha(t) := r_\alpha(t) - \gamma(\alpha)$$

Then for all $\lambda \in [0, \lambda^*_\alpha)$, $\xi_\alpha \in L^1_\lambda$.

Remark 3.26. *Note that for all $z \in \mathbb{C}$ with $\Re(z) > 0$, we have*

$$\begin{aligned}\widehat{\xi}_\alpha(z) &= \widehat{r}_\alpha(z) - \frac{\gamma(\alpha)}{z}\\ &= \frac{\widehat{K}_\alpha(z)}{z\widehat{H}_\alpha(z)} - \frac{\widehat{K^{\nu^\infty_\alpha}_\alpha}(z)}{z\widehat{H}_\alpha(z)}\\ &\overset{(2.8)}{=} \frac{\widehat{H^{\nu^\infty_\alpha}_\alpha}(z) - \widehat{H}_\alpha(z)}{\widehat{H}_\alpha(z)}.\end{aligned}$$

The right-hand side is an holomorphic function on $\Re(z) > -\lambda^*_\alpha$. However, this information on the Laplace transform of ξ_α ***is in general not*** *sufficient to deduce that $\xi_\alpha \in L^1_\lambda$ for $\lambda < \lambda^*_\alpha$. Consider for instance the following example, taken from [Wid41, Ch. 2]. Let*

$$\forall t \geq 0, \quad x(t) = \sin(e^t).$$

Its Laplace transform is given by, for all $z \in \mathbb{C}$ with $\Re(z) > 0$

$$\widehat{x}(z) = \int_0^\infty e^{-zt} \sin(e^t) dt = \int_1^\infty \frac{\sin(u)}{u^{z+1}} du = \cos(1) - (z+1) \int_1^\infty \frac{\cos(u)}{u^{z+2}} du.$$

The last equality is obtained by an integration by parts. Note that the integrand on the right hand side converges absolutely on $\Re(z) > -1$ and moreover $z \mapsto \int_1^\infty \frac{\cos(u)}{u^{z+2}} du$ is holomorphic on $\Re(z) > -1$. We deduce that $z \mapsto \widehat{x}(z)$ is holomorphic on $\Re(z) > -1$. A further integration by parts show that in fact $\widehat{x}$ is entire. However, $x \notin L^1(\mathbb{R}_+)$.

Proposition 3.25 can be deduced from general theorems of the Volterra equations theory. For instance, one can apply [GLS90, Th. 2.4, Chap. 7]. However, this last result is written for general measure kernels in weighted spaces and its proof is somehow difficult to follow. In our setting, the proof given by [GLS90] simplifies and we give it here for completeness.

Proof of Proposition 3.25. Let σ_- and σ_+ be any real numbers such that:

$$-\lambda^*_\alpha < \sigma_- < 0 < \sigma_+ < \infty.$$

We first extend r_α, K_α and H_α to the whole line by defining for all t in $\mathbb{R}$, $r_\alpha(t) := r_\alpha(t)\mathbb{1}_{\mathbb{R}_+}(t)$, $K_\alpha(t) := K_\alpha(t)\mathbb{1}_{\mathbb{R}_+}(t)$ and $H_\alpha(t) := H_\alpha(t)\mathbb{1}_{\mathbb{R}_+}(t)$. We have from (3.12)

$$\forall t \in \mathbb{R}, \quad r_\alpha(t) = K_\alpha(t) + \int_{\mathbb{R}} K_\alpha(t-u) r_\alpha(u) du. \tag{3.15}$$

For any $\triangle \in \mathbb{R}$, we also define $r_{\alpha,\triangle}(t) := e^{-\triangle t} r_\alpha(t)$, $K_{\alpha,\triangle}(t) := e^{-\triangle t} K_\alpha(t)$. Note that $K_{\alpha,\sigma_-} \in L^1(\mathbb{R})$ and that $\forall y \in \mathbb{R}$, $\widehat{K}_{\alpha,\sigma_-}(iy) = \widehat{K}_\alpha(\sigma_- + iy) \neq 1$ (by definition of λ^*_α). We can apply Theorem 3.24: there exists $\xi_{\alpha,\sigma_-} \in L^1(\mathbb{R})$ such that

$$\forall t \in \mathbb{R}, \quad \xi_{\alpha,\sigma_-}(t) = K_{\alpha,\sigma_-}(t) + \int_{\mathbb{R}} K_{\alpha,\sigma_-}(t-u) \xi_{\alpha,\sigma_-}(u) du. \tag{3.16}$$

We define $\xi_\alpha(t) := e^{\sigma_- t} \xi_{\alpha,\sigma_-}(t)$. We have $\int_{\mathbb{R}} |\xi_\alpha(u)| e^{-\sigma_- u} du < \infty$ and (3.16) reads

$$\forall t \in \mathbb{R}, \quad \xi_\alpha(t) = K_\alpha(t) + \int_{\mathbb{R}} K_\alpha(t-u) \xi_\alpha(u) du.$$

From (3.16), and using the continuity of K_{α,σ_-}, one deduces that ξ_α is continuous. Note that the function ξ_α is not null on $\mathbb{R}_-$ (see formula (3.18) just below).

We now regularize the functions ξ_{α,σ_-} and r_{α,σ_+} in order to apply the Fourier inversion formula. Let $\psi \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R})$ be a non-negative function vanishing outside of $[-1,1]$ such that $\int \psi = 1$ and let $\psi_n(t) = n\psi(nt)$. We define for all $t \in \mathbb{R}$ and $n \in \mathbb{N}^*$:

$$\xi^n_{\alpha,\sigma_-}(t) := e^{-\sigma_- t} \int_{\mathbb{R}} \psi_n(t-u) \xi_\alpha(u) du \quad \text{and} \quad r^n_{\alpha,\sigma_+}(t) := e^{-\sigma_+ t} \int_{\mathbb{R}} \psi_n(t-u) r_\alpha(u) du.$$

From (3.15) and (3.16) we have

$$\xi^n_{\alpha,\sigma_-} \in L^1(\mathbb{R}), \quad \widehat{\xi^n_{\alpha,\sigma_-}}(iy) = \left[\widehat{\psi_n}\frac{\widehat{K_\alpha}}{1-\widehat{K_\alpha}}\right](iy+\sigma_-),$$
$$r^n_{\alpha,\sigma_+} \in L^1(\mathbb{R}), \quad \widehat{r^n_{\alpha,\sigma_+}}(iy) = \left[\widehat{\psi_n}\frac{\widehat{K_\alpha}}{1-\widehat{K_\alpha}}\right](iy+\sigma_+).$$

Thanks to the regularization by ψ_n, the functions $y \mapsto \widehat{\xi^n_{\alpha,\sigma_-}}(iy)$ and $y \mapsto \widehat{r^n_{\alpha,\sigma_+}}(iy)$ also belong to $L^1(\mathbb{R})$. So we can use the Fourier inverse formula (see for instance [Hör90, Th. 7.1.10]) to get

$$\xi^n_{\alpha,\sigma_-}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyt}\left[\widehat{\psi_n}\frac{\widehat{K_\alpha}}{1-\widehat{K_\alpha}}\right](iy+\sigma_-)dy$$

and

$$r^n_{\alpha,\sigma_+}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{iyt}\left[\widehat{\psi_n}\frac{\widehat{K_\alpha}}{1-\widehat{K_\alpha}}\right](iy+\sigma_+)dy.$$

After the changes of variable $z = iy + \sigma_-$ and $z = iy + \sigma_+$, one has

$$\xi^n_{\alpha,\sigma_-}(t)e^{\sigma_- t} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_- - iT}^{\sigma_- + iT} e^{zt}\widehat{\psi_n}(z)\frac{\widehat{K}_\alpha(z)}{1-\widehat{K}_\alpha(z)}dz$$

and

$$r^n_{\alpha,\sigma_+}(t)e^{\sigma_+ t} = \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma_+ - iT}^{\sigma_+ + iT} e^{zt}\widehat{\psi_n}(z)\frac{\widehat{K}_\alpha(z)}{1-\widehat{K}_\alpha(z)}dz.$$

Let Γ_T be the closed curve in the complex plane composed of four straight lines that join the points $\sigma_- - iT$, $\sigma_- + iT$, $\sigma_+ + iT$, and $\sigma_+ - iT$ in the anti-clockwise direction. It follows from the residue theorem that

$$\int_{\Gamma_T} e^{zt}\widehat{\psi_n}(z)\frac{\widehat{K}_\alpha(z)}{1-\widehat{K}_\alpha(z)}dz \overset{(3.13)}{=} \int_{\Gamma_T} e^{zt}\widehat{\psi_n}(z)\frac{\widehat{K}_\alpha(z)}{z\widehat{H}_\alpha(z)}dz = 2\pi i\widehat{\psi_n}(0)\frac{\widehat{K}_\alpha(0)}{\widehat{H}_\alpha(0)} = 2\pi i\gamma(\alpha). \quad (3.17)$$

The last equality follows from

$$\widehat{H}_\alpha(0) = \int_0^\infty H_\alpha(t)dt = \int_0^\infty \exp\left(-\int_0^t f(\varphi^\alpha_u)du\right)dt \overset{(3.10)}{=} \frac{1}{\gamma(\alpha)}.$$

By Lemma 3.23, one can find a constant $T_0 > 0$ such that for all z in the strip $\Re(z) \in [\sigma_-, \sigma_+]$, we have

$$|\Im(z)| \geq T_0 \implies |\widehat{K}_\alpha(z)| \leq 1/2.$$

Moreover, there exists a constant M_n such that for all z in the strip $\Re(z) \in [\sigma_-, \sigma_+]$, $z \neq 0$ we have

$$\left|\widehat{\psi_n}(z)\right| \leq \frac{M_n}{z}.$$

We deduce that

$$\lim_{T\to\pm\infty}\int_{\sigma_- + iT}^{\sigma_+ + iT} e^{zt}\widehat{\psi_n}(z)\frac{\widehat{K}_\alpha(z)}{1-\widehat{K}_\alpha(z)}dz = 0.$$

Therefore we can take the limit $T \to \infty$ in (3.17) and obtain for all $n \geq 1$

$$\forall t \in \mathbb{R}, \quad r^n_{\alpha,\sigma_+}(t)e^{\sigma_+ t} - \xi^n_{\alpha,\sigma_-}e^{\sigma_- t}(t) = \gamma(\alpha).$$

Letting $n \to \infty$ (and using the continuity of r_α and ξ_α), we find that

$$\forall t \in \mathbb{R}, \quad r_\alpha(t) = \gamma(\alpha) + \xi_\alpha(t). \tag{3.18}$$

The proposition is proved by choosing $\sigma_- = -\lambda$. □

Remark 3.27. *The speed of convergence obtained in this result is optimal if $\lambda^*_\alpha < f(\sigma_\alpha)$ (i.e. $\widehat{H}_\alpha$ has at least one complex zero with $\Re(z) > -f(\sigma_\alpha)$) in the sense that*

$$\forall \lambda > \lambda^*_\alpha, \; r_\alpha - \gamma(\alpha) \notin L^1_\lambda.$$

Indeed, assume that $\lambda^*_\alpha < f(\sigma_\alpha)$ and choose σ_- such that $-f(\sigma_\alpha) < \sigma_- < -\lambda^*_\alpha$. The previous proof can be mimicked except that the residues of equation (3.17) now involves terms of the order $e^{-\lambda^*_\alpha t}$ - corresponding to the roots of $\widehat{H}_a$ with real part equal to $-\lambda^*_\alpha$.

3.5.2 Convergence from initial condition ν

We now come back to the general case of any initial condition $\nu \in \mathcal{P}(\mathbb{R}_+)$ such that $\nu(f^2) < \infty$ and give the proof of Proposition 3.17.

Proof of Proposition 3.17. Note that we only consider here convolutions on $[0, t]$, denoted by "$*$" (and no more the convolution on $\mathbb{R}$). Let $r^\nu_\alpha(t) = \mathbb{E}\, f(Y^{\alpha,\nu}_t)$ with $\mathcal{L}(Y_0) = \nu$. The function r^ν_α is the unique solution of the Volterra equation

$$r^\nu_\alpha = K^\nu_\alpha + K_\alpha * r^\nu_\alpha.$$

If we choose ν to be the invariant probability measure ν^∞_α, we get $\gamma(\alpha) = K^{\nu^\infty_\alpha}_\alpha + K_\alpha * \gamma(\alpha)$ and

$$r^\nu_\alpha - \gamma(\alpha) = K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha + K_\alpha * (r^\nu_\alpha - \gamma(\alpha)).$$

We can solve this equation in terms of r_α, the resolvent of K_α. Lemma 2.22 yields

$$\begin{aligned} r^\nu_\alpha - \gamma(\alpha) &= K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha + r_\alpha * (K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha) \\ &= K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha + \xi_\alpha * (K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha) + \gamma(\alpha) * (K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha), \end{aligned}$$

where $r_\alpha = \xi_\alpha + \gamma(\alpha)$, see (3.18), is the solution of the Volterra equation $r_\alpha = K_\alpha + K_\alpha * r_\alpha$. Using (2.8), we have $\gamma(\alpha) * K^\nu_\alpha = \gamma(\alpha)(1 - H^\nu_\alpha)$ and thus

$$r^\nu_\alpha - \gamma(\alpha) = K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha + \gamma(\alpha)(H^{\nu^\infty_\alpha}_\alpha - H^\nu_\alpha) + \xi_\alpha * (K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha).$$

We now denote by A any constant only depending on λ, f, b and α and which may change from line to line. It is clear that for any $0 < \lambda < f(\sigma_\alpha)$

$$|H^{\nu^\infty_\alpha}_\alpha - H^\nu_\alpha|(t) \leq \int_0^\infty H^x_\alpha(t)|\nu - \nu^\infty_\alpha|(dx) \leq \int_0^\infty H_\alpha(t)|\nu - \nu^\infty_\alpha|(dx) \leq Ae^{-\lambda t}\int_0^\infty |\nu - \nu^\infty_\alpha|(dx).$$

Similarly, for any $0 < \lambda < f(\sigma_\alpha)$,

$$\begin{aligned}
|K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha|(t) &\leq \int_0^\infty f(\varphi^\alpha_t(x)) H^x_\alpha(t) |\nu - \nu^\infty_\alpha|(dx) \\
&\leq \int_0^\infty f(x + (C_b + \alpha)t) H_\alpha(t) |\nu - \nu^\infty_\alpha|(dx) \\
&\leq C_f \int_0^\infty [1 + f(x) + f((C_b + \alpha)t)] H_\alpha(t) |\nu - \nu^\infty_\alpha|(dx) \\
&\leq A e^{-\lambda t} \int_0^\infty (1 + f(x)) |\nu - \nu^\infty_\alpha|(dx).
\end{aligned}$$

We used here Assumption 3.1 and 3.2(a). Let now $0 < \lambda < \lambda^*_\alpha$. Using $\xi_\alpha \in L^1_\lambda$, it holds that

$$|\xi_\alpha * (K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha)|(t) \leq \int_0^t |\xi_\alpha(t-u)| |K^\nu_\alpha - K^{\nu^\infty_\alpha}_\alpha|(u) du \leq A e^{-\lambda t} \int_0^\infty (1 + f(x)) |\nu - \nu^\infty_\alpha|(dx).$$

Combining the three estimates, one deduces that

$$|r^\nu_\alpha(t) - \gamma(\alpha)| \leq A e^{-\lambda t} \int_0^\infty (1 + f(x)) |\nu - \nu^\infty_\alpha|(dx).$$

□

3.6 Long time behavior with a general drift

In this section, we generalize the results obtained in Section 3.5 to non constant currents. We consider the process (1.9) driven by a current $\boldsymbol{a}$. We assume that $\boldsymbol{a}$ converges exponentially fast to α. We seek to prove that the jump rate of this process converges to $\gamma(\alpha)$ and estimate the speed of convergence. This "perturbation" analysis is useful to study the long time behavior of the solution of the nonlinear McKean-Vlasov equation (1.2). We consider $\boldsymbol{a} \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$ such that

Assumption 3.28. *1. $\sup_{t \geq 0} a_t \leq \bar{\alpha}$ for some constant $\bar{\alpha} > 0$.*

2. There exist $\alpha \geq 0$, $C \geq 0$ and $\lambda \in (0, \min(\lambda^*_\alpha, f(\sigma_0)))$, where σ_0 and λ^*_α are defined by (3.6) and (3.14), such that

$$\forall t \geq 0, \quad |a_t - \alpha| \leq C e^{-\lambda t}. \tag{3.19}$$

Note that the values of C and λ are important in this analysis. Any mention of C and λ in this section refers to these two constants.

Let $r^\nu_{\boldsymbol{a}}(t, s) = \mathbb{E} f(Y^{\boldsymbol{a},\nu}_{t,s})$, where $Y^{\boldsymbol{a},\nu}_{t,s}$ is the solution of (1.9) driven by the current $\boldsymbol{a}$ and starting at time s with law ν. The goal of this section is to prove that if C is small enough, then there exists an explicit constant D such that

$$\forall t \geq s \geq 0, \quad |r^\nu_{\boldsymbol{a}}(t, s) - \gamma(\alpha)| \leq D e^{-\lambda(t-s)} ,$$

where $\gamma(\alpha)$ is given by (3.5). Note that the exponential decay rate λ is preserved. We make efforts to keep track of the constant D and to relate it to C. As in Section 3.5 it is useful to split the study in two parts: the case where the initial condition is a Dirac mass at 0 and the general case. We thus consider the unique solution $r_{\boldsymbol{a}}$ of the following Volterra equation:

$$r_{\boldsymbol{a}} = K_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r_{\boldsymbol{a}}. \tag{3.20}$$

It is also useful to introduce a Banach space adapted to this non-homogeneous setting.

3.6.1 A Banach algebra

Recall that $\Delta := \{(t,s) \in \mathbb{R}^2,\ t \geq s\}$ (see (2.11)).

Definition 3.29. *A function $\kappa :\ \Delta \to \mathbb{R}$ is said to be a Volterra Kernel with weight $\lambda \in \mathbb{R}_+$ if κ is Borel measurable and $||\kappa||^1_\lambda < \infty$ with*

$$||\kappa||^1_\lambda := \operatorname{ess\,sup}_{t\geq 0} \int_{\mathbb{R}_+} |\kappa(t,s)| e^{\lambda(t-s)} ds.$$

We denote by $\mathcal{V}^1_\lambda$ the set of Volterra kernels with weight λ. We also define for $\kappa \in \mathcal{V}^1_\lambda$:

$$||\kappa||^\infty_\lambda := \operatorname{ess\,sup}_{t,s\geq 0} |\kappa(t,s) e^{\lambda(t-s)}| \in \mathbb{R}_+ \cup \{+\infty\}.$$

Proposition 3.30. *The space $(\mathcal{V}^1_\lambda, ||\cdot||^1_\lambda)$ is a Banach algebra. Furthermore, for all $a, b \in \mathcal{V}^1_\lambda$, $||a * b||^1_\lambda \leq ||a||^1_\lambda ||b||^1_\lambda$.*

Proposition 3.30 is proved in [GLS90], Theorem 2.4 and Proposition 2.7 (i) of Chapter 9.

Lemma 3.31 (Connection with the time homogeneous setting)**.** *Let $g \in L^1_\lambda$. We define*

$$\forall t, s \in \Delta, \quad \tilde{g}(t,s) := g(t-s).$$

Then $\tilde{g} \in \mathcal{V}^1_\lambda$ and $||g||^1_\lambda = ||\tilde{g}||^1_\lambda$.

This result allows us to consider elements of L^1_λ as elements of $\mathcal{V}^1_\lambda$. **Note that the algebra L^1_λ is commutative whereas $\mathcal{V}^1_\lambda$ is not.**

3.6.2 The perturbation method

Define $\bar{K}_{\boldsymbol{a}} := K_{\boldsymbol{a}} - K_\alpha$ and $\bar{H}_{\boldsymbol{a}} := H_{\boldsymbol{a}} - H_\alpha$.

Lemma 3.32. *Grant Assumptions 3.1, 3.2 and 3.28. Then, there exists a continuous non-negative and integrable function η such that for all $t \geq s \geq 0$, one has*

$$|\bar{K}_{\boldsymbol{a}}(t,s)| \leq C e^{-\lambda t} \eta(t-s),$$
$$|\bar{H}_{\boldsymbol{a}}(t,s)| \leq C e^{-\lambda t} \eta(t-s).$$

The function η only depends on $b, \bar{\alpha}, f$ and λ (in particular it does not depend on C). Furthermore, we can choose η such that $||\eta||_1$ is a non-decreasing function of $\bar{\alpha}$.

Proof. Here, to simplify the notations, we write $\varphi^{\boldsymbol{a}}_{t,s}$ for $\varphi^{\boldsymbol{a}}_{t,s}(0)$. We have

$$\bar{K}_{\boldsymbol{a}}(t,s) = f(\varphi^{\boldsymbol{a}}_{t,s}) \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s})du\right) - f(\varphi^{\alpha}_{t,s}) \exp\left(-\int_s^t f(\varphi^{\alpha}_{u,s})du\right).$$

So

$$\begin{aligned} |\bar{K}_{\boldsymbol{a}}(t,s)| &\leq |f(\varphi^{\boldsymbol{a}}_{t,s}) - f(\varphi^{\alpha}_{t,s})| \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s})du\right) \\ &\quad + f(\varphi^{\alpha}_{t,s}) \left|\exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s})du\right) - \exp\left(-\int_s^t f(\varphi^{\alpha}_{u,s})du\right)\right| \\ &=: M_1 + M_2. \end{aligned}$$

Assumptions 3.1, 3.2 and (3.19) give

$$\begin{aligned} |f(\varphi^{\boldsymbol{a}}_{t,s}) - f(\varphi^{\alpha}_{t,s})| &\leq C_f(1 + f(\varphi^{\bar{\alpha}}_{t-s})) \left|\varphi^{\boldsymbol{a}}_{t,s} - \varphi^{\alpha}_{t,s}\right| \leq C_f(1 + f(\varphi^{\bar{\alpha}}_{t-s}))C_\varphi \int_s^t |a_u - \alpha| du \\ &\leq C_f C_\varphi (1 + f(\varphi^{\bar{\alpha}}_{t-s}))C \int_s^t e^{-\lambda u} du \leq C e^{-\lambda t} C_f C_\varphi (1 + f(\varphi^{\bar{\alpha}}_{t-s})) \frac{e^{\lambda(t-s)}}{\lambda}. \end{aligned}$$

Moreover choosing $\lambda' \in (\lambda, f(\sigma_0))$ and using the fact that $f(\varphi^0_u) \to f(\sigma_0)$ as $u \to \infty$, one obtains

$$\begin{aligned} \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s})du\right) &\leq \exp\left(-\int_s^t f(\varphi^{0}_{u,s})du\right) = \exp\left(-\int_0^{t-s} f(\varphi^{0}_{u})du\right) \\ &\leq D(b,f,\lambda') e^{-\lambda'(t-s)}, \end{aligned}$$

for some finite constant $D(b,f,\lambda')$. Let $A_1(u) := \frac{D(b,f,\lambda')}{\lambda} e^{-(\lambda'-\lambda)u} C_f C_\varphi (1 + f(\varphi^{\bar{\alpha}}_{t-s}))$, we have

$$M_1 \leq C e^{-\lambda t} A_1(t-s),$$

and $A_1 \in L^1(\mathbb{R}_+)$. Moreover, for $A, B \geq 0$, we have $|e^{-A} - e^{-B}| \leq e^{-\min(A,B)}|A - B|$. So,

$$\begin{aligned} M_2 &\leq f(\varphi^{\bar{\alpha}}_{t,s}) \exp\left(-\int_0^{t-s} f(\varphi^0_u)du\right) \left|\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}) - f(\varphi^{\alpha}_{u,s})du\right| \\ &\leq f(\varphi^{\bar{\alpha}}_{t,s}) D(b,f,\lambda') e^{-\lambda'(t-s)} C_f (1 + f(\varphi^{\bar{\alpha}}_{t,s})) \int_s^t |\varphi^{\boldsymbol{a}}_{u,s} - \varphi^{\alpha}_{u,s}| du. \end{aligned}$$

One has

$$\int_s^t |\varphi^{\boldsymbol{a}}_{u,s} - \varphi^{\alpha}_{u,s}| du \leq C_\varphi \int_s^t \int_s^u |a_\theta - \alpha| d\theta du \leq C C_\varphi \int_s^t \int_s^u e^{-\lambda\theta} d\theta du \leq C e^{-\lambda t} \frac{C_\varphi}{\lambda}(t-s) e^{\lambda(t-s)}.$$

Consequently $M_2 \leq C e^{-\lambda t} A_2(t-s)$ with

$$A_2(u) := D(b,f,\lambda') e^{-(\lambda'-\lambda)u} f(\varphi^{\bar{\alpha}}_{t,s}) C_f (1 + f(\varphi^{\bar{\alpha}}_{t,s})) \frac{C_\varphi}{\lambda} u.$$

It holds that $A_2 \in L^1(\mathbb{R}_+)$ and setting $\eta := A_1 + A_2$ completes the proof for $\bar{K}_{\boldsymbol{a}}$. The same computations give a similar result for $\bar{H}_{\boldsymbol{a}}$. □

These estimates are sharp enough to give the following result:

Lemma 3.33. *Grant Assumptions 3.1, 3.2 and 3.28. Let η be the function given by Lemma 3.32. Denote by 1 the kernel $\mathbb{1}_{t\geq s}$. Then*

1. $\bar{K}_{\boldsymbol{a}} \in \mathcal{V}^1_\lambda$ *and* $||\bar{K}_{\boldsymbol{a}}||^1_\lambda \leq C||\eta||_1$.
2. $\bar{K}_{\boldsymbol{a}} * 1 \in \mathcal{V}^1_\lambda$ *and* $||\bar{K}_{\boldsymbol{a}} * 1||^1_\lambda \leq \frac{C||\eta||_1}{\lambda}$.

The exact same estimates holds for $\bar{H}_{\boldsymbol{a}}$ and $\bar{H}_{\boldsymbol{a}} * 1$.

Proof. Using Lemma 3.32, we have

$$||\bar{K}_{\boldsymbol{a}}||^1_\lambda := \sup_{t\geq 0} \int_0^t |\bar{K}_{\boldsymbol{a}}|(t,s)e^{\lambda(t-s)}ds \leq \sup_{t\geq 0} \int_0^t Ce^{-\lambda s}\eta(t-s)ds \leq C||\eta||_1,$$

proving point 1. For point 2, we have $\forall t \geq s \geq 0$, $(\bar{K}_{\boldsymbol{a}} * 1)(t,s) := \int_s^t \bar{K}_{\boldsymbol{a}}(t,u)du$. And Lemma 3.32 gives

$$||\bar{K}_{\boldsymbol{a}} * 1||^1_\lambda = \sup_{t\geq 0} \int_0^t |\bar{K}_{\boldsymbol{a}} * 1|(t,s)e^{\lambda(t-s)}ds \leq \sup_{t\geq 0} \int_0^t Ce^{-\lambda t}||\eta||_1 e^{\lambda(t-s)}ds = \frac{C||\eta||_1}{\lambda}.$$

□

Proposition 3.34. *Grant Assumptions 3.1, 3.2. Assume $\boldsymbol{a}$ satisfies Assumption 3.28 and that the constant C is small enough:*

$$\beta := C||\eta||_1 \left(1 + \lambda^{-1}\right)\left(1 + ||\xi_\alpha||^1_\lambda + \gamma(\alpha)\right) < 1. \tag{3.21}$$

Define $\Delta_K := \bar{K}_{\boldsymbol{a}} + \xi_\alpha * \bar{K}_{\boldsymbol{a}} - \gamma(\alpha)\bar{H}_{\boldsymbol{a}}$ and let Δ_r be the solution of the Volterra equation

$$\Delta_r = \Delta_K + \Delta_K * \Delta_r. \tag{3.22}$$

Then

1. $\Delta_K \in \mathcal{V}^1_\lambda$ *with* $||\Delta_K||^1_\lambda \leq \beta$ *and* $\Delta_K * 1 \in \mathcal{V}^1_\lambda$ *with* $||\Delta_K * 1||^1_\lambda \leq \beta$.
2. $\Delta_r \in \mathcal{V}^1_\lambda$ *with* $||\Delta_r||^1_\lambda \leq \frac{\beta}{1-\beta}$ *and* $\Delta_r * 1 \in \mathcal{V}^1_\lambda$ *with* $||\Delta_r * 1||^1_\lambda \leq \frac{\beta}{1-\beta}$.
3. *Consider $r_{\boldsymbol{a}}(t,s)$ the jump rate associated to the current $\boldsymbol{a}$. Then*

$$r_{\boldsymbol{a}} = r_\alpha + \Delta_r + \Delta_r * r_\alpha. \tag{3.23}$$

Consequently, we have $r_{\boldsymbol{a}} = \gamma(\alpha) + \xi_{\boldsymbol{a}}$ with

$$\xi_{\boldsymbol{a}} := \xi_\alpha + \Delta_r + \Delta_r * \xi_\alpha + \gamma(\alpha)(\Delta_r * 1) \in \mathcal{V}^1_\lambda.$$

Furthermore,

$$||\xi_{\boldsymbol{a}}||^1_\lambda \leq ||\xi_\alpha||^1_\lambda + \frac{\beta}{1-\beta}[1 + ||\xi_\alpha||^1_\lambda + \gamma(\alpha)].$$

Proof. By Lemma 3.33, we have $||\Delta_K||_\lambda^1 \le \beta < 1$. Consequently equation (3.22) admits a unique solution $\Delta_r \in \mathcal{V}_\lambda^1$ satisfying $||\Delta_r||_\lambda^1 \le \frac{\beta}{1-\beta}$. The kernel $\Delta_r * 1$ satisfies the following Volterra equation

$$\Delta_r * 1 = (\Delta_K * 1) + \Delta_K * (\Delta_r * 1) \tag{3.24}$$

with $\Delta_K * 1 = (\bar{K}_{\boldsymbol{a}} * 1) + \xi_\alpha * (\bar{K}_{\boldsymbol{a}} * 1) + \gamma(\alpha)(\bar{H}_{\boldsymbol{a}} * 1)$. It follows from Lemma 3.33 that $\Delta_K * 1 \in \mathcal{V}_\lambda^1$ and $||\Delta_K * 1||_\lambda^1 \le \beta$. From $||\Delta_K||_\lambda^1 < 1$, one gets that equation (3.24) has its solution in $\mathcal{V}_\lambda^1$ and

$$\Delta_r * 1 \in \mathcal{V}_\lambda^1, \quad ||\Delta_r * 1||_\lambda^1 \le \frac{\beta}{1-\beta}.$$

It remains to check that $r_{\boldsymbol{a}}$ given by (3.23) is indeed the solution of (3.20). We set $\rho := r_\alpha + \Delta_r + \Delta_r * r_\alpha$. We thus have to check that $r_{\boldsymbol{a}} = \rho$. To this aim, we prove that ρ solves (3.20). Using (3.22), one has

$$\begin{aligned}
\Delta_K * \rho &= \Delta_K * r_\alpha + (\Delta_r - \Delta_K) + (\Delta_r - \Delta_K) * r_\alpha \\
&= \Delta_r * r_\alpha + \Delta_r - \Delta_K \\
&= \rho - r_\alpha - \Delta_K,
\end{aligned}$$

i.e. ρ satisfies

$$\rho = r_\alpha + \Delta_K + \Delta_K * \rho. \tag{3.25}$$

Using Proposition 3.25 and (2.8), we have $\Delta_K = \bar{K}_{\boldsymbol{a}} + r_\alpha * \bar{K}_{\boldsymbol{a}}$. Eq. (3.25) gives

$$\rho - (\bar{K}_{\boldsymbol{a}} + r_\alpha * \bar{K}_{\boldsymbol{a}}) * \rho = r_\alpha + \bar{K}_{\boldsymbol{a}} + r_\alpha * \bar{K}_{\boldsymbol{a}}.$$

We multiply this equation by K_α on the left and obtain, using that $K_\alpha * r_\alpha = r_\alpha * K_\alpha = r_\alpha - K_\alpha$:

$$K_\alpha * \rho - r_\alpha * \bar{K}_{\boldsymbol{a}} * \rho = r_\alpha - K_\alpha + r_\alpha * \bar{K}_{\boldsymbol{a}}.$$

The relation $\bar{K}_{\boldsymbol{a}} = K_{\boldsymbol{a}} - K_\alpha$ yields

$$K_\alpha * \rho - r_\alpha * \bar{K}_{\boldsymbol{a}} * \rho = r_\alpha * K_{\boldsymbol{a}},$$

or equivalently

$$\Delta_K * \rho = K_{\boldsymbol{a}} * \rho - r_\alpha * K_{\boldsymbol{a}}.$$

We substitute this equality in (3.25) and finally obtain

$$\rho = K_{\boldsymbol{a}} + K_{\boldsymbol{a}} * \rho.$$

By uniqueness (Lemma 2.21 with $\nu = \delta_0$), one deduces that $\rho = r_{\boldsymbol{a}}$. The end of the proof follows easily. $\square$

Remark 3.35. *Let us explain how the formula (3.23) was derived. The algebra $\mathcal{V}_\lambda^1$ does not have any neutral element (in fact the neutral element would be a Dirac distribution) but assume for the sake of this heuristic that I is a neutral element of the algebra (i.e. $k * I = I * k = k\ \forall k \in \mathcal{V}_\lambda^1$). Equation (3.20) can be rewritten as*

$$(I - K_{\boldsymbol{a}}) * (I + r_{\boldsymbol{a}}) = I. \tag{3.26}$$

In particular (taking $\boldsymbol{a} \equiv \alpha$), we have $(I-K_\alpha)*(I+r_\alpha) = (I+r_\alpha)*(I-K_\alpha) = I$. Furthermore,

$$I - K_{\boldsymbol{a}} = (I - K_\alpha) * (I - (I + r_\alpha) * \bar{K}_{\boldsymbol{a}}),$$

with $\bar{K}_{\boldsymbol{a}} = K_{\boldsymbol{a}} - K_\alpha \in \mathcal{V}^1_\lambda$. Equation (3.26) becomes $(I-K_\alpha)*(I-(I+r_\alpha)*\bar{K}_{\boldsymbol{a}})*(I+r_{\boldsymbol{a}}) = I$. We multiply by $I+r_\alpha$ on the left of each side, and we get $(I-(I+r_\alpha)*\bar{K}_{\boldsymbol{a}})*(I+r_{\boldsymbol{a}}) = I+r_\alpha$.

We now expand this equation - the neutral element I disappears and obtain:

$$r_{\boldsymbol{a}} - (\bar{K}_{\boldsymbol{a}} + r_\alpha * \bar{K}_{\boldsymbol{a}}) * r_{\boldsymbol{a}} = r_\alpha + \bar{K}_{\boldsymbol{a}} + r_\alpha * \bar{K}_{\boldsymbol{a}}.$$

Using the definition of Δ_K we obtain $r_{\boldsymbol{a}} = r_\alpha + \Delta_K * r_{\boldsymbol{a}} + \Delta_K$. Solving this equation in terms of Δ_r the resolvent of Δ_K we have $r_{\boldsymbol{a}} = r_\alpha + \Delta_K + \Delta_r * (r_\alpha + \Delta_K)$. It gives the desired formula.

We now come back to an arbitrary initial condition ν and prove the main result of this section.

Proposition 3.36. *Grant Assumptions 3.1, 3.2. Let $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\nu(f^2) < \infty$. Let $\boldsymbol{a} \in \mathcal{C}(\mathbb{R}; \mathbb{R}_+)$, and consider $(Y^{\boldsymbol{a},\nu}_{t,s})_{t \geq s}$ the solution of (1.9) driven by $\boldsymbol{a}$ and with initial distribution ν at time s. Let $r^\nu_{\boldsymbol{a}}(t,s) = \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s})$. Assume $\boldsymbol{a}$ satisfies Assumption 3.28 and that the constant C satisfies the inequality (3.21) for some $\beta \in (0,1)$. Then it holds that*

$$\forall t \geq s \geq 0, \; |r^\nu_{\boldsymbol{a}}(t,s) - \gamma(\alpha)| \leq D e^{-\lambda(t-s)},$$

with

$$D := \frac{1 + \beta\gamma(\alpha) + ||\xi_\alpha||^1_\lambda}{1-\beta} ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda + \gamma(\alpha)||H^\nu_{\boldsymbol{a}}||^\infty_\lambda.$$

Proof. The kernel $r^\nu_{\boldsymbol{a}}$ solves the Volterra equation $r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r^\nu_{\boldsymbol{a}}$. By Lemma 2.22, its solution is

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + r_{\boldsymbol{a}} * K^\nu_{\boldsymbol{a}}.$$

Using Proposition 3.34, we know that $r_{\boldsymbol{a}} = \gamma(\alpha) + \xi_{\boldsymbol{a}}$, with $\xi_{\boldsymbol{a}} \in \mathcal{V}^1_\lambda$. Furthermore using that $\gamma(\alpha) * K^\nu_{\boldsymbol{a}} = \gamma(\alpha)[1 - H^\nu_{\boldsymbol{a}}]$, we deduce that:

$$r^\nu_{\boldsymbol{a}} = \gamma(\alpha) + K^\nu_{\boldsymbol{a}} + \xi_{\boldsymbol{a}} * K^\nu_{\boldsymbol{a}} - \gamma(\alpha) H^\nu_{\boldsymbol{a}}.$$

Using that $\lambda < f(\sigma_0)$ (Assumption 3.28) we deduce

$$||H^\nu_{\boldsymbol{a}}||^\infty_\lambda = \sup_{t,s} H^\nu_{\boldsymbol{a}}(t,s) e^{\lambda(t-s)} < \infty, \qquad \text{and} \qquad ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda = \sup_{t,s} K^\nu_{\boldsymbol{a}}(t,s) e^{\lambda(t-s)} < \infty.$$

We obtain for all $t \geq s$

$$\begin{aligned} |r^\nu_{\boldsymbol{a}}(t,s) - \gamma(\alpha)| e^{\lambda(t-s)} &\leq ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda + \gamma(\alpha)||H^\nu_{\boldsymbol{a}}||^\infty_\lambda + e^{\lambda(t-s)} \int_s^t |\xi_{\boldsymbol{a}}|(t,u) K^\nu_{\boldsymbol{a}}(u,s) du \\ &\leq ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda + \gamma(\alpha)||H^\nu_{\boldsymbol{a}}||^\infty_\lambda + ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda \int_s^t |\xi_{\boldsymbol{a}}|(t,u) e^{\lambda(t-u)} du \\ &\leq ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda + \gamma(\alpha)||H^\nu_{\boldsymbol{a}}||^\infty_\lambda + ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda ||\xi_{\boldsymbol{a}}||^1_\lambda. \end{aligned}$$

Using the estimate of $||\xi_{\boldsymbol{a}}||^1_\lambda$ given by Proposition 3.34, we deduce the result. □

3.7 Long time behavior for small interactions: proof of Theorem 3.7

3.7.1 Some uniform estimates

We now turn to the proof of Theorem 3.7. It is convenient to first extend the results obtained in Section 3.5: we need uniform estimates in the input current α. In this section, we grant Assumptions 3.1 and 3.2.

Lemma 3.37. *Let $\bar{\alpha} > 0$. It holds that*

$$\inf_{\alpha \in [0,\bar{\alpha}]} \lambda^*_\alpha > 0.$$

Proof. We define the function g related to the first zero of $\widehat{H}_\alpha$ by

$$\forall \alpha \in [0,\bar{\alpha}], \quad g(\alpha) := -\sup\{\Re(z) |\ \widehat{H}_\alpha(z) = 0,\ \Re(z) > -f(\sigma_0)\}.$$

By convention, $g(\alpha) = f(\sigma_0)$ if $\widehat{H}_\alpha$ is not null on $\Re(z) > -f(\sigma_0)$. By definition of λ^*_α and by the results of Section 3.5 we know that $g(\alpha) \in (0, \lambda^*_\alpha]$. So, to prove the lemma, it suffices to show the following result:
Claim The function g is lower semi-continuous, that is

$$\forall \alpha_0 \in [0,\bar{\alpha}], \quad \liminf_{\alpha \to \alpha_0} g(\alpha) \geq g(\alpha_0).$$

Proof of the claim. Choose $\alpha_0 \in [0,\bar{\alpha}]$. We have $g(\alpha_0) > 0$. Fix $\lambda \in (0, g(\alpha_0))$. Thanks to Lemma 3.23, one can find $R > 0$, such that for all $\alpha \in [0,\bar{\alpha}]$, for all z with $\Re(z) \in [-\lambda, 0]$ and $\Im(z) \notin [-R, R]$, we have $\widehat{H}_\alpha(z) \neq 0$. Denote $U = \{z \in \mathbb{C}, \Re(z) \in [-\lambda, 0], |\Im(z)| \leq R\}$. By definition of $g(\alpha_0)$, we have $\widehat{H}_{\alpha_0} \neq 0$ on U and the continuity of $z \mapsto \widehat{H}_{\alpha_0}(z)$ yields $\inf_{z \in U} |\widehat{H}_{\alpha_0}(z)| > 0$. Moreover, $(\alpha, z) \mapsto \hat{H}_\alpha(z)$ is continuous on $[0,\bar{\alpha}] \times U$, so one can find $\delta > 0$ such that for all $|\alpha - \alpha_0| \leq \delta$, $z \in U$, we have $|\widehat{H}_\alpha(z)| \neq 0$. So $g(\alpha) \geq \lambda$. We have proved that $\forall \lambda \in (0, g(\alpha_0))$, $\liminf_{\alpha \to \alpha_0} g(\alpha) \geq \lambda$. It ends the proof. □

Proposition 3.38 (Whole-line Paley-Wiener, an extension)**.** *Let $\bar{\alpha} > 0$ and for all $\alpha \in [0,\bar{\alpha}]$, let $k_\alpha \in L^1(\mathbb{R};\mathbb{R})$. Assume that*

1. *$\exists \eta \in L^1(\mathbb{R};\mathbb{R}_+)$ s.t. $\forall \alpha \in [0,\bar{\alpha}]$, $\forall 0 < \epsilon < 1, \forall t \in \mathbb{R}$, $\quad |k_\alpha(t) - k_\alpha(t-\epsilon)| \leq \epsilon \eta(t)$.*
2. *$\exists \theta \in L^1(\mathbb{R};\mathbb{R}_+)$ s.t. $\forall \alpha \in [0,\bar{\alpha}]$, $\forall t \in \mathbb{R}$, $\quad |k_\alpha(t)| \leq \theta(t)$.*
3. *$\forall \alpha \in [0,\bar{\alpha}]$, $\forall y \in \mathbb{R}$ let $\widehat{k}_\alpha(iy) = \int_{\mathbb{R}} e^{-iyt} k_\alpha(t) dt$. We assume that*

$$\inf_{\alpha \in [0,\bar{\alpha}], y \in \mathbb{R}} |1 - \widehat{k}_\alpha(iy)| > 0.$$

Then for all $\alpha \in [0, \bar{\alpha}]$*, there exists a function* $x_\alpha \in L^1(\mathbb{R}; \mathbb{R})$ *satisfying the equation*

$$x_\alpha(t) = k_\alpha(t) + \int_{\mathbb{R}} k_\alpha(t-s) x_\alpha(s) ds$$

and

$$\sup_{\alpha \in [0,\bar{\alpha}]} ||x_\alpha||_{L^1} < \infty.$$

Proof. We follow the proof of Theorem 4.3 in [GLS90, Chap. 2] and stress the differences. Let $\zeta(t) := \frac{1}{\pi t^2}(1-\cos(t))$ be the Fejer kernel; its Fourier transform is $\widehat{\zeta}(iy) = (1-|y|)\mathbb{1}_{\{|y|\leq 1\}}$. For any $p \geq 1$, set $\zeta_p(t) := p\zeta(pt)$ and $\forall \alpha \in [0, \bar{\alpha}]$

$$k_\alpha^{\infty,p}(t) := k_\alpha(t) - \int_{\mathbb{R}} \zeta_p(t-s) k_\alpha(s) ds.$$

Claim 1 There is an integer $p \geq 1$ such that $\forall \alpha \in [0, \bar{\alpha}]$, $\forall |y| \geq p$, we have

$$||k_\alpha^{\infty,p}||_{L^1} \leq 1/2 \quad \text{and} \quad \widehat{k}_\alpha^{\infty,p}(iy) = \widehat{k}_\alpha(iy).$$

Proof of the claim. It is clear that with this choice of ζ, $\forall |y| \geq p$, $\widehat{k}_\alpha^{\infty,p}(iy) = \widehat{k}_\alpha(iy)$. Moreover, using $\int_{\mathbb{R}} \zeta_p(s) ds = 1$, we find

$$\begin{aligned} ||k_\alpha^{\infty,p}||_{L^1} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} k_\alpha(t)\zeta_p(s) - k_\alpha(t-s)\zeta_p(s) ds \right| dt \\ &\leq \int_{\mathbb{R}} \zeta(u) \int_{\mathbb{R}} |k_\alpha(t) - k_\alpha(t - \frac{u}{p})| dt du. \end{aligned}$$

We used the Tonelli-Fubini Theorem (everything is non-negative). Let $R > 0$ such that $\int_{\mathbb{R}\setminus[-R,R]} \zeta(u) du \leq \frac{1}{8||\theta||_{L^1}}$. It follows that

$$\begin{aligned} ||k_\alpha^{\infty,p}||_{L^1} &\leq 1/4 + \int_{-R}^{R} \zeta(u) \int_{\mathbb{R}} |k_\alpha(t) - k_\alpha(t - \frac{u}{p})| dt du \\ &\leq 1/4 + \int_{-R}^{R} \left(\int_{\mathbb{R}} |\frac{u}{p}| \eta(t) dt \right) du \\ &\leq 1/4 + \frac{R^2}{p} ||\eta||_{L^1}. \end{aligned}$$

The claim is proved by choosing an integer $p \geq 4R^2 ||\eta||_{L^1}$.

Using the same idea, we define $\beta(t) := 4\zeta(2t) - \zeta(t) = \frac{1}{\pi t^2}(\cos t - \cos 2t)$. Note that $\forall |y| \leq 1$, we have $\widehat{\beta}(iy) = 1$. Then for all $\delta > 0$, we set $\beta_\delta(t) := \delta\beta(\delta t)$ and

$$\forall y_0 \in \mathbb{R}, \forall t \geq 0, \quad k_\alpha^{y_0,\delta}(t) := \int_{\mathbb{R}} (\beta_\delta(t-s) - \beta_\delta(t)) e^{iy_0(t-s)} k_\alpha(s) ds.$$

Claim 2 Given $\epsilon > 0$, one can find a constant $\delta > 0$ such that: $\forall y_0 \in \mathbb{R}, \forall \alpha \in [0, \bar{\alpha}]$,

$$\forall |y - y_0| \leq \delta, \quad \widehat{k}_\alpha(iy) = \widehat{k}_\alpha(iy_0) + \widehat{k_\alpha^{y_0,\delta}}(iy) \quad \text{and} \quad ||k_\alpha^{y_0,\delta}||_{L^1} \leq \frac{\epsilon}{2}.$$

Proof of the claim. By definition of $k_\alpha^{y_0,\delta}$, it holds that

$$\forall y \in \mathbb{R}, \quad \widehat{k_\alpha^{y_0,\delta}}(iy) = \widehat{\beta}_\delta(i(y - y_0))(\widehat{k}_\alpha(iy) - \widehat{k}_\alpha(iy_0)).$$

Moreover, $\widehat{\beta}_\delta(iy) = 1$ if $|y| \leq \delta$ and consequently the first point of the claim is satisfied. Furthermore,

$$\begin{aligned}\int_{\mathbb{R}} |k_\alpha^{y_0,\delta}(t)| dt &\leq \int_{\mathbb{R}} |k_\alpha(s)| \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dt ds \\ &\leq \int_{\mathbb{R}} \theta(s) \int_{\mathbb{R}} |\beta(t - \delta s) - \beta(t)| dt ds.\end{aligned}$$

The right hand side does not depend on y_0 nor α and goes to zero as δ goes to zero. This proves the second point of the claim.

Let $p \geq 1$ be given by Claim 1. It follows from Claim 1 that $\forall \alpha \in [0, \bar{\alpha}]$, the equation

$$x_\alpha^\infty(t) = k_\alpha(t) + \int_{\mathbb{R}} k_\alpha^{\infty,p}(t - s) x_\alpha^\infty(ds)$$

has a unique solution $x_\alpha^\infty \in L^1(\mathbb{R})$ with $||x_\alpha^\infty||_{L^1} \leq 2||\theta||_{L^1}$. Moreover, we have

$$\forall \alpha \in [0, \bar{\alpha}], \forall |y| \geq p, \quad \widehat{x}_\alpha^\infty(iy) = \frac{\widehat{k}_\alpha(iy)}{1 - \widehat{k}_\alpha(iy)}.$$

Similarly, we define $\epsilon := \inf_{\alpha \in [0,\bar{\alpha}], y \in \mathbb{R}} |1 - \widehat{k}_\alpha(iy)| > 0$ and apply the second claim. Given $y_0 \in \mathbb{R}$ and $\alpha \in [0, \bar{\alpha}]$, let $A_\alpha^{y_0} = \frac{1}{1 - \widehat{k}_\alpha(iy_0)}$. We have $1 - \widehat{k}_\alpha(iy) = 1 - \widehat{k}_\alpha(iy_0) - \widehat{k_\alpha^{y_0,\delta}}(iy) = \frac{1}{A_\alpha^{y_0}}(1 - A_\alpha^{y_0} \widehat{k_\alpha^{y_0,\delta}}(iy))$. So,

$$\forall |y - y_0| \leq \delta, \quad \frac{\widehat{k}_\alpha(iy)}{1 - \widehat{k}_\alpha(iy)} = \frac{A_\alpha^{y_0} \widehat{k}_\alpha(iy)}{1 - A_\alpha^{y_0} \widehat{k_\alpha^{y_0,\delta}}(iy)}.$$

Using $||A_\alpha^{y_0} k_\alpha^{y_0,\delta}||_{L^1} \leq 1/2$, we can define the solution of

$$x_\alpha^{y_0}(t) = A_\alpha^{y_0} k_\alpha(t) + \int_{\mathbb{R}} A_\alpha^{y_0} k_\alpha^{y_0,\delta}(t - s) x_\alpha^{y_0}(s) ds$$

and we find

$$||x_\alpha^{y_0}||_{L^1} \leq \frac{2}{\epsilon} ||\theta||_{L^1}.$$

Consequently, for all y with $|y - y_0| \leq \delta$ we have

$$\widehat{x}_\alpha^{y_0}(iy) = \frac{\widehat{k}_\alpha(iy)}{1 - \widehat{k}_\alpha(iy)}.$$

Furthermore, still following [GLS90], one can find an integer $m > 0$ such that: $\forall \alpha \in [0, \bar{\alpha}]$, $\forall j \in \mathbb{Z}, |j| \leq mp$, there exists a function $x_\alpha^{j/m} \in L^1(\mathbb{R})$ with $||x_\alpha^{j/m}||_{L^1} \leq \frac{2}{\epsilon}||\theta||_{L^1}$ such that

$$\forall |y - j/m| \leq 1/m, \quad \widehat{x_\alpha^{j/m}}(iy) = \frac{\widehat{k}_\alpha(iy)}{1 - \widehat{k}_\alpha(iy)}.$$

We define $\psi_j(t) = \frac{1}{m} e^{-ijt/m} \zeta(t/m)$. We have $||\psi_j||_{L^1} = 1$. Its Fourier transform is given by

$$\widehat{\psi}_j(iy) = \begin{cases} 0 & \text{if } |y - j/m| > 1/m \\ 1 - m|y - j/m| & \text{otherwise.} \end{cases}$$

We set

$$x_\alpha(t) = \sum_{|j| \leq mp} \int_{\mathbb{R}} \psi_j(t-s)(x_\alpha^{j/m} - x_\alpha^\infty)(s) ds + x_\alpha^\infty(t).$$

It is clear that $x_\alpha \in L^1(\mathbb{R})$ and that

$$\sup_{\alpha \in [0, \bar{\alpha}]} ||x_\alpha||_{L^1} \leq mp \left(\frac{2}{\epsilon} ||\theta||_{L^1} + 2||\theta||_{L^1} \right) + 2||\theta||_{L^1} < \infty.$$

With this choice of ψ_j, $\forall y \in \mathbb{R}, \widehat{x}_\alpha(iy) = \frac{\widehat{k}_\alpha(iy)}{1 - \widehat{k}_\alpha(iy)}$ and by uniqueness of the Fourier transform, we conclude that x_α is the solution of

$$x_\alpha(t) = k_\alpha(t) + \int_{\mathbb{R}} k_\alpha(t-s) x_\alpha(s) ds.$$

It ends the proof. □

As a consequence of the previous result, we have

Corollary 3.39. *Let $\bar{\alpha} > 0$, define $\lambda^* = \inf_{\alpha \in [0, \bar{\alpha}]} \lambda_\alpha^*$ ($\lambda^* > 0$ by Lemma 3.37). Let $0 < \lambda < \lambda^*$ and consider r_α the solution of the Volterra equation $r_\alpha = K_\alpha + K_\alpha * r_\alpha$. Let $\xi_\alpha := r_\alpha - \gamma(\alpha)$. We have*

$$\sup_{\alpha \in [0, \bar{\alpha}]} ||\xi_\alpha||_\lambda^1 < \infty.$$

Proof. Recall (see proof of Proposition 3.25) that $\xi_\alpha(t) = e^{-\lambda t} \xi_{\alpha, -\lambda}(t)$ and so $||\xi_\alpha||_\lambda^1 = ||\xi_{\alpha, -\lambda}||_{L^1}$. We now prove that Proposition 3.38 applies to $\xi_{\alpha, -\lambda}$. Indeed, it solves

$$\xi_{\alpha, -\lambda}(t) = K_{\alpha, -\lambda}(t) + \int_{\mathbb{R}} K_{\alpha, -\lambda}(t-s) \xi_{\alpha, -\lambda}(s),$$

with $K_{\alpha, -\lambda}(t) := e^{\lambda t} K_\alpha(t) \mathbb{1}_{\{t \geq 0\}}$. It remains to show that $K_{\alpha, -\lambda}$ fulfills the assumptions of Proposition 3.38.

1. We use $\sup_{\alpha \in [0, \bar{\alpha}]} K_\alpha(t) \leq f(\varphi_t^{\bar{\alpha}}(0)) H_0(t)$ and $\sup_{\alpha \in [0, \bar{\alpha}]} |\varphi_t^\alpha(0) - \varphi_{t-\epsilon}^\alpha(0)| \leq \epsilon(C_b + \bar{a})$.

2. For all $t \geq 0$ and $\alpha \in [0, \bar{\alpha}]$, we have

$$K_{\alpha,-\lambda}(t) \leq \theta(t) := e^{\lambda t} f(C_t^{\bar{\alpha}}) H_0(t) \mathbb{1}_{\mathbb{R}_+}(t) \in L^1(\mathbb{R}).$$

3. We have $\widehat{K_{\alpha,-\lambda}}(iy) = \widehat{K}_\alpha(-\lambda + iy)$. We conclude by Lemmas 3.37 and 3.23.

$\square$

Finally, we give a uniform in time bound of the jump rate of (1.9), using similar arguments that in Proposition 2.26. It is here that we require Assumption 3.2(b) to hold.

Proposition 3.40. *Grant Assumptions 3.1 and 3.2 and let $J \geq 0$ be fixed. Given any $\kappa \geq 0$, there is a constant $\bar{\alpha} \geq \kappa$ (only depending on b, f, J and κ) such that for all $\nu \in \mathcal{P}(\mathbb{R}_+)$ with $\nu(f^2) < \infty$ and all $s \geq 0$, it holds that*

$$\forall \boldsymbol{a} \in \mathcal{C}([s,\infty); \mathbb{R}_+), \quad \left\{ \sup_{t \geq s} a_t \leq \bar{\alpha} \text{ and } J\nu(f) \leq \bar{\alpha} \right\} \implies \sup_{t \geq s} J r^\nu_{\boldsymbol{a}}(t,s) \leq \bar{\alpha}.$$

Moreover, $\bar{\alpha}$ can be chosen to be an increasing function of J and κ.

Proof. Assume $\sup_{t \geq s} a_t \leq \bar{\alpha}$ for some $\bar{\alpha} > 0$ that we specify later. Applying the Itô formula and taking expectations yields

$$\forall t \geq s, \quad \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}) = \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{s,s}) + \int_s^t \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{u,s})[b(Y^{\boldsymbol{a},\nu}_{u,s}) + a_u] du - \int_s^t \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{u,s}) du.$$

Lemma 2.25 implies that $t \mapsto \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s})$ is $\mathcal{C}^1$ and

$$\forall t \geq s, \ \frac{d}{dt}\mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}) = \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s})(b(Y^{\boldsymbol{a},\nu}_{t,s}) + a_t) - \mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{t,s}).$$

Using (3.2), the Cauchy-Schwarz inequality gives

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}) &\leq \left\{ [\bar{\alpha} + C_b]\, \mathbb{E}\, f'(Y^{\boldsymbol{a},\nu}_{t,s}) - \frac{1}{2}\mathbb{E}\, f^2(Y^{\boldsymbol{a},\nu}_{t,s}) \right\} - \frac{1}{2}\mathbb{E}^2\, f(Y^{\boldsymbol{a},\nu}_{t,s}) \\ &\leq \frac{1}{2}[2\psi(\bar{\alpha} + C_b) - \mathbb{E}^2\, f(Y^{\boldsymbol{a},\nu}_{t,s})],\end{aligned}$$

where in the last line, we used Assumption 3.2(b). Setting $M(\bar{\alpha}) := \sqrt{2\psi(\bar{\alpha} + C_b)}$ and using Lemmas 2.27 and 2.28 we conclude that

$$\nu(f) \leq M(\bar{\alpha}) \implies [\forall t \geq s \ \ \mathbb{E}\, f(Y^{\boldsymbol{a},\nu}_{t,s}) \leq M(\bar{\alpha})].$$

To complete the proof, we need to check that for any $\kappa \geq 0$, any $J \geq 0$, there is a constant $\bar{\alpha} \geq \kappa$ such that $JM(\bar{\alpha}) \leq \bar{\alpha}$. This follows easily from Assumption 3.2(b), which gives

$$\lim_{\theta \to \infty} \frac{J\sqrt{2\psi(\theta)}}{\theta} = 0.$$

It is clear that $\bar{\alpha}(J)$ can be chosen to be a non-decreasing function of J and κ. This ends the proof. $\square$

3.7.2 Proof of Theorem 3.7

We are now in position to prove Theorem 3.7.

- **Step 1** Recall that equation (2.14) gives

$$\frac{d}{dt}\,\mathbb{E}\, f(X_t) \le \frac{1}{2}[\bar{r}(J)^2 - \mathbb{E}^2\, f(X_t)],$$

  where $(X_t)_{t\ge 0}$ is the solution of the nonlinear equation (1.2) and the function $J \mapsto \bar{r}(J)$ is non-decreasing. Using Proposition 3.40 with $\kappa := J\bar{r}(J)+1$, there is a non-decreasing function $J \mapsto \bar{\alpha}(J)$ such that:

$$\forall J, s \ge 0,\ \forall \boldsymbol{a} \in \mathcal{C}([s,\infty);\mathbb{R}_+),\ [\sup_{t\ge s} a_t \le \bar{\alpha}(J) \text{ and } J\nu(f) \le \bar{\alpha}(J)] \implies \sup_{t\ge s} J r^{\nu}_{\boldsymbol{a}}(t,s) \le \bar{\alpha}(J).$$

  Moreover, it holds that $\forall J \ge 0,\ J\bar{r}(J) < \bar{\alpha}(J)$.

- **Step 2** Let $J_m > 0$ be given by Proposition 3.6. Define

$$\lambda^* := \inf_{\alpha \in [0,\bar{\alpha}(J_m)]} \lambda^*_\alpha.$$

  Lemma 3.37 gives $\lambda^* > 0$. We now fix λ such that $0 < \lambda < \lambda^*$.

- **Step 3**

  - Using Corollary 3.39, we know that the solution of the Volterra equation $r_\alpha = K_\alpha + K_\alpha * r_\alpha$ is $r_\alpha = \gamma(\alpha) + \xi_\alpha$ with $\xi_\alpha \in L^1_\lambda$ and that:

$$\xi^\infty(J) := \sup_{\alpha\in[0,\bar{\alpha}(J)]} ||\xi_\alpha||^1_\lambda < \infty.$$

    It is clear that $J \mapsto \xi^\infty(J)$ is non-decreasing (as $J \mapsto \bar{\alpha}(J)$ is non-decreasing).
  - One can find a function $k^\infty : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, non-decreasing with respect to its two parameters, such that for all $\boldsymbol{a} \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}_+)$ we have:

$$\sup_{t\ge 0} a_t \le \bar{\alpha} \implies ||K^\nu_{\boldsymbol{a}}||^\infty_\lambda \le k^\infty(\nu(f),\bar{\alpha}) < \infty.$$

    Moreover, one can find a constant h^∞ (only depending on λ, b and f) such that for all $\boldsymbol{a} \in \mathcal{C}(\mathbb{R}_+;\mathbb{R}_+)$, we have

$$||H^\nu_{\boldsymbol{a}}||^\infty_\lambda \le h^\infty.$$

    These two points follow from $\lambda < f(\sigma_0)$, Assumption 3.1 and Remark 3.16(a).
  - The function $\eta_{\bar{\alpha}}$ of Lemma 3.32 satisfies

$$||\eta_{\bar{\alpha}}||_1 < \infty,\ \ \bar{\alpha} \mapsto ||\eta_{\bar{\alpha}}||_1 \text{ is non-decreasing,}$$

    and consequently the function $J \mapsto ||\eta_{\bar{\alpha}(J)}||_1$ is non-decreasing.

- Finally the normalization γ is a non-decreasing function of α (see (3.10)) and it follows that
$$\forall \alpha \in [0, \bar{\alpha}(J)], \quad \gamma(\alpha) \leq \gamma(\bar{\alpha}(J)).$$

- **Step 4** Let ν be a probability measure such that $\nu(f) \leq \bar{r}(J_m) + 1$. Recall that for all $J \in (0, J_m)$, the equation $\alpha\gamma^{-1}(\alpha) = J$ has a unique solution $\alpha^*(J) \in [0, \bar{\alpha}(J_m)]$. We now apply Proposition 3.36 with $\beta = 1/2$. Define:
$$C(J) := \frac{1}{2||\eta_{\bar{\alpha}(J)}||_1 (1 + \lambda^{-1}) \left(1 + \xi^\infty(J) + \gamma(\bar{\alpha}(J))\right)}$$
$$D(J) := 2 \left(1 + \gamma(\bar{\alpha}(J)) + \xi^\infty(J)\right) k^\infty(\bar{r}(J_m) + 1, \bar{\alpha}(J)) + \gamma(\bar{\alpha}(J)) h^\infty.$$
From Step 3, it is clear that the functions $J \mapsto \frac{1}{C(J)}$ and $J \mapsto D(J)$ are non-decreasing. Consequently, we can find a constant $J^* \in (0, J_m)$ such that
$$\forall J \in [0, J^*], \quad \frac{J D(J)}{C(J)} \leq 1.$$
Proposition 3.36 tells us that for every $0 \leq J \leq J^*$, given any $\boldsymbol{a} \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$ with $\sup_{t \geq 0} a_t \leq \bar{\alpha}(J)$ and such that
$$\forall t \geq 0, \quad |a_t - \alpha^*(J)| \leq C(J) e^{-\lambda t},$$
it holds
$$\forall t \geq 0, \quad |J r^\nu_{\boldsymbol{a}}(t, 0) - \alpha^*(J)| \leq J D(J) e^{-\lambda t} \leq C(J) e^{-\lambda t}.$$

- **Step 5** Let now $J \in (0, J^*]$ be fixed (the case $J = 0$ is already treated by Proposition 3.17). We assume the initial condition ν of (1.2) satisfies $J\nu(f) \leq \bar{\alpha}(J)$ and that $\nu(f) \leq \bar{r}(J_m) + 1$ (we shall come back to the general case in Step 6). We define recursively $\boldsymbol{a}^n \in \mathcal{C}(\mathbb{R}_+; \mathbb{R}_+)$ by
$$\forall t \geq 0, \quad a^0(t) := \alpha^*(J) \quad \text{and} \quad \forall n \geq 0, \quad a^{n+1}(t) := J r^\nu_{\boldsymbol{a}^n}(t, 0). \tag{3.27}$$
From Step 4 and by induction, it holds that:
$$\forall n \geq 0, \ \forall t \geq 0, \quad |a^n(t) - \alpha^*(J)| \leq C(J) e^{-\lambda t}.$$
We deduce that
$$\begin{aligned}\forall t \geq 0, \quad |\mathbb{E} f(X_t) - \gamma(\alpha^*(J))| &\leq |\mathbb{E} f(X_t) - r^\nu_{\boldsymbol{a}^n}(t, 0)| + \frac{1}{J} \left| a^{n+1}(t) - \alpha^*(J) \right| \\ &\leq \frac{1}{J} |J \, \mathbb{E} f(X_t) - a^{n+1}(t)| + \frac{C(J)}{J} e^{-\lambda t}.\end{aligned}$$
The Picard iteration studied in Section 2.6 shows that
$$\forall t \geq 0, \quad \lim_{n \to \infty} |J \, \mathbb{E} f(X_t) - a^n(t)| = 0.$$
We have proved that
$$\forall t \geq 0, \quad |\mathbb{E} f(X_t) - \gamma(\alpha^*(J))| \leq \frac{C(J)}{J} e^{-\lambda t}.$$

- **Step 6** We now prove that there exists $s \geq 0$ such that $\mathbb{E}\, f(X_s) \leq \min(\frac{\bar{\alpha}(J)}{J}, \bar{r}(J_m)+1)$. By Step 1, we have $\limsup \mathbb{E}\, f(X_t) \leq \bar{r}(J)$. Since $\bar{r}(J) < \bar{\alpha}(J)/J$ and since $\bar{r}(J) \leq \bar{r}(J_m)$, the conclusion follows. Consequently, Step 5 can be applied to the process $(X_t)_{t \geq s}$ starting with $\nu = \mathcal{L}(X_s)$. This proves the convergence of the jump rate.

The convergence of the law of X_t to the invariant measure then follows from Proposition 3.15. This ends the proof of Theorem 3.7.

Remark 3.41. *There is some freedom in the above construction of the constants λ and J^*. We can choose any λ in $[0, \lambda^*)$ and the value of J^* depends both on λ and on a parameter $\beta \in (0,1)$, here chosen to be equal to $1/2$ (see Step 4). We may optimize this construction to get either J^* or λ as large as possible.*

3.8 Discussions and perspectives

Studying the long time behavior of a McKean-Vlasov SDE is generally a difficult task. One can study it by considering the long time behavior of the finite particle system (1.1) and then apply the propagation of chaos to extend the results to the McKean-Vlasov equation (1.2). This strategy has been developed in [Ver06; BGM10] for diffusive problems. The long time behavior of the particle system (1.1) has been studied in [DO16; HKL18] (again in a slightly different setting but the methods could be adapted to our case): the authors proved that the particle system is Harris-ergodic and consequently converges weakly to its unique invariant probability measure. However, transferring the long time behavior of the particle system to the McKean-Vlasov equation is possible if the propagation of chaos holds *uniformly in time*. In [DGLP15; FL16], the propagation of chaos is only proved on compact time interval $[0, T]$ and their estimates diverge as T goes to infinity. A natural (and difficult) question is the following: in the case of small enough interaction parameters J, is uniform in time propagation of chaos holds?

Coupling methods are also used to study the long time behavior of SDEs. In [BCGMZ13], the authors have studied the TCP (a linear PDMP) which is close to (1.2). The size of the jumps is $-x/2$ in the TCP and $-x$ in our setting, x being the position of the process just before the jump. The main difference is the nonlinearity: we failed to adapt their methods when the interactions are non-zero ($J > 0$). Recently, in [LM20], the authors were able to implement such coupling argument on the nonlinear SDE (1.2). It would be interesting to see if this argument could be adapted in the setting of weak-enough interactions.

Butkovsky studied in [But14] the long time behavior of some McKean-Vlasov diffusion SDE of the form:

$$\forall t \geq 0,\ X_t = X_0 + \int_0^t [b_1(X_u) + \epsilon b_2(X_u, \mu_u)]\, du + W_t,\ \mu_u = \mathcal{L}(X_u), \tag{3.28}$$

where $(W_t)_{t \geq 0}$ is a Brownian motion. Here the drift terms b_1 and b_2 are assumed to be globally Lipschitz and b_2 is assumed uniformly bounded with respect to its two parameters.

The author proved that if the parameter ϵ is small enough, (3.28) has a unique invariant probability measure which is globally stable. The case $\epsilon > 0$ (and small) is treated as a perturbation of the case $\epsilon = 0$ using a Girsanov transform. It could be interesting to see how this method could be adapted to SDE driven by Poisson measures, but we did not pursue this path.

Of course, the main question is: what happens when the interactions J **are not** small? We answer partially this question in the following Chapter, by describing the long time behavior of the solution of (1.2) assuming one starts with an initial condition ν close to a given invariant measure of (1.2).

Chapter 4

Local stability of the stationary solutions

Consider an invariant probability measure of the McKean-Vlasov equation (1.2). We give a sufficient condition to ensure the local stability of this invariant measure. Our criteria involves the location of the zeros of an explicit holomorphic function associated to the considered invariant probability measure. We prove that when all the complex zeros have negative real part, local stability holds. The material of this Chapter is available as a preprint [Cor20].

4.1 Introduction

Consider (X^ν_t) the solution of the McKean-Vlasov SDE (1.2)

$$X^\nu_t = X^\nu_0 + \int_0^t b(X^\nu_u)du + J\int_0^t \mathbb{E}\, f(X^\nu_u)du - \int_0^t \int_{\mathbb{R}_+} X^\nu_{u-} \mathbb{1}_{\{z \leq f(X^\nu_{u-})\}} \mathbf{N}(du, dz),$$

where $\mathcal{L}(X^\nu_0) = \nu$.

Understanding the long time behavior of the solution of (1.2) for an arbitrary interaction parameter J is a difficult open question. We are interested here in the following sub-problem: given an invariant probability measure of (1.2), at which condition this invariant measure is locally stable? That is, if we start from an initial condition ν "close" to the invariant probability measure ν_∞, does the solution of (1.2) converge to ν_∞? We assume that the initial condition ν belongs to

$$\mathcal{M}(f^2) := \{\nu \in \mathcal{P}(\mathbb{R}_+),\ \int_{\mathbb{R}_+} f^2(x)\nu(dx) < +\infty\}, \tag{4.1}$$

and we equip $\mathcal{M}(f^2)$ with the following weighted total variation distance

$$\forall \nu, \mu \in \mathcal{M}(f^2), \quad d(\nu, \mu) := \int_{\mathbb{R}_+} [1 + f^2(x)]|\nu - \mu|(dx). \tag{4.2}$$

By Proposition 3.9, the invariant measures of (1.2) belongs to $\mathcal{M}(f^2)$. Consider $(X^{\nu_\infty}_t)_{t\geq 0}$ the solution of (1.2) starting from an invariant measure ν_∞. Then, the mean-field interaction $\alpha := J\,\mathbb{E}\, f(X^{\nu_\infty}_t)$ is constant. We denote by ν^∞_α the invariant measure corresponding to a current $\alpha > 0$. Our main result, Theorem 4.13, gives a sufficient condition for ν^∞_α to be locally stable. Our condition involves the location of the roots of an explicit holomorphic function associated to ν_α^∞. When all the roots of this function have negative real part, we prove that the invariant measure is locally stable, in a precise sense. Furthermore, in Theorem 4.14, we prove that this last criteria is satisfied if

$$\inf_{x \in \mathbb{R}_+} f(x) + b'(x) \geq 0. \tag{4.3}$$

To be more precise, we only need a local version of (4.3): in the above inequality, we can replace $\mathbb{R}_+$ by the support of the invariant measure. We significantly generalize the result of [FL16] and [DV21], valid only for $b \equiv 0$. This local approach is a first step to study the static and dynamic bifurcations of (1.2), such as the Hopf bifurcations, leading to periodic solutions (see Chapter 5). We now detail the main arguments leading to the proof of Theorem 4.13.

Perturbation of constant currents. Let us recall some results from Chapters 2 and 3. Given a non-negative bounded measurable "external current" $\boldsymbol{a} \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, we consider $Y_{t,s}^{\boldsymbol{a},\nu}$ the solution of the linear non-homogeneous SDE (1.9) starting with law ν at time s and driven by the current $\boldsymbol{a}$. Let $r_{\boldsymbol{a}}^\nu(t,s) := \mathbb{E}\, f(Y_{t,s}^{\boldsymbol{a},\nu})$ be the associated spiking rate. We recall that it solves the Volterra integral equation (1.14) and that $Y_{t,0}^{\boldsymbol{a},\nu}$ is a solution of (1.2) if and only if (1.11) holds, that is

$$\forall t \geq 0, \quad a_t = J r_{\boldsymbol{a}}^\nu(t,0).$$

In Section 3.5, we studied the long time behavior of $(Y_{t,0}^{\alpha,\nu})$, the solution of (1.9) when $\boldsymbol{a}$ is constant: $a_t \equiv \alpha > 0$. We proved that $Y_{t,0}^{\alpha,\nu}$ converges in law to its invariant probability measure ν_α^∞, where ν_α^∞ has the explicit expression (3.4). The convergence holds at an exponential rate. More precisely, denote by $\mathcal{B}(\mathbb{R}_+;\mathbb{R})$ the Borel-measurable functions from $\mathbb{R}_+$ to $\mathbb{R}$ and define for any $\lambda \geq 0$ the Banach space

$$L_\lambda^\infty := \{h \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}),\ ||h||_\lambda^\infty < \infty\}, \quad \text{with} \quad ||h||_\lambda^\infty := \operatorname*{ess\,sup}_{t \geq 0} |h_t| e^{\lambda t}. \tag{4.4}$$

Let $\gamma(\alpha) = \nu_\alpha^\infty(f)$ be the mean number of jumps per unit of time under this invariant measure. We can find a constant $\lambda_\alpha^* > 0$ (defined by (3.14)) such that for all $\lambda \in (0, \lambda_\alpha^*)$ and all $\nu \in \mathcal{M}(f^2)$:

$$r_\alpha^\nu(t,0) - \gamma(\alpha) \in L_\lambda^\infty.$$

Then, in Section 3.6, we extended this result to non constant current of the form

$$a_t = \alpha + h_t,$$

where $\boldsymbol{h}$ belongs to L_λ^∞, $\lambda < \lambda_\alpha^*$. More specifically, we proved that there exists $\delta > 0$ such that for all $\boldsymbol{h} \in L_\lambda^\infty$ with $||\boldsymbol{h}||_\lambda^\infty < \delta$ and all $\nu \in \mathcal{M}(f^2)$, one has

$$r_{\alpha+\boldsymbol{h}}^\nu(t,0) - \gamma(\alpha) \in L_\lambda^\infty.$$

The implicit function theorem. We apply the implicit function theorem to the function

$$\Phi(\nu, \boldsymbol{h}) := J r_{\alpha+\boldsymbol{h}}^\nu(\cdot, 0) - (\alpha + \boldsymbol{h}),$$

which maps $\mathcal{M}(f^2) \times L_\lambda^\infty$ to L_λ^∞. Obviously one has

$$\Phi(\nu_\alpha^\infty, 0) = 0.$$

By inspecting carefully the perturbative argument of Section 3.6, one can prove that the function $\boldsymbol{h} \mapsto \Phi(\nu, \boldsymbol{h})$ is Fréchet differentiable on the Banach space L^∞_λ. We then compute $D_h\Phi(\nu^\infty_\alpha, 0)$, the Fréchet derivative of Φ at the point $(\nu^\infty_\alpha, 0)$. The key point is that $D_h\Phi(\nu^\infty_\alpha, 0)$ is a convolution

$$\forall c \in L^\infty_\lambda, \quad \left[D_h\Phi(\nu^\infty_\alpha, 0) \cdot c\right](t) = -c(t) + J\int_0^t \Theta_\alpha(t-u)c(u)du.$$

Here, the function $\Theta_\alpha : \mathbb{R}_+ \to \mathbb{R}$ has a simple expression in terms of the invariant measure ν^∞_α (see (4.8)). In order to proceed, we use the following ruse: given $\boldsymbol{h} \in L^\infty_\lambda$, we extend it to $\mathbb{R}$ by setting $h_t = 0$ for $t \in \mathbb{R}_-$. It then holds that

$$H^{\nu^\infty_\alpha}_{\alpha+\boldsymbol{h}}(t, 0) = \gamma(\alpha)\int_{-\infty}^0 H^{\delta_0}_{\alpha+\boldsymbol{h}}(t, u)du.$$

This formula, proved in Lemma 4.50, has a simple probabilistic interpretation which relies both on the fact that ν^∞_α is the invariant measure of $(Y^{\alpha,\nu}_{t,0})$ and on the fact that the process is reset to 0 just after a jump. The advantage of this representation is to eliminate the specific shape of the invariant measure ν^∞_α. We then study the inversibility of this linear mapping: it gives a criteria of stability in term of the location of the zeros of the holomorpic function discussed above. It is worth noting that the implicit function theorem provides an explicit Newton's type approximation scheme, which differs from the "standard" Picard iteration scheme (3.27) we used in Chapters 2 and 3. Remark 4.35 emphasis the difference between the two schemes. We prove that this Newton scheme converges to some $\boldsymbol{h}(\nu) \in L^\infty_\lambda$, provided that ν is sufficiently close to ν^∞_α. This limit $\boldsymbol{h}(\nu)$ satisfies

$$\Phi(\nu, \boldsymbol{h}(\nu)) = 0,$$

and so $\alpha + \boldsymbol{h}(\nu)$ solves (1.11). This proves that the nonlinear interactions $J\,\mathbb{E}\,f(X^\nu_u)$ of (1.2) converge to the constant current α at an exponential rate, provided that the law of the initial condition X^ν_0 is sufficiently close to ν^∞_α: it gives the stability of ν^∞_α.

The layout of this chapter is as follows. Our main results are given in Section 4.2. In Section 4.3, we give different interpretations of the result: we study the Fokker-Planck equation (1.3), *linearized* around the invariant measure ν^∞_α. Additionally, we give connections with the L-derivative and the Linear Functional Derivative. This section can be read independently. In Section 4.4, we introduce a functional analysis framework and give estimates on the kernels (1.13). Section 4.5 is devoted to the proof of Proposition 4.12, which shows the well-posedness of our stability criteria. Finally, Sections 4.6 and 4.7 are devoted to the proofs of our main results (Theorem 4.14 and 4.13).

4.2 Notations and results

We assume that:

Assumption 4.1. *The drift $b : \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{C}^2$, with $b(0) \geq 0$ and*

$$\sup_{x\geq 0} |b'(x)| + |b''(x)| < \infty.$$

Assumption 4.2. *The function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is $\mathcal{C}^2$, strictly increasing, with $f(0) = 0$, $\sup_{x\geq 1} \frac{|f''(x)|}{f(x)} < \infty$ and there exists a constant C_f such that*

4.2(a) for all $x, y \geq 0$, $f(xy) \leq C_f(1 + f(x))(1 + f(y))$.

4.2(b) for all $A > 0$, $\sup_{x\geq 0} Af'(x) - f(x) < \infty$.

4.2(c) for all $x \geq 0$, $|b(x)| \leq C_f(1 + f(x))$.

Remark 4.3. *If a non-decreasing function f satisfies Assumption 4.2(a), there exists another constant C_f such that for all*

$$\forall x, y \geq 0, \quad f(x + y) \leq C_f(1 + f(x) + f(y)).$$

So under Assumptions 4.1 and 4.2, the Assumptions 2.3 and 2.5 hold. In particular, Theorem 2.8, Proposition 3.9 as well as Proposition 3.25 apply, and the constant λ^*_α, defined by (3.14) is strictly positive.

Remark 4.4. *Let $b_0 \geq 0$, $b_1 \in \mathbb{R}$ and $p \geq 1$. Then the following functions b and f satisfy Assumptions 4.1 and 4.2:*

$$\forall x \geq 0, \quad b(x) = b_0 + b_1 x, \quad \text{and} \quad f(x) = x^p.$$

In particular b does not need to be bounded and one may have $b(0) = 0$.

Consider an invariant measure ν_∞ of (1.2). If $(X^{\nu_\infty}_t)$ starts from the law ν_∞, its jumps rate $t \mapsto \mathbb{E}\, f(X^{\nu_\infty}_t)$ is constant. Define

$$\alpha := J\, \mathbb{E}\, f(X^{\nu_\infty}_t).$$

We say that ν_∞ is non-trivial if $\alpha > 0$. For such α, define

$$\sigma_\alpha := \inf\{x \geq 0 : b(x) + \alpha = 0\}, \quad \text{with} \quad \inf \emptyset = \infty.$$

Because $b(0) + \alpha > 0$, one has $\sigma_\alpha \in \mathbb{R}^*_+ \cup \{+\infty\}$. Mimicking the proof of Lemma 3.12, we find

Proposition 4.5. *Let f and b such that Assumptions 4.1 and 4.2 hold. The non-trivial invariant measures of (1.2) are $\{\nu^\infty_\alpha \mid \alpha \in (0, \infty), \alpha = J\gamma(\alpha)\}$, where ν^∞_α is given by (3.4)*

$$\nu^\infty_\alpha(dx) := \frac{\gamma(\alpha)}{b(x) + \alpha} \exp\left(-\int_0^x \frac{f(y)}{b(y) + \alpha} dy\right) \mathbb{1}_{[0,\sigma_\alpha)}(x) dx$$

and $\gamma(\alpha)$ is the normalizing factor, given by (3.5).

Note that in Lemma 3.12, it is assumed that $b(0) > 0$, and so δ_0, the Dirac measure at 0, is not an invariant probability measure. Here, one may have $b(0) = 0$ and if it is the case, δ_0 is the trivial invariant probability measure of (1.2).

In this work, we focus on the stability of the non-trivial invariant measures, which have the above explicit expression. For $\alpha > 0$, we define $J(\alpha)$ to be the corresponding interaction parameter:

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)}. \tag{4.5}$$

Assumption 4.6. *The constant $\alpha > 0$ satisfies one of the following non-degeneracy condition:*

$$\sigma_\alpha < \infty \quad \textit{and} \quad b'(\sigma_\alpha) < 0 \tag{4.6}$$

$$\textbf{\textit{or}} \qquad \sigma_\alpha = \infty \quad \textit{and} \quad \inf_{x \geq 0} b(x) + \alpha > 0. \tag{4.7}$$

If $\sigma_\alpha < \infty$ we have a technical restriction on the size of the support of the initial datum:

Definition 4.7. *Define*

$$\tilde{\sigma}_\alpha := \inf\{x > \sigma_\alpha \mid b(x) + \alpha = 0\}, \qquad \textit{with} \qquad \inf \emptyset = +\infty,$$
$$\mathcal{S}_\alpha := \{[0, \beta],\ \sigma_\alpha \leq \beta < \tilde{\sigma}_\alpha\},$$

with the convention that $\mathcal{S}_\alpha := \{\mathbb{R}_+\}$ when $\sigma_\alpha = +\infty$.

Remark 4.8. *Note that due to (4.6), if $\sigma_\alpha < \infty$ one has $\sigma_\alpha < \tilde{\sigma}_\alpha$ (and $\tilde{\sigma}_\alpha = +\infty$ if $\sigma_\alpha = +\infty$). In particular, for all $\sigma_\alpha < x < \tilde{\sigma}_\alpha$, it holds that $b(x) + \alpha < 0$. Any $S \in \mathcal{S}_\alpha$ is invariant by the dynamics in the following sense: given $\lambda > 0$ we can find $\delta > 0$ small enough such that for all $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$ one has*

$$x \in S \implies \left[\forall t \geq s, \quad Y^{\alpha + \boldsymbol{h}, \delta_x}_{t,s} \in S\right].$$

We exploit this property in Section 4.7.3.

Given $S \in \mathcal{S}_\alpha$, we denote by $\mathcal{M}_S(f^2)$ the set of probability measure with support included in S and such that $\int_S f^2(x)\mu(dx) < \infty$. We equip $\mathcal{M}_S(f^2)$ with the distance (4.2).

Definition 4.9. *Let $\lambda > 0$. An invariant measure ν^∞_α of (1.2) is said to be locally exponentially stable with rate λ if for all $S \in \mathcal{S}_\alpha$ and all $\epsilon > 0$, there exists $\rho > 0$ such that*

$$\forall \nu \in \mathcal{M}_S(f^2), \quad d(\nu, \nu^\infty_\alpha) < \rho \implies \sup_{t \geq 0} |J(\alpha)\, \mathbb{E}\, f(X^\nu_t) - \alpha| e^{\lambda t} < \epsilon\ ,$$

where (X^ν_t) is the solution of (1.2) starting with law ν.

Remark 4.10. *Once it is known that the current $J(\alpha)\,\mathbb{E}\, f(X^\nu_t)$ converges to the constant α at an exponential rate, it holds that (X^ν_t) converges in law to ν^∞_α at an exponential rate. Indeed, it is straightforward to adapt the proof of Proposition 3.15 to our Assumptions 4.1, 4.2 and 4.6.*

Definition 4.11. *Given $\alpha > 0$, let ν^∞_α be the corresponding invariant measure and define*

$$\forall t \geq 0, \quad \Theta_\alpha(t) := \int_0^\infty \left[\frac{d}{dx} r^x_\alpha(t)\right] \nu^\infty_\alpha(dx). \tag{4.8}$$

Proposition 4.12. *Under Assumptions 4.1, 4.2 and 4.6, it holds that for all $\lambda \in (0, \lambda^*_\alpha)$, the function $t \mapsto e^{\lambda t}\Theta_\alpha(t)$ belongs to $L^1(\mathbb{R}_+)$.*

The proof is given in Section 4.5. We can thus consider $\widehat{\Theta}_\alpha(z)$, the Laplace transform of Θ_α, defined for all $z \in \mathbb{C}$ with $\Re(z) > -\lambda^*_\alpha$.

Theorem 4.13. *Consider a non-trivial invariant measure ν_α^∞ of (1.2), for some $\alpha > 0$. Grant Assumptions 4.1, 4.2 and 4.6. Define the "abscissa" of the first zero of $J(\alpha)\widehat{\Theta}_\alpha - 1$ to be:*

$$\lambda'_\alpha := - \sup_{z\in\mathbb{C}, \Re(z) > -\lambda^*_\alpha} \{\Re(z) \mid J(\alpha)\widehat{\Theta}_\alpha(z) = 1\}.$$

It holds that $\lambda'_\alpha \in [-\infty, \lambda^*_\alpha]$. Assume that

$$\lambda'_\alpha > 0. \tag{4.9}$$

Then for all $\lambda \in (0, \lambda'_\alpha)$, ν_α^∞ is locally exponentially stable with rate λ, in the sense of Definition 4.9. That is, for all $S \in \mathcal{S}_\alpha$ and all $\epsilon > 0$, there exists $\rho > 0$ such that

$$\forall \nu \in \mathcal{M}_S(f^2), \quad d(\nu, \nu_\alpha^\infty) < \rho \implies \sup_{t\geq 0} |J(\alpha)\,\mathbb{E}\, f(X_t^\nu) - \alpha| e^{\lambda t} < \epsilon,$$

where (X_t^ν) is the solution of (1.2) starting with initial law ν.

The proof is given in Section 4.7. We now give a sufficient condition for (4.9) to hold, namely

$$\inf_{x\in[0,\sigma_\alpha)} f(x) + b'(x) \geq 0. \tag{4.10}$$

Theorem 4.14. *Consider f and b satisfying Assumptions 4.1 and 4.2.*

1. *Let $\alpha > 0$ such that Assumption 4.6 holds. Assume furthermore that the condition (4.10) is satisfied. Then the non-trivial invariant measure ν_α^∞ is locally exponentially stable, in the sense of Definition 4.9.*

2. *Assume that $\inf_{x\geq 0} f(x) + b'(x) \geq 0$. Then for all $J > 0$ the nonlinear equation (1.2) has exactly one non-trivial invariant measure (which is locally exponentially stable).*

The proof is given in Section 4.6.

Remark 4.15. *This result generalizes the case $b \equiv 0$, which is well-known. When $b \equiv 0$, (1.2) has two invariant measures: a trivial one (δ_0, the Dirac mass at zero) and a non-trivial one. The trivial invariant measure δ_0 is known to be unstable, whereas the non-trivial invariant measure is stable (see [FL16], Proposition 11 and [DV21]). Given the assumptions of Theorem 4.14, the question of the* ***global*** *convergence to the unique invariant measure is left open. The situation where*

$$\forall x \geq 0, \quad f(x) + b'(x) = 0$$

is an interesting limit case for which the invariant probability measure is the uniform distribution on $[0, \sigma_\alpha]$.

4.3 Interpretations of the result

4.3.1 The linearized Fokker-Planck equation near the equilibrium

The objective of this section is to provide a **heuristic** view point about the stability criteria (4.9) through a linearized analysis of the PDE (1.3). Let $g \in \mathcal{C}^1(\mathbb{R}_+;\mathbb{R})$ be a compactly supported test function. The Itô's formula applied to (1.2) gives

$$\frac{d}{dt}\mathbb{E}\, g(X_t) = \mathbb{E}\, g'(X_t)\left[b(X_t) + J(\alpha)\,\mathbb{E}\, f(X_t)\right] + \mathbb{E}\left[g(0) - g(X_t)\right] f(X_t). \tag{4.11}$$

In other words, if $\nu(t, dx)$ is the law of X_t, it solves the Fokker-Planck PDE (1.3). Consider now an invariant measure ν_α^∞ of (1.2), for some $\alpha > 0$. Using that $\langle \nu_\alpha^\infty, f\rangle = \gamma(\alpha)$, one has

$$\partial_x\left[(b(x) + \alpha)\nu_\alpha^\infty\right] + f(x)\nu_\alpha^\infty(x) = \gamma(\alpha)\delta_0(dx).$$

Define $\phi(t, dx) := \nu(t, dx) - \nu_\alpha^\infty(x)dx$, it solves

$$\begin{aligned}\partial_t\phi(t, dx) &+ \partial_x\left[(b(x) + \alpha)\phi(t, dx)\right] + f(x)\phi(t, dx)\\ &+ J(\alpha)\langle\phi(t), f\rangle\partial_x\phi(t, dx) + J(\alpha)\langle\phi(t), f\rangle\partial_x\nu_\alpha^\infty(dx) = \langle\phi(t), f\rangle\delta_0(dx).\end{aligned}$$

We used again the notation

$$\langle\phi(t), f\rangle := \int_0^\infty f(x)\phi(t, dx).$$

The term $J(\alpha)\langle\phi(t), f\rangle\partial_x\phi(t, dx)$ is of **second order** in ϕ. By neglecting it, we obtain the **linearized Fokker-Planck** equation

$$\partial_t\phi(t, dx) + \partial_x\left[(b(x) + \alpha)\phi(t, dx)\right] + J(\alpha)\langle\phi(t), f\rangle\partial_x\nu_\alpha^\infty(dx) + f(x)\phi(t, dx) = \langle\phi(t), f\rangle\delta_0(dx). \tag{4.12}$$

This equation can be written

$$\partial_t\phi = \mathcal{L}_\alpha^*\phi + \mathcal{B}_\alpha\phi,$$

with

$$\begin{aligned}\mathcal{L}_\alpha^*\phi &:= -\partial_x\left[(b + \alpha)\phi\right] - f\phi + \langle\phi, f\rangle\delta_0,\\ \mathcal{B}_\alpha\phi &:= -J(\alpha)\langle\phi, f\rangle\partial_x\nu_\alpha^\infty.\end{aligned}$$

First note that the operator $\mathcal{B}_\alpha\phi$ is defined such that for any test function $g \in \mathcal{C}^1(\mathbb{R}_+)$, it holds that

$$\langle\mathcal{B}_\alpha\phi, g\rangle := J(\alpha)\langle\phi, f\rangle\langle\nu_\alpha^\infty, g'\rangle.$$

In particular, the operator $\partial_x\nu_\alpha^\infty$ is defined such that

$$\langle\partial_x\nu_\alpha^\infty, g\rangle = -\langle\nu_\alpha^\infty, g'\rangle. \tag{4.13}$$

Second, remark that $\mathcal{L}_\alpha^*$ is the generator of the process corresponding to an isolated neuron subject to a constant current equal to α. Let $(T_\alpha(t))_{t\geq 0}$ be the Markov semi-group generated by $\mathcal{L}_\alpha^*$. Using the Duhamel's principle (see for instance [EN00, Chapter III, Corollary 1.7]), the solution of the linearized equation (4.12) satisfies

$$\phi(t) = T_\alpha(t)\phi(0) - J(\alpha)\int_0^t \langle \phi(s), f\rangle T_\alpha(t-s)\partial_x \nu_\alpha^\infty ds.$$

Integrating f against this equation, one obtains a closed integral equation for $\langle \phi(t), f\rangle$

$$\langle \phi(t), f\rangle = \langle T_\alpha(t)\phi(0), f\rangle - J(\alpha)\int_0^t \langle \phi(s), f\rangle \langle T_\alpha(t-s)\partial_x \nu_\alpha^\infty, f\rangle ds.$$

Claim: One has for all $t \geq 0$, $-\langle T_\alpha(t)\partial_x\nu_\alpha^\infty, f\rangle = \Theta_\alpha(t)$.
Proof of the claim: Let $(Y_t^{\alpha,x})$ be the solution of the SDE (1.9), with constant current α and starting with law δ_x at $t = 0$. For all ν, it holds that

$$\begin{aligned}\langle T_\alpha(t)\nu, f\rangle &= \int_0^\infty \mathbb{E}\, f(Y_t^{\alpha,x})\nu(dx)\\ &= \int_0^\infty r_\alpha^x(t)\nu(dx)\\ &= \langle \nu, r_\alpha^\cdot(t)\rangle.\end{aligned}$$

We shall see that for a fixed value of t, the function $x \mapsto r_\alpha^x(t)$ is $\mathcal{C}^1$ (see the proof of Proposition 4.12). Using (4.13) (with $g(x) = r_\alpha^x(t)$), we finally have

$$\langle T_\alpha(t)\partial_x\nu_\alpha^\infty, f\rangle = -\int_0^\infty \frac{d}{dx} r_\alpha^x(t)\nu_\alpha^\infty(dx),$$

and the claim follows.

Consequently, $\langle \phi(t), f\rangle$ solves the convolution Volterra equation

$$\langle \phi(t), f\rangle = \int_0^\infty r_\alpha^x(t)\phi(0)(dx) + J(\alpha)\int_0^t \Theta_\alpha(t-s)\langle \phi(s), f\rangle ds. \tag{4.14}$$

Claim: For all $\lambda \in (0, \lambda_\alpha^*)$, the function $t \mapsto e^{\lambda t}\int_0^\infty r_\alpha^x(t)\phi(0)(dx)$ belongs to $L^1(\mathbb{R}_+)$.
Proof of the claim: Because $r_\alpha^x(t)$ is the jump rate of an isolated neuron subject to a constant current α, one has $\lim_{t\infty} r_\alpha^x(t) = \gamma(\alpha) = \nu_\alpha^\infty(f)$ exponentially fast. More precisely, define

$$\xi_\alpha^x(t) := r_\alpha^x(t) - \gamma(\alpha),$$

Proposition 3.17 yields

$$\forall \lambda \in (0, \lambda_\alpha^*), \quad e^{\lambda t}\xi_\alpha^x(t) \in L^1(\mathbb{R}_+).$$

Recall that $\phi(0)$ is the difference of two probability measures so $\int_0^\infty \gamma(\alpha)\phi(0)(dx) = 0$. So for all $\lambda \in (0, \lambda_\alpha^*)$,

$$t \mapsto e^{\lambda t}\int_0^\infty r_\alpha^x(t)\phi(0)(dx) = e^{\lambda t}\int_0^\infty \xi_\alpha^x(t)\phi(0)(dx) \in L^1(\mathbb{R}_+).$$

Consequently, $e^{\lambda t}\langle \phi(t), f\rangle$ solves a Volterra integral equation where both the "forcing term" $t \mapsto e^{\lambda t}\int_0^\infty r^x_\alpha(t)\phi(0)(dx)$ and the "kernel" $t \mapsto J(\alpha)e^{\lambda t}\Theta_\alpha(t)$ belongs to $L^1(\mathbb{R}_+)$. The condition for $e^{\lambda t}\langle \phi(t), f\rangle$ to belongs to $L^1(\mathbb{R}_+)$ is exactly (4.9) [see GLS90, Chapter 2]. If (4.9) holds then

$$\forall \lambda \in (0, \lambda'_\alpha), \quad t \mapsto e^{\lambda t}\langle \phi(t), f\rangle \in L^1(\mathbb{R}_+).$$

This gives the linear stability of the invariant measure ν^∞_α. From this point of view, Theorem 4.13 is a *Principle of Linearized Stability*: it legitimates the linearization of the Fokker-Planck equation above, in the sense that stability of the linearized equation implies stability of the nonlinear Fokker-Planck equation.

4.3.2 Connection with the L-derivative and the Linear Functional Derivative

Let $\mathcal{P}_2(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$ having a second moment. Consider a function $\mathcal{P}_2(\mathbb{R}) \ni \mu \mapsto u(\mu) \in \mathbb{R}$ that we seek to differentiate with respect to μ. We first review briefly two notions of differentiability of functions of probability measures: the L-derivative and the Linear Functional Derivative. We follow the presentation of [CD18].

The L-derivative

Consider $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ a Hilbert space defined over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and define the *lifted* function $\tilde{u}$

$$\forall Z \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}), \quad \tilde{u}(Z) := u(\mathcal{L}(Z)).$$

Because $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ is a Banach space, we can define the Fréchet derivative of $\tilde{u}$. This Fréchet derivative serves as the definition of the L-derivative.

Definition 4.16. *A function $u : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to be L-differentiable at $\mu_0 \in \mathcal{P}_2(\mathbb{R})$ if there exists a random variable $Z_0 \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ with $\mathcal{L}(Z_0) = \mu_0$ and such that $\tilde{u}$ is Fréchet-differentiable at Z_0.*

If $\tilde{u}$ is Fréchet-differentiable at Z_0, let $D\tilde{u}(Z_0)$ its derivative, it can see it as an element of $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ (that is, we identify $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ and its dual). Let μ, μ_0 in $\mathcal{P}_2(\mathbb{R})$. We then have

$$u(\mu) = u(\mu_0) + \mathbb{E}\, D\tilde{u}(Z_0)(Z - Z_0) + o\left(\sqrt{\mathbb{E}(Z - Z_0)^2}\right),$$

as $\sqrt{\mathbb{E}(Z - Z_0)^2}$ goes to zero, provided that $\mathcal{L}(Z) = \mu$ and $\mathcal{L}(Z_0) = \mu_0$. A key result of the theory is that the element $D\tilde{u}(Z_0)$ can be represented by a deterministic measurable function, denoted $\partial_\mu u(\mu_0)(\cdot) : \mathbb{R} \mapsto \mathbb{R}$, such that

$$\mathbb{P}(d\omega) - a.s., \quad D\tilde{u}(Z_0)(\omega) = \partial_\mu u(\mu_0)(Z_0(\omega)).$$

So we have

$$u(\mu) = u(\mu_0) + \mathbb{E}\, \partial_\mu(\mu_0)(Z_0)(Z - Z_0) + o\left(\sqrt{\mathbb{E}(Z - Z_0)^2}\right), \tag{4.15}$$

as $\sqrt{\mathbb{E}(Z - Z_0)^2}$ goes to zero, provided that $\mathcal{L}(Z) = \mu$ and $\mathcal{L}(Z_0) = \mu_0$. The function $\partial_\mu u(\mu_0)(\cdot)$ is the L-derivative of u at μ_0. Of course, much work is needed to prove that this function exists and does not depend on the specific choice of the $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R})$ space. We refer to [CD18, Ch. 5.2] for the complete construction of this object.

The Linear Functional Derivative

We quote [CD18, Ch. 5.4]:

Definition 4.17. *A function $u : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is said to have a Linear Functional Derivative if there exists a function*

$$\frac{\delta u}{\delta m} : \mathcal{P}_2(\mathbb{R}) \times \mathbb{R} \ni (m, x) \mapsto \frac{\delta u}{\delta m}(m)(x) \in \mathbb{R},$$

continuous (for the product topology, the space $\mathcal{P}_2(\mathbb{R})$ being equipped with the 2-Wasserstein distance, denoted W_2), such that, for any bounded subset $\mathcal{K} \subset \mathcal{P}_2(\mathbb{R})$, the function $x \mapsto \frac{\delta u}{\delta m}(m)(x)$ is at most of quadratic growth in x uniformly in m for $m \in \mathcal{K}$, and such that for all $m, m' \in \mathcal{P}_2(\mathbb{R})$, it holds that

$$u(m') - u(m) = \int_0^1 \int_{\mathbb{R}} \frac{\delta u}{\delta m}(tm' + (1-t)m)(x)(m' - m)(dx)dt.$$

Under some regularity conditions on $\frac{\delta u}{\delta m}$ (see [CD18, Prop. 5.44 and Prop. 5.48]), it holds that

$$u(m') - u(m) = \int_{\mathbb{R}} \frac{\delta u}{\delta m}(m)(x)[m' - m](dx) + o\big(W_2(m', m)\big),$$

as $W_2(m', m)$ goes to zero. Furthermore, the function u is L-differentiable with:

$$\partial_\mu(\mu)(\cdot) = \partial_x \frac{\delta u}{\delta m}(\mu)(\cdot).$$

That is, the L-derivative is the gradient of the Linear Functional Derivative.

The interpretation

Let $t \geq 0$ be fixed, define

$$u_t : \mathcal{P}_2(\mathbb{R}) \ni \nu \mapsto J(\alpha)\, \mathbb{E}\, f(X_t^\nu),$$

where (X_t^ν) is the solution of the nonlinear equation (1.2), starting at time 0 with law ν. We compute the Linear Functional Derivative and the L-derivative of u_t at ν_α^∞, assuming these two objects exist. Consider $\Omega_\alpha : \mathbb{R}_+ \to \mathbb{R}$ the resolvent of $J(\alpha)\Theta_\alpha$, that is the solution of the Volterra integral equation

$$\forall t \geq 0, \quad \Omega_\alpha(t) = J(\alpha)\Theta_\alpha(t) + J(\alpha) \int_0^t \Theta_\alpha(t-u)\Omega_\alpha(u)du. \tag{4.16}$$

Using Lemma 2.22, we can solve (4.14) in terms of Ω_α:

$$\langle \phi(t), f\rangle = \int_0^\infty r_\alpha^x(t)\phi(0)(dx) + \int_0^t \Omega_\alpha(t-u)\int_0^\infty r_\alpha^x(u)\phi(0)(dx)du.$$

That is, we have (consider $\phi(0) = \nu - \nu_\alpha^\infty$)

$$u_t(\nu) - u_t(\nu_\alpha^\infty) \approx \int_{\mathbb{R}_+} J(\alpha)\left[r_\alpha^x(t) + \int_0^t \Omega_\alpha(t-u)r_\alpha^x(u)du\right](\nu - \nu_\alpha^\infty)(dx).$$

In other words, it holds that

$$\frac{\delta u_t}{\delta m}(\nu_\alpha^\infty)(x) = J(\alpha)r_\alpha^x(t) + \left(\Omega_\alpha * J(\alpha)r_\alpha^x\right)(t)$$

and so

$$\partial_\mu u_t(\nu_\alpha^\infty)(x) = J(\alpha)\frac{d}{dx}r_\alpha^x(t) + \left(\Omega_\alpha * J(\alpha)\frac{d}{dx}r_\alpha^x\right)(t).$$

In particular, consider Z_0 a random variable with $\mathcal{L}(Z_0) = \nu_\alpha^\infty$ and let $Z = Z_0 + \epsilon$, for some deterministic $\epsilon > 0$. Let $\nu_{\alpha,\epsilon}^\infty := \mathcal{L}(Z) = \mathcal{L}(Z_0+\epsilon)$. Eq. (4.15) gives:

$$u_t(\nu_{\alpha,\epsilon}^\infty) - u_t(\nu_\alpha^\infty) = \epsilon\,\mathbb{E}\,\partial_\mu u_t(\nu_\alpha^\infty)(Z_0) + o(\epsilon)\,.$$

Furthermore we have:

$$\begin{aligned}\mathbb{E}\,\partial_\mu u_t(\nu_\alpha^\infty)(Z_0) &= \int_{\mathbb{R}_+}\left[J(\alpha)\frac{d}{dx}r_\alpha^x(t) + \left(\Omega_\alpha * J(\alpha)\frac{d}{dx}r_\alpha^x\right)(t)\right]\nu_\alpha^\infty(dx)\\ &= J(\alpha)\Theta_\alpha(t) + J(\alpha)\left(\Omega_\alpha * \Theta_\alpha\right)(t)\\ &\overset{(4.16)}{=} \Omega_\alpha(t).\end{aligned}$$

To summarize, it holds that

$$\lim_{\epsilon\downarrow 0}\frac{J(\alpha)\,\mathbb{E}\,f(X_t^{\nu_{\alpha,\epsilon}^\infty}) - \alpha}{\epsilon} = \Omega_\alpha(t), \tag{4.17}$$

where $\nu_{\alpha,\epsilon}^\infty = \mathcal{L}(Z_0+\epsilon)$ is the invariant measure translated by ϵ. Now, if ν_α^∞ is locally stable, (4.17) suggests that Ω_α goes to zero exponentially fast: there exists some $\lambda > 0$ such that

$$\sup_{t\geq 0}|\Omega_\alpha(t)|e^{\lambda t} < \infty.$$

So, its Laplace transform is well-defined on $\Re(z) > -\lambda$: $z \mapsto \widehat{\Omega}_\alpha(z)$ is holomorphic and so has no poles on $\Re(z) > -\lambda$. Note that (4.16) yields

$$\forall z \in \mathbb{C}, \Re(z) > -\lambda, \quad \widehat{\Omega}_\alpha(z) = \frac{J(\alpha)\widehat{\Theta}_\alpha(z)}{1 - J(\alpha)\widehat{\Theta}_\alpha(z)}.$$

So, the poles of $\widehat{\Omega}_\alpha$ are exactly the zeros of $1 - J(\alpha)\widehat{\Theta}_\alpha$. Overall, (4.17) gives another explanation of the criteria (4.9), i.e. local stability occurs when all the complex zeros of $1 - J(\alpha)\widehat{\Theta}_\alpha$ have negative real parts.

4.4 Preliminaries

4.4.1 Notations

Given $t \geq s \geq 0$ and $\boldsymbol{a} \in L^\infty(\mathbb{R}_+; \mathbb{R}_+)$, we consider $\varphi^{\boldsymbol{a}}_{t,s}(x)$ the solution of (2.3). Recall that we have explicit expressions of $H^\nu_{\boldsymbol{a}}$ and $K^\nu_{\boldsymbol{a}}$ (see (2.4) and (2.5)). As before, to shorten notations, we write: $r_{\boldsymbol{a}}(t,s) := r^{\delta_0}_{\boldsymbol{a}}(t,s)$, $K_{\boldsymbol{a}}(t,s) := K^{\delta_0}_{\boldsymbol{a}}(t,s)$, $H_{\boldsymbol{a}}(t,s) := H^{\delta_0}_{\boldsymbol{a}}(t,s)$. When the current $\boldsymbol{a}$ is constant and equals to α, (1.9) is homogeneous and we write for all $t \geq 0$:

$$Y^{\alpha,\nu}_t := Y^{\boldsymbol{a},\nu}_{t,0}, \quad r^\nu_\alpha(t) := r^\nu_{\boldsymbol{a}}(t,0), \quad K^\nu_\alpha(t) := K^\nu_{\boldsymbol{a}}(t,0), \quad H^\nu_\alpha(t) := H^\nu_{\boldsymbol{a}}(t,0), \quad \varphi^\alpha_t(x) := \varphi^{\boldsymbol{a}}_{t,0}(x).$$

Recall that in that case, the operation "$*$", defined by (2.6), corresponds to the classical convolution. Finally given two real numbers A and B we denote by $A \wedge B$ the minimum between A and B and by $A \vee B$ the maximum.

4.4.2 The Banach algebra

One key ingredient of the proof of Theorem 4.13 is the choice of adapted Banach spaces. In addition to (4.4) we make use of L^1_λ and $\mathcal{V}^1_\lambda$, respectively given by Definitions 3.20 and 3.29. That is, given $\Delta := \{(t,s) \in \mathbb{R}^2,\ t \geq s\}$ (see (2.11)) and $\lambda \geq 0$,

$$\mathcal{V}^1_\lambda := \{\kappa \in \mathcal{B}(\Delta;\mathbb{R}),\ ||\kappa||^1_\lambda < \infty\}, \quad \text{with} \quad ||\kappa||^1_\lambda := \sup_{t \geq 0} \int_0^t |\kappa(t,s)| e^{\lambda(t-s)} ds.$$
$$L^1_\lambda := \{h \in \mathcal{B}(\mathbb{R}_+;\mathbb{R}),\ ||h||^1_\lambda < \infty\}, \quad \text{with} \quad ||h||^1_\lambda := \int_0^\infty |h_t| e^{\lambda t} dt.$$

Recall that for any $a, b \in \mathcal{V}^1_\lambda$, $a * b \in \mathcal{V}^1_\lambda$ with $||a * b||^1_\lambda \leq ||a||^1_\lambda \cdot ||b||^1_\lambda$. Moreover, note that if $a \in \mathcal{V}^1_\lambda$ and $b \in L^\infty_\lambda$ then $a * b \in L^\infty_\lambda$ with $||a * b||^\infty_\lambda \leq ||a||^1_\lambda \cdot ||b||^\infty_\lambda$. Finally, if $c \in L^1_\lambda$, then $\Delta \ni (t,s) \mapsto c(t-s)$ belongs to $\mathcal{V}^1_\lambda$ and the norms coincide. This allows us to see an element of L^1_λ as an element of $\mathcal{V}^1_\lambda$. Recall that the algebra L^1_λ is commutative (for the convolution '$*$' operator) whereas $\mathcal{V}^1_\lambda$ is not.

For any $h \in L^\infty_\lambda$ and $\rho > 0$, we denote by $B^\infty_\lambda(h,\rho)$ the open ball

$$B^\infty_\lambda(h,\rho) := \{c \in L^\infty_\lambda,\ ||c-h||^\infty_\lambda < \rho\}. \tag{4.18}$$

Lemma 4.18. *The following functions*

$$\begin{array}{rcl} \mathcal{V}^1_\lambda \times \mathcal{V}^1_\lambda & \to & \mathcal{V}^1_\lambda \\ (a,b) & \mapsto & a * b, \end{array} \qquad \begin{array}{rcl} \mathcal{V}^1_\lambda \times L^\infty_\lambda & \to & L^\infty_\lambda \\ (a,b) & \mapsto & a * b \end{array}$$

are $\mathcal{C}^1$, with differential given by

$$(h,k) \mapsto a * k + h * b.$$

Proof. One has $(a+h)*(b+k) = a*b + a*k + h*b + h*k$ and moreover

$$||h*k||_\lambda^1 \leq ||h||_\lambda^1 ||k||_\lambda^1 = o\big((||h||_\lambda^1 + ||k||_\lambda^1)\big)\,.$$

The second result is proved similarly. □

One denotes by $B_\lambda^1(0,1)$ the following open ball of $\mathcal{V}_\lambda^1$

$$B_\lambda^1(0,1) := \{\kappa \in \mathcal{V}_\lambda^1,\ ||\kappa||_\lambda^1 < 1\}.$$

Lemma 4.19. *The function*

$$\begin{array}{rccl} R: & B_\lambda^1(0,1) & \to & \mathcal{V}_\lambda^1 \\ & \kappa & \mapsto & \sum_{n\geq 1} \kappa^{(*)n} \end{array}$$

is $\mathcal{C}^1$ *and for all* $c \in \mathcal{V}_\lambda^1$

$$D_\kappa R(\kappa)\cdot c = c + R(\kappa)*c + c*R(\kappa) + R(\kappa)*c*R(\kappa).$$

Proof. First, note that the series converges normally. So R is well defined. Note that the result is easily proved when $\kappa = 0$, for which $D_\kappa R(0)\cdot c = c$. Second, remark that $R(\kappa)$ is the resolvent of κ, that is, it solves the Volterra equation

$$R(\kappa) = \kappa + \kappa * R(\kappa). \tag{4.19}$$

Moreover, κ and $R(\kappa)$ always commute: $\kappa * R(\kappa) = R(\kappa)*\kappa$. Using (4.19), we have

$$\begin{aligned} R(\kappa+c) &= (\kappa+c) + (\kappa+c)*R(\kappa+c) \\ &= (\kappa + c + c*R(\kappa+c)) + \kappa * R(\kappa+c). \end{aligned}$$

So, using Lemma 2.22, we find that

$$\begin{aligned} R(\kappa+c) &= (\kappa + c + c*R(\kappa+c)) + R(\kappa)*(\kappa+c+c*R(\kappa+c)) \\ &= R(\kappa) + c + R(\kappa)*c + (c + R(\kappa)*c)*R(\kappa+c). \end{aligned}$$

Let $\Delta_c := c + R(\kappa)*c$. We have shown that $R(\kappa+c)$ solves the Volterra integral equation

$$R(\kappa+c) = R(\kappa) + \Delta_c + \Delta_c * R(\kappa+c).$$

Assume that $||c||_\lambda^1$ is small enough, such that $\Delta_c \in B_\lambda^1(0,1)$. We find, using Lemma 2.22 that

$$\begin{aligned} R(\kappa+c) &= R(\kappa) + \Delta_c + R(\Delta_c)*(R(\kappa)+\Delta_c) \\ &= R(\kappa) + \Delta_c + \Delta_c*(R(\kappa)+\Delta_c) + o\big(||\Delta_c||_\lambda^1\big) \\ &= R(\kappa) + c + R(\kappa)*c + c*R(\kappa) + R(\kappa)*c*R(\kappa) + o\big(||c||_\lambda^1\big)\,. \end{aligned}$$

We used that $R(c) = c + o(c)$ as $||c||_\lambda^1$ goes to zero. This ends the proof. □

4.4.3 Results on the deterministic flow

Lemma 4.20 (Differentiability of the flow)**.** *Let $b \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |b'(x)| + |b''(x)| < +\infty$. Let $x \in \mathbb{R}$, $s \geq 0$, $\lambda > 0$. Consider $\alpha > 0$ and $\boldsymbol{h} \in L^\infty_\lambda$.*

4.20(a) *The equation*

$$\forall t \geq s, \quad \varphi_t = x + \int_s^t [b(\varphi_u) + \alpha + h_u]\, du$$

has a unique continuous solution on $[s, +\infty)$. We denote it by $\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$. Moreover setting $L := \sup_{x\in\mathbb{R}} |b'(x)|$, one has

$$\forall \boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^\infty_\lambda,\ \forall t \geq s, \quad \left|\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)\right| \leq \int_s^t e^{L(t-u)} |\tilde{h}_u - h_u| du. \tag{4.20}$$

4.20(b) *The function $x \mapsto \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$ is $\mathcal{C}^1(\mathbb{R};\mathbb{R})$. Let $U^{\alpha+\boldsymbol{h}}_{t,s}(x) := \frac{d}{dx}\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$, one has*

$$U^{\alpha+\boldsymbol{h}}_{t,s}(x) = \exp\left(\int_s^t b'(\varphi^{\alpha+\boldsymbol{h}}_{\theta,s}(x))d\theta\right). \tag{4.21}$$

When $\boldsymbol{h} \equiv 0$, the above formula simplifies to

$$U^{\alpha}_{t,s}(x) = \frac{b(\varphi^\alpha_{t-s}(x)) + \alpha}{b(x) + \alpha}. \tag{4.22}$$

4.20(c) *The function $L^\infty_\lambda \ni \boldsymbol{h} \mapsto \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \in \mathbb{R}$ is $\mathcal{C}^1$ and for all $c \in L^\infty_\lambda$,*

$$D_h \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot c := \int_s^t c_u \exp\left(\int_u^t b'(\varphi^{\alpha+\boldsymbol{h}}_{\theta,s}(x))d\theta\right) du. \tag{4.23}$$

Moreover setting $L := \sup_{x\in\mathbb{R}} |b'(x)|$ and $M := \sup_{x\in\mathbb{R}} |b''(x)|$ one has for all $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^\infty_\lambda$, for all $x \in \mathbb{R}$ and for all $t \geq s$

$$\left|\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) - D_h\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h})\right| \leq \frac{M}{2L^3}\left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^\infty_\lambda e^{-\lambda s} e^{L(t-s)}\right]^2. \tag{4.24}$$

The proof of this lemma is postponed in Section 4.8.

Remark 4.21. *If in addition to the assumptions of the lemma, we have almost everywhere $b(0) + \alpha + \boldsymbol{h} \geq 0$, then the flow stays on $\mathbb{R}_+$, i.e.*

$$\forall t \geq s,\ \forall x \geq 0, \quad \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \geq 0.$$

Lemma 4.22 (Asymptotic of the flow when $\sigma_\alpha < \infty$)**.** *Grant Assumption 4.1. Let $\alpha > 0$, assume that $\sigma_\alpha < \infty$ and that (4.6) holds. Define $\ell_\alpha := -b'(\sigma_\alpha)$ ($\ell_\alpha > 0$ by (4.6)). Consider $S \in \mathcal{S}_\alpha$.*

4.22(a) *There exists a constant C (only depending on b, α and S) such that for all $x \in S$,*

$$|\varphi_t^\alpha(x) - \sigma_\alpha| + \left|\frac{d}{dt}\varphi_t^\alpha(x)\right| \leq Ce^{-\ell_\alpha t}.$$

Moreover, there exists a constant c (only depending on b and α) such that

$$|\varphi_t^\alpha(0) - \sigma_\alpha| + \left|\frac{d}{dt}\varphi_t^\alpha(0)\right| \geq ce^{-\ell_\alpha t}.$$

4.22(b) *Let $\mu \in (0, \ell_\alpha)$. There exists constants $\delta_\mu, C_\mu > 0$ (only depending on b, α, μ and S) such that for all $\boldsymbol{h} \in L^\infty_\mu$ with $||\boldsymbol{h}||^\infty_\mu < \delta_\mu$ one has*

$$\forall x \in S,\ \forall t \geq s, \quad |\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x) - \varphi_{t,s}^\alpha(x)| \leq C_\mu ||\boldsymbol{h}||^\infty_\mu e^{-\mu t}, \tag{4.25}$$

$$|\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x) - \sigma_\alpha| \leq C_\mu e^{-\mu(t-s)}. \tag{4.26}$$

Let $\lambda \geq \mu$. For all $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^\infty_\lambda$, one has for all $x \in S$ and $t \geq s$

$$|\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)| \leq C_\mu \int_s^t |\tilde{h}_u - h_u| du \tag{4.27}$$

and

$$|\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x) - D_h\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h})| \leq C_\mu \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^\infty_\lambda e^{-\lambda s}\right]^2. \tag{4.28}$$

Again, the proof of this lemma is postponed in Section 4.8.

4.4.4 Estimates on the kernels H and K

Lemma 4.23. *Grant Assumptions 4.1 and 4.2. Let $\alpha, \delta > 0$. Let $\lambda \geq 0$ and $\boldsymbol{h} \in L^\infty_\lambda$ such that $||\boldsymbol{h}||^\infty_\lambda < \delta$ and such that for almost all $t \geq 0$, $b(0) + h_t + \alpha \geq 0$. One has:*

4.23(a) *For all $x \geq 0$*

$$\forall t \geq s, \quad \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x) \leq \left[x + \frac{b(0) + \alpha + \delta}{L}\right] e^{L(t-s)}.$$

4.23(b) *There exists a constant C only depending on f, b and α and δ such that*

$$\forall x \geq 0, \forall t \geq s, \quad f(\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)) \leq C(1 + f(x))e^{pL(t-s)}.$$

In these inequalities, L is the Lipschitz constant of b and $p > 0$ is given by (2.2).

Proof. The first point is easily proved using that for all $x \geq 0$, $|b(x)| \leq b(0) + Lx$ and Grönwall's Lemma. To prove the second point, we denote by C any constant only depending on b, f and α and that may change from line to line. Using that f is non-decreasing, we have

$$\begin{aligned} f(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) &\leq f((x+\beta)e^{L(t-s)}) \qquad \text{By 4.23(a) with } \beta := (b(0)+\alpha+\delta)/L. \\ &\overset{4.2(a)}{\leq} C[1+f(x+\beta)][1+f(e^{L(t-s)})] \\ &\overset{(2.2)}{\leq} C[1+f(x+\beta)]e^{pL(t-s)}. \\ &\overset{\text{Rk. } 4.3}{\leq} C[1+f(x)]e^{pL(t-s)}. \end{aligned}$$

□

Lemma 4.24. *Let $b : \mathbb{R}_+ \to \mathbb{R}$ satisfying Assumption 4.1. Let $\alpha > 0$. Consider $\delta, \lambda > 0$ and $\sigma \in (0, \sigma_\alpha)$. There exists a constant $T > 0$ (only depending on $b, \alpha, \lambda, \delta$ and σ) such that*

$$\left(\boldsymbol{h} \in L^\infty_\lambda,\ \operatorname*{ess\,inf}_{t\geq 0} h_t \geq -(b(0)+\alpha),\ ||\boldsymbol{h}||^\infty_\lambda < \delta\right) \implies \inf_{x\geq 0} \inf_{\substack{s\geq 0\\ t\geq s+T}} \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \geq \sigma.$$

Proof. Because $\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \geq \varphi^{\alpha+\boldsymbol{h}}_{t,s}(0)$, it suffices to prove the result for $x = 0$. Because b is continuous and because $\sigma < \sigma_\alpha$, one has $\kappa := \inf_{x\in[0,\sigma]} b(x) + \alpha > 0$. There exists T_0 such that for all $t \geq T_0$, one has $|h_t| \leq \delta e^{-\lambda t} \leq \delta e^{-\lambda T_0} \leq \kappa/2$ and so

$$\forall t \geq T_0, \forall x \in [0,\sigma], \quad b(x) + \alpha + h_t \geq \kappa/2.$$

So, it suffices to choose $T := T_0 + \frac{2\sigma}{\kappa}$ to end the proof. □

Lemma 4.25. *Grant Assumptions 4.1 and 4.2. Let $\alpha,\ \delta > 0$. Let $\lambda \in (0, f(\sigma_\alpha))$. There is a constant $C > 0$ (only depending on b, f, α and λ) such that for all $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$ satisfying $\inf_{t\geq 0} h_t \geq -(b(0)+\alpha)$, one has*

$$\forall x \in \mathbb{R}_+,\ \forall t \geq s, \quad H^x_{\alpha+\boldsymbol{h}}(t,s) \leq Ce^{-\lambda(t-s)}.$$

and

$$\forall x \in \mathbb{R}_+,\ \forall t \geq s, \quad K^x_{\alpha+\boldsymbol{h}}(t,s) \leq C(1+f(x))e^{-\lambda(t-s)}.$$

Proof. Define for $t \geq s$:

$$G_x(t) := e^{\lambda(t-s)} H^x_{\alpha+\boldsymbol{h}}(t,s).$$

By Lemma 4.24, there exists a constant T (only depending on b, f, α, λ and δ) such that for all t, s with $t - s \geq T$ and for all $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$, one has

$$\inf_{x\geq 0} f(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) \geq \lambda.$$

It follows that for all $t \geq s$, $\int_s^t f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x))du \geq (t-s-T)\lambda$, and so

$$G_x(t) = e^{\lambda(t-s)} \exp\left(-\int_s^t f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x))du\right) \leq e^{T\lambda} =: A_0.$$

This proves the first inequality. Moreover, define for $t \geq s$ and $x \geq 0$:

$$F_x(t) := e^{\lambda(t-s)} K^x_{\alpha+\boldsymbol{h}}(t,s) - \lambda e^{\lambda(t-s)} H^x_{\alpha+\boldsymbol{h}}(t,s),$$

which satisfies for all $t \geq s$, $F_x(t) = -\frac{d}{dt} G_x(t)$. By the first point, to prove the second inequality, it suffices to show that F_x is upper bounded by $C(1+f(x))$ for some constant C. We have

$$F'_x(t) = \Big\{ f'(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) \left[b(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) + \alpha + h_t \right] \\ -\lambda^2 + 2\lambda f(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) - f^2(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) \Big\} e^{\lambda(t-s)} H^x_{\alpha+\boldsymbol{h}}(t,s).$$

By Assumption 4.2(b), one has

$$\sup_{y \geq 0} f'(y)[b(y) + \alpha + \delta] + 2\lambda f(y) - f^2(y) < \infty.$$

Hence, there exists a constant A_1 (only depending on b, f, α, λ and δ) such that for all $x \geq 0$, for all $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$,

$$\sup_{t \geq s,\ x \geq 0} F'_x(t) \leq A_1.$$

We conclude using the Landau inequality: let $\eta := \sqrt{\frac{2A_0}{A_1}}$. Consider t, s with $t \geq s + \eta$. By the Mean value theorem, there exists $\zeta \in [t-\eta, t]$ such that

$$F_x(\zeta) = \frac{G_x(t-\eta) - G_x(t)}{\eta}.$$

So $|F_x(\zeta)| \leq \frac{2A_0}{\eta}$. We deduce that

$$F_x(t) = F_x(\zeta) + \int_\zeta^t F'_x(\theta) d\theta \leq \frac{2A_0}{\eta} + A_1 \eta = 2\sqrt{2A_0 A_1}.$$

Finally, using Lemma 4.23(b), there exists a constant C (only depending on b, f, α, δ and η) such that

$$\forall x \geq 0, \quad \sup_{\substack{s \geq 0 \\ s \leq t \leq s+\eta}} f(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) \leq C(1 + f(x)).$$

Altogether, this proves the result. $\square$

4.5 Proof of Proposition 4.12

Define for all $t \geq 0$

$$\Psi_\alpha(t) := -\int_0^{\sigma_\alpha} \frac{d}{dx} H^x_\alpha(t) \nu^\infty_\alpha(x) dx. \tag{4.29}$$

Lemma 4.26. *Grant Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Then for all $\lambda \in (0, f(\sigma_\alpha))$, the function Ψ_α belongs to $L^1_\lambda \cap L^\infty_\lambda$. Moreover, $\Psi_\alpha(0) = 0$.*

Proof. First note that for all $x \geq 0$, one has $H^x_\alpha(0) = 1$ and so $\frac{d}{dx} H^x_\alpha(0) = 0$ and $\Psi_\alpha(0) = 0$.
Claim: one has for all $t, x \geq 0$:

$$\frac{d}{dx} H^x_\alpha(t) = -H^x_\alpha(t) \frac{f(\varphi^\alpha_t(x)) - f(x)}{b(x) + \alpha}.$$

Proof of the claim. From

$$H^x_\alpha(t) = \exp\left(-\int_0^t f(\varphi^\alpha_u(x)) du\right),$$

we deduce by the dominated convergence theorem that for any fixed $t \geq 0$, the function $x \mapsto H^x_\alpha(t)$ is $\mathcal{C}^1$ with

$$\frac{d}{dx} H^x_\alpha(t) = -H^x_\alpha(t) \int_0^t f'(\varphi^\alpha_u(x)) \frac{d}{dx} \varphi^\alpha_u(x) du.$$

By Lemma 4.20, one has

$$\frac{d}{dx} \varphi^\alpha_u(x) = \frac{b(\varphi^\alpha_u(x)) + \alpha}{b(x) + \alpha}.$$

So,

$$\int_0^t f'(\varphi^\alpha_u(x)) \frac{d}{dx} \varphi^\alpha_u(x) du = \frac{f(\varphi^\alpha_t(x)) - f(x)}{b(x) + \alpha}.$$

This ends the proof of the claim.
Note that the integrand of (4.29) has a constant sign (because f is increasing). Plugging the explicit expression of ν^∞_α (equation (3.4)), we find

$$\begin{aligned}
\Psi_\alpha(t) &= \gamma(\alpha) \int_0^{\sigma_\alpha} H^x_\alpha(t) \frac{f(\varphi^\alpha_t(x)) - f(x)}{(b(x) + \alpha)^2} \exp\left(-\int_0^x \frac{f(y)}{b(y) + \alpha} dy\right) dx \\
&= \gamma(\alpha) \int_0^\infty \exp\left(-\int_0^t f(\varphi^\alpha_{\theta+u}(0)) d\theta\right) \frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha} H_\alpha(u) du.
\end{aligned}$$

To obtain the last equality we made first the change of variable $x = \varphi^\alpha_u(0)$ and then $y = \varphi^\alpha_\theta(0)$.
Hence, we find:

$$\Psi_\alpha(t) = \gamma(\alpha) \int_0^\infty H_\alpha(t+u) \frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha} du. \tag{4.30}$$

We now distinguish between the two cases $\sigma_\alpha < \infty$ and $\sigma_\alpha = \infty$.
Case $\sigma_\alpha = \infty$. Denote by L the Lipschitz constant of b, one has using Lemma 4.23(b)

$$\forall t \geq 0, \quad f(\varphi^\alpha_t(0)) \leq C e^{pLt}.$$

Hence, (4.7) gives the existence of a constant C such that

$$\frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha} \leq C f(\varphi^\alpha_{t+u}(0)) \leq C e^{pL(t+u)}.$$

Let $\lambda > 0$ and $\epsilon > 0$. By Lemma 4.25 (with $f(\sigma_\alpha) = \infty$), there is another constant C_ϵ such that

$$H_\alpha(t+u) \leq C_\epsilon e^{-(\lambda+\epsilon+pL)(t+u)},$$

and so $\Psi_\alpha(t) \le C_\epsilon e^{-(\lambda+\epsilon)t}$. This proves that $\Psi_\alpha \in L^1_\lambda \cap L^\infty_\lambda$ for all $\lambda > 0$.
Case $\sigma_\alpha < \infty$. Let $\ell_\alpha := -b'(\sigma_\alpha)$. Assumption 4.6 yields $\ell_\alpha > 0$. Let $\lambda \in (0, f(\sigma_\alpha))$. By Lemma 4.22, there is a constant $C > 0$ (that may change from line to line) such that

$$\forall u \ge 0, \quad b(\varphi^\alpha_u(0)) + \alpha = \frac{d}{du}\varphi^\alpha_u(0) \ge C e^{-\ell_\alpha u}.$$

Using moreover that

$$f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0)) = \int_{\varphi^\alpha_u(0)}^{\varphi^\alpha_{t+u}(0)} f'(\theta)d\theta \overset{\text{Ass. } 4.2(b)}{\le} C(1+f(\sigma_\alpha))\left|\varphi^\alpha_{t+u}(0) - \varphi^\alpha_u(0)\right| \overset{\text{Lem. } 4.22}{\le} Ce^{-\ell_\alpha u},$$

we deduce that there exists another constant C such that

$$\frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha} \le C.$$

Let $\epsilon \in (0, f(\sigma_\alpha) - \lambda)$. By Lemma 4.25 there exists a constant C_ϵ such that

$$\forall t, \forall u, \quad H_\alpha(t+u) \le C_\epsilon e^{-(\lambda+\epsilon)(t+u)}.$$

Finally, we have $\Psi_\alpha(t) \le C_\epsilon e^{-(\lambda+\epsilon)t}$, hence $\Psi_\alpha(t) \in L^1_\lambda \cap L^\infty_\lambda$ as required. This ends the proof. □

Similarly to (4.29), define

$$\forall t \ge 0, \quad \Xi_\alpha(t) := \int_0^{\sigma_\alpha} \frac{d}{dx} K^x_\alpha(t)\nu^\infty_\alpha(x)dx. \tag{4.31}$$

Lemma 4.27. *Grant Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Then for all $\lambda \in (0, f(\sigma_\alpha))$, the function Ξ_α belongs to $L^1_\lambda \cap L^\infty_\lambda$. Moreover one has*

$$\Xi_\alpha(t) = \frac{d}{dt}\Psi_\alpha(t). \tag{4.32}$$

Proof. The proof is similar to the one of the previous lemma. We find

$$\begin{aligned}\Xi_\alpha(t) = -\gamma(\alpha)&\int_0^\infty K_\alpha(t+u)\frac{f(\varphi^\alpha_{t+u}(0)) - f(\varphi^\alpha_u(0))}{b(\varphi^\alpha_u(0)) + \alpha}du \\ &+ \gamma(\alpha)\int_0^\infty H_\alpha(t+u) f'(\varphi^\alpha_{t+u}(0))\frac{b(\varphi^\alpha_{t+u}(0)) + \alpha}{b(\varphi^\alpha_u(0)) + \alpha}du.\end{aligned}$$

Using similar arguments, for all $\lambda \in (0, f(\sigma_\alpha))$, Ξ_α belongs to $L^1_\lambda \cap L^\infty_\lambda$. Finally, using that for all $x \ge 0$

$$K^x_\alpha(t) = -\frac{d}{dt}H^x_\alpha(t),$$

eq. (4.32) follows. □

We now give a proof of Proposition 4.12.

Proof of Proposition 4.12. First, by (2.12), we have for all $x \geq 0$

$$r_\alpha^x = K_\alpha^x + r_\alpha * K_\alpha^x.$$

This proves that $x \mapsto r_\alpha^x(t)$ is $\mathcal{C}^1$ and

$$\frac{d}{dx} r_\alpha^x = \frac{d}{dx} K_\alpha^x + r_\alpha * \left[\frac{d}{dx} K_\alpha^x\right].$$

Integrating this equality with respect to $\nu_\alpha^\infty(dx)$, we find that

$$\Theta_\alpha = \Xi_\alpha + r_\alpha * \Xi_\alpha. \tag{4.33}$$

Consider $\lambda \in (0, \lambda_\alpha^*)$. Proposition 3.25 gives

$$\xi_\alpha := r_\alpha - \gamma(\alpha) \in L^1_\lambda.$$

We have

$$\Theta_\alpha = \Xi_\alpha + \gamma(\alpha) * \Xi_\alpha + \xi_\alpha * \Xi_\alpha,$$

and because

$$\gamma(\alpha) * \Xi_\alpha(t) = \gamma(\alpha) \int_0^t \Xi_\alpha(s) ds = \gamma(\alpha) \left(\Psi_\alpha(t) - \Psi_\alpha(0)\right) = \gamma(\alpha)\Psi_\alpha(t),$$

we deduce that

$$\Theta_\alpha = \Xi_\alpha + \gamma(\alpha)\Psi_\alpha + \xi_\alpha * \Xi_\alpha.$$

Hence $\Theta_\alpha \in L^1_\lambda$, which ends the proof. □

Remark 4.28. *Using (4.33), we have, for any $z \in \mathbb{C}$ with $\Re(z) > 0$*

$$\begin{aligned}
\widehat{\Theta}_\alpha(z) &= \widehat{\Xi}_\alpha(z)\left[1 + \widehat{r}_\alpha(z)\right] \\
&= \widehat{\Xi}_\alpha(z)\left[1 + \frac{\widehat{K}_\alpha(z)}{1 - \widehat{K}_\alpha(z)}\right] \quad (\textit{using } r_\alpha = K_\alpha + K_\alpha * r_\alpha) \\
&= \frac{\widehat{\Xi}_\alpha(z)}{z\widehat{H}_\alpha(z)} \quad (\textit{using } \widehat{K}_\alpha(z) = 1 - z\widehat{H}_\alpha(z)) \\
&= \frac{\widehat{\Psi}_\alpha(z)}{\widehat{H}_\alpha(z)} \quad (\textit{using } \Psi_\alpha(0) = 0).
\end{aligned} \tag{4.34}$$

The left hand side and the right hand side being two holomorphic functions on $\Re(z) > -\lambda_\alpha^*$, the equality is valid on $\Re(z) > -\lambda_\alpha^*$ and so the equation $J(\alpha)\widehat{\Theta}_\alpha(z) = 1$ is equivalent to

$$J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = 0. \tag{4.35}$$

In this new formulation of (4.9), the stability is given by the location of the roots of a holomorphic function which is explicitly known in term of f, b and α.

4.6 Proof of Theorem 4.14

Assume that

$$\liminf_{x \uparrow \sigma_\alpha} f(x) + b'(x) \geq 0. \tag{4.36}$$

Under (4.36), we can integrate by parts Ψ_α and Ξ_α:

Lemma 4.29. *Consider f and b satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 is satisfied. Assume furthermore that (4.36) holds. Then:*

1. *The following limit exists and is finite*

$$\nu_\alpha^\infty(\sigma_\alpha) := \lim_{x \uparrow \sigma_\alpha} \nu_\alpha^\infty(x) < \infty.$$

2. *Define $C_\alpha := \frac{b(0)+\alpha}{\gamma(\alpha)} \nu_\alpha^\infty(\sigma_\alpha)$ and*

$$\Upsilon_\alpha(t) := C_\alpha H_\alpha^{\sigma_\alpha}(t) + \int_0^\infty H_\alpha(t+u) \left[f(\varphi_u^\alpha(0)) + b'(\varphi_u^\alpha(0)) \right] \frac{b(0)+\alpha}{b(\varphi_u^\alpha(0)) + \alpha} du. \tag{4.37}$$

It holds that for all $t \geq 0$

$$\Psi_\alpha(t) = \frac{\gamma(\alpha)}{b(0)+\alpha} \left[H_\alpha(t) - \Upsilon_\alpha(t) \right]. \tag{4.38}$$

3. *Define $\Lambda_\alpha(t) := -\frac{d}{dt} \Upsilon_\alpha(t)$. One has for all $t \geq 0$*

$$\Lambda_\alpha(t) = C_\alpha K_\alpha^{\sigma_\alpha}(t) + \int_0^\infty K_\alpha(t+u) \left[f(\varphi_u^\alpha(0)) + b'(\varphi_u^\alpha(0)) \right] \frac{b(0)+\alpha}{b(\varphi_u^\alpha(0)) + \alpha} du. \tag{4.39}$$

Moreover, for all $t \geq 0$

$$\Xi_\alpha(t) = \frac{\gamma(\alpha)}{b(0)+\alpha} \left[\Lambda_\alpha(t) - K_\alpha(t) \right]. \tag{4.40}$$

Remark 4.30 (A probabilistic interpretation of C_α and Λ_α)**.** *Recall that by Lemma 4.26: $\Psi_\alpha(0) = 0$. Using (4.32), we find $\Xi_\alpha(t) = \frac{d}{dt} \Psi_\alpha(t)$ whence $\int_0^\infty \Xi_\alpha(t) = 0$. So (4.40) yields*

$$\int_0^\infty \Lambda_\alpha(t) dt = 1.$$

Assume now that (4.10) holds, such that for all $t \geq 0$, $\Lambda_\alpha(t) \geq 0$. We deduce that $\Lambda_\alpha(t)$ is a probability density function. The interpretation is the following. Consider τ_1 the first jump time of a Poisson process with time-dependent intensity $u \mapsto f(\varphi_u^\alpha(0)) + b'(\varphi_u^\alpha(0))$. The law of τ_1 is

$$\mathcal{L}(\tau_1)(du) = \infty \mathbb{P}(\tau_1 = \infty) + (f + b')(\varphi_u^\alpha(0)) \exp\left(- \int_0^u (f + b')(\varphi_\theta^\alpha(0)) d\theta \right) du.$$

Consider then τ_2 the first jump time of a second Poisson process with time-dependent intensity $f(\varphi^{\alpha}_{t+\tau_1}(0))$. The law of τ_2 is given by

$$\mathcal{L}(\tau_2)(dt) = \mathbb{P}(\tau_1 = \infty) K^{\sigma_\alpha}_{\alpha}(t) + \int_0^\infty K^{\varphi^{\alpha}_{u}(0)}_{\alpha}(t)(f+b')(\varphi^{\alpha}_{u}(0)) \exp\left(-\int_0^u (f+b')(\varphi^{\alpha}_{\theta}(0)) d\theta\right) du.$$

In view of (4.21) and (4.22), (4.39) can be rewritten:

$$\Lambda_\alpha(t) = C_\alpha K^{\sigma_\alpha}_{\alpha}(t) + \int_0^\infty K^{\varphi^{\alpha}_{u}(0)}_{\alpha}(t)(f+b')(\varphi^{\alpha}_{u}(0)) \exp\left(-\int_0^u (f+b')(\varphi^{\alpha}_{\theta}(0)) d\theta\right) du.$$

Using that $1 = \int_0^\infty \mathcal{L}(\tau_2)(dt) = \int_0^\infty \Lambda_\alpha(t)dt$, we deduce that

$$C_\alpha = \mathbb{P}(\tau_1 = \infty)$$

and so $\Lambda_\alpha(t)$ is the density of τ_2:

$$\mathcal{L}(\tau_2)(dt) = \Lambda_\alpha(t) dt.$$

Proof of Lemma 4.29. To prove Point 1, we use the explicit formula of the invariant measure (3.4). When $\sigma_\alpha = +\infty$, we have $\nu^\infty_\alpha(\sigma_\alpha) = 0$. The result follows from $\inf_{x \geq 0} b(x) + \alpha > 0$ and from $\liminf_{x\to\infty} f(x) > 0$ (in particular there is no need of (4.36) when $\sigma_\alpha = \infty$). Assume now $\sigma_\alpha < \infty$. Define for all $x \in [0, \sigma_\alpha)$

$$G_\alpha(x) := \frac{f(x)}{b(x)+\alpha} - \frac{1}{\sigma_\alpha - x}.$$

We claim that:

$$\lim_{x \uparrow \sigma_\alpha} \nu^\infty_\alpha(x) = -\frac{\gamma(\alpha)}{b'(\sigma_\alpha)\sigma_\alpha} \exp\left(-\int_0^{\sigma_\alpha} G_\alpha(y) dy\right) < \infty.$$

Indeed

$$b(x) + \alpha = -b'(\sigma_\alpha)(\sigma_\alpha - x) + \mathcal{O}(\sigma_\alpha - x)^2 \quad \text{as } x \to \sigma_\alpha,\ x < \sigma_\alpha,$$

so

$$\frac{f(x)}{b(x)+\alpha} = -\frac{f(\sigma_\alpha)}{b'(\sigma_\alpha)} \frac{1}{\sigma_\alpha - x} + \mathcal{O}(1) \quad \text{as } x \to \sigma_\alpha,\ x < \sigma_\alpha.$$

We then have

$$\begin{aligned}
\nu^\infty_\alpha(x) &= \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\int_0^x G_\alpha(y)dy\right) \exp\left(-\int_0^x \frac{dy}{\sigma_\alpha - y}\right) \\
&= \left[-\frac{\gamma(\alpha)}{b'(\sigma_\alpha)(\sigma_\alpha - x)} + \mathcal{O}(1)\right] \exp\left(-\int_0^x G_\alpha(y)dy\right) \frac{\sigma_\alpha - x}{\sigma_\alpha} \quad \text{as} \quad x \to \sigma_\alpha,\ x < \sigma_\alpha. \\
&= \left[-\frac{\gamma(\alpha)}{b'(\sigma_\alpha)\sigma_\alpha} + o(1)\right] \exp\left(-\int_0^{\sigma_\alpha} G_\alpha(y)dy\right) \quad \text{as} \quad x \to \sigma_\alpha,\ x < \sigma_\alpha.
\end{aligned}$$

Note that when $f(\sigma_\alpha) + b'(\sigma_\alpha) > 0$, we have $-\frac{f(\sigma_\alpha)}{b'(\sigma_\alpha)} > 1$ and so $\lim_{x\to\sigma_\alpha} G_\alpha(x) = \infty$ and $\nu^\infty_\alpha(\sigma_\alpha) = 0$. When $f(\sigma_\alpha) + b'(\sigma_\alpha) = 0$, we have $-\frac{f(\sigma_\alpha)}{b'(\sigma_\alpha)} = 1$ and so $\lim_{x\to\sigma_\alpha} G_\alpha(x) < \infty$,

which proves that $G_\alpha(x)$ is integrable between 0 and σ_α.
To prove Point 2, we integrate by parts the right-hand side of (4.29). By Point 1, one has

$$\Psi_\alpha(t) = \frac{\gamma(\alpha)}{b(0)+\alpha}\left[H_\alpha(t) - C_\alpha H_\alpha^{\sigma_\alpha}(t)\right] + \int_0^{\sigma_\alpha} H_\alpha^x(t)\frac{d}{dx}\nu_\alpha^\infty(x)dx.$$

Differentiating (3.4) with respect to x, one gets for all $x \in [0, \sigma_\alpha)$

$$\frac{d}{dx}\nu_\alpha^\infty(x) = -\gamma(\alpha)\frac{f(x)+b'(x)}{(b(x)+\alpha)^2}\exp\left(-\int_0^x \frac{f(y)}{b(y)+\alpha}dy\right).$$

We now make the change of variables $y = \varphi_\theta^\alpha(0)$ and $x = \varphi_u^\alpha(0)$ and obtain

$$\Psi_\alpha(t) = \frac{\gamma(\alpha)}{b(0)+\alpha}\left[H_\alpha(t) - C_\alpha H_\alpha^{\sigma_\alpha}(t)\right] - \gamma(\alpha)\int_0^\infty H_\alpha^{\varphi_u^\alpha(0)}(t)\frac{f(\varphi_u^\alpha(0)) + b'(\varphi_u^\alpha(0))}{b(\varphi_u^\alpha(0))+\alpha}H_\alpha(u)du.$$

Using that $H_\alpha^{\varphi_u^\alpha(0)}(t)H_\alpha(u) = H_\alpha(t+u)$ we obtain the stated formula. Recall now that $\Xi_\alpha(t) = \frac{d}{dt}\Psi_\alpha(t)$ so to prove Point 3, it suffices to differentiate Point 2 with respect to t. $\square$

Proof of Theorem 4.14, first point. It suffices to verify that under the additional Assumption (4.10), the criteria of stability (4.9) holds. Consider Λ_α and Υ_α given by (4.39) and (4.37). By Lemma 4.26 and 4.27, it holds that $\Psi_\alpha, \Xi_\alpha \in L^\infty_\lambda \cap L^1_\lambda$, for all $\lambda < f(\lambda_\alpha^*)$. The same holds for H_α and K_α. From (4.38) and (4.40), we deduce that for all $\lambda < f(\lambda_\alpha^*)$, $\Lambda_\alpha, \Upsilon_\alpha \in L^\infty_\lambda \cap L^1_\lambda$. In view of

$$\Upsilon_\alpha(t) = \int_t^\infty \Lambda_\alpha(v)dv,$$

an integration by parts of the Laplace transform of $\Lambda_\alpha(t)$ shows that for all $z \in \mathbb{C}$ with $\Re(z) > -f(\sigma_\alpha)$

$$\widehat{\Lambda}_\alpha(z) = 1 - z\widehat{\Upsilon}_\alpha(z).$$

Here we used the fact that $\int_0^\infty \Lambda_\alpha(v)dv = 1$. Similarly we have

$$\widehat{K}_\alpha(z) = 1 - z\widehat{H}_\alpha(z).$$

From (4.40), it holds that for all $z \in \mathbb{C}$ with $\Re(z) > -f(\sigma_\alpha)$

$$\widehat{\Xi}_\alpha(z) = \frac{\gamma(\alpha)z}{b(0)+\alpha}\left[\widehat{H}_\alpha(z) - \widehat{\Upsilon}_\alpha(z)\right].$$

Using (4.34) we have for all $z \in \mathbb{C}$ with $\Re(z) > -\lambda_\alpha^*$

$$\widehat{\Theta}_\alpha(z) = \frac{\gamma(\alpha)}{b(0)+\alpha}\frac{\widehat{H}_\alpha(z) - \widehat{\Upsilon}_\alpha(z)}{\widehat{H}_\alpha(z)}.$$

We deduce that the equation $J(\alpha)\widehat{\Theta}_\alpha(z) = 1$ on $\Re(z) > -\lambda_\alpha^*$ is equivalent to

$$b(0)\widehat{H}_\alpha(z) + \alpha\widehat{\Upsilon}_\alpha(z) = 0.$$

Note that $z = 0$ is not a solution because

$$b(0)\widehat{H}_\alpha(0) + \alpha\widehat{\Upsilon}_\alpha(0) = b(0)\int_0^\infty H_\alpha(t)dt + \alpha\int_0^\infty \Upsilon_\alpha(t)dt > 0.$$

Hence, to check that (4.9) holds, it suffices to find $\lambda'_\alpha > 0$ such that the equation

$$b(0)\widehat{K}_\alpha(z) + \alpha\widehat{\Lambda}_\alpha(z) = b(0) + \alpha \tag{4.41}$$

has no solution on $\Re(z) > -\lambda'_\alpha, z \neq 0$ (we have to eliminate $z = 0$ which is solution of (4.41)). First, equation (4.41) has no solution for $\Re(z) > 0$ because:

$$\Re(z) > 0 \implies |b(0)\widehat{K}_\alpha(z) + \alpha\widehat{\Lambda}_\alpha(z)| < b(0)|\widehat{K}_\alpha(z)| + \alpha|\widehat{\Lambda}_\alpha(z)| < b(0) + \alpha.$$

Now if $z = iw$ with $w > 0$ it holds that

$$\Re\left[b(0)(1 - \widehat{K}_\alpha(iw)) + \alpha(1 - \widehat{\Lambda}_\alpha(iw))\right] = \int_0^\infty [1 - \cos(wt)](b(0)K_\alpha(t) + \alpha\Lambda_\alpha(t))dt.$$

Because for $t \in \mathbb{R}_+$, $1 - \cos(wt) > 0$, the right hand side is null only if almost everywhere

$$b(0)K_\alpha(t) + \alpha\Lambda_\alpha(t) = 0.$$

This leads to a contradiction because for all $t > 0$, $K_\alpha(t) > 0$ and $\Lambda_\alpha(t) \geq 0$. Following the argument of Lemma 3.23, the solutions of (4.41) are within a cone and are isolated. We deduce that

$$\lambda'_\alpha := -\sup\{\Re(z) \mid z \in \mathbb{C}^*,\ \Re(z) > -\lambda^*_\alpha,\ \text{equation (4.41) holds}\}$$

is strictly positive. This ends the proof of the first point. □

Proof of Theorem 4.14, second point. Assume that $\inf_{x\geq 0} f(x) + b'(x) \geq 0$. By Proposition 4.5, to show uniqueness of the non-trivial invariant measure, it suffices to prove that the continuous function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is strictly increasing on $\mathbb{R}^*_+$. Note that by (4.21) and (4.22), we have

$$\forall t \geq 0, \quad [b(\varphi^\alpha_t(0)) + \alpha]\exp\left(-\int_0^t b'(\varphi^\alpha_u(0))du\right) = b(0) + \alpha.$$

We deduce that for all $\alpha > 0$

$$\begin{aligned}
\frac{\alpha}{\gamma(\alpha)} &\overset{(3.10)}{=} \alpha\int_0^\infty H_\alpha(t)dt \\
&= \frac{\alpha}{b(0)+\alpha}\int_0^\infty [b(\varphi^\alpha_t(0)) + \alpha]\exp\left(-\int_0^t b'(\varphi^\alpha_u(0))du\right)H_\alpha(t)dt \\
&= \frac{\alpha}{b(0)+\alpha}\int_0^\infty [b(\varphi^\alpha_t(0)) + \alpha]\exp\left(-\int_0^t (f+b')(\varphi^\alpha_u(0))du\right)dt
\end{aligned}$$

The changes of variable $\theta = \varphi^\alpha_u(0)$ and $x = \varphi^\alpha_t(0)$ shows that

$$\frac{\alpha}{\gamma(\alpha)} = \frac{\alpha}{b(0)+\alpha}\int_0^{\sigma_\alpha} \exp\left(-\int_0^x \frac{(f+b')(\theta)}{b(\theta)+\alpha}d\theta\right)dx.$$

Note that the function $\alpha \mapsto \frac{\alpha}{b(0)+\alpha}$ is non-decreasing and $\alpha \mapsto \sigma_\alpha$ is strictly increasing. Moreover, because $f + b' \geq 0$, for all fixed x, the function

$$\alpha \mapsto \exp\left(- \int_0^x \frac{(f+b')(\theta)}{b(\theta)+\alpha} d\theta \right)$$

is non-decreasing. It ends the proof. $\square$

4.7 Proof of Theorem 4.13

4.7.1 Structure of the proof

Let $\alpha > 0$. We recall that $J(\alpha) := \frac{\alpha}{\gamma(\alpha)} > 0$. Let ν_α^∞ be the corresponding invariant measure. Define:

$$\forall \nu \in \mathcal{M}(f^2), \forall \boldsymbol{h} \in L^\infty_\lambda, \quad \Phi(\nu, \boldsymbol{h}) := J(\alpha) r^\nu_{\alpha+\boldsymbol{h}} - (\alpha + \boldsymbol{h}). \tag{4.42}$$

Proposition 4.31. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda^*_\alpha > 0$ be given by (3.14). Then for all $\lambda \in (0, \lambda^*_\alpha)$, there exists a constant $\delta > 0$ (only depending on b, f, α and λ) such that*

$$\forall \nu \in \mathcal{M}(f^2), \forall \boldsymbol{h} \in B^\infty_\lambda(0,\delta), \quad \Phi(\nu, \boldsymbol{h}) \in L^\infty_\lambda.$$

Such result was proved in Chapter 3 (see Proposition 3.36), with slightly different assumptions. We recall the main steps and adapt the proof to our assumptions in Section 4.7.2.

Proposition 4.32. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda \in (0, \lambda^*_\alpha)$ and $S \in \mathcal{S}_\alpha$. There exists $\delta > 0$ (only depending on b, f, α, λ and S) such that*

1. *The function $\Phi : \mathcal{M}_S(f^2) \times B^\infty_\lambda(0,\delta) \to L^\infty_\lambda$ is continuous.*
2. *For a fixed $\nu \in \mathcal{M}_S(f^2)$, the function $\Phi(\nu, \cdot)$ is Fréchet differentiable at $\boldsymbol{h} \in B^\infty_\lambda(0,\delta)$. We denote by $D_h\Phi(\nu, \boldsymbol{h}) \in \mathcal{L}(L^\infty_\lambda, L^\infty_\lambda)$ its derivative.*
3. *The function $(\nu, \boldsymbol{h}) \mapsto D_h\Phi(\nu, \boldsymbol{h})$ is continuous.*

The proof is given in Section 4.7.3. We are looking for the zeros of Φ: if $\Phi(\nu, \boldsymbol{h}) = 0$, then $\boldsymbol{a} := \alpha + \boldsymbol{h}$ solves (1.11) and $X^\nu_t = Y^{\boldsymbol{a},\nu}_{t,0}$ solves (1.2). So

$$a_t = \alpha + h_t = J(\alpha)\mathbb{E} f(X^\nu_t),$$

and

$$h_t = J(\alpha)\mathbb{E} f(X^\nu_t) - \alpha \in L^\infty_\lambda.$$

The strategy is thus to apply the implicit function theorem. We have $\Phi(\nu^\infty_\alpha, 0) = 0$. Consider the differential of Φ with respect to the external current $\boldsymbol{h}$ at the point $(\nu, \boldsymbol{h}) = (\nu^\infty_\alpha, 0)$:

$$\begin{array}{rccc} D_h\Phi(\nu^\infty_\alpha, 0): & L^\infty_\lambda & \to & L^\infty_\lambda \\ & c & \mapsto & -c + J(\alpha) D_h r^{\nu^\infty_\alpha}_\alpha \cdot c \end{array}$$

Proposition 4.33. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda^*_\alpha > 0$ be given by Proposition 4.31. Then it holds that*

$$\forall c \in L^\infty_\lambda, \quad D_h\Phi(\nu^\infty_\alpha, 0) \cdot c = -c + J(\alpha)\Theta_\alpha * c,$$

where the function $\Theta_\alpha : \mathbb{R}_+ \to \mathbb{R}$ is given by (4.8).

We prove this proposition in Section 4.7.4.

Proposition 4.34. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Assume moreover that (4.9) holds. Let Ω_α be the resolvent of $J(\alpha)\Theta_\alpha$, that is the solution of (4.16). Then for all $\lambda' \in (0, \lambda'_\alpha)$, Ω_α belongs to $L^1_{\lambda'}$, and the linear operator $D_h\Phi(\nu^\infty_\alpha, 0) \in \mathcal{L}(L^\infty_{\lambda'}; L^\infty_{\lambda'})$ is invertible, with inverse given by*

$$[D_h\Phi(\nu^\infty_\alpha, 0)]^{-1} : \begin{array}{ccc} L^\infty_{\lambda'} & \to & L^\infty_{\lambda'} \\ c & \mapsto & -c - \Omega_\alpha * c. \end{array} \tag{4.43}$$

Proof. First, by Lemma 2.20 eq. (4.16) has a unique solution. So $\Omega_\alpha(t)$ is well-defined on $\mathbb{R}_+$. We extend Θ_α and Ω_α to $\mathbb{R}$ by setting

$$\forall t \in \mathbb{R}, \quad \Theta_\alpha(t) := \Theta_\alpha(t)\mathbb{1}_{\mathbb{R}_+}(t) \quad \text{and} \quad \Omega_\alpha(t) := \Omega_\alpha(t)\mathbb{1}_{\mathbb{R}_+}(t).$$

Let $\lambda' \in (0, \lambda'_\alpha)$. From (4.16), it holds that for all $t \in \mathbb{R}$,

$$\Omega_\alpha(t)e^{\lambda' t} = J(\alpha)\Theta_\alpha(t)e^{\lambda' t} + \int_{\mathbb{R}} J(\alpha)\Theta_\alpha(t-u)e^{\lambda'(t-u)}\Omega_\alpha(u)e^{\lambda' u}du.$$

We apply the Whole-line Paley-Wiener theorem (Theorem 3.24) with $k(t) := \Theta_\alpha(t)e^{\lambda' t}$. It holds that for all $y \in \mathbb{R}$

$$\widehat{k}(iy) = J(\alpha)\int_0^\infty e^{-iyt+\lambda' t}\Theta_\alpha(t)dt = J(\alpha)\widehat{\Theta}_\alpha(iy - \lambda') \overset{\lambda'<\lambda'_\alpha}{\neq} 1.$$

So $\Omega_\alpha(t)e^{\lambda' t} \in L^1(\mathbb{R})$. That is $\Omega_\alpha \in L^1_{\lambda'}$. Consider now $c \in L^\infty_{\lambda'}$. The eq. $D_h\Phi(\nu^\infty_\alpha, 0) \cdot d = c$ writes

$$d = -c + J(\alpha)\Theta_\alpha * d.$$

Solving this using the resolvent Ω_α (see Lemma 2.22), we find

$$d = -c - \Omega_\alpha * c.$$

By Lemma 4.18, it holds that $d \in L^\infty_{\lambda'}$. Moreover if $d = 0$, then $c = 0$. Overall $\Phi(\nu^\infty_\alpha, 0)$ is invertible, with inverse given by (4.43). This ends the proof. □

Consequently, if (4.9) holds, we can define the following iteration scheme:

$$\boldsymbol{h}_0 := 0, \quad \boldsymbol{h}_{n+1} = \boldsymbol{h}_n - [D_h\Phi(\nu^\infty_\alpha, 0)]^{-1} \cdot \Phi(\nu, \boldsymbol{h}_n). \tag{4.44}$$

Equivalently, setting $\boldsymbol{a}_n := \boldsymbol{h}_n + \alpha$ one has

$$\begin{aligned}\boldsymbol{a}_{n+1} &= \alpha + \boldsymbol{h}_{n+1} \\ &= \alpha + \boldsymbol{h}_n - (-\Phi(\nu, \boldsymbol{h}_n) - \Omega_\alpha * \Phi(\nu, \boldsymbol{h}_n)) \\ &= J(\alpha) r^\nu_{\boldsymbol{a}_n} + \Omega_\alpha * (J(\alpha) r^\nu_{\boldsymbol{a}_n} - \boldsymbol{a}_n),\end{aligned}$$

and so

$$\boldsymbol{a}_0 := \alpha, \quad \boldsymbol{a}_{n+1} = J(\alpha) r^\nu_{\boldsymbol{a}_n} + \Omega_\alpha * (J(\alpha) r^\nu_{\boldsymbol{a}_n} - \boldsymbol{a}_n). \tag{4.45}$$

Remark 4.35. *This scheme is actually a refinement of the "standard" Picard scheme (3.27):*

$$\boldsymbol{a}_{n+1} = J(\alpha) r^\nu_{\boldsymbol{a}_n}, \quad \boldsymbol{a}_0 := \alpha.$$

This Picard scheme may not converge if $J(\alpha)$ is not small enough. Note that (4.44) is an approximation of the Newton scheme, which would be:

$$\boldsymbol{h}_0 := 0, \quad \boldsymbol{h}_{n+1} := \boldsymbol{h}_n - [D_h \Phi(\nu, \boldsymbol{h}_n)]^{-1} \cdot \Phi(\nu, \boldsymbol{h}_n).$$

We prefer to use (4.44) for simplicity (by doing so we lost in the speed of convergence of the scheme, but it does not matter here).

We now prove that the scheme (4.44) converges to some $\boldsymbol{h}(\nu) \in L^\infty_{\lambda'}$ with $\Phi(\nu, \boldsymbol{h}(\nu)) = 0$. This gives the proof of Theorem 4.13.

Proof of Theorem 4.13. Let $0 < \lambda' < \lambda'_\alpha$. We have $\boldsymbol{h}_{n+1} = T_\nu(\boldsymbol{h}_n)$, with:

$$\begin{array}{rccl} T_\nu : & L^\infty_{\lambda'} & \to & L^\infty_{\lambda'} \\ & \boldsymbol{h} & \mapsto & \boldsymbol{h} - [D_h \Phi(\nu^\infty_\alpha, 0)]^{-1} \cdot \Phi(\nu, \boldsymbol{h}) \end{array}$$

Claim. Let $\epsilon > 0$ be fixed. We can find small enough $\rho, \rho' > 0$ with $\rho' < \epsilon$ such that $d(\nu, \nu^\infty_\alpha) < \rho$ implies

$$T_\nu(\overline{B^\infty_{\lambda'}(0, \rho')}) \subset \overline{B^\infty_{\lambda'}(0, \rho')}.$$

Indeed we have

$$D_h T_\nu(\boldsymbol{h}) = I - [D_h \Phi(\nu^\infty_\alpha, 0)]^{-1} D_h \Phi(\nu, \boldsymbol{h}),$$

which is close to zero because $(\nu, \boldsymbol{h}) \mapsto D_h \Phi(\nu, \boldsymbol{h})$ is continuous at $(\nu^\infty_\alpha, 0)$. It follows that for ρ and ρ' small enough, we have

$$\forall \nu \in \mathcal{M}_S(f^2), \forall \boldsymbol{h} \in L^\infty_{\lambda'}, \quad d(\nu, \nu^\infty_\alpha) < \rho \text{ and } ||\boldsymbol{h}||^\infty_{\lambda'} \le \rho' \implies |||D_h T_\nu(\boldsymbol{h})|||^\infty_{\lambda'} \le \tfrac{1}{2}. \tag{4.46}$$

Without loss of generality, such ρ' can be chosen smaller that ϵ. Moreover, for ρ small enough

$$||T_\nu(0)||^\infty_{\lambda'} \le |||[D_h \Phi(\nu^\infty_\alpha, 0)]^{-1}|||^\infty_{\lambda'} \; ||\Phi(\nu, 0)||^\infty_{\lambda'} \le \tfrac{\rho'}{2}.$$

It follows that if $d(\nu, \nu^\infty_\alpha) < \rho$ and $||\boldsymbol{h}||^\infty_{\lambda'} \le \rho'$, then

$$\begin{aligned}||T_\nu(\boldsymbol{h})||^\infty_{\lambda'} &\le ||T_\nu(\boldsymbol{h}) - T_\nu(0)||^\infty_{\lambda'} + ||T_\nu(0)||^\infty_{\lambda'} \\ &\le \tfrac{1}{2} ||\boldsymbol{h}||^\infty_{\lambda'} + \tfrac{\rho'}{2} \\ &\le \rho'.\end{aligned}$$

We use that $\overline{B^\infty_{\lambda'}(0,\rho')} \ni \boldsymbol{h} \mapsto T_\nu(\boldsymbol{h})$ is $\frac{1}{2}$-Lipschitz (as a consequence of (4.46)). It follows that T_ν has a unique fixed point $\boldsymbol{h}(\nu) \in L^\infty_{\lambda'}$ such that $||\boldsymbol{h}(\nu)||^\infty_{\lambda'} \leq \rho' < \epsilon$. Moreover we have

$$\lim_{n\to\infty} ||\boldsymbol{h}_n - \boldsymbol{h}(\nu)||^\infty_{\lambda'} = 0.$$

This fixed point satisfies $[D_h\Phi(\nu^\infty_\alpha, 0)]^{-1}\cdot\Phi(\nu, \boldsymbol{h}(\nu)) = 0$. So Proposition 4.34 yields $\Phi(\nu, \boldsymbol{h}(\nu)) = 0$. Consequently we have

$$J(\alpha)\,\mathbb{E}\, f(X^\nu_t) = \alpha + \boldsymbol{h}(\nu).$$

We deduce that ν^∞_α is exponentially stable, in the sense of Definition 4.9.

Remark 4.36. *This construction follows precisely the standard proof of the implicit function theorem. At any step n, the Picard iteration $\boldsymbol{h}_{n+1} = T_\nu(\boldsymbol{h}_n)$ is continuous in ν. We know in this case that the fixed point $\boldsymbol{h}(\nu)$ is itself continuous in ν because by (4.46), the Lipschitz constant of T_ν is uniformly bounded in ν.*

□

4.7.2 Proof of Proposition 4.31

We follow the proof of Proposition 3.36 and stress the differences, due to our different assumptions. For instance, in Proposition 3.36, we assumed b to be bounded and assume the uniform in time control (3.3) of the deterministic flow. Such assumption is replaced here by Assumption 4.6.

Given $\alpha, \lambda > 0$ and $\boldsymbol{h} \in L^\infty_\lambda$ we write:

$$\begin{aligned} \boldsymbol{h} &\mapsto \bar{H}^\alpha_{\boldsymbol{h}} := H_{\alpha+\boldsymbol{h}} - H_\alpha \\ \boldsymbol{h} &\mapsto \bar{K}^\alpha_{\boldsymbol{h}} := K_{\alpha+\boldsymbol{h}} - K_\alpha \end{aligned} \tag{4.47}$$

Lemma 4.37. *Consider f and b satisfying Assumptions 4.1, 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$. There exists a constant $\delta > 0$ and a function $\eta \in L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ such that for all $(t,s) \in \Delta$ and $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$, it holds*

$$|\bar{H}^\alpha_{\boldsymbol{h}}|(t,s) \leq ||\boldsymbol{h}||^\infty_\lambda e^{-\lambda t}\eta(t-s),$$

$$|\bar{K}^\alpha_{\boldsymbol{h}}|(t,s) \leq ||\boldsymbol{h}||^\infty_\lambda e^{-\lambda t}\eta(t-s).$$

In particular, $\bar{H}^\alpha_{\boldsymbol{h}} * 1 \in \mathcal{V}^1_\lambda$ and

$$||\bar{H}^\alpha_{\boldsymbol{h}}||^1_\lambda \leq ||\boldsymbol{h}||^\infty_\lambda ||\eta||_1, \quad ||\bar{H}^\alpha_{\boldsymbol{h}} * 1||^1_\lambda \leq \frac{||\boldsymbol{h}||^\infty_\lambda ||\eta||_1}{\lambda}.$$

The same inequalities holds for $\bar{K}^\alpha_{\boldsymbol{h}}$.

Remark 4.38. *The constant δ and the function η only depend on α, b, f and λ. This result is a generalization of Lemma 3.32, (with $\boldsymbol{a} = \alpha + h$). Among the differences, we now assume that the perturbation $\boldsymbol{h}$ is small ($||\boldsymbol{h}||^\infty_\lambda < \delta$).*

Proof. We prove only the result for $\bar{H}^{\alpha}_{\boldsymbol{h}}$. Using the inequality $|e^{-A} - e^{-B}| \leq e^{-A\wedge B}|A - B|$, valid for all $A, B \geq 0$, we have

$$|\bar{H}^{\alpha}_{\boldsymbol{h}}|(t,s) \leq \exp\left(-\int_s^t f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)) \wedge f(\varphi^{\alpha}_{u,s}(0))du\right) \int_s^t \left| f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)) - f(\varphi^{\alpha}_{u,s}(0))\right| du.$$

Let $\lambda \in (0, f(\sigma_\alpha))$. We distinguish the cases $\sigma_\alpha = +\infty$ and $\sigma_\alpha < \infty$.
Case $\sigma_\alpha = \infty$. We choose

$$\delta := \frac{1}{2}\left[\inf_{x\geq 0} b(x) + \alpha\right] > 0.$$

Let $\boldsymbol{h} \in L^\infty_\lambda$, with $||\boldsymbol{h}||^\infty_\lambda < \delta$. For all $u \in [s,t]$, it holds that

$$\left| f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)) - f(\varphi^{\alpha}_{u,s}(0))\right| = \left| \int_{\varphi^{\alpha}_{u,s}(0)}^{\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)} f'(\theta)d\theta\right| \overset{\text{Ass. 4.2}(b)}{\leq} Cf(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0) \vee \varphi^{\alpha}_{u,s}(0)) \left|\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0) - \varphi^{\alpha}_{u,s}(0)\right|.$$

So, using that $\delta < \alpha$, Lemma 4.23 yields the existence of a constant C such that

$$\left| f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)) - f(\varphi^{\alpha}_{u,s}(0))\right| \leq Ce^{pL(u-s)} \left|\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0) - \varphi^{\alpha}_{u,s}(0)\right|.$$

Moreover, by Lemma 4.20 we have

$$\left|\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0) - \varphi^{\alpha}_{u,s}(0)\right| \leq \int_s^u e^{L(u-\theta)}|h_\theta|d\theta \leq \frac{||\boldsymbol{h}||^\infty_\lambda}{L} e^{-\lambda s} e^{L(u-s)}.$$

We deduce that there exists another constant C such that

$$\int_s^t \left| f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)) - f(\varphi^{\alpha}_{u,s}(0))\right| du \leq C||\boldsymbol{h}||^\infty_\lambda e^{-\lambda s} e^{(p+1)L(t-s)} = C||\boldsymbol{h}||^\infty_\lambda e^{-\lambda t} e^{((p+1)L+\lambda)(t-s)}.$$

To conclude, note that

$$\frac{d}{dt}\varphi^{\alpha+\boldsymbol{h}}_{t,s}(0) \geq \tfrac{\delta}{2},$$

and so $\varphi^{\alpha+\boldsymbol{h}}_{t,s}(0) \geq \frac{\delta(t-s)}{2}$: we can find a constant C (only depending on b, δ, α and λ) such that

$$\exp\left(-\int_s^t f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0)) \wedge f(\varphi^{\alpha}_{u,s}(0))du\right) \leq Ce^{-((p+1)L+\lambda+1)(t-s)},$$

and the result follows.
Case $\sigma_\alpha < \infty$. Define $\mu := \lambda \wedge \ell_\alpha/2$. By Lemma 4.22, there exists constants $\delta > 0$ and C_μ such that for all $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\mu < \delta$ one has

$$|\varphi^{\alpha+\boldsymbol{h}}_{t,s}(0) - \sigma_\alpha| \leq C_\mu e^{-\mu(t-s)}, \tag{4.48}$$

and

$$|\varphi^{\alpha+\boldsymbol{h}}_{t,s}(0) - \varphi^{\alpha}_{t,s}(0)| \leq C_\mu \int_s^t |h_u|du. \tag{4.49}$$

Let $\boldsymbol{h} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$. Because $\mu \leq \lambda$, one has $\boldsymbol{h} \in L^\infty_\mu$. Let $\epsilon := (f(\sigma_\alpha) - \lambda)/2$. By (4.48) and by continuity of f at σ_α, there exists another constant C_μ such that

$$|\bar{H}^{\alpha}_{\boldsymbol{h}}(t,s)| \leq C_\mu e^{-(\lambda+\epsilon)(t-s)} \int_s^t \left|\varphi^{\alpha+\boldsymbol{h}}_{u,s}(0) - \varphi^{\alpha}_{u,s}(0)\right| du.$$

Moreover by (4.49) one has

$$\int_s^t \left|\varphi_{u,s}^{\alpha+\boldsymbol{h}}(0) - \varphi_{u,s}^{\alpha}(0)\right| du \leq C_\mu \int_s^t \int_s^u |h_\theta| d\theta du$$
$$\leq C_\mu ||\boldsymbol{h}||_\lambda^\infty e^{-\lambda t} e^{\lambda(t-s)} \frac{(t-s)^2}{2}.$$

Altogether there exists another constant C_μ such that

$$|\bar{H}_{\boldsymbol{h}}^\alpha(t,s)| \leq C_\mu ||\boldsymbol{h}||_\lambda^\infty e^{-\lambda t}(t-s)^2 e^{-\epsilon(t-s)}.$$

This ends the proof. □

Proposition 4.39. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda \in (0, \lambda_\alpha^*)$. There exists a constant $\delta > 0$ (only depending on b, f, α and λ) such that for any $\boldsymbol{h} \in L_\lambda^\infty$ with $||\boldsymbol{h}||_\lambda^\infty < \delta$:*

1. *The following function $V_{\boldsymbol{h}}^\alpha$ belongs to $\mathcal{V}_\lambda^1$:*
$$V_{\boldsymbol{h}}^\alpha := \bar{K}_{\boldsymbol{h}}^\alpha + \xi_\alpha * \bar{K}_{\boldsymbol{h}}^\alpha - \gamma(\alpha)\bar{H}_{\boldsymbol{h}}^\alpha \in \mathcal{V}_\lambda^1. \tag{4.50}$$
2. *Let $Q_{\boldsymbol{h}}^\alpha$ be the solution of the Volterra equation*
$$Q_{\boldsymbol{h}}^\alpha = V_{\boldsymbol{h}}^\alpha + V_{\boldsymbol{h}}^\alpha * Q_{\boldsymbol{h}}^\alpha, \tag{4.51}$$
It holds that $Q_{\boldsymbol{h}}^\alpha \in \mathcal{V}_\lambda^1$.
3. *The function*
$$\xi_{\alpha+\boldsymbol{h}}(t,s) := r_{\alpha+\boldsymbol{h}}(t,s) - \gamma(\alpha)$$
satisfies $\xi_{\alpha+\boldsymbol{h}} \in \mathcal{V}_\lambda^1$. Moreover one has the explicit decomposition
$$\xi_{\alpha+\boldsymbol{h}} = \xi_\alpha + Q_{\boldsymbol{h}}^\alpha + Q_{\boldsymbol{h}}^\alpha * \xi_\alpha + \gamma(\alpha)(Q_{\boldsymbol{h}}^\alpha * 1). \tag{4.52}$$

Proof. This is Proposition 3.34 with $\boldsymbol{a} := \alpha + h$, $\Delta_K := V_{\boldsymbol{h}}^\alpha$ and $\Delta_r := Q_{\boldsymbol{h}}^\alpha$. Note that δ has to be chosen smaller than the δ of Lemma 4.37 and such that

$$\delta < \alpha \wedge \frac{1}{||\eta||_1(1+\lambda^{-1})(1+||\xi_\alpha||_\lambda^1 + \gamma(\alpha))},$$

where η is given in Lemma 4.37. □

Finally, we consider a general initial condition $\nu \in \mathcal{M}(f^2)$.

Proposition 4.40. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda \in (0, \lambda_\alpha^*)$. Consider $\delta > 0$ be given by the previous proposition. For all $\boldsymbol{h} \in L_\lambda^\infty$ such that $||\boldsymbol{h}||_\lambda^\infty < \delta$ and for all $\nu \in \mathcal{M}(f^2)$ define:*

$$\xi_{\alpha+\boldsymbol{h}}^\nu(t) := r_{\alpha+\boldsymbol{h}}^\nu(t) - \gamma(\alpha).$$

It holds that $\xi_{\alpha+\boldsymbol{h}}^\nu \in L_\lambda^\infty$. Moreover, we have the explicit decomposition

$$\xi_{\alpha+\boldsymbol{h}}^\nu = K_{\alpha+\boldsymbol{h}}^\nu - \gamma(\alpha)H_{\alpha+\boldsymbol{h}}^\nu + \xi_{\alpha+\boldsymbol{h}} * K_{\alpha+\boldsymbol{h}}^\nu. \tag{4.53}$$

Again, this is Proposition 3.36 with $\boldsymbol{a} := \alpha + \boldsymbol{h}$. In particular, (4.53) yields

$$\Phi(\nu, \boldsymbol{h}) = J(\alpha)\xi^\nu_{\alpha+\boldsymbol{h}} - \boldsymbol{h} \in L^\infty_\lambda,$$

which ends the proof of Proposition 4.31.

4.7.3 Regularity of Φ: Proof of Proposition 4.32

Continuity of $\nu \mapsto \Phi(\nu, h)$.

Proposition 4.41. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda \in (0, \lambda^*_\alpha)$ and fix $\boldsymbol{h} \in L^\infty_\lambda$ such that $||\boldsymbol{h}||^\infty_\lambda < \delta$, where δ is given by Proposition 4.31. The function*

$$\mathcal{M}(f^2) \ni \nu \mapsto \Phi(\nu, \boldsymbol{h}) \in L^\infty_\lambda$$

is continuous.

Proof. Let $\boldsymbol{a} := \alpha + \boldsymbol{h}$. Fix $\mu, \nu \in \mathcal{M}(f^2)$. Solving the Volterra equation (2.7) in term of its resolvent $r_{\boldsymbol{a}}$ gives

$$r^\nu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} + r_{\boldsymbol{a}} * K^\nu_{\boldsymbol{a}}.$$

It follows that

$$r^\nu_{\boldsymbol{a}} - r^\mu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}} + r_{\boldsymbol{a}} * (K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}}).$$

Using that $r_{\boldsymbol{a}} = \gamma(\alpha) + \xi_{\boldsymbol{a}}$, where $\xi_{\boldsymbol{a}} \in \mathcal{V}^1_\lambda$, we have

$$r^\nu_{\boldsymbol{a}} - r^\mu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}} + \gamma(\alpha) * (K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}}) + \xi_{\boldsymbol{a}} * (K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}}).$$

Moreover the identity

$$1 * K^\nu_{\boldsymbol{a}} = 1 - H^\nu_{\boldsymbol{a}},$$

yields

$$r^\nu_{\boldsymbol{a}} - r^\mu_{\boldsymbol{a}} = K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}} - \gamma(\alpha)(H^\nu_{\boldsymbol{a}} - H^\mu_{\boldsymbol{a}}) + \xi_{\boldsymbol{a}} * (K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}}).$$

To conclude we use:
Claim: There exists a constant $C > 0$ only depending on b, f, α, λ and δ such that

$$|H^\nu_{\boldsymbol{a}} - H^\mu_{\boldsymbol{a}}|(t) + |K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}}|(t) \le Ce^{-\lambda t} d(\nu, \mu).$$

Proof of the Claim. By Lemma 4.25, there is a constant $C > 0$ such that for all $x \ge 0$ and $t \ge 0$

$$H^x_{\boldsymbol{a}}(t) \le Ce^{-\lambda t} \quad \text{and} \quad K^x_{\boldsymbol{a}}(t) \le C(1 + f(x))e^{-\lambda t}.$$

So

$$|H^\nu_{\boldsymbol{a}} - H^\mu_{\boldsymbol{a}}|(t) = \left| \int_0^\infty H^x_{\boldsymbol{a}}(t)\nu(dx) - \int_0^\infty H^x_{\boldsymbol{a}}(t)\mu(dx) \right| \le \int_0^\infty H^x_{\boldsymbol{a}}(t)|\nu - \mu|(dx) \le Ce^{-\lambda t} d(\nu, \mu).$$

Similarly,

$$|K^\nu_{\boldsymbol{a}} - K^\mu_{\boldsymbol{a}}|(t) \le \int_0^\infty K^x_{\boldsymbol{a}}(t)|\nu - \mu|(dx) \le Ce^{-\lambda t} d(\nu, \mu).$$

This ends the proof. $\square$

Differentiability of $h \mapsto \Phi(\nu, h)$.

Lemma 4.42. *Consider b and f satisfying Assumptions 4.1 and 4.2. Consider $\alpha > 0$ and $\lambda \in (0, f(\sigma_\alpha))$. Let $x \geq 0$ and $t \geq s$ be fixed. The function $L^\infty_\lambda \ni \boldsymbol{h} \mapsto H^x_{\alpha+\boldsymbol{h}}(t,s) \in \mathbb{R}$ is $\mathcal{C}^1$ and*

$$\forall c \in L^\infty_\lambda, \quad \left[D_h H^x_{\alpha+\boldsymbol{h}} \cdot c\right](t,s) = -H^x_{\alpha+\boldsymbol{h}}(t,s) \int_s^t f'(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) \left[D_h \varphi^{\alpha+\boldsymbol{h}}_{u,s}(x) \cdot c\right] du. \quad (4.54)$$

Similarly $L^\infty_\lambda \ni \boldsymbol{h} \mapsto K^x_{\alpha+\boldsymbol{h}}(t,s) \in \mathbb{R}$ is $\mathcal{C}^1$ and

$$\forall c \in L^\infty_\lambda, \quad \left[D_h K^x_{\alpha+\boldsymbol{h}} \cdot c\right] = -\frac{d}{dt}\left[D_h H^x_{\alpha+\boldsymbol{h}} \cdot c\right](t,s). \quad (4.55)$$

Proof. The result follows from Lemma 4.20(c), from the fact that f is $\mathcal{C}^1$ and from the explicit expressions of H and K. It suffices to apply the chain rule for Fréchet derivatives. □

Lemma 4.43. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$ and $S \in \mathcal{S}_\alpha$. There exists $\delta > 0$ and a function $\eta \in L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ (both only depending on b, f, α, λ and S) such that for all $\nu \in \mathcal{M}_S(f^2)$, for all $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^\infty_\lambda$ with $||\boldsymbol{h}||^\infty_\lambda < \delta$ and $||\boldsymbol{h} - \tilde{\boldsymbol{h}}||^\infty_\lambda < \delta/2$, one has*

$$\begin{aligned}\forall x \in S, \forall t \geq s, \quad &\left|H^x_{\alpha+\tilde{\boldsymbol{h}}}(t,s) - H^x_{\alpha+\boldsymbol{h}}(t,s) - D_h H^x_{\alpha+\boldsymbol{h}}(t,s) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h})\right| \\ &\leq \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^\infty_\lambda e^{-\lambda s}\right]^2 \left(1 + f^2(x)\right) e^{-\lambda(t-s)} \eta(t-s), \quad (4.56)\end{aligned}$$

where $D_h H^x_{\alpha+\boldsymbol{h}}$ is given by (4.54). A similar result holds for $K^x_{\alpha+\boldsymbol{h}}$.

Proof. Let $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^\infty_\lambda$ such that $||\boldsymbol{h}||^\infty_\lambda < \delta$ and $||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^\infty_\lambda < \delta/2$, where δ will be specified later. Fix $t \geq s$. We use the following inequality, valid for every $A, B \in \mathbb{R}$:

$$|e^{-B} - e^{-A} + (B-A)e^{-A}| \leq (B-A)^2 \left(e^{-A} + e^{-B}\right), \quad (4.57)$$

with

$$A := \int_s^t f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) du \quad \text{and} \quad B := \int_s^t f(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)) du.$$

The Taylor formula gives for all $u \geq s$

$$\begin{aligned} f(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)) =& f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) + f'(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) \left[\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)\right] \\ &+ \int_{\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)}^{\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)} (\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x) - v) f''(v) dv \\ =& f(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) + f'(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) \left[D_h \varphi^{\alpha+\boldsymbol{h}}_{u,s}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h})\right] \\ &+ f'(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)) \left[\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{u,s}(x) - D_h \varphi^{\alpha+\boldsymbol{h}}_{u,s}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h})\right] \\ &+ \int_{\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)}^{\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)} (\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x) - v) f''(v) dv. \end{aligned}$$

So

$$B - A = \int_s^t f'(\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x)) D_h \varphi_{u,s}^{\alpha+\boldsymbol{h}}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}) du + \epsilon_1(t,s) + \epsilon_2(t,s),$$

with:

$$\epsilon_1(t,s) := \int_s^t f'(\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x)) \left[\varphi_{u,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{u,s}^{\alpha+\boldsymbol{h}}(x) - D_h \varphi_{u,s}^{\alpha+\boldsymbol{h}}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h})\right] du,$$

$$\epsilon_2(t,s) := \int_s^t \int_{\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x)}^{\varphi_{u,s}^{\alpha+\tilde{\boldsymbol{h}}}(x)} (\varphi_{u,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - v) f''(v) dv du.$$

Note that $e^{-A} = H^x_{\alpha+\boldsymbol{h}}(t,s)$ and $e^{-B} = H^x_{\alpha+\tilde{\boldsymbol{h}}}(t,s)$. We deduce from (4.57) that

$$\left| H^x_{\alpha+\tilde{\boldsymbol{h}}}(t,s) - H^x_{\alpha+\boldsymbol{h}}(t,s) - D_h H^x_{\alpha+\boldsymbol{h}}(t,s) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}) \right| \leq$$
$$(B-A)^2 (e^{-A} + e^{-B}) + e^{-A}|\epsilon_1(t,s)| + e^{-A}|\epsilon_2(t,s)|.$$

We denote by C any constant that may depend on b, f, λ, δ and S and may change from line to line. We distinguish the case $\sigma_\alpha = \infty$ and $\sigma_\alpha < \infty$.
Case $\sigma_\alpha = \infty$. Let $\delta := \frac{1}{2}\inf_{x\geq 0} b(x) + \alpha$. First, using Assumption 4.2(b) and Lemma 4.23, there exists a constant C such that

$$\forall x \geq 0, \forall u \geq s, \quad \left| f'(\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x)) \right| \leq C[1 + f(x)] e^{pL(u-s)}.$$

So

$$|\epsilon_1(t,s)| \leq C[1+f(x)] e^{pL(t-s)} \int_s^t \left| \varphi_{u,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{u,s}^{\alpha+\boldsymbol{h}}(x) - D_h \varphi_{u,s}^{\alpha+\boldsymbol{h}}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}) \right| du$$
$$\overset{(4.24)}{\leq} C[1+f(x)] e^{pL(t-s)} \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||_\lambda^\infty e^{-\lambda s} \right]^2 e^{2L(t-s)}.$$

By Lemma 4.25, for all $\theta > 0$ we can find a constant C (that also depends on θ) such that

$$\forall t \geq s, \quad \sup_{x \geq 0} H^x_{\alpha+\boldsymbol{h}}(t,s) \leq C e^{-\theta(t-s)},$$

which implies that there exists C such that

$$e^{-A}|\epsilon_1(t,s)| \leq C \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||_\lambda^\infty \right]^2 [1+f(x)] e^{-(\lambda+1)(t-s)}.$$

Secondly, we have for all $v \in [\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x), \varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x)]$,

$$|f''(v)| \overset{(4.2)}{\leq} C(1+f(v)) \leq C(1+f(x)) e^{Lp(t-s)},$$

and so using (4.20) we deduce that

$$e^{-A}|\epsilon_2(t,s)| \leq C \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||_\lambda^\infty e^{-\lambda s} \right]^2 [1+f(x)] e^{-(\lambda+1)(t-s)}.$$

Finally we have by (4.20)

$$|B-A| \leq C(1+f(x))e^{Lp(t-s)}\int_s^t \left|\varphi_{u,s}^{\alpha+\tilde{\boldsymbol{h}}}(x)-\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x)\right| du \leq C(1+f(x))\left[||\tilde{\boldsymbol{h}}-\boldsymbol{h}||_\lambda^\infty e^{-\lambda s}\right]e^{L(p+1)(t-s)}.$$

So there exists another constant C such that

$$(B-A)^2(e^{-A}+e^{-B}) \leq C\left[||\tilde{\boldsymbol{h}}-\boldsymbol{h}||_\lambda^\infty e^{-\lambda s}\right]^2[1+f(x)]e^{-(\lambda+1)(t-s)}.$$

This ends to proof.
Case $\sigma_\alpha < \infty$. Define $\mu := \lambda \wedge \ell_\alpha/2$. By Lemma 4.22, the exists a constant $\delta > 0$ and C such that for all $\boldsymbol{h} \in L_\lambda^\infty$ with $||\boldsymbol{h}||_\mu^\infty < \delta$ one has

$$\forall x \in S, \forall t \geq s, \quad |\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)-\sigma_\alpha| \leq Ce^{-\mu(t-s)}.$$

Using (4.28), there exists C such that

$$|\epsilon_1(t,s)| \leq C(t-s)\left[||\tilde{\boldsymbol{h}}-\boldsymbol{h}||e^{-\lambda s}\right]^2.$$

Using (4.27), we deduce that the same inequality is satisfied by $|\epsilon_2(t,s)|$. Moreover, let $\epsilon \in (\lambda, f(\sigma_\alpha))$, there exists a constant C (that also depends on ϵ) such that

$$\forall x \in S, \forall t \geq s, \quad H_{\alpha+\boldsymbol{h}}^x(t,s)+H_{\alpha+\tilde{\boldsymbol{h}}}^x(t,s) \leq Ce^{-(\lambda+\epsilon)(t-s)}.$$

Finally, by (4.27)

$$(B-A)^2 \leq C\left[||\tilde{\boldsymbol{h}}-\boldsymbol{h}||_\lambda^\infty e^{-\lambda s}\right]^2(t-s)^2.$$

Combining the estimates, the result follows. □

Lemma 4.44. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$ and $S \in \mathcal{S}_\alpha$. There exists $\delta > 0$ and a function $\eta \in L^1 \cap L^\infty(\mathbb{R}_+, \mathbb{R}_+)$ (both only depending on b, f, α, λ and S) such that for all $\boldsymbol{h}, c \in L_\lambda^\infty$ with $||\boldsymbol{h}||_\lambda^\infty < \delta$*

$$\forall x \in S, \forall t \geq s, \quad \left|\left[D_h H_{\alpha+\boldsymbol{h}}^x \cdot c\right](t,s)\right| \leq (1+f(x))\left[||c||_\lambda^\infty e^{-\lambda s}\right]e^{-\lambda(t-s)}\eta(t-s).$$

Moreover, for all $\boldsymbol{h}, \tilde{\boldsymbol{h}}, c \in L_\lambda^\infty$ with $||\boldsymbol{h}||_\lambda^\infty \vee ||\tilde{\boldsymbol{h}}||_\lambda^\infty < \delta$ and for all $x \in S$

$$\left|\left[D_h H_{\alpha+\tilde{\boldsymbol{h}}}^x \cdot c\right](t,s)-\left[D_h H_{\alpha+\boldsymbol{h}}^x \cdot c\right](t,s)\right| \leq \left(1+f^2(x)\right)\left[||c||_\lambda^\infty ||\tilde{\boldsymbol{h}}-\boldsymbol{h}||_\lambda^\infty e^{-2\lambda s}\right]e^{-\lambda(t-s)}\eta(t-s).$$

Similar inequalities holds for $D_h K_{\alpha+\boldsymbol{h}}^x$.

Proof. Let $\delta > 0$ be given by Lemma 4.43. We denote by C any constant only depending on b, f, α, λ and S. We start with the first inequality. Let $c \in L_\lambda^\infty$.
Case $\sigma_\alpha = \infty$. We have, using Assumption 4.2(b) and (4.23)

$$\begin{aligned}\left|\int_s^t f'(\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x))\left[D_h\varphi_{u,s}^{\alpha+\boldsymbol{h}}(x)\cdot c\right]du\right| &\leq C(1+f(x))e^{Lp(t-s)}\int_s^t\int_s^u |c_\theta|\exp\left(\int_\theta^u b'(\varphi_{v,s}^{\alpha+\boldsymbol{h}}(x))dv\right)d\theta du \\ &\leq C(1+f(x))\left[||c||_\lambda^\infty e^{-\lambda s}\right]e^{(p+1)L(t-s)}.\end{aligned}$$

So there exists a constant C such that

$$\left|\left[D_h H^x_{\alpha+\boldsymbol{h}}\cdot c\right](t,s)\right| \le C(1+f(x))||c||^\infty_\lambda e^{-(\lambda+1)(t-s)}.$$

Case $\sigma_\alpha < \infty$. The proof is similar. We use that $\exp\left(\int_\theta^u b'(\varphi^{\alpha+\boldsymbol{h}}_{v,s}(x))dv\right)$ is bounded (because $b'(\sigma_\alpha) < 0$).

We now prove the second inequality. The triangular inequality yields

$$\begin{aligned}
&\left|\left[D_h H^x_{\alpha+\tilde{\boldsymbol{h}}}\cdot c\right](t,s) - \left[D_h H^x_{\alpha+\boldsymbol{h}}\cdot c\right](t,s)\right| \\
&\le \left|H^x_{\alpha+\tilde{\boldsymbol{h}}}(t,s) - H^x_{\alpha+\boldsymbol{h}}(t,s)\right| \left|\int_s^t f'(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x))\left[D_h\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)\cdot c\right]du\right| \\
&\quad + H^x_{\alpha+\boldsymbol{h}}(t,s)\int_s^t \left|f'(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)) - f'(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x))\right|\left[D_h\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)\cdot c\right]du \\
&\quad + H^x_{\alpha+\boldsymbol{h}}(t,s)\int_s^t \left|f'(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x))\right|\left|D_h\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)\cdot c - D_h\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)\cdot c\right|du \\
&=: A_1 + A_2 + A_3.
\end{aligned}$$

Case $\sigma_\alpha = +\infty$. First one has

$$\begin{aligned}
\left|\int_s^t f'(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x))\left[D_h\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)\cdot c\right]du\right| &\le C(1+f(x))e^{pL(t-s)}\int_s^t\left|D_h\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)\cdot c\right|du \\
&\le C(1+f(x))\left[||c||^\infty_\lambda e^{-\lambda s}\right]e^{L(p+1)(t-s)}.
\end{aligned}$$

Moreover, following the same arguments of Lemma 4.37, for all $\theta \ge 0$ there exists a constant C (also depending on θ) such that

$$\forall x \ge 0, \forall t \ge s, \quad \left|H^x_{\alpha+\tilde{\boldsymbol{h}}}(t,s) - H^x_{\alpha+\boldsymbol{h}}(t,s)\right| \le C(1+f(x))\left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^\infty_\lambda e^{-\lambda s}\right]e^{-\theta(t-s)}.$$

We deduce that A_1 satisfies the inequality stated in the lemma. For A_2, we have using Assumption 4.2

$$\begin{aligned}
\left|f'(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)) - f'(\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x))\right| &\le C(1+f(x))e^{Lp(t-s)}\left|\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)\right| \\
&\le C(1+f(x))\left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^\infty_\lambda e^{-\lambda s}\right]e^{L(p+1)(t-s)}.
\end{aligned}$$

So, A_2 also satisfied the stated inequality. Finally, for A_3, we have

$$A_3 \le H^x_{\alpha+\boldsymbol{h}}(t,s)\ C(1+f(x))e^{Lp(t-s)}\int_s^t\left|D_h\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{u,s}(x)\cdot c - D_h\varphi^{\alpha+\boldsymbol{h}}_{u,s}(x)\cdot c\right|du.$$

Moreover by (4.23) one has

$$\left|D_h\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}(x)\cdot c - D_h\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)\cdot c\right| \le \int_s^t |c_u|\left|\exp\left(\int_u^t b'(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{\theta,s}(x))d\theta\right) - \exp\left(\int_u^t b'(\varphi^{\alpha+\boldsymbol{h}}_{\theta,s}(x))d\theta\right)\right|du$$

Using the inequality $|e^A - e^B| \le |A-B|(e^A + e^B))$ one obtains

$$\begin{aligned}
\left|\exp\left(\int_u^t b'(\varphi^{\alpha+\boldsymbol{h}}_{\theta,s}(x))d\theta\right) - \exp\left(\int_u^t b'(\varphi^{\alpha+\boldsymbol{h}}_{\theta,s}(x))d\theta\right)\right| &\le Ce^{L(t-u)}\int_u^t|\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{\theta,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{\theta,s}(x)|d\theta \\
&\overset{(4.20)}{\le} Ce^{2L(t-s)}||\boldsymbol{h} - \tilde{\boldsymbol{h}}||^\infty_\lambda e^{-\lambda s}
\end{aligned}$$

So
$$\left|D_h\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x)\cdot c - D_h\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)\cdot c\right| \leq C\left[||c||_\lambda^\infty||\boldsymbol{h}-\tilde{\boldsymbol{h}}||_\lambda^\infty e^{-2\lambda s}\right]e^{2L(t-s)}.$$
We deduce that A_3 also satisfies the inequality stated in the lemma. This ends the proof.
Case $\sigma_\alpha < \infty$. The proof is similar using, as before, the estimates of Lemma 4.22. □

Lemma 4.45. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ be such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$ and $S \in \mathcal{S}_\alpha$. There exists $\delta > 0$ (only depending on b, f, α, λ and S) such that for all $\nu \in \mathcal{M}_S(f^2)$, the following functions are Fréchet differentiable*

$$\begin{array}{rcl} B_\lambda^\infty(0,\delta) & \to & L_\lambda^\infty \\ \boldsymbol{h} & \mapsto & \left[t \mapsto H_{\alpha+\boldsymbol{h}}^\nu(t,0)\right], \end{array} \qquad \begin{array}{rcl} B_\lambda^\infty(0,\delta) & \to & L_\lambda^\infty \\ \boldsymbol{h} & \mapsto & \left[t \mapsto K_{\alpha+\boldsymbol{h}}^\nu(t,0)\right]. \end{array}$$

Moreover, the functions $\mathcal{M}_S(f^2)\times B_\lambda^\infty(0,\delta) \ni (\nu,\boldsymbol{h}) \mapsto D_h H_{\alpha+\boldsymbol{h}}^\nu \in \mathcal{L}(L_\lambda^\infty, L_\lambda^\infty)$ and $(\nu,\boldsymbol{h}) \mapsto D_h K_{\alpha+\boldsymbol{h}}^\nu$ are continuous.

Proof. Lemma 4.43 (with $s = 0$) proves the result for $\nu = \delta_x$. By integrating the inequality (4.56) with respect to ν, the result is extended to any $\nu \in \mathcal{M}_S(f^2)$. The continuity of $(\nu,\boldsymbol{h}) \mapsto D_h H_{\alpha+\boldsymbol{h}}^\nu$ follows from the second estimate of Lemma 4.44. The proof for $K_{\alpha+\boldsymbol{h}}^\nu$ is similar. □

Similarly we have

Lemma 4.46. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$. There exists $\delta > 0$ (only depending on b, f, α and λ) such that the following functions are $\mathcal{C}^1$:*

$$\begin{array}{rcl} B_\lambda^\infty(0,\delta) & \to & \mathcal{V}_\lambda^1 \\ \boldsymbol{h} & \mapsto & \left[(t,s) \mapsto H_{\alpha+\boldsymbol{h}}(t,s)\right], \end{array} \qquad \begin{array}{rcl} B_\lambda^\infty(0,\delta) & \to & \mathcal{V}_\lambda^1 \\ \boldsymbol{h} & \mapsto & \left[(t,s) \mapsto K_{\alpha+\boldsymbol{h}}(t,s)\right]. \end{array}$$

and

$$\begin{array}{rcl} B_\lambda^\infty(0,\delta) & \to & \mathcal{V}_\lambda^1 \\ \boldsymbol{h} & \mapsto & \left[(t,s) \mapsto (\bar{H}_h^\alpha * 1)(t,s)\right], \end{array} \qquad \begin{array}{rcl} B_\lambda^\infty(0,\delta) & \to & \mathcal{V}_\lambda^1 \\ \boldsymbol{h} & \mapsto & \left[(t,s) \mapsto (\bar{K}_{\boldsymbol{h}}^\alpha * 1)(t,s)\right]. \end{array}$$

Proof. The proof for the first two functions follows immediately from Lemma 4.43. We prove the result for $\bar{H}_{\boldsymbol{h}}^\alpha * 1$ (recall that $\bar{H}_{\boldsymbol{h}}^\alpha$ is defined by (4.47)). Note that Lemma 4.18 cannot be applied because $1 \notin \mathcal{V}_\lambda^1$. Nevertheless, by Lemma 4.43, there exists $\delta > 0$ and $\eta \in L^1 \cap L^\infty(\mathbb{R}_+,\mathbb{R}_+)$ such that for all $\boldsymbol{h},\tilde{\boldsymbol{h}} \in L_\lambda^\infty$ with $||\boldsymbol{h}||_\lambda^\infty < \delta$ and $||\tilde{\boldsymbol{h}}-\boldsymbol{h}||_\lambda^\infty < \delta/2$ one has, for all $t \geq u$:

$$\left|H_{\alpha+\tilde{\boldsymbol{h}}}(t,u) - H_{\alpha+\boldsymbol{h}}(t,u) - D_h H_{\alpha+\boldsymbol{h}}(t,u)\cdot(\tilde{\boldsymbol{h}}-\boldsymbol{h})\right| \leq \left[||\tilde{\boldsymbol{h}}-\boldsymbol{h}||_\lambda^\infty e^{-\lambda u}\right]^2 e^{-\lambda(t-u)}\eta(t-u).$$

Let $t \geq s$. We integrate this inequality with respect to u on $[s,t]$ obtain

$$\begin{aligned}
&\left|\left(\bar{H}^{\alpha}_{\tilde{\boldsymbol{h}}} * 1\right)(t,s) - \left(\bar{H}^{\alpha}_{\boldsymbol{h}} * 1\right)(t,s) - \left(D_h H_{\alpha+\boldsymbol{h}} \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}) * 1\right)(t,s)\right| \\
&\quad \leq \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^{\infty}_{\lambda}\right]^2 e^{-\lambda t} \int_s^t e^{-\lambda u} \eta(t-u) du \\
&\quad \leq \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^{\infty}_{\lambda}\right]^2 e^{-\lambda t} e^{-\lambda s} ||\eta||_1.
\end{aligned}$$

So,

$$||\bar{H}^{\alpha}_{\tilde{\boldsymbol{h}}} * 1 - \bar{H}^{\alpha}_{\boldsymbol{h}} * 1 - \left(D_h H_{\alpha+\boldsymbol{h}} \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}) * 1\right) ||^1_{\lambda} \leq \left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||^{\infty}_{\lambda}\right]^2 \frac{||\eta||_1}{2\lambda}.$$

This proves that $\boldsymbol{h} \mapsto \bar{H}^{\alpha}_{\boldsymbol{h}}$ is Fréchet differentiable. The continuity of the derivative follows from Lemma 4.44: it suffices to similarly integrate the estimates for u between s and t. $\square$

Lemma 4.47. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ such that Assumption 4.6 holds. There exists $\delta > 0$, only depending on b, f, α and λ, such that the following function is $\mathcal{C}^1$*

$$\begin{array}{rcl}
B^{\infty}_{\lambda}(0,\delta) & \to & \mathcal{V}^1_{\lambda} \\
\boldsymbol{h} & \mapsto & \xi_{\alpha+\boldsymbol{h}} := r_{\alpha+\boldsymbol{h}} - \gamma(\alpha).
\end{array}$$

Proof. The proof relies on formula (4.52). First, one proves that the function $\boldsymbol{h} \mapsto V^{\alpha}_{\boldsymbol{h}}$ is $\mathcal{C}^1$ from $B^{\infty}_{\lambda}(0,\delta)$ to $\mathcal{V}^1_{\lambda}$. This follows from its explicit expression (4.50):

$$V^{\alpha}_{\boldsymbol{h}} = \bar{K}^{\alpha}_{\boldsymbol{h}} + \xi_{\alpha} * \bar{K}^{\alpha}_{\boldsymbol{h}} - \gamma(\alpha)\bar{H}^{\alpha}_{\boldsymbol{h}}$$

We use here Lemma 4.46. Now, it is clear from Lemma 4.37 that for all $\boldsymbol{h} \in L^{\infty}_{\lambda}$ with $||\boldsymbol{h}||^{\infty}_{\lambda} < \delta$ one has

$$||V^{\alpha}_{\boldsymbol{h}}||^1_{\lambda} \leq ||\eta||^1 \left[1 + ||\xi_{\alpha}||^1_{\lambda} + \gamma(\alpha)\right] ||\boldsymbol{h}||^{\infty}_{\lambda} < 1.$$

Using Lemma 4.19 we deduce that the function

$$\boldsymbol{h} \mapsto R(V^{\alpha}_{\boldsymbol{h}}) = Q^{\alpha}_{\boldsymbol{h}}$$

is $\mathcal{C}^1$. It remains to check that $\boldsymbol{h} \mapsto Q^{\alpha}_{\boldsymbol{h}} * 1$ is also $\mathcal{C}^1$. From (4.51), we have

$$Q^{\alpha}_{\boldsymbol{h}} * 1 = V^{\alpha}_{\boldsymbol{h}} * 1 + Q^{\alpha}_{\boldsymbol{h}} * (V^{\alpha}_{\boldsymbol{h}} * 1),$$

and so using Lemma 4.18, it suffices to show that

$$\boldsymbol{h} \mapsto V^{\alpha}_{\boldsymbol{h}} * 1 = (\bar{K}^{\alpha}_{\boldsymbol{h}} * 1) + \xi_{\alpha} * (\bar{K}^{\alpha}_{\boldsymbol{h}} * 1) - \gamma(\alpha)(\bar{H}^{\alpha}_{\boldsymbol{h}} * 1)$$

is $\mathcal{C}^1$. This is a consequence of Lemma 4.46. Finally, (4.52) ends the proof. $\square$

To end the proof of Proposition 4.32, it remains to show that:

Lemma 4.48. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$ and $S \in \mathcal{S}_\alpha$. There exists $\delta > 0$ small enough (δ only depending on b, f, α, λ and S) such that for all $\nu \in \mathcal{M}_S(f^2)$, the following function is Fréchet differentiable*

$$\begin{array}{rcl} B^\infty_\lambda(0,\delta) & \to & L^\infty_\lambda \\ \boldsymbol{h} & \mapsto & \xi^\nu_{\alpha+\boldsymbol{h}}, \end{array}$$

with a differential at point $\boldsymbol{h}$ given by, for all $c \in L^\infty_\lambda$:

$$D_h\xi^\nu_{\alpha+\boldsymbol{h}} \cdot c = D_h K^\nu_{\alpha+\boldsymbol{h}} \cdot c - \gamma(\alpha) D_h H^\nu_{\alpha+\boldsymbol{h}} \cdot c + \xi_{\alpha+\boldsymbol{h}} * \left[D_h K^\nu_{\alpha+\boldsymbol{h}} \cdot c\right] + \left[D_h \xi_{\alpha+\boldsymbol{h}} \cdot c\right] * K^\nu_{\alpha+\boldsymbol{h}}. \tag{4.58}$$

Moreover, the function

$$\begin{array}{rcl} \mathcal{M}_S(f^2) \times B^\infty_\lambda(0,\delta) & \to & \mathcal{L}(L^\infty_\lambda, L^\infty_\lambda) \\ (\nu, \boldsymbol{h}) & \mapsto & D_h\xi^\nu_{\alpha+\boldsymbol{h}} \end{array}$$

is continuous.

Proof. Recall (4.53)

$$\xi^\nu_{\alpha+\boldsymbol{h}} = K^\nu_{\alpha+\boldsymbol{h}} - \gamma(\alpha) H^\nu_{\alpha+\boldsymbol{h}} + \xi_{\alpha+\boldsymbol{h}} * K^\nu_{\alpha+\boldsymbol{h}}.$$

Using Lemmas 4.18, 4.45 and 4.47, we deduce that $\boldsymbol{h} \mapsto \xi^\nu_{\alpha+\boldsymbol{h}}$ is Fréchet differentiable, with a derivative given by (4.58). The continuity of $(\nu, \boldsymbol{h}) \mapsto D_h\xi^\nu_{\alpha+\boldsymbol{h}}$ then follows by Lemmas 4.45 and 4.47. □

4.7.4 Proof of Proposition 4.33

In this section we grant Assumptions 4.1 and 4.2 and consider $\alpha > 0$ such that Assumption 4.7 holds. We fix $\lambda \in (0, \lambda^*_\alpha)$ and $S \in \mathcal{S}_\alpha$. By Proposition 4.32, there exists $\delta > 0$ (only depending on b, f, α, λ and S) such that for all $\boldsymbol{h} \in L^\infty_\lambda$, with $||\boldsymbol{h}||^\infty_\lambda < \delta$, the jump rate starting from $\nu \in \mathcal{M}_S(f^2)$ satisfies

$$r^\nu_{\alpha+\boldsymbol{h}} = \gamma(\alpha) + \xi^\nu_{\alpha+\boldsymbol{h}},$$

for some function $\boldsymbol{h} \mapsto \xi^\nu_{\alpha+\boldsymbol{h}} \in \mathcal{C}^1(B^\infty_\lambda(0,\delta), L^\infty_\lambda)$. We write $\nu_\infty := \nu^\infty_\alpha$ to simplify the notation. The aim of this section is to compute explicitly $D_h\xi^{\nu_\infty}_\alpha$, the Fréchet derivative of $\boldsymbol{h} \mapsto \xi^{\nu_\infty}_{\alpha+\boldsymbol{h}}$ at $\boldsymbol{h} = 0$. We have

$$r^{\nu_\infty}_{\alpha+\boldsymbol{h}} = K^{\nu_\infty}_{\alpha+\boldsymbol{h}} + K_{\alpha+\boldsymbol{h}} * r^{\nu_\infty}_{\alpha+\boldsymbol{h}}.$$

In particular, using that $\gamma(\alpha) \equiv r^{\nu_\infty}_\alpha$, taking $\boldsymbol{h} = 0$ gives

$$\gamma(\alpha) = K^{\nu_\infty}_\alpha + K_\alpha * \gamma(\alpha).$$

So we deduce that $\xi^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t)$ solves

$$\xi^{\nu_\infty}_{\alpha+\boldsymbol{h}} = G_\alpha(\boldsymbol{h}) + K_{\alpha+\boldsymbol{h}} * \xi^{\nu_\infty}_{\alpha+\boldsymbol{h}}, \tag{4.59}$$

with

$$G_\alpha(\boldsymbol{h}) := (K^{\nu_\infty}_{\alpha+\boldsymbol{h}} - K^{\nu_\infty}_\alpha) + (K_{\alpha+\boldsymbol{h}} - K_\alpha) * \gamma(\alpha). \tag{4.60}$$

Definition 4.49. *Given $s \in \mathbb{R}_+$ and $\boldsymbol{h} \in L^\infty(\mathbb{R}_+)$, we denote by $\boldsymbol{h}_{[s]}$ the function*

$$\forall t \in \mathbb{R}, \quad \boldsymbol{h}_{[s]}(t) := h_t \mathbb{1}_{\{t \geq s\}}.$$

Lemma 4.50. *Consider b and f such that Assumptions 4.1 and 4.2 hold. Let $\alpha > 0$ and $\boldsymbol{h} \in L^\infty(\mathbb{R}_+)$ with $||\boldsymbol{h}||_\infty < \alpha$. For all $t \geq s \geq 0$ we have*

$$H^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t,s) = \gamma(\alpha) \int_{-\infty}^s H_{\alpha+\boldsymbol{h}_{[s]}}(t,u)du.$$

Similarly, we have

$$K^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t,s) = \gamma(\alpha) \int_{-\infty}^s K_{\alpha+\boldsymbol{h}_{[s]}}(t,u)du.$$

Proof. First note that the second equality is obtained by taking the derivative of the first equality with respect to t. To prove the first equality, we show that:
Claim: for all $T \geq 0$,

$$H^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t,s) = \gamma(\alpha) \int_{-T}^s H_{\alpha+\boldsymbol{h}_{[s]}}(t,u)du + H^{\nu_\infty}_{\alpha+\boldsymbol{h}_{[s]}}(t,-T).$$

Proof of the claim. We rely on a probabilistic argument. Consider $(Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{u,-T})_{u\in[-T,t]}$ the solution of (1.9) starting with law ν_∞ at time $-T$ and driven by then current $\boldsymbol{h}_{[s]}$. At time s, one has $Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{s,-T} \overset{\mathcal{L}}{=} \nu^\infty_\alpha$. So

$$H^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t,s) = \mathbb{P}(Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{\cdot,-T} \text{ does not jump between } s \text{ and } t).$$

Let τ be the time of the last jump before s:

$$\tau := \sup\{-T < u < s \mid Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{u-,-T} \neq Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{u,-T}\},$$

with the convention that $\tau = -T$ if there is no jump between $-T$ and s. We have

$$\begin{aligned} H^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t,s) &= \mathbb{E}\left[\mathbb{P}(Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{\cdot,-T} \text{ does not jump between } s \text{ and } t \mid \tau)\right] \\ &= \mathbb{E}\left[H^{\varphi^\alpha_{s,\tau}(0)}_{\alpha+\boldsymbol{h}_{[s]}}(t,s)\mathbb{1}_{\{\tau > -T\}}\right] + H^{\nu_\infty}_{\alpha+\boldsymbol{h}_{[s]}}(t,-T). \end{aligned}$$

For $u \in [-T,s]$, the jump rate $\mathbb{E}\, f(Y^{\alpha+\boldsymbol{h}_{[s]},\nu_\infty}_{u,-T})$ is constant and equal to $\gamma(\alpha) = \nu_\infty(f)$. So, using Lemma 2.16, the law of τ is

$$\mathcal{L}(\tau)(du) = \gamma(\alpha) H_\alpha(s,u)\mathbb{1}_{(-T,s]}(u)du + H^{\nu_\infty}_\alpha(s,-T)\delta_{-T}(du).$$

Consequently, using $\boldsymbol{h}_{[s]}(u) = 0$ for $u < s$ we have

$$\begin{aligned} H^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t,s) &= \gamma(\alpha)\int_{-T}^s H^{\varphi^\alpha_{s,u}(0)}_{\alpha+\boldsymbol{h}_{[s]}}(t,s)H_\alpha(s,u)du + H^{\nu_\infty}_{\alpha+\boldsymbol{h}_{[s]}}(t,-T). \\ &= \gamma(\alpha)\int_{-T}^s H^{\varphi^\alpha_{s,u}(0)}_{\alpha+\boldsymbol{h}_{[s]}}(t,s)H_{\alpha+\boldsymbol{h}_{[s]}}(s,u)du + H^{\nu_\infty}_{\alpha+\boldsymbol{h}_{[s]}}(t,-T). \end{aligned}$$

Finally, for all $u \le s \le t$ and $\tilde{\boldsymbol{h}} \in L^\infty(\mathbb{R})$ with $||\tilde{\boldsymbol{h}}||_\infty < \alpha$ one has (by the Markov property at time s)

$$H_{\alpha+\tilde{\boldsymbol{h}}}(t,u) = H_{\alpha+\tilde{\boldsymbol{h}}}(s,u) H_{\alpha+\tilde{\boldsymbol{h}}}^{\varphi_{s,u}^{\alpha+\tilde{\boldsymbol{h}}}(0)}(t,s).$$

Using this identity with $\tilde{\boldsymbol{h}} = \boldsymbol{h}_{[s]}$, the claim follows. It suffices then to let T goes to infinity to obtain the stated formula, using that

$$\limsup_{T\to\infty} H^{\nu_\infty}_{\alpha+\boldsymbol{h}_{[s]}}(t,-T) \le \limsup_{T\to\infty} H^{\nu_\infty}_{\alpha}(s,-T) = 0.$$

□

Similarly to the definition of G_α (eq. (4.60)), let:

$$L_\alpha(\boldsymbol{h}) := (H^{\nu_\infty}_{\alpha+\boldsymbol{h}} - H^{\nu_\infty}_{\alpha}) + (H_{\alpha+\boldsymbol{h}} - H_\alpha) * \gamma(\alpha).$$

Lemma 4.51. *Consider b and f satisfying Assumptions 4.1 and 4.2. Let $\alpha > 0$ such that Assumption 4.6 holds. Let $\lambda \in (0, f(\sigma_\alpha))$. There exists $\delta > 0$ (only depending on b, f, α and λ) such that the functions $\boldsymbol{h} \mapsto L_\alpha(\boldsymbol{h})$ and $\boldsymbol{h} \mapsto G_\alpha(\boldsymbol{h})$ are $\mathcal{C}^1(B^\infty_\lambda(0,\delta), L^\infty_\lambda)$. Moreover one has*

$$L_\alpha(\boldsymbol{h})(t) = -1 + \gamma(\alpha)\int_{-\infty}^t H_{\alpha+\boldsymbol{h}}(t,u)du$$

and

$$G_\alpha(\boldsymbol{h})(t) = -\gamma(\alpha) + \gamma(\alpha)\int_{-\infty}^t K_{\alpha+\boldsymbol{h}}(t,u)du.$$

Finally it holds that for all $c \in L^\infty_\lambda$

$$[D_h G_\alpha(\boldsymbol{h}) \cdot c](t) = -\frac{d}{dt}[D_h L_\alpha(\boldsymbol{h}) \cdot c](t).$$

Remark 4.52. *In these formulas, the perturbation $\boldsymbol{h}$ is extended to $\mathbb{R}$ by setting $h_t := 0$ for $t < 0$. In other words, we have $\boldsymbol{h} \equiv \boldsymbol{h}_{[0]}$.*

Proof. By Lemmas 4.45 and 4.46, the functions $G_\alpha(\boldsymbol{h})$ and $L_\alpha(\boldsymbol{h})$ are $\mathcal{C}^1$. Moreover we have using Lemma 4.50 with $s = 0$:

$$H^{\nu_\infty}_{\alpha+\boldsymbol{h}}(t) + (H_{\alpha+\boldsymbol{h}} * \gamma(\alpha))(t) = \gamma(\alpha)\int_{-\infty}^t H_{\alpha+\boldsymbol{h}}(t,u)du.$$

Setting $\boldsymbol{h} \equiv 0$ in this equality, one obtains for all $t \ge 0$

$$H^{\nu_\infty}_{\alpha}(t) + (H_\alpha * \gamma(\alpha))(t) = \gamma(\alpha)\int_{-\infty}^t H_\alpha(t-u)du = \gamma(\alpha)\int_0^\infty H_\alpha(u)du = 1.$$

This proves the first identity. The second identity is proved similarly. Finally note that

$$\forall t \ge 0, \quad \frac{d}{dt}L_\alpha(\boldsymbol{h})(t) = -G_\alpha(\boldsymbol{h})(t),$$

and so the equality on the Fréchet derivatives follows. □

Lemma 4.53. *Let $\lambda \in (0, f(\sigma_\alpha))$. The derivative of $\boldsymbol{h} \mapsto G_\alpha(\boldsymbol{h})$ at $\boldsymbol{h} = 0$ is, for all $c \in L^\infty_\lambda$:*

$$[D_h G_\alpha(0) \cdot c](t) = \Xi_\alpha * c(t) = \int_0^t \Xi_\alpha(t-u) c_u du,$$

where the function Ξ_α is given (4.31).

Proof. Here, h is null so we are in the time homogeneous setting and so $\varphi^\alpha_{t,s}(0) = \varphi^\alpha_{t-s}(0)$, etc. By Lemma 4.51 and eq. (4.54), one has for all $c \in L^\infty_\lambda$,

$$\begin{aligned}
[D_h L_\alpha(0) \cdot c](t) &= \gamma(\alpha) \int_{-\infty}^t [D_h H_\alpha \cdot c](t,s) ds \\
&= -\gamma(\alpha) \int_{-\infty}^t H_\alpha(t-s) \int_s^t f'(\varphi^\alpha_{u-s}(0)) \left[D_h \varphi^\alpha_{u-s}(0) \cdot c\right] duds \\
&= -\gamma(\alpha) \int_{-\infty}^t H_\alpha(t-s) \int_s^t f'(\varphi^\alpha_{u-s}(0)) \int_s^u c_\theta \exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0)) dv\right) d\theta duds.
\end{aligned}$$

To obtain the last equality we use (4.23) with $\boldsymbol{h} \equiv 0$. So, by Fubini:

$$[D_h L_\alpha(0) \cdot c](t) = -\gamma(\alpha) \int_{-\infty}^t c_\theta \left[\int_{-\infty}^\theta H_\alpha(t-s) \int_\theta^t f'(\varphi^\alpha_{u-s}(0)) \exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0)) dv\right) duds\right] d\theta.$$

Using

$$\exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0)) dv\right) = \exp\left(\int_{\theta-s}^{u-s} b'(\varphi^\alpha_v(0)) dv\right) = \frac{b(\varphi^\alpha_{u-s}(0)) + \alpha}{b(\varphi^\alpha_{\theta-s}(0)) + \alpha},$$

we deduce that

$$\int_\theta^t f'(\varphi^\alpha_{u-s}(0)) \exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0)) dv\right) du = \frac{f(\varphi^\alpha_{t-s}(0)) - f(\varphi^\alpha_{\theta-s}(0))}{b(\varphi^\alpha_{\theta-s}(0)) + \alpha}.$$

By the change of variable $s = \theta - v$ one gets

$$\begin{aligned}
[D_h L_\alpha(0) \cdot c](t) &= -\gamma(\alpha) \int_{-\infty}^t c_\theta \left[\int_0^\infty H_\alpha((t-\theta)+v) \frac{f(\varphi^\alpha_{(t-\theta)+v}(0)) - f(\varphi^\alpha_v(0))}{b(\varphi^\alpha_v(0)) + \alpha} dv\right] d\theta \\
&= -(\Psi_\alpha * c)(t).
\end{aligned}$$

where $\Psi_\alpha(t)$ is given by (4.30). We use here that $c_\theta = 0$ for all $\theta < 0$. Finally, we deduce from Lemma 4.51 and from $\Psi_\alpha(0) = 0$ that

$$[D_h G_\alpha(0) \cdot c](t) = -\frac{d}{dt}[D_h L_\alpha(0) \cdot c](t) = \Xi_\alpha * c(t),$$

with

$$\Xi_\alpha(t) = -\frac{d}{dt}\Psi_\alpha(t) = \int_0^{\sigma_\alpha} \frac{d}{dx} K^x_\alpha(t) \nu^\infty_\alpha(x) dx.$$

This ends the proof. □

We now give the proof of Proposition 4.33.

Proof of Proposition 4.33. We compute the Fréchet derivative of (4.59) at $\boldsymbol{h} \equiv 0$ and obtain for all $c \in L^\infty_\lambda$:

$$D_h \xi^{\nu_\infty}_\alpha \cdot c = \Xi_\alpha * c + K_\alpha * (D_h \xi^{\nu_\infty}_\alpha \cdot c)\,.$$

We used here $\xi^{\nu_\infty}_\alpha = 0$. We apply now Lemma 2.22 and obtain

$$D_h \xi^{\nu_\infty}_\alpha \cdot c = \Xi_\alpha * c + r_\alpha * \Xi_\alpha * c.$$

To conclude, it suffices to note that $\Xi_\alpha + r_\alpha * \Xi_\alpha = \Theta_\alpha$ (see (4.33)). □

4.8 Proof of Lemmas 4.20 and 4.22

Proof of Lemma 4.20

Proof. Our assumptions are very close to the classical setting, except we only assume that $\boldsymbol{h} \in L^\infty_\lambda$. The drift b is assumed to be globally Lipschitz continuous. Thus, a classical fixed point argument on the space $\mathcal{C}([s,T];\mathbb{R})$ applies. For $T > s$, we introduce the function $F : \mathcal{C}([s,T];\mathbb{R}) \to \mathcal{C}([s,T];\mathbb{R})$ defined by

$$F_{x,\boldsymbol{h}}(\psi)(t) := x + \int_s^t b(\psi_u) du + \int_s^t (\alpha + h_u) du.$$

The Banach space $\mathcal{C}([s,T];\mathbb{R})$ is equipped with the infinite norm on $[s,T]$. The function $(x, \boldsymbol{h}, \psi) \mapsto F_{x,\boldsymbol{h}}(\psi)$ is $\mathcal{C}^1$. Moreover, for n large enough, the n-fold iteration of $F_{x,\boldsymbol{h}}$ is contracting, with a constant of contraction independent of x and $\boldsymbol{h}$. We deduce that $F_{x,\boldsymbol{h}}$ has a unique fixed point $\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$ and that the function $(x, \boldsymbol{h}) \mapsto \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$ is $\mathcal{C}^1$. We refer to [Hal69, Th. 3.3] for more details. The estimate (4.20) is obtained using Grönwall's Lemma. The function $U^{\alpha+\boldsymbol{h}}_{t,s}(x) := \frac{d}{dx}\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$ solves the *linear variational equation*

$$\frac{d}{dt} U^{\alpha+\boldsymbol{h}}_{t,s}(x) = b'(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) U^{\alpha+\boldsymbol{h}}_{t,s}(x),$$

with $U^{\alpha+\boldsymbol{h}}_{s,s}(x) = 1$. The explicit solution of this ODE is given by (4.21). When $\boldsymbol{h} \equiv 0$, note that $t \mapsto b(\varphi^\alpha_t(x)) + \alpha$ satisfies the same ODE, and so (4.22) follows by uniqueness.

Similarly, $D_h \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot c$ solves the *linear variational equation*

$$\forall t \geq s, \quad \frac{d}{dt}\left[D_h \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot c\right] = b'(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) \left[D_h \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot c\right] + c_t \tag{4.61}$$
$$D_h \varphi^{\alpha+\boldsymbol{h}}_{s,s}(x) \cdot c = 0,$$

whose explicit solution is given by (4.23). We now prove that (4.24) holds. Let

$$\forall t \geq s, \quad y_t := \varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) - D_h \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}).$$

One has

$$\dot{y}_t = b(\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}(x)) + \tilde{h}_t - b(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) - h_t - \left[b'(\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)) D_h \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) \cdot (\tilde{\boldsymbol{h}} - \boldsymbol{h}) + \tilde{h}_t - h_t\right]$$

So

$$\dot{y}_t = b'(\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x))y_t + \epsilon_x(t,s),$$

with

$$\epsilon_x(t,s) := \int_{\varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)}^{\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x)} b''(u)(\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - u)du.$$

Solving this ODE, we find

$$y_t = \int_s^t U_{t,u}^{\alpha+\boldsymbol{h}}(x)\epsilon_x(u,s)du.$$

By (4.21) one has

$$|y_t| \leq \int_s^t e^{L(t-u)}|\epsilon_x(u,s)|du.$$

Using that $M := \sup_{x\in\mathbb{R}} |b''(x)| < \infty$, one has

$$|\epsilon_x(t,s)| \leq \frac{M}{2}|\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)|^2.$$

Moreover (4.20) yields

$$|\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)| \leq ||\tilde{\boldsymbol{h}} - \boldsymbol{h}||_\lambda^\infty e^{-\lambda s}\frac{e^{L(t-s)}}{L},$$

and so finally

$$|y_t| \leq \frac{M}{2L^3}\left[||\tilde{\boldsymbol{h}} - \boldsymbol{h}||_\lambda^\infty e^{-\lambda s}e^{L(t-s)}\right]^2.$$

□

Proof of Lemma 4.22

Proof. We only prove the second point. We start with (4.25). Recall that $\ell_\alpha := -b'(\sigma_\alpha) > 0$. Given $\mu \in (0, \ell_\alpha)$, consider $\boldsymbol{h} \in L_\mu^\infty$ and fix $s \geq 0$ and $x \in S$. For all $t \geq s$, set $y(t) := \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x) - \varphi_{t,s}^\alpha(x)$. It solves for all $t \geq s$

$$\begin{aligned}\dot{y}(t) &= b(\varphi_{t,s}^\alpha(x) + y(t)) - b(\varphi_{t,s}^\alpha(x)) + h_t \\ &= b'(\varphi_{t,s}^\alpha(x))y(t) + h_t + \epsilon_x(t, y(t)),\end{aligned}$$

with $\epsilon_x(t,y(t)) := \int_{\varphi_{t,s}^\alpha(x)}^{\varphi_{t,s}^\alpha(x)+y(t)} b''(u)(\varphi_{t,s}^\alpha(x) + y(t) - u)du$. So

$$y(t) = \int_s^t U_{t,u}^\alpha(x)h_u du + \int_s^t U_{t,u}^\alpha(x)\epsilon_x(u, y(u))du.$$

Let $M := \sup_{x\geq 0} |b''(x)|$. One has $|\epsilon_x(t,y(t))| \leq \frac{M}{2}y^2(t)$. Eq. (4.21) yields

$$U_{t,u}^\alpha(x) = \exp\left(\int_0^{t-u} b'(\varphi_v^\alpha(x))dv\right)$$

Let $\theta := (\ell_\alpha - \mu)/2$. It follows by Point 1 that there exists a constant C_μ such that $|U^\alpha_{t,u}(x)| \leq C_\mu e^{-(\mu+\theta)(t-u)}$. So

$$|y(t)| \leq C_\mu \int_s^t e^{-(\mu+\theta)(t-u)} |h_u| du + \frac{MC_\mu}{2} \int_s^t e^{-(\mu+\theta)(t-u)} y^2(u) du.$$

Note that

$$\begin{aligned} C_\mu \int_s^t e^{-(\mu+\theta)(t-u)} |h_u| du &\leq C_\mu ||\boldsymbol{h}||^\infty_\mu e^{-(\mu+\theta)t} \int_s^t e^{\theta u} du \\ &\leq \frac{2C_\mu}{\theta} ||\boldsymbol{h}||^\infty_\mu e^{-\mu t}. \end{aligned}$$

Consider the deterministic time

$$t_0 := \inf\{t \geq s : \; MC_\mu |y(t)| \geq \theta\}.$$

For all $t \in [s, t_0]$ one has

$$|y(t)| \leq \frac{2C_\mu}{\theta} ||\boldsymbol{h}||^\infty_\mu e^{-\mu t} + \frac{\theta}{2} \int_s^t e^{-(\mu+\theta)(t-u)} |y(u)| du.$$

We now use a Grönwall Lemma for Volterra integral equation (see [GLS90, Th. 8.2 p. 257]) with the kernel $k(t) = \frac{\theta}{2} e^{-(\mu+\theta)t}$. The solution of the Volterra equation $p = k + k * p$ is $p(t) = \frac{\theta}{2} e^{-(\mu+\theta/2)t}$. So

$$\begin{aligned} |y(t)| &\leq \frac{2C_\mu}{\theta} ||\boldsymbol{h}||^\infty_\mu \left[e^{-\mu t} + \int_s^t p(t-u) e^{-\mu u} du \right] \\ &\leq \frac{2C_\mu}{\theta} ||\boldsymbol{h}||^\infty_\mu \left[e^{-\mu t} + \frac{\theta}{2} \int_s^t e^{-(\mu+\theta/2)(t-u)} e^{-\mu u} du \right]. \\ &\leq \frac{4C_\mu}{\theta} ||\boldsymbol{h}||^\infty_\mu e^{-\mu t}. \end{aligned}$$

Consequently if

$$||\boldsymbol{h}||^\infty_\mu \leq \delta_\mu := \frac{\theta^2}{4MC_\mu^2},$$

one has $t_0 = +\infty$, which ends the proof. Formula (4.26) then follows by (4.25) and by Point 1. We now prove (4.27). Using that $\ell_\alpha > 0$, we deduce the existence of $\kappa > 0$ such that

$$\forall x \in \mathbb{R}_+, \quad |x - \sigma_\alpha| \leq \kappa \implies b'(x) \leq 0.$$

By Point 1, there exists t_0 such that for all $x \in S$ and for all $t \geq t_0$, $|\varphi^\alpha_t(x) - \sigma_\alpha| \leq \kappa/2$. Moreover, by (4.25), there exists t_1 such that for all $\boldsymbol{h} \in L^\infty_\mu$ with $||\boldsymbol{h}||^\infty_\mu < \delta_\mu$:

$$\forall x \in S, \; \forall t \geq t_1, \quad |\varphi^{\alpha+\boldsymbol{h}}_{t,s}(x) - \varphi^\alpha_{t,s}(x)| \leq \kappa/2.$$

Let $t^* = \max(t_0, t_1)$ and given $\boldsymbol{h}, \tilde{\boldsymbol{h}} \in L^\infty_\mu$, let $z(t) := \varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}(x) - \varphi^{\alpha+\boldsymbol{h}}_{t,s}(x)$. For all $t \geq s + t^*$, one has

$$\dot{z}(t) = \tilde{h}_t - h_t + \int_{\varphi^{\alpha+\boldsymbol{h}}_{t,s}}^{\varphi^{\alpha+\tilde{\boldsymbol{h}}}_{t,s}} b'(u) du \leq |\tilde{h}_t - h_t|.$$

The same holds for $-\dot{z}(t)$, and so

$$\forall t \geq s + t^*, \quad |z(t)| \leq |z(s + t^*)| + \int_{s+t^*}^{t} |\tilde{h}_u - h_u| du.$$

To conclude, it suffices to use (4.20): for all $t \geq s$ one has

$$|z(t)| \leq e^{L(t-s)} \int_s^t |\tilde{h}_u - h_u| du,$$

and so for $t = s + t^*$:

$$|z(s + t^*)| \leq e^{Lt^*} \int_s^{s+t^*} |\tilde{h}_u - h_u| du.$$

We deduce that for all $t \geq s$

$$|\varphi_{t,s}^{\alpha+\tilde{\boldsymbol{h}}}(x) - \varphi_{t,s}^{\alpha+\boldsymbol{h}}(x)| \leq e^{Lt^*} \int_s^t |\tilde{h}_u - h_u| du,$$

which ends the proof. Finally, we mimic the proof of (4.24) to obtain (4.28), using (4.27). □

4.9 Discussions and perspectives

We studied the local stability of the invariant measures of (1.2). To do so, one method is to use the Fokker-Planck equation (1.4). A difficulty with these nonlinear transport equations is that the nonlinear flow (that is the family of functions $\nu \mapsto T_t(\nu) := \mathcal{L}(X_t^\nu)$, where $\mathcal{L}(X_0^\nu) = \nu$ and $t \geq 0$) is usually not differentiable with respect to the initial condition (see [DV21]). So, techniques for nonlinear semi-group theory are difficult to apply. A second difficulty with (1.4) is the boundary condition (1.5), which is nonlinear with respect to the solution. To overcome these problems, we used the Volterra integral equation (1.14). By defining appropriate Banach spaces, we reduced the problem of the local stability of an invariant measure to the application of the implicit function theorem. We emphasis that our criteria involving (4.8) is not specific to (1.2) and to the Volterra equation (1.14). Consider for instance the following McKean-Vlasov diffusion in $\mathbb{R}^d$:

$$dX_t = b(X_t)dt + \mathbb{E}\, f(X_t)dt + dW_t, \tag{4.62}$$

starting from some $\nu \in \mathcal{P}(\mathbb{R}^d)$. Here $f, b \in \mathcal{C}(\mathbb{R}^d; \mathbb{R}^d)$ are some continuous functions and (W_t) is a standard Brownian motion in $\mathbb{R}^d$. Given $\alpha \in \mathbb{R}^d$, consider the following linear SDE

$$dY_t^{\alpha,x} = b(Y_t^{\alpha,x})dt + \alpha dt + dW_t, \tag{4.63}$$

starting at time 0 from the deterministic point $x \in \mathbb{R}^d$. Assume that b and f are such (4.62) has a unique path-wise solution. Assume furthermore that (4.63) has a unique invariant measure, denoted ν_α^∞ and that ν_α^∞ is globally attractive:

$$\forall x \in \mathbb{R}^d, \quad \mathcal{L}(Y_t^{\alpha,x}) \xrightarrow[t\to\infty]{\mathcal{L}} \nu_\alpha^\infty.$$

Define $\gamma(\alpha) := \nu^\infty_\alpha(f)$. Assume that for some $\alpha \in \mathbb{R}^d$, $\gamma(\alpha) = \alpha$, such that ν^∞_α is an invariant measure of (4.62). Define for all $x \in \mathbb{R}^d$

$$r^x_\alpha(t) := \mathbb{E}\, f(Y^{\alpha,x}_t),$$

and let Θ_α be (compare with (4.8))

$$\forall t \geq 0, \quad \Theta_\alpha(t) := \int_{\mathbb{R}^d} \nabla r^x_\alpha(t) \nu^\infty_\alpha(dx).$$

Here, $\nabla r^x_\alpha(t)$ is the Jacobian matrix of the function $x \mapsto r^x_\alpha(t)$. In particular $\Theta_\alpha(t)$ is a $d \times d$ matrix. Let I_d be the $d \times d$ identity matrix. We claim that the invariant measure ν^∞_α of (4.63) is locally stable if the complex roots of the equation

$$\det\left(\widehat{\Theta}_\alpha(z) - I_d\right) = 0$$

are all located on the left-half plane, "det" denoting the determinant of the $d \times d$ matrix. It would be interesting to prove rigorously such result, possibly using the connection with the L-derivative described in Section 4.3.2. Such tools are for instance used in [AD20] to obtain global in time estimates for nonlinear diffusions. Finally, it would be interesting to study more complex interactions. Consider for instance the McKean-Vlasov SDE

$$dX_t = b(X_t)dt + \int_{\mathbb{R}^d} f(X_t, y)\mu_t(dy)dt + dW_t, \quad \mu_t = \mathcal{L}(X_t),$$

where $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$. What stability criteria can be written in this case? What is the nature of the spectrum? We suspect that the spectrum is not necessarily discrete as before. The particular case where the interaction is symmetric (that is $f(x, y) = \tilde{f}(x - y)$ for some function $\tilde{f} : \mathbb{R}^d \to \mathbb{R}^d$) has been intensively studied, see for instance [BRTV98; BRV98; HT12; Tug13; Tug14].

Going back to (1.2), we mention few extensions of this work. First, assume $b(0) = 0$, such that the Dirac measure δ_0 is an invariant probability measure of (1.2). We claim that δ_0 is locally stable if:

$$b'(0) + Jf'(0) < 0. \tag{4.64}$$

This can be seen by adapting slightly the proof of Theorem 4.13. Let $J \geq 0$ be fixed. Similarly to (4.42), define

$$\forall \nu \in \mathcal{M}(f^2), \forall \boldsymbol{h} \in L^\infty_\lambda, \quad \Phi(\nu, \boldsymbol{h}) := J r^\nu_{\boldsymbol{h}} - \boldsymbol{h}. \tag{4.65}$$

We have for all $c \in L^\infty_\lambda$, $D_h\Phi(\delta_0, 0) \cdot c = JD_h r_0 \cdot c - c$. The Volterra equation (1.14) gives

$$r_{\boldsymbol{h}} = K_{\boldsymbol{h}} + K_{\boldsymbol{h}} * r_{\boldsymbol{h}},$$

and so, using that $K_0 = 0$ (because $f(0) = b(0) = 0$), we have

$$D_h r_0 \cdot c = D_h K_0 \cdot c.$$

Using the explicit expression of $K_{\boldsymbol{h}}$, given by (2.5) and (4.21), we find that

$$\left[D_h K_0 \cdot c\right](t) = f'(0) D_h \varphi^0_t \cdot c = (\Theta_0 * c)(t),$$

with
$$\forall t \geq 0, \quad \Theta_0(t) := f'(0)e^{b'(0)t}.$$
So, the equation $J\widehat{\Theta_0}(z) = 1$ has a unique solution given by $z = Jf'(0) - b'(0)$, which is located on the left half-plane if and only if (4.64) holds. Of course, the study of the stability of δ_0 is much simpler than the study of the stability of the non-trivial invariant measures. More direct methods are available to do so: we refer to [RT16, Prop. 8] and to [LM20, Prop. 3.2] for probabilistic proofs in a slightly different framework. Finally, we mention an extension of this work to excitatory and inhibitory neurons. Consider the McKean-Vlasov SDE (2.20). Assume that the functions $b_e, b_i : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz and $f_e, f_i : \mathbb{R} \to \mathbb{R}_+$ are null on $\mathbb{R}_-$. Because X^e_0, X^i_0, $\mathbf{N}_e$ and $\mathbf{N}_i$ are all independent, it holds that for all t, X^e_t and X^i_t are independent. In particular, the invariant measures of (2.20) are of the form $\mu \otimes \nu$, where μ, ν are two probability measures on $\mathbb{R}$. We say that such an invariant probability measure is non-trivial if
$$\nu(f_e) > 0 \quad \text{and} \quad \mu(f_i) > 0.$$
Otherwise, at least one of the two neurons never spikes under $\mu \otimes \nu$ (and so we can study separately the spiking neuron). Given $\alpha > -b_e(0)$ and $\beta > -b_i(0)$, define for all $x \in \mathbb{R}$:
$$\nu^{\infty,e}_\alpha(x) := \frac{\gamma_e(\alpha)}{b_e(x) + \alpha} \exp\left(-\int_0^t \frac{f_e(y)}{b_e(y)+\alpha} dy\right) \mathbb{1}_{\{x \in [0, \sigma^e_\alpha)\}}, \quad \sigma^e_\alpha := \inf\{x \geq 0, b_e(x) + \alpha = 0\}.$$
and
$$\nu^{\infty,i}_\beta(x) := \frac{\gamma_i(\alpha)}{b_i(x) + \beta} \exp\left(-\int_0^t \frac{f_i(y)}{b_i(y)+\beta} dy\right) \mathbb{1}_{\{x \in [0, \sigma^i_\beta)\}}, \quad \sigma^i_\beta := \inf\{x \geq 0, b_i(x) + \beta = 0\}.$$
In these equations, $\gamma_e(\alpha)$ and $\gamma_i(\beta)$ are the normalizing factors. We furthermore assume that
$$\inf_{x \in \mathbb{R}_-} b_e(x) \geq b_e(0) \quad \text{and} \quad \inf_{x \in \mathbb{R}_-} b_i(x) \geq b_i(0). \tag{4.66}$$
Under this additional assumption, the non-trivial invariant measures of (2.20) are $\nu^{\infty,e}_\alpha \otimes \nu^{\infty,i}_\beta$, where $\alpha, \beta \in \mathbb{R}$ solves the following nonlinear 2D system
$$\begin{cases} \alpha &= J_{ee}\gamma_e(\alpha) + J_{ie}\gamma_i(\beta), \quad \alpha > -b_e(0) \\ \beta &= J_{ei}\gamma_e(\alpha) + J_{ii}\gamma_i(\beta), \quad \beta > -b_i(0). \end{cases}$$
We define $\Theta^e_\alpha(t)$ and $\Theta^i_\beta(t)$ similarly to (4.8). Consider $\nu^{\infty,e}_\alpha \otimes \nu^{\infty,i}_\beta$ a non-trivial invariant measure of (2.20). This invariant probability measure is locally stable if all the complex roots of
$$\det \begin{pmatrix} -1 + J_{ee}\widehat{\Theta^e_\alpha}(z) & J_{ie}\widehat{\Theta^i_\beta}(z) \\ J_{ei}\widehat{\Theta^e_\alpha}(z) & -1 + J_{ii}\widehat{\Theta^i_\beta}(z) \end{pmatrix} = 0,$$
are located on the left half-plane, "det" denoting the determinant of the 2×2 matrix. A further study of such excitatory/inhibitory mean-field model (both theoretically and numerically) would be interesting. The case where (4.66) is not satisfied is particularly intriguing. For instance, if $b_e(x) = b_i(x) = 1 + x$, a fraction of the mass of an invariant measure is possibly located on $\mathbb{R}_-$. Because f_e, f_i are null on $\mathbb{R}_-$, this fraction of neurons does not spike anymore. This fraction might depends on the initial condition, making the analysis more challenging. Finally, in the next Chapter, we study situations where (4.9) is broken for some interaction parameter J. We shall see that this typically leads to periodic solutions through an Hopf bifurcation.

Chapter 5

Periodic solutions via Hopf bifurcations

We give sufficient conditions such that the McKean-Vlasov equation (1.2) admits periodic solutions, through a Hopf bifurcation. Our spectral conditions involve the location of the roots of the explicit holomorphic function described in Chapter 4. The proof relies on two main ingredients. First, we introduce a discrete time Markov Chain modeling the phases of the successive spikes of a neuron. The invariant measure of this Markov Chain is related to the shape of the periodic solutions. Secondly, we use the Lyapunov-Schmidt method to obtain self-consistent oscillations. The material of this Chapter is available as a preprint [CTV20b].

5.1 Introduction

Consider the McKean-Vlasov SDE (1.2)

$$X_t = X_0 + \int_0^t b(X_u)du + J\int_0^t \mathbb{E}\, f(X_u)du - \int_0^t \int_{\mathbb{R}_+} X_{u-}\mathbb{1}_{\{z\leq f(X_{u-})\}}\mathbf{N}(du,dz).$$

We recall that $\mathbf{N}(du, dz)$ is a Poisson measure on $\mathbb{R}_+^2$ with intensity the Lebesgue measure $dudz$, and is independent of the initial condition X_0 of law ν. As before, the deterministic constant J is non-negative. We assume that the functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $b : \mathbb{R}_+ \to \mathbb{R}$ satisfy Assumptions 4.1 and 4.2. Furthermore, we assume that the law of the initial condition belongs to

$$\mathcal{M}(f^2) := \{\nu \in \mathcal{P}(\mathbb{R}_+), \quad \int_{\mathbb{R}_+} f^2(x)\nu(dx) < \infty\}.$$

Under these assumptions, (1.2) has a unique path-wise solution (see Theorem 2.8). We study the existence of periodic solution of (1.2), that is:

Definition 5.1. *A family of probability measures $(\nu(t))_{t\in[0,T]}$ is said to be a T-periodic solution of (1.2) if*

1. $\nu(0) \in \mathcal{M}(f^2)$.
2. *For all $t \in [0,T]$, $\nu(t) = \mathcal{L}(X_t)$ where $(X_t)_{t\in[0,T]}$ is the solution of (1.2) starting from $X_0 \sim \nu(0)$.*

3. *It holds that $\nu(T) = \nu(0)$.*

In this case, we can obviously extend $(\nu(t))$ for $t \in \mathbb{R}$ by periodicity. Considering now the solution $(X_t)_{t\geq 0}$ of (1.2) defined for $t \geq 0$, it remains true that $\nu(t) = \mathcal{L}(X_t)$ for any $t \geq 0$. Moreover, we can also consider the solution of (1.2) defined on $[t_0, +\infty)$ for any $t_0 \in \mathbb{R}$ with initial condition $\mathcal{L}(X_{t_0}) = \nu(t_0)$.

Equivalently, such $(\nu(t))_{t\in[0,T]}$ is a periodic solution of the Fokker-Planck PDE (1.3). We give sufficient conditions for the existence of a Hopf bifurcation around an invariant probability measure of (1.2).

Similarly, given $\boldsymbol{a} : \mathbb{R} \to \mathbb{R}_+$ a continuous and T-periodic function, we say that $(\nu(t))_{t\in[0,T]}$ is a T-periodic solution of the non-homogeneous linear equation (1.9) if for all $t \in [0, T]$, it holds that $\nu(t) = \mathcal{L}(Y^{\boldsymbol{a},\nu(0)}_{t,0})$, where $Y^{\boldsymbol{a},\nu(0)}_{t,0}$ is the solution of (1.9) starting at time 0 with law $\nu(0)$.

There is a qualitative difference between the particle systems, solution of (1.1), and the solution of the limit equation (1.2): for a fixed value of N, the particle system is Harris ergodic (see [DO16], where this result is proved under stronger assumptions on b and f) and so it admits a unique, globally attractive, invariant measure. In particular, there are no stable oscillations when the number of particles is finite. For the limit equation however, the long time behavior is richer: for fixed values of the parameters there can be multiple invariant measures (see Chapter 6) and, as shown here, there can exist periodic solutions. One example is the following: consider for all $x \geq 0$, $f(x) = x^{10}$, $b(x) = 2 - 2x$. Numerically, we find a Hopf bifurcation for $J = 0.71$. The periodic solution for $J = 0.8$ is reported Figure 5.1. We refer to Chapter 6, Section 6.4.3 for detailed analysis of this example.

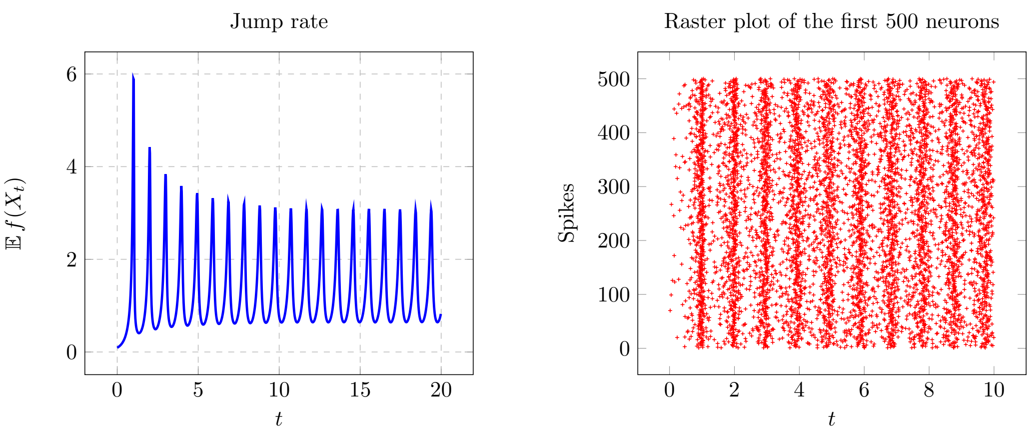
In [LM20], the authors study a "metastable" behavior of the particle system. They give examples of drifts b and rate functions f where the particle system follows the long time behavior of the mean-field model for an exponential large time, before finally converging to its (unique) invariant probability measure.

In [DV17], numerical evidences are given for the existence of a Hopf bifurcation in a close setting: the dynamics between the jumps is (as in [DGLP15]) given by

$$\dot{X}_t = -(X_t - \mathbb{E}\, X_t) + J\, \mathbb{E}\, f(X_t).$$

In particular the potentials of each neuron are attracted to their common mean. This last phenomenon models the behavior of "electrical synapses", while $J\,\mathbb{E}\,f(X_t)$ models the chemical synapses. Oscillations with both electrical and chemical synapses is also studied in a different model in [PDRDM19]. In this work, the mean-field equation is a 2D-ODE and so the analysis of the Hopf bifurcation is standard. Finally, oscillations with multi-populations, in particular with both excitatory and inhibitory neurons have been extensively studied in neuroscience. For instance in [DL17], it is shown that multi-populations of mean-field Hawkes processes can oscillate. Again, the dynamics is reduced to a finite dimension ODE.

It is well-known that the long time behavior of McKean-Vlasov SDEs can be significantly different from markovian SDEs. In [Sch85b] and [Sch85a], the author give simple examples of such nonlinear SDEs which oscillate. Again, in these examples, the dynamic can be reduced to an ordinary differential equation. To go beyond ODEs, the framework of Delay differential equation is often used: see for instance [Sta87] for the study of Hopf bifurcations for such equations, based on the Lyapunov-Schmidt method. In [LP20a; LP20b] the authors study periodic solutions of a McKean-Vlasov SDE using a slow-fast approach. Another approach is to use the center manifold theory to reduce infinite dimensional problem to manifold of finite dimension: we refer to [HI11] (see also [GPPP12] for an application to some McKean-Vlasov SDE). Finally, in [Kie12] an abstract framework is presented to study Hopf bifurcations for some classes of regular PDEs. Even though our proof is not based on the PDE (1.3), we follow the methodology of [Kie12] to obtain our main result.



(a) (b)

Figure 5.1: Consider the following example where for all $x \geq 0$, $f(x) = x^{10}$, $b(x) = 2 - 2x$ and $J = 0.8$. Using a Monte-Carlo method, we simulate the particle system with $N = 8 \cdot 10^5$ neurons, starting at $t = 0$ with i.i.d. uniformly distributed random variables on $[0, 1]$. "Stable" oscillations appear. (a) Empirical mean number of spikes per unit of time. (b) Each red cross corresponds to a spike of one of the first 500 neurons (spike raster plot). See Chapter 6, Section 6.4.3 for a detailed analysis of this example.

Recall that by Proposition 4.5 the (non-trivial) invariant measures of (1.2) are $\{\nu^\infty_\alpha \mid \alpha > 0,\ \alpha = J\gamma(\alpha)\}$, where for all $\alpha > 0$, ν^∞_α is given by (3.4). The constant $\gamma(\alpha)$ satisfies $\nu^\infty_\alpha(f) = \gamma(\alpha)$. For all $\alpha > 0$, let $J(\alpha)$ be defined by (4.5):

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)}.$$

Let ν^∞_α be an invariant measure of (1.2). Consider $\Theta_\alpha(t)$ defined by (4.8):

$$\forall t \geq 0, \quad \Theta_\alpha(t) := \int_0^\infty \left[\frac{d}{dx} r^x_\alpha(t)\right] \nu^\infty_\alpha(dx).$$

We recall that $r^x_\alpha(t) = \mathbb{E}\, f(Y^{\alpha,\delta_x}_{t,0})$, where $Y^{\alpha,\delta_x}_{t,0}$ is the solution of (1.9) driven by the constant current α and starting at time 0 with law δ_x. By Theorem 4.13, the invariant measure ν^∞_α is locally stable if the complex zeros of the equation

$$J(\alpha)\widehat{\Theta}_\alpha(z) = 1$$

have negative real part.

Under Assumptions 4.1 and 4.2, the function $\alpha \mapsto J(\alpha)$ is $\mathcal{C}^2$ on $\mathbb{R}_+^*$. Assume J is small enough such that for some $\alpha > 0$ one has $J(\alpha)||\Theta_\alpha||_1 < 1$, and so ν_α^∞ is locally stable. There are two "canonical" ways to break (4.9) at some *bifurcation point* α_0: either there exists some $\tau_0 > 0$ such that $J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\pm\frac{i}{\tau_0}) = 1$ or $J(\alpha_0)\widehat{\Theta}_{\alpha_0}(0) = 1$. The first case is the subject of this chapter: we give explicit sufficient conditions to have a Hopf bifurcation.

In the second case, the following lemma shows that $J'(\alpha_0) = 0$. So, at least in the non-degenerate case where $J''(\alpha_0) \neq 0$, the function $\alpha \mapsto J(\alpha)$ is not strictly monotonic in the neighborhoods of α_0: this is a static bifurcation which typically leads to *bistability* (or *multistability*, etc.).

Lemma 5.2. *Under Assumptions 4.1 and 4.2, it holds that for all* $\alpha > 0$,

$$J'(\alpha) = \frac{1 - J(\alpha)\widehat{\Theta}_\alpha(0)}{\gamma(\alpha)}.$$

Proof. First, recall that $J(\alpha) = \frac{\alpha}{\gamma(\alpha)}$. So it suffices to show that $\gamma'(\alpha) = \widehat{\Theta}_\alpha(0)$. By (3.10), one has $\gamma(\alpha)^{-1} = \widehat{H}_\alpha(0)$. So we have to prove that

$$\frac{d}{d\alpha}\widehat{H}_\alpha(0) = -\frac{\widehat{\Theta}_\alpha(0)}{[\gamma(\alpha)]^2}.$$

It holds that

$$\frac{d}{d\alpha}\varphi_t^\alpha(0) = \int_0^t \frac{b(\varphi_t^\alpha(0)) + \alpha}{b(\varphi_u^\alpha(0)) + \alpha} du.$$

So, using Fubini, we have

$$\begin{aligned}
\frac{d}{d\alpha}H_\alpha(t) &= -H_\alpha(t)\int_0^t f'(\varphi_u^\alpha(0))\frac{d}{d\alpha}\varphi_u^\alpha(0)du \\
&= -H_\alpha(t)\int_0^t\int_\theta^t \frac{f'(\varphi_u^\alpha(0))\left[b(\varphi_u^\alpha(0)) + \alpha\right]}{b(\varphi_\theta^\alpha(0)) + \alpha} du d\theta \\
&= -H_\alpha(t)\int_0^t \frac{f(\varphi_t^\alpha(0)) - f(\varphi_\theta^\alpha(0))}{b(\varphi_\theta^\alpha(0)) + \alpha} d\theta.
\end{aligned}$$

Consider Ψ_α defined by (4.29). We have

$$\begin{aligned}
\frac{d}{d\alpha}\widehat{H}_\alpha(0) &= \int_0^\infty \frac{d}{d\alpha}H_\alpha(t)dt \\
&= -\int_0^\infty H_\alpha(t)\int_0^t \frac{f(\varphi_t^\alpha(0)) - f(\varphi_\theta^\alpha(0))}{b(\varphi_\theta^\alpha(0)) + \alpha} d\theta dt \\
&= -\int_0^\infty\int_0^\infty H_\alpha(u+\theta)\frac{f(\varphi_{u+\theta}^\alpha(0)) - f(\varphi_\theta^\alpha(0))}{b(\varphi_\theta^\alpha(0)) + \alpha} d\theta du \qquad \text{(using Fubini} \\
&\qquad \text{and the change of variables } u = t - \theta). \\
&= -\frac{\widehat{\Psi}_\alpha(0)}{\gamma(\alpha)}.
\end{aligned}$$

To obtain the last line, we used (4.30). Using Remark 4.28 (with $z = 0$), we have $\widehat{\Theta}_\alpha(0) = \frac{\widehat{\Psi}_\alpha(0)}{\widehat{H}_\alpha(0)} = \gamma(\alpha)\widehat{\Psi}_\alpha(0)$ and so $\frac{d}{d\alpha}\widehat{H}_\alpha(0) = -\frac{\widehat{\Theta}_\alpha(0)}{[\gamma(\alpha)]^2}$ as required. This ends the proof. □

The chapter is structured as follows: in Section 5.2, we state the spectral assumptions and the main result, Theorem 5.9. We give a layout of its proof at the end of Section 5.2. In Section 5.3, we give the proof of Theorem 5.9.

5.2 Assumptions and main result

Consider $\nu^\infty_{\alpha_0}$ an invariant measure of (1.2), for some $\alpha_0 > 0$. The interaction parameter is $J = J(\alpha_0)$. We assume that the stability criterion (4.9) is not satisfied for α_0:

Assumption 5.3. *Assume that there exist $\alpha_0 > 0$ and $\tau_0 > 0$ such that*

$$J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\tfrac{i}{\tau_0}) = 1 \quad \textit{and} \quad \frac{d}{dz}\widehat{\Theta}_{\alpha_0}(\tfrac{i}{\tau_0}) \neq 0.$$

Assumption 5.4 (Non-resonance condition)**.** *Assume that for all $n \in \mathbb{Z}\backslash\{-1, 1\}$,*

$$J(\alpha_0)\widehat{\Theta}_{\alpha_0}(\tfrac{in}{\tau_0}) \neq 1.$$

Remark 5.5 (Local uniqueness of the invariant measure in the neighborhood of α_0)**.** *Under Assumption 5.4, we have in particular $J(\alpha_0)\widehat{\Theta}_\alpha(0) \neq 1$ and so, by Lemma 5.2*

$$J'(\alpha_0) \neq 0.$$

Fix J in the neighborhood of $J(\alpha_0)$. Recall that the values of α such that ν^∞_α is an invariant measure of (1.2) are precisely the solutions of $J(\alpha) = J$. So, in the neighborhood of $\alpha = \alpha_0$, the invariant measure of (1.2) is (locally) unique.

Lemma 5.6. *Under Assumption 5.3, there exist $\eta_0, \varrho_0 > 0$ and a function $\mathfrak{Z}_0 \in \mathcal{C}^1((\alpha_0 - \eta_0, \alpha_0 + \eta_0); \mathbb{C})$ with $\mathfrak{Z}_0(\alpha_0) = \frac{i}{\tau_0}$ such that for all $z \in \mathbb{C}$ with $|z - \frac{i}{\tau_0}| < \varrho_0$ and for all $\alpha > 0$ with $|\alpha - \alpha_0| < \eta_0$ we have*

$$J(\alpha)\widehat{\Theta}_\alpha(z) = 1 \iff z = \mathfrak{Z}_0(\alpha). \tag{5.1}$$

Proof. Application of the implicit function theorem to $(\alpha, z) \mapsto J(\alpha)\widehat{\Theta}_\alpha(z) - 1$. □

Assumption 5.7. *Assume that $\alpha \mapsto \mathfrak{Z}_0(\alpha)$ crosses the imaginary part with non-vanishing speed, that is*

$$\Re\mathfrak{Z}_0'(\alpha_0) \neq 0, \quad \textit{where} \quad \mathfrak{Z}_0'(\alpha) = \frac{d}{d\alpha}\mathfrak{Z}_0(\alpha).$$

Remark 5.8. *Using (5.1), Assumption 5.7 is equivalent to*

$$\Re\left(\frac{\frac{\partial}{\partial\alpha}\left(J(\alpha)\widehat{\Theta}_\alpha\right)\Big|_{\alpha=\alpha_0}(\frac{i}{\tau_0})}{J(\alpha_0)\frac{\partial}{\partial z}\widehat{\Theta}_{\alpha_0}(\frac{i}{\tau_0})}\right) \neq 0.$$

Our main result is the following.

Theorem 5.9. *Consider b, f satisfying Assumptions 4.1 and 4.2. Let $\alpha_0, \tau_0 > 0$ be such that 4.6, 5.3 and 5.4 and 5.7 hold. Then, there exists a family of $2\pi\tau_v$-periodic solutions of (1.2), parametrized by $v \in (-v_0, v_0)$, for some $v_0 > 0$. More precisely, there exists a continuous curve $\{(\nu_v(\cdot), \alpha_v, \tau_v),\ v \in (-v_0, v_0)\}$ such that*

1. *For all $v \in (-v_0, v_0)$, $(\nu_v(t))_{t\in\mathbb{R}}$ is a $2\pi\tau_v$-periodic solution of (1.2) with $J = J(\alpha_v)$.*
2. *The curve passes through $(\nu^\infty_{\alpha_0}, \alpha_0, \tau_0)$ at $v = 0$. In particular we have for all $t \in \mathbb{R}$, $\nu_0(t) \equiv \nu^\infty_{\alpha_0}$.*
3. *The "periodic current" $\boldsymbol{a}_v$, defined by*
$$t \mapsto a_v(t) := J(\alpha_v) \int_{\mathbb{R}_+} f(x)\nu_v(t, dx), \tag{5.2}$$
is continuous and $2\pi\tau_v$-periodic. Moreover, its mean over one period is α_v:
$$\frac{1}{2\pi\tau_v} \int_0^{2\pi\tau_v} a_v(u)du = \alpha_v.$$
4. *Furthermore, v is the amplitude of the first harmonic of a_v, that is for all $v \in (-v_0, v_0)$*
$$\frac{1}{2\pi\tau_v} \int_0^{2\pi\tau_v} a_v(u)\cos(u/\tau_v)du = v \quad \text{and} \quad \frac{1}{2\pi\tau_v} \int_0^{2\pi\tau_v} a_v(u)\sin(u/\tau_v)du = 0.$$

Every other periodic solution in a neighborhood of $\nu^\infty_{\alpha_0}$ is obtained from a phase-shift of one such ν_v. More precisely, there exist small enough constants $\epsilon_0, \epsilon_1 > 0$ (only depending on b, f, α_0 and τ_0) such that if $(\nu(t))_{t\in\mathbb{R}}$ is any $2\pi\tau$-periodic solution of (1.2) for some value of $J > 0$ such that
$$|\tau - \tau_0| < \epsilon_0 \quad \text{and} \quad \sup_{t\in[0,2\pi\tau]} \left| J \int_{\mathbb{R}_+} f(x)\nu(t, dx) - \alpha_0 \right| < \epsilon_1,$$
then there exists a shift $\theta \in [0, 2\pi\tau)$ and $v \in (-v_0, v_0)$ such that $J = J(\alpha_v)$ and
$$\forall t \in \mathbb{R}, \quad \nu(t) \equiv \nu_v(t + \theta).$$

Remark 5.10. *Given the "periodic current" $\boldsymbol{a}_\nu$ defined by (5.2), the shape of the solution is known explicitly: for all $v \in (-v_0, v_0)$, it holds that*
$$\nu_v = \tilde{\nu}_{\boldsymbol{a}_v},$$
where $\tilde{\nu}_{\boldsymbol{a}_v}$, defined by (5.22) below, is known explicitly in terms of b, f and $\boldsymbol{a}_v$.

Notation 5.11. *For $T > 0$, we denote by C^0_T the space of continuous and T-periodic functions from $\mathbb{R}$ to $\mathbb{R}$ and by $C^{0,0}_T$ the subspace of centered functions*
$$C^{0,0}_T := \{h \in C^0_T, \quad \int_0^T h(t)dt = 0\}.$$

We now give an outline of the proof of Theorem 5.9. The proof is divided in two main parts.

The first part is devoted to the study of an isolated neuron subject to a periodic external current. That is, given $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi\tau}$, we study the jump rate of an isolated neuron driven by $\boldsymbol{a}$. We give in Section 5.3.1 estimates on the kernels $K_{\boldsymbol{a}}$ and $H_{\boldsymbol{a}}$. We want to characterize the "asymptotic" jump rate of a neuron driven by this external periodic current. That is, informally

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a}}(t) = \lim_{k\in\mathbb{N},\ k\to\infty} r_{\boldsymbol{a}}(t, -2\pi k\tau).$$

In order to characterize such limit $\rho_{\boldsymbol{a}}$, we introduce in Section 5.3.2 a discrete-time Markov Chain corresponding to the phases of the successive spikes of the neuron driven by $\boldsymbol{a}$. We prove that this Markov Chain has a unique invariant measure, which is proportional to $\rho_{\boldsymbol{a}}$. This serves as a definition of $\rho_{\boldsymbol{a}}$. Given this periodic jump rate $\rho_{\boldsymbol{a}} \in C^0_{2\pi\tau}$, we give in Section 5.3.3 an explicit description of the associated time-periodic probability densities, that we denote $(\tilde{\nu}_{\boldsymbol{a}}(t))_{t\in[0,2\pi\tau]}$. Consequently, to find a $2\pi\tau$-periodic solution of (1.2), it is equivalent to find $\boldsymbol{a} \in C^0_{2\pi\tau}$ such that

$$\boldsymbol{a} = J\rho_{\boldsymbol{a}}. \tag{5.3}$$

One classical difficulty with Hopf bifurcation is that the period $2\pi\tau$ itself is unknown: τ varies when the interaction parameter J varies. To address this problem, we make in Section 5.3.4 a change of time to only consider 2π-periodic functions. We define

$$\forall \boldsymbol{d} \in C^0_{2\pi}, \forall \tau > 0, \quad \rho_{\boldsymbol{d},\tau} = \mathcal{T}_\tau(\rho_{\mathcal{T}_{1/\tau}(\boldsymbol{d})}), \quad \text{with} \quad \forall t \geq 0,\ \mathcal{T}_\tau(\boldsymbol{d})(t) := d(\tau t). \tag{5.4}$$

We shall see that this change of time has a simple probabilistic interpretation by scaling b, f and $\boldsymbol{d}$ appropriately. In Section 5.3.5, we prove that the function $C^0_{2\pi} \times \mathbb{R}^*_+ \ni (\boldsymbol{d}, \tau) \mapsto \rho_{\boldsymbol{d},\tau} \in C^0_{2\pi}$ is $\mathcal{C}^2$-Fréchet differentiable. Furthermore, if the mean over one period of $\boldsymbol{d}$ is α, that is if $\boldsymbol{d} = \alpha + h$ for some $h \in C^{0,0}_{2\pi}$, we prove that the mean number of spikes over one period only depends on α, namely

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha+h,\tau}(u)du = \gamma(\alpha). \tag{5.5}$$

In the second part of the proof, we find self-consistent periodic solutions using the Lyapunov-Schmidt method. We introduce in Section 5.3.6 the following functional

$$C^{0,0}_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ \ni (h, \alpha, \tau) \mapsto G(h, \alpha, \tau) := (\alpha + h) - J(\alpha)\rho_{\alpha+h,\tau}.$$

Using (5.5), this functional takes values in $C^{0,0}_{2\pi}$. The roots of G, described by Proposition 5.27, match with the periodic solutions of (1.2). For instance if $G(h, \alpha, \tau) = 0$, we set $\boldsymbol{a} := \mathcal{T}_\tau(\alpha+h)$ which solves (5.3) with $J = J(\alpha)$ and so it can be used to define a periodic solution of (1.2). Conversely, to any periodic solution of (1.2), we can associate a root of G. So Theorem 5.9 is equivalent to Proposition 5.27. Sections 5.3.7, 5.3.8, 5.3.9 and 5.3.10 are then devoted to the proof of Proposition 5.27. In Section 5.3.7, we prove that the linear operator $D_hG(0, \alpha, \tau)$ can be written using a convolution involving Θ_α, given by (4.8). We then follow the method of [Kie12, Ch. I.8]. In Section 5.3.8, we study the range and the kernel of $D_hG(0, \alpha_0, \tau_0)$: we prove that under the spectral Assumptions 5.3 and 5.4, $D_hG(0, \alpha_0, \tau_0)$ is a Fredholm operator of index zero, with a kernel of dimension two. The problem of finding the roots of G is a priori of infinite dimension (h belongs to $C^{0,0}_{2\pi}$). In Section 5.3.9 we apply the Lyapunov-Schmidt method to obtain an equivalent problem of dimension two. Finally in Section 5.3.10 we study the reduced 2D-problem.

5.3 Proof of Theorem 5.9

5.3.1 Preliminaries

Let $T > 0$, $s \in \mathbb{R}$ and $\boldsymbol{a} \in C^0_T$ such that

$$\inf_{t\in[0,T]} a_t > -b(0). \tag{5.6}$$

For $x \geq 0$, we consider $\varphi^{\boldsymbol{a}}_{t,s}(x)$ the solution of the ODE (2.3).

$$\begin{aligned}\frac{d}{dt}\varphi^{\boldsymbol{a}}_{t,s}(x) &= b(\varphi^{\boldsymbol{a}}_{t,s}(x)) + a_t\\ \varphi^{\boldsymbol{a}}_{s,s}(x) &= x.\end{aligned}$$

By Assumption 4.1, this ODE has a unique solution. Recall moreover that the kernels $H^\nu_{\boldsymbol{a}}(t,s)$ and $K^\nu_{\boldsymbol{a}}(t,s)$, defined by (1.13), have the explicit expressions (2.4) and (2.5):

$$\begin{aligned}H^\nu_{\boldsymbol{a}}(t,s) &= \int_{\mathbb{R}_+} \exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx),\\ K^\nu_{\boldsymbol{a}}(t,s) &= \int_{\mathbb{R}_+} f(\varphi^{\boldsymbol{a}}_{t,s}(x))\exp\left(-\int_s^t f(\varphi^{\boldsymbol{a}}_{u,s}(x))du\right)\nu(dx).\end{aligned}$$

The function $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ belongs to $\mathcal{C}^1((-\infty,t];\mathbb{R}_+)$ and

$$\frac{d}{ds}\varphi^{\boldsymbol{a}}_{t,s}(0) = -\left[b(0)+a_s\right]\exp\left(\int_s^t b'(\varphi^{\boldsymbol{a}}_{\theta,s}(0))d\theta\right). \tag{5.7}$$

In particular, under the assumption (5.6), $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is strictly decreasing on $(-\infty,t]$, for all t. Let $\sigma_{\boldsymbol{a}}(t)$ be given by (2.19), that is:

$$\sigma_{\boldsymbol{a}}(t) := \lim_{s\to-\infty}\varphi^{\boldsymbol{a}}_{t,s}(0) \in \mathbb{R}^*_+ \cup \{+\infty\}. \tag{5.8}$$

Given $d \in C^0_T$ and $\eta > 0$, we define the following open balls of C^0_T:

$$B^T_\eta(d) := \{a \in C^0_T,\quad \sup_{t\in[0,T]}|a_t - d_t| < \eta\}. \tag{5.9}$$

Lemma 5.12. *Let $T > 0$ and $b : \mathbb{R}_+ \to \mathbb{R}$ such that Assumption 4.1 holds. Let $\alpha_0 > 0$ satisfying Assumption 4.6. There exists $\eta_0 > 0$ such that for all $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$, it holds that*

1. *If $\sigma_{\alpha_0} = \infty$, then for all $t \in [0,T]$, $\sigma_{\boldsymbol{a}}(t) = +\infty$.*
2. *If $\sigma_{\alpha_0} < \infty$, then the function $t \mapsto \sigma_{\boldsymbol{a}}(t)$ belongs to C^0_T and*

$$\inf_{\boldsymbol{a}\in B^T_{\eta_0}(\alpha_0)}\ \inf_{t\in[0,T]}\sigma_{\boldsymbol{a}}(t) > 0.$$

Proof. Assume first that $\sigma_{\alpha_0} = \infty$, and let $\eta_0 := \frac{1}{2}\inf_{x\geq 0} b(x) + \alpha_0$, which is strictly positive by assumption. Then it holds that

$$\inf_{t\geq 0}\inf_{x\geq 0} b(x) + a_t \geq \frac{\eta_0}{2},$$

and so

$$\varphi^{\boldsymbol{a}}_{t,s}(0) \geq \frac{\eta_0}{2}(t-s). \tag{5.10}$$

Letting s tend to $-\infty$, we deduce that $\sigma_{\boldsymbol{a}}(t) = +\infty$.
Assume now that $\sigma_{\alpha_0} < \infty$. Using (4.6), we can apply the implicit function theorem to the function

$$(x,\epsilon) \mapsto b(x) + \alpha_0 + \epsilon$$

at the point $(\sigma_{\alpha_0}, 0)$. We deduce that there exists a constant $\epsilon_1 > 0$ and a function $\epsilon \mapsto \tilde{\sigma}_{\alpha_0+\epsilon}$, which belongs to $\mathcal{C}^1([0,\epsilon_1];\mathbb{R}^*_+)$ and such that

$$\forall \epsilon \in [0,\epsilon_1], \quad b(\tilde{\sigma}_{\alpha_0+\epsilon}) + \alpha_0 + \epsilon = 0.$$

In addition, we have $\sigma_{\alpha_0} = \tilde{\sigma}_{\alpha_0}$. By definition, $\sigma_{\alpha_0+\epsilon} := \inf\{x \geq 0 \mid b(x) + \alpha_0 + \epsilon = 0\}$ and so $\sigma_{\alpha_0+\epsilon} \leq \tilde{\sigma}_{\alpha_0+\epsilon}$. We have $b'(\sigma_{\alpha_0}) < 0$. Because b is $\mathcal{C}^1$, we can find $0 < \epsilon_0 < \epsilon_1$ and $\underline{b}_0 > 0$ such that

$$\forall \epsilon \in [0,\epsilon_0], \forall u \in [\sigma_{\alpha_0+\epsilon}, \tilde{\sigma}_{\alpha_0+\epsilon}], \quad b'(u) \leq -\underline{b}_0.$$

So

$$0 = b(\tilde{\sigma}_{\alpha_0+\epsilon}) + \alpha_0 + \epsilon = b(\tilde{\sigma}_{\alpha_0+\epsilon}) - b(\sigma_{\alpha_0+\epsilon}) = \int_{\sigma_{\alpha_0+\epsilon}}^{\tilde{\sigma}_{\alpha_0+\epsilon}} b'(u)du \leq -\underline{b}_0(\tilde{\sigma}_{\alpha_0+\epsilon} - \sigma_{\alpha_0+\epsilon}).$$

We deduce that for all $\epsilon \in [0,\epsilon_0]$, $\tilde{\sigma}_{\alpha_0+\epsilon} = \sigma_{\alpha_0+\epsilon}$. We choose $\eta_0 := \frac{\epsilon_0}{2}$. Let $\boldsymbol{a} \in C^0_T$ such that $\sup_{t\in[0,T]} |a_t - \alpha_0| \leq \eta_0$. By Grönwall's inequality, we have

$$\forall t \geq s, \quad \varphi^{\boldsymbol{a}}_{t,s}(0) \leq \varphi^{\alpha_0+\epsilon_0}_{t,s}(0) \leq \sigma_{\alpha_0+\epsilon_0}.$$

In particular $\sigma_{\boldsymbol{a}}(t) < \infty$. We prove that this function is right-continuous in t. We fix $t \geq s$ and $\epsilon \in [0,\epsilon_0]$, we have

$$\begin{aligned}\varphi^{\boldsymbol{a}}_{t+\epsilon,s}(0) - \varphi^{\boldsymbol{a}}_{t,s}(0) &= \varphi^{\boldsymbol{a}}_{t+\epsilon,t}(\varphi^{\boldsymbol{a}}_{t,s}(0)) - \varphi^{\boldsymbol{a}}_{t,s}(0)\\ &= \int_t^{t+\epsilon} b(\varphi^{\boldsymbol{a}}_{t+u,s}(0))du + \int_t^{t+\epsilon} a_u du.\end{aligned}$$

So if $A_0 := \sup_{x\in[0,\sigma_{\alpha_0+\epsilon_0}]} |b(x)| < \infty$, we deduce that

$$|\varphi^{\boldsymbol{a}}_{t+\epsilon,s}(0) - \varphi^{\boldsymbol{a}}_{t,s}(0)| \leq (A_0 + ||\boldsymbol{a}||_\infty)\epsilon. \tag{5.11}$$

Letting s tend to $-\infty$ we deduce that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is right-continuous. Left-continuity is proved similarly. Using $\varphi^{\boldsymbol{a}}_{t+T,s+T}(0) = \varphi^{\boldsymbol{a}}_{t,s}(0)$, we deduce that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is T-periodic. Finally, because $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is strictly decreasing, and takes value 0 when $s = t$, we deduce that $\sigma_{\boldsymbol{a}}(t) > 0$. More precisely, let

$$m_0 := -\min_{x\in[0,\sigma_{\alpha_0+\epsilon_0}]} b'(x).$$

It holds that $m_0 > 0$. Moreover, using (5.7), we deduce that

$$\frac{d}{ds}\varphi^{\boldsymbol{a}}_{t,s}(0) \leq -(b(0)+\alpha_0-\eta_0)e^{-m_0(t-s)},$$

and so

$$\forall s \leq t, \quad \varphi^{\boldsymbol{a}}_{t,s}(0) \geq (b(0)+\alpha_0-\eta_0)\frac{1-e^{-m_0(t-s)}}{m_0}. \tag{5.12}$$

It ends the proof. □

Inspecting the proof of Lemma 5.12, we deduce that

Lemma 5.13. *Grant Assumptions 4.1 and 4.2. Let $\alpha_0 > 0$ such that Assumption 4.6 holds. There exist $\lambda_0, \eta_0, s_0 > 0$ (only depending on α_0 and b) such that for all $T > 0$, for all $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$, it holds that*

$$\forall t, s, \quad t - s \geq s_0 \implies \varphi^{\boldsymbol{a}}_{t,s}(0) \geq \lambda_0.$$

Moreover, if $\sigma_{\alpha_0} = \infty$, λ_0 can be chosen arbitrarily large. Finally, it holds that

$$\sup_{T>0} \sup_{\boldsymbol{a}\in B^T_{\eta_0}(\alpha_0)} \sup_{t\geq s} \left[H_{\boldsymbol{a}}(t,s) + K_{\boldsymbol{a}}(t,s)\right] e^{f(\lambda_0)(t-s)} < \infty.$$

Proof. Case $\sigma_{\alpha_0} < \infty$: the lower bound of the flow follows from (5.12). The bounds on H and K then follow directly from their explicit expressions (2.4) and (2.5) and the upper bound $f(\varphi^{\boldsymbol{a}}_{t,s}(0)) \leq f(\sigma_{\alpha_0+\epsilon_0})$.
Case $\sigma_{\alpha_0} = \infty$: the lower bound of the flow is a consequence of (5.10). Similarly, the bound on H follows from (2.4). Using (2.2) and the global Lipschitz property of b (say with constant L), there exists a constant C such that

$$f(\varphi^{\boldsymbol{a}}_{t,s}(0)) \leq Ce^{Lp(t-s)}.$$

The bound on K follows. □

5.3.2 Study of the non-homogeneous linear equation

In this section, we study the asymptotic jump rate of an "isolated" neuron driven by a periodic continuous function. Grant Assumptions 4.1, 4.2 and let $\alpha_0 > 0$ such that Assumption 4.6 holds. Let $\lambda_0, \eta_0 > 0$ be given by Lemma 5.13 and $T > 0$. Consider $\boldsymbol{a} \in B^T_{\eta_0}(\alpha_0)$ and let $r_{\boldsymbol{a}}$ be the solution of the Volterra equation $r_{\boldsymbol{a}} = K_{\boldsymbol{a}} + K_{\boldsymbol{a}} * r_{\boldsymbol{a}}$. We consider the following limit

$$\forall t \in [0,T], \quad \rho_{\boldsymbol{a}}(t) = \lim_{k\to+\infty} r_{\boldsymbol{a}}(t, -kT).$$

The goal of this section is to show that $\rho_{\boldsymbol{a}}$ is well defined and to study some of its properties. First, (1.14) and (2.10) write

$$\forall t \in \mathbb{R}, \quad r_{\boldsymbol{a}}(t,-kT) = K_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^{t} K_{\boldsymbol{a}}(t,s) r_{\boldsymbol{a}}(s,-kT)ds,$$

$$1 = H_{\boldsymbol{a}}(t,-kT) + \int_{-kT}^{t} H_{\boldsymbol{a}}(t,s) r_{\boldsymbol{a}}(s,-kT)ds.$$

Letting $k \to \infty$, we find that $\rho_{\boldsymbol{a}}$ has to solve

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a}}(t) = \int_{-\infty}^{t} K_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds. \tag{5.13}$$

$$1 = \int_{-\infty}^{t} H_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds. \tag{5.14}$$

Note that if $\rho_{\boldsymbol{a}}$ is a solution of (5.13), then it automatically holds that the function $t \mapsto \int_{-\infty}^{t} H_{\boldsymbol{a}}(t,s)\rho_{\boldsymbol{a}}(s)ds$ is constant (its derivative is null). In Lemma 5.15 below, we prove that the solutions of eq. (5.13) form a linear space of dimension 1. Consequently (5.13) together with (5.14) have a unique solution: this will serve as the definition of $\rho_{\boldsymbol{a}}$.

A probabilistic interpretation of (5.13) and (5.14)

Let x be a T-periodic solution of (5.13). We have for all $t \in [0,T]$

$$\begin{aligned}
x(t) &= \int_{-\infty}^{T} K_{\boldsymbol{a}}(t,s)x(s)ds, \quad \text{(recall } K_{\boldsymbol{a}}(t,s) = 0 \text{ for } s > t) \\
&= \sum_{k \geq 0} \int_{-kT}^{T-kT} K_{\boldsymbol{a}}(t,s)x(s)ds \\
&= \sum_{k \geq 0} \int_{0}^{T} K_{\boldsymbol{a}}(t,u-kT)x(u)du \quad \text{(by the change of variable } u = s + kT).
\end{aligned}$$

Define for all $t, s \in [0,T]$

$$K_{\boldsymbol{a}}^{T}(t,s) := \sum_{k \geq 0} K_{\boldsymbol{a}}(t, s-kT).$$

Note that by Lemma 5.13 we have normal convergence:

$$\forall t, s \in [0,T], \quad K_{\boldsymbol{a}}(t, s-kT) \leq C e^{-f(\lambda_0)kT},$$

for some constant C only depending on $b, f, \alpha_0, \eta_0, \lambda_0$ and T. We deduce that x solves

$$x(t) = \int_{0}^{T} K_{\boldsymbol{a}}^{T}(t,s)x(s)ds. \tag{5.15}$$

Using that $\boldsymbol{a}$ is T-periodic, we have

$$\forall t \geq s, \quad K_{\boldsymbol{a}}(t+T, s+T) = K_{\boldsymbol{a}}(t,s). \tag{5.16}$$

Moreover, $K_{\boldsymbol{a}}$ is a probability density so

$$\forall s \in \mathbb{R}, \quad \int_{s}^{\infty} K_{\boldsymbol{a}}(t,s)dt = 1. \tag{5.17}$$

From (5.16) and (5.17), we deduce that

$$\forall s \in [0,T], \quad \int_{0}^{T} K_{\boldsymbol{a}}^{T}(t,s)dt = 1. \tag{5.18}$$

In view of (5.18), $K_{\boldsymbol{a}}^T(\cdot, s)$ can be seen as the transition probability kernel of a Markov Chain acting on the continuous space $[0, T]$. The interpretation of this Markov Chain is the following. Let $(Y_t^{\nu,\boldsymbol{a}})_{t\geq 0}$ be the solution of (1.9), starting at time 0 with law ν and driven by the T-periodic current $\boldsymbol{a}$. Define $(\tau_i)_{i\geq 1}$ the times of the successive jumps of $(Y_t^{\boldsymbol{a},\nu})_{t\geq 0}$. Let $\phi_i \in [0, T)$ and $\Delta_i \in \mathbb{N}$ be defined by:

$$\phi_i := \tau_i - \left\lfloor \frac{\tau_i}{T} \right\rfloor T, \quad \tau_{i+1} - \tau_i =: \Delta_{i+1} T + \phi_{i+1} - \phi_i. \tag{5.19}$$

That is, ϕ_i is the *phase* of the i-ith jump, while Δ_i is the number of "revolutions" between τ_{i-1} and τ_i:

$$\Delta_i = \#\{k \in \mathbb{N}, \quad kT \in [\tau_{i-1}, \tau_i)\}.$$

In other words, if one considers that a period is a "lap", Δ_i is the number of times we cross the start line of the lap between two spikes.

Then, $(\phi_i, \Delta_i)_{i\geq 0}$ is Markov, with a transition probability given by

$$\forall A \in \mathcal{B}([0,T]),\ \forall n \in \mathbb{N}, \quad \mathbb{P}(\phi_{i+1} \in A, \Delta_{i+1} = n | \phi_i) = \int_A K_{\boldsymbol{a}}(t + nT, \phi_i) dt.$$

In particular, $(\phi_i)_{i\geq 0}$ is Markov, with a transition probability given by $K_{\boldsymbol{a}}^T$. With some slight abuse of notations, we also write $K_{\boldsymbol{a}}^T$ for the linear operator which maps $y \in L^1([0,T])$ to

$$K_{\boldsymbol{a}}^T(y) := t \mapsto \int_0^T K_{\boldsymbol{a}}^T(t,s) y(s) ds \in L^1([0,T]). \tag{5.20}$$

Lemma 5.14. *Let $\boldsymbol{a} \in C_T^0$. The linear operator $K_{\boldsymbol{a}}^T : L^1([0,T]) \to L^1([0,T])$ is a compact operator. Moreover, if $y \in L^1([0,T])$, then $K_{\boldsymbol{a}}^T(y) \in C_T^0$.*

Proof. First, the function $[0,T]^2 \ni (t,s) \mapsto K_{\boldsymbol{a}}^T(t,s)$ is (uniformly) continuous. Let $\epsilon > 0$, there exists $\eta > 0$ such that

$$|t - t'| + |s - s'| \leq \eta \implies |K_{\boldsymbol{a}}^T(t,s) - K_{\boldsymbol{a}}^T(t', s')| \leq \epsilon.$$

It follows that

$$\left|K_{\boldsymbol{a}}^T(y)(t) - K_{\boldsymbol{a}}^T(y)(t')\right| \leq \int_0^T \left|K_{\boldsymbol{a}}^T(t,s) - K_{\boldsymbol{a}}^T(t',s)\right| |y(s)|\, ds \leq \epsilon ||y||_1,$$

and so the function $K_{\boldsymbol{a}}^T(y)$ is continuous. Note that

$$\forall s \in [0,T], \quad K_{\boldsymbol{a}}^T(T,s) = K_{\boldsymbol{a}}^T(0,s),$$

and so $K_{\boldsymbol{a}}^T(y)$ is T-periodic. This shows that $K_{\boldsymbol{a}}^T(y) \in C_T^0$. To prove that $K_{\boldsymbol{a}}^T$ is a compact operator, we use the Weierstrass approximation Theorem: there exists a sequence of polynomial functions $(t,s) \mapsto P_n(t,s)$ such that $\sup_{t,s\in[0,T]} \left|P_n(t,s) - K_{\boldsymbol{a}}^T(t,s)\right| \to_n 0$ as $n \to \infty$. For each $n \in \mathbb{N}$, the linear operator $L^1([0,T]) \ni y \mapsto P_n(y) := t \mapsto \int_0^T P_n(t,s) y(s) ds$ is of finite-rank. Moreover, the sequence P_n converges to $K_{\boldsymbol{a}}^T$ for the norm operator, and so $K_{\boldsymbol{a}}^T$ is a compact operator (as the limit of finite-rank operators, see [Bre11, Ch. 6]). $\square$

Lemma 5.15. *Let $\boldsymbol{a} \in C_T^0$. The Markov Chain $(\phi_i)_{i \geq 0}$ with transition probability kernel $K_{\boldsymbol{a}}^T$ has a unique invariant probability measure $\pi_{\boldsymbol{a}} \in C_T^0$. Consequently, the solutions of (5.13) in C_T^0 span a vector space of dimension 1.*

Proof. The proof follows directly from the Krein–Rutman Theorem, which is a generalization of the Perron-Frobenius Theorem for compact operators. We give the proof for the sake of completeness, and because some of its elements will be reused later. Let $(K_{\boldsymbol{a}}^T)' : L^\infty([0,T]) \to L^\infty([0,T])$ be the dual operator of $K_{\boldsymbol{a}}^T$. We have:

$$\forall v \in L^\infty([0,T]), \quad (K_{\boldsymbol{a}}^T)'(v) = t \mapsto \int_0^T K_{\boldsymbol{a}}^T(s,t)v(s)ds.$$

From (5.18), we deduce that 1 is an eigenvalue of $(K_{\boldsymbol{a}}^T)'$ (its associated eigenvector is $\mathbf{1}$, the constant function equal to 1). By the Fredholm alternative [Bre11, Th. 6.6] , we have $\dim\, N(I - K_{\boldsymbol{a}}^T) = \dim\, N(I - (K_{\boldsymbol{a}}^T)')$ and so there exists $\pi \in L^1([0,T])$ such that:

$$\pi = K_{\boldsymbol{a}}^T(\pi), \; ||\pi||_1 = 1.$$

We now prove that π can be chosen positive. Let $\delta := \inf_{t,s\in[0,T]} K_{\boldsymbol{a}}^T(t,s)$. The kernel $K_{\boldsymbol{a}}^T$ is positive and continuous on $[0,T]^2$ so $\delta > 0$. We write π_+ for the positive part of π and π_- for its negative part and define $\beta := \min(||\pi_+||_1, ||\pi_-||_1)$. We have $K_{\boldsymbol{a}}^T(\pi_+)(t) \geq \delta\beta T$ and $K_{\boldsymbol{a}}^T(\pi_-)(t) \geq \delta\beta T$. Consequently

$$\begin{aligned}
||K_{\boldsymbol{a}}^T(\pi)||_1 =& ||K_{\boldsymbol{a}}^T(\pi_+) - K_{\boldsymbol{a}}^T(\pi_-)||_1 \\
\leq& ||K_{\boldsymbol{a}}^T(\pi_+) - \delta\beta T||_1 + ||K_{\boldsymbol{a}}^T(\pi_-) - \delta\beta T||_1 \\
\leq& ||\pi||_1 - 2\delta\beta T.
\end{aligned}$$

But the identity $K_{\boldsymbol{a}}^T(\pi) = \pi$ implies that $\beta = 0$ and so either π_+ or π_- is null. So π has a constant sign and may be chosen positive. Note moreover that

$$\pi(t) = \int_0^T K_{\boldsymbol{a}}^T(t,s)\pi(s)ds \geq \delta \int_0^T \pi(s)ds \geq \delta.$$

Finally, if π_1 and π_2 are two non-negative solutions of (5.15) with $||\pi_1||_1 = ||\pi_2||_1 = 1$, then $\pi_3 := \pi_1 - \pi_2$ also solves (5.15) and has a constant sign. Consequently, $||\pi_3||_1 = |||\pi_1||_1 - ||\pi_2||_1| = 0$ and we deduce that $\pi_3 = 0$, proving that the space of solutions in $L^1([0,T])$ of (5.15) is of dimension 1. Finally Lemma 5.14 gives the continuity of π and $\pi(T) = \pi(0)$. Consequently π can be extended to C_T^0 and solves (5.13). This ends the proof. ∎

We define for all $\theta \in \mathbb{R}$ the following shift operator

$$\begin{array}{rccc} S_\theta : & C_T^0 & \to & C_T^0 \\ & x & \mapsto & (x(t+\theta))_t. \end{array}$$

Corollary 5.16. *Given $\boldsymbol{a} \in C_T^0$, eq. (5.13) and (5.14) have a unique solution $\rho_{\boldsymbol{a}} \in C_T^0$. Moreover, it holds that for all $\theta \in \mathbb{R}$,*

$$\rho_{S_\theta(\boldsymbol{a})} = S_\theta(\rho_{\boldsymbol{a}}). \tag{5.21}$$

Proof. By Lemma 5.15, the solution $\rho_{\boldsymbol{a}}$ of eq. (5.13) and (5.14) is

$$\rho_{\boldsymbol{a}} = \frac{\pi_{\boldsymbol{a}}}{c_{\boldsymbol{a}}},$$

where $\pi_{\boldsymbol{a}}$ is the invariant measure (on $[0, T]$) of the Markov Chain with transition probability kernel $K^T_{\boldsymbol{a}}$ and $c_{\boldsymbol{a}}$ is given by

$$c_{\boldsymbol{a}} := \int_{-\infty}^{t} H_{\boldsymbol{a}}(t, s)\pi_{\boldsymbol{a}}(s)ds.$$

Note that $c_{\boldsymbol{a}}$ is constant in time. Define for all $t, s \in [0, T]$:

$$H^T_{\boldsymbol{a}}(t, s) := \sum_{k\geq 0} H_{\boldsymbol{a}}(t, s - kT).$$

We have $c_{\boldsymbol{a}} = H^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}})$. Moreover, we have

$$\forall t, s, \theta \in \mathbb{R}, \quad \varphi^{S_\theta(\boldsymbol{a})}_{t,s}(0) = \varphi^{\boldsymbol{a}}_{t+\theta,s+\theta}(0),$$

because both sides satisfy the same ODE with the same initial condition at $t = s$. We deduce from (2.4) and (2.5) that

$$H_{S_\theta(\boldsymbol{a})}(t, s) = H_{\boldsymbol{a}}(t + \theta, s + \theta), \quad K_{S_\theta(\boldsymbol{a})}(t, s) = K_{\boldsymbol{a}}(t + \theta, s + \theta).$$

So $S_\theta(\rho_{\boldsymbol{a}})$ solves (5.13) and (5.14), where the kernels are replaced by $K_{S_\theta(\boldsymbol{a})}$ and $H_{S_\theta(\boldsymbol{a})}$. By uniqueness it follows that $\rho_{S_\theta(\boldsymbol{a})} = S_\theta(\rho_{\boldsymbol{a}})$. □

Remark 5.17. *Using that* $\int_0^T \pi_{\boldsymbol{a}}(s)ds = 1$, *we find that the average number of spikes over one period* $[0, T]$ *is given by*

$$\frac{1}{T}\int_0^T \rho_{\boldsymbol{a}}(s)ds = \frac{1}{c_{\boldsymbol{a}}T}.$$

The probabilistic interpretation of $c_{\boldsymbol{a}}$ *is the following: remembering the Markov chain defined by* (5.19), *we have*

$$\mathbb{P}(\Delta_{i+1} > k|\phi_i) = H_{\boldsymbol{a}}((k + 1)T, \phi_i),$$

and so, if $\mathcal{L}(\phi_i) = \pi_{\boldsymbol{a}}$, *we deduce that*

$$\mathbb{E}\,\Delta_{i+1} = \mathbb{E}\,\mathbb{E}(\Delta_{i+1}|\phi_i) = \mathbb{E}\left[\sum_{k\geq 0}\mathbb{P}(\Delta_{i+1} > k|\phi_i)\right] = H^T_{\boldsymbol{a}}(\pi_{\boldsymbol{a}}) = c_{\boldsymbol{a}}.$$

In other words, $c_{\boldsymbol{a}}$ *is the expected number of "revolutions" between two successive spikes, assuming the phase of each spike follows its invariant measure* $\pi_{\boldsymbol{a}}$*.We shall see in Proposition 5.26 that* $c_{\boldsymbol{a}}$ *only depends on the mean of* $\boldsymbol{a}$*. Furthermore, it holds that for* $\boldsymbol{a} \equiv \alpha > 0$

$$c_\alpha = H^T_\alpha(1/T) = \frac{1}{T}\int_0^\infty H_\alpha(t)dt = \frac{1}{T\gamma(\alpha)},$$

and so for all t, $\rho_\alpha(t) = \gamma(\alpha)$.

5.3.3 Shape of the solutions

Let $\boldsymbol{a} \in C^0_T$ such that (5.6) holds. Let $\sigma_{\boldsymbol{a}}(t)$ be defined by (5.8), such that $s \mapsto \varphi^{\boldsymbol{a}}_{t,s}(0)$ is a bijection from $(-\infty, t]$ to $[0, \sigma_{\boldsymbol{a}}(t))$. We denote by $x \mapsto \beta^{\boldsymbol{a}}_t(x)$ its inverse. Note that $t \mapsto \sigma_{\boldsymbol{a}}(t)$ is T-periodic and

$$\forall t \in \mathbb{R}, \forall x \in [0, \sigma_{\boldsymbol{a}}(t)), \quad \beta^{\boldsymbol{a}}_{t+T}(x) = \beta^{\boldsymbol{a}}_t(x) + T.$$

Using that $\varphi^{\boldsymbol{a}}_{t,t}(0) = 0$, we have $\beta^{\boldsymbol{a}}_t(0) = t$.

Notation 5.18. *Given $\boldsymbol{a} \in C^0_T$, we define for all $t \in \mathbb{R}$*

$$\tilde{\nu}_{\boldsymbol{a}}(t,x) := \frac{\rho_{\boldsymbol{a}}(\beta^{\boldsymbol{a}}_t(x))}{b(0) + a(\beta^{\boldsymbol{a}}_t(x))} \exp\left(- \int_{\beta^{\boldsymbol{a}}_t(x)}^t (f + b')(\varphi^{\boldsymbol{a}}_{\theta, \beta^{\boldsymbol{a}}_t(x)}(0)) d\theta \right) \mathbb{1}_{[0, \sigma_{\boldsymbol{a}}(t))}(x), \tag{5.22}$$

where $\rho_{\boldsymbol{a}}$ is the unique solution of the eq. (5.13) *and* (5.14)*.*

By the change of variables $u = \beta^{\boldsymbol{a}}_t(x)$, one obtains that for any non-negative measurable test function g

$$\int_0^\infty g(x) \tilde{\nu}_{\boldsymbol{a}}(t,x) dx = \int_{-\infty}^t g(\varphi^{\boldsymbol{a}}_{t,u}(0)) \rho_{\boldsymbol{a}}(u) H_{\boldsymbol{a}}(t,u) du. \tag{5.23}$$

Note moreover that when $\boldsymbol{a}$ is constant and equal to $\alpha > 0$ ($\boldsymbol{a} \equiv \alpha$), (5.22) matches with the definition of the invariant measure ν^∞_α given by (3.4):

$$\forall t \in \mathbb{R}, \quad \sigma_\alpha(t) = \sigma_\alpha \quad \text{and} \quad \tilde{\nu}_\alpha(t) = \nu^\infty_\alpha.$$

The main result of this section is

Proposition 5.19. *Let $\boldsymbol{a} \in C^0_T$ such that $\inf_{t \in \mathbb{R}} a_t > -b(0)$. It holds that $(\tilde{\nu}_{\boldsymbol{a}}(t, \cdot))_t$ is the unique T-periodic solution of* (1.9)*.*

Proof. **Existence**. We first prove that $\tilde{\nu}_{\boldsymbol{a}}(t, \cdot)$ is indeed a T-periodic solution. We follow the same strategy as in Proposition 3.9. First note that, by (5.23), one has

$$\int_0^\infty f(x) \tilde{\nu}_{\boldsymbol{a}}(t,x) dx = \int_{-\infty}^t K_{\boldsymbol{a}}(t,u) \rho_{\boldsymbol{a}}(u) du = \rho_{\boldsymbol{a}}(t).$$

Consider the solution $(Y^{\boldsymbol{a}, \tilde{\nu}_{\boldsymbol{a}}(0)}_{t,0})$ of (1.9) starting with law $\tilde{\nu}_{\boldsymbol{a}}(0)$ at time $s = 0$ and let $r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0) = \mathbb{E} f(Y^{\boldsymbol{a}, \tilde{\nu}_{\boldsymbol{a}}(0)}_{t,0})$.
Claim: It holds that for all $t \geq 0$, $r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0) = \rho_{\boldsymbol{a}}(t)$.

Proof of the Claim. Recall that $r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0)$ is the unique solution of the Volterra equation

$$r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0) = K^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0) + K_{\boldsymbol{a}} * r^{\tilde{\nu}_{\boldsymbol{a}}(0)}_{\boldsymbol{a}}(t,0).$$

So, to prove the claim is suffices to show that $\rho_{\boldsymbol{a}}$ also solves this equation. For all $u \leq 0 \leq t$, one has

$$K^{\varphi^{\boldsymbol{a}}_{0,u}(0)}_{\boldsymbol{a}}(t,0) H_{\boldsymbol{a}}(0,u) = K_{\boldsymbol{a}}(t,u).$$

Consequently, we deduce from (5.23) that

$$K_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t,0) = \int_{-\infty}^{0} K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du.$$

So

$$\rho_{\boldsymbol{a}}(t) = \int_{-\infty}^{t} K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du = K_{\boldsymbol{a}}^{\tilde{\nu}_{\boldsymbol{a}}(0)}(t,0) + \int_{0}^{t} K_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du,$$

and the conclusion follows. □

Finally, using Proposition 2.19 and the claim, we deduce that for any non-negative measurable function g

$$\mathbb{E}\, g(Y_{t,0}^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)}) = \int_0^t g(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du + \int_0^\infty g(\varphi_{t,0}^{\boldsymbol{a}}(x))H_{\boldsymbol{a}}^x(t,0)\tilde{\nu}_{\boldsymbol{a}}(0,x)dx.$$

By (5.23) (with $t=0$ and $g(x) = g(\varphi_{t,0}^{\boldsymbol{a}}(x))H_{\boldsymbol{a}}^x(t,0)$), the second term is equal to

$$\int_{-\infty}^{0} g(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du,$$

and so

$$\forall t \geq 0, \quad \mathbb{E}\, g(Y_{t,0}^{\boldsymbol{a},\tilde{\nu}_{\boldsymbol{a}}(0)}) = \int_{-\infty}^{t} g(\varphi_{t,u}^{\boldsymbol{a}}(0))H_{\boldsymbol{a}}(t,u)\rho_{\boldsymbol{a}}(u)du \overset{(5.23)}{=} \int_0^\infty g(x)\tilde{\nu}_{\boldsymbol{a}}(t,x)dx.$$

This ends the proof of the existence.

Uniqueness. Consider $(\nu(t))_{t\in[0,T]}$ a T-periodic solution of (1.9) and define $\rho(t) = \mathbb{E}\, f(Y_{t,0}^{\boldsymbol{a},\nu(0)})$. The function ρ is T-periodic. Moreover, it holds that for all $k \geq 0$, $\rho(t) = \mathbb{E}\, f(Y_{t,-kT}^{\boldsymbol{a},\nu(0)})$ and so (1.14) and (2.10) yields

$$\begin{aligned}\rho(t) &= K_{\boldsymbol{a}}^{\nu(0)}(t,-kT) + \int_{-kT}^{t} K_{\boldsymbol{a}}(t,u)\rho(u)du \\ 1 &= H_{\boldsymbol{a}}^{\nu(0)}(t,-kT) + \int_{-kT}^{t} H_{\boldsymbol{a}}(t,u)\rho(u)du.\end{aligned}$$

Letting k go to infinity, we deduce that ρ solves (5.13) and (5.14). By uniqueness, we deduce that for all t, $\rho(t) = \rho_{\boldsymbol{a}}(t)$ (and so ρ is continuous). Finally define τ_t the time of the last spike of $Y_{t,-kT}^{\boldsymbol{a},\nu(0)}$ before t (with the convention that $\tau_t = -kT$ if there is no spike between $-kT$ and t). The law of τ_t is

$$\mathcal{L}(\tau_t)(du) = \delta_{-kT}(du)H_{\boldsymbol{a}}^{\nu(0)}(t,-kT) + \rho_{\boldsymbol{a}}(u)H_{\boldsymbol{a}}(t,u)du.$$

Consequently, for any non-negative test function g

$$\begin{aligned}\mathbb{E}\, g(Y_{t,-kT}^{\boldsymbol{a},\nu(0)}) &= \mathbb{E}\, g(Y_{t,-kT}^{\boldsymbol{a},\nu(0)} \mathbb{1}_{\tau_t=-kT}) + \mathbb{E}\, g(\varphi_{t,\tau_t}^{\boldsymbol{a}}(0))\mathbb{1}_{\tau_t\in(-kT,t]} \\ &= \int_0^\infty g(\varphi_{t,-kT}^{\boldsymbol{a}}(x))H_{\boldsymbol{a}}^x(t,-kT)\nu(0)(dx) + \int_{-kT}^{t} g(\varphi_{t,u}^{\boldsymbol{a}}(0))\rho_{\boldsymbol{a}}(u)H_{\boldsymbol{a}}(t,u)du.\end{aligned}$$

Using that $\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,-kT}) = \mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,0})$ and letting again k to infinity we deduce that

$$\mathbb{E}\, g(Y^{\boldsymbol{a},\nu(0)}_{t,0}) = \int_{-\infty}^{t} g(\varphi^{\boldsymbol{a}}_{t,u}(0))\rho_{\boldsymbol{a}}(u) H_{\boldsymbol{a}}(t,u)du.$$

So for all t, $\nu(t) \equiv \tilde{\nu}_{\boldsymbol{a}}(t)$. □

5.3.4 Reduction to 2π-periodic functions

Convention: For now on, we prefer to work with the *reduced period* τ, such that

$$T =: 2\pi\tau, \quad \tau > 0.$$

Consider $\boldsymbol{d} \in C^0_{2\pi\tau}$ and let $\boldsymbol{a}$ be the 2π-periodic function defined by:

$$\forall t \in \mathbb{R}, \quad a(t) := d(\tau t).$$

We define

$$\forall t \in \mathbb{R}, \quad \rho_{\boldsymbol{a},\tau}(t) := \rho_{\boldsymbol{d}}(\tau t),$$

where $\rho_{\boldsymbol{d}}$ is the unique solution of (5.13) and (5.14) (with kernels $K_{\boldsymbol{d}}$ and $H_{\boldsymbol{d}}$). Because $\rho_{\boldsymbol{d}}$ is $2\pi\tau$-periodic, $\rho_{\boldsymbol{a},\tau}$ is 2π-periodic. Note that when $\boldsymbol{a} \equiv \alpha$ is constant we have

$$\forall \tau > 0, \forall t \in \mathbb{R}, \quad \rho_{\alpha,\tau}(t) = \gamma(\alpha). \tag{5.24}$$

To better understand how $\rho_{\boldsymbol{a},\tau}$ depends on τ, consider $(Y^{\boldsymbol{d},\nu}_{t,s})$ the solution of (1.9), starting with law ν and driven by $\boldsymbol{d}$. Note that for all $t \geq s$

$$\begin{aligned}
Y^{\boldsymbol{d},\nu}_{\tau t,\tau s} &= Y^{\boldsymbol{d},\nu}_{\tau s,\tau s} + \int_{\tau s}^{\tau t} b(Y^{\boldsymbol{d},\nu}_{u,\tau s})du + \int_{\tau s}^{\tau t} d_u du - \int_{\tau s}^{\tau t}\int_{\mathbb{R}_+} Y^{\boldsymbol{d},\nu}_{u-,\tau s} \mathbb{1}_{\{\tau z \leq \tau f(Y^{\boldsymbol{d},\nu}_{u-,\tau s})\}} \mathbf{N}(du,dz) \\
&= Y^{\boldsymbol{d},\nu}_{\tau s,\tau s} + \int_{s}^{t} \tau b(Y^{\boldsymbol{d},\nu}_{\tau u,\tau s})du + \int_{s}^{t} \tau a_u du - \int_{s}^{t}\int_{\mathbb{R}_+} Y^{\boldsymbol{d},\nu}_{\tau u-,\tau s} \mathbb{1}_{\{z \leq \tau f(Y^{\boldsymbol{d},\nu}_{\tau u-,\tau s})\}} \tilde{\mathbf{N}}(du,dz).
\end{aligned}$$

Here, $\tilde{\mathbf{N}} := \mathbf{N} \circ g^{-1}$ is the push-forward measure of $\mathbf{N}$ by the function

$$g(t,z) := (\tau t, z/\tau).$$

Note that $\tilde{\mathbf{N}}(du,dz)$ is again a Poisson measure of intensity $dudz$, and so $(Y^{\boldsymbol{d},\nu}_{\tau t,\tau s})$ is a (weak) solution of (1.9) for $\tilde{f} := \tau f$, $\tilde{b} := \tau b$ and $\tilde{\boldsymbol{a}} := \tau \boldsymbol{a}$. So, in particular (taking $\nu = \delta_0$), if we define:

$$\begin{aligned}
&\frac{d}{dt}\varphi^{\boldsymbol{a},\tau}_{t,s}(0) = \tau b(\varphi^{\boldsymbol{a},\tau}_{t,s}(0)) + \tau a(t); \quad \varphi^{\boldsymbol{a},\tau}_{s,s}(0) = 0, \\
&H_{\boldsymbol{a},\tau}(t,s) := \exp\left(-\int_s^t \tau f(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))du\right), \\
&K_{\boldsymbol{a},\tau}(t,s) := \tau f(\varphi^{\boldsymbol{a},\tau}_{t,s}(0)) \exp\left(-\int_s^t \tau f(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))du\right),
\end{aligned} \tag{5.25}$$

we have

Lemma 5.20. *Let $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi}$. Set, for all $t \in \mathbb{R}$, $d(t) := a(\frac{t}{\tau})$. Then it holds that*

$$\forall t \geq s, \quad H_{\boldsymbol{a},\tau}(t,s) = H_{\boldsymbol{d}}(\tau t, \tau s) \quad \text{and} \quad K_{\boldsymbol{a},\tau}(t,s) = \tau K_{\boldsymbol{d}}(\tau t, \tau s).$$

In view of this result, we deduce that $\rho_{\boldsymbol{a},\tau}$ solves

$$\rho_{\boldsymbol{a},\tau}(t) = \int_{-\infty}^{t} K_{\boldsymbol{a},\tau}(t,s)\rho_{\boldsymbol{a},\tau}(s)ds, \quad 1 = \tau \int_{-\infty}^{t} H_{\boldsymbol{a},\tau}(t,s)\rho_{\boldsymbol{a},\tau}(s)ds, \tag{5.26}$$

or equivalently, setting

$$\forall t,s \in [0,2\pi], \quad K^{2\pi}_{\boldsymbol{a},\tau}(t,s) := \sum_{k\geq 0} K_{\boldsymbol{a},\tau}(t, s-2\pi k) \quad \text{and} \quad H^{2\pi}_{\boldsymbol{a},\tau}(t,s) := \sum_{k\geq 0} H_{\boldsymbol{a},\tau}(t, s-2\pi k), \tag{5.27}$$

one has, using the same operator notation as in (5.20)

$$\rho_{\boldsymbol{a},\tau} = K^{2\pi}_{\boldsymbol{a},\tau}(\rho_{\boldsymbol{a},\tau}), \quad 1 = \tau H^{2\pi}_{\boldsymbol{a},\tau}(\rho_{\boldsymbol{a},\tau}).$$

Note that $\rho_{\cdot,\tau}$ and $\rho_{\cdot}$ are linked by (5.4). Consequently eq. (5.26) define a unique 2π-periodic continuous function

$$\rho_{\boldsymbol{a},\tau} = \frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}, \tag{5.28}$$

where $\pi_{\boldsymbol{a},\tau}$ is the unique invariant measure of the Markov Chain with transition probability kernel $K^{2\pi}_{\boldsymbol{a},\tau}$ and $c_{\boldsymbol{a},\tau}$ is the constant given by

$$c_{\boldsymbol{a},\tau} := \tau H^{2\pi}_{\boldsymbol{a},\tau}(\pi_{\boldsymbol{a},\tau}).$$

5.3.5 Regularity of ρ

The goal of this section is to study the regularity of $\rho_{\boldsymbol{a},\tau}$ with respect to $\boldsymbol{a}$ and τ. For $\eta_0 > 0$, recall that $B^{2\pi}_{\eta_0}$ is the open ball of $C^0_{2\pi}$ defined by (5.9). The main result of this section is

Proposition 5.21. *Grant Assumptions 4.1, 4.2 and let $\alpha_0 > 0$ such that Assumption 4.6 holds. Let $\tau_0 > 0$. There exists $\epsilon_0, \eta_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the function*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & C^0_{2\pi} \\ (\boldsymbol{a}, \tau) & \mapsto & \rho_{\boldsymbol{a},\tau} \end{array}$$

is $\mathcal{C}^2$ Fréchet differentiable.

The proof of Proposition 5.21 relies on (5.28) and on Lemma 5.24 below, which states that the function $(\boldsymbol{a}, \tau) \mapsto \pi_{\boldsymbol{a},\tau}$ is $\mathcal{C}^2$. Recall Notation 5.11

$$C^{0,0}_{2\pi} := \{u \in C^0_{2\pi} | \int_0^{2\pi} u(s)ds = 0\}.$$

Let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}$ and $\tau > 0$. Because $\int_0^{2\pi} \pi_{\boldsymbol{a},\tau}(u)du = 1$, the space $C^0_{2\pi}$ can be decomposed in the following way

$$C^0_{2\pi} = \mathrm{Span}(\pi_{\boldsymbol{a},\tau}) \oplus C^{0,0}_{2\pi}.$$

We denote by $K^{2\pi}_{\boldsymbol{a},\tau}|_{C^0_{2\pi}}$ the restriction of $K^{2\pi}_{\boldsymbol{a},\tau}$ to $C^0_{2\pi}$ (recall that the linear operator $h \mapsto K^{2\pi}_{\boldsymbol{a},\tau}h$ it defined for all $h \in L^1([0, 2\pi])$). Similarly, we denote by $I|_{C^0_{2\pi}}$ the identity operator on $C^0_{2\pi}$. Given a linear operator L, we denote by $N(L)$ its kernel (null-space) and by $R(L)$ its range.

Lemma 5.22. *Grant Assumptions 4.1 and 4.2, let $\alpha_0 > 0$ such that Assumption 4.6 holds and let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)$, where $\eta_0 > 0$ is given by Lemma 5.13. It holds that*

$$N(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}|_{C^0_{2\pi}}) = \mathrm{Span}(\pi_{\boldsymbol{a},\tau}) \quad \textit{and} \quad R(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}|_{C^0_{2\pi}}) = C^{0,0}_{2\pi}.$$

Proof. We proved in Lemma 5.15 that $N(I - K^{2\pi}_{\boldsymbol{a},\tau}) = \mathrm{Span}(\pi_{\boldsymbol{a},\tau})$. It remains to show that $R(I|_{C^0_{2\pi}} - K^{2\pi}_{\boldsymbol{a},\tau}|_{C^0_{2\pi}}) = C^{0,0}_{2\pi}$. By the Fredholm alternative, we have

$$R(I - K^{2\pi}_{\boldsymbol{a},\tau}) = N(I - (K^{2\pi}_{\boldsymbol{a},\tau})')^{\perp},$$

where $(K^{2\pi}_{\boldsymbol{a},\tau})' \in \mathcal{L}\left(L^\infty([0, 2\pi]); L^\infty([0, 2\pi])\right)$ is the dual operator of $K^{2\pi}_{\boldsymbol{a},\tau} \in \mathcal{L}\left(L^1([0, 2\pi]); L^1([0, 2\pi])\right)$. In the proof of Lemma 5.15, it is shown that

$$\mathbf{1} \in N(I - (K^{2\pi}_{\boldsymbol{a},\tau})'),$$

where $\mathbf{1}$ denotes the constant function equal to 1 on $[0, 2\pi]$. The Fredholm alternative [Bre11, Th. 6.6] yields

$$\dim N(I - (K^{2\pi}_{\boldsymbol{a},\tau})') = \dim N(I - K^{2\pi}_{\boldsymbol{a},\tau}) = 1.$$

So

$$N(I - (K^{2\pi}_{\boldsymbol{a},\tau})') = \mathrm{Span}(\mathbf{1}).$$

It follows that

$$R(I - K^{2\pi}_{\boldsymbol{a},\tau}) = \mathrm{Span}(\mathbf{1})^{\perp} = \{u \in L^1([0, 2\pi]) | \int_0^{2\pi} u(s)ds = 0\}.$$

Finally, using that for $h \in L^1([0, 2\pi])$, one has $K^{2\pi}_{\boldsymbol{a},\tau}h \in C^0_{2\pi}$, one obtains the result for the restrictions to $C^0_{2\pi}$. □

As a consequence, the linear operator $I - K^{2\pi}_{\boldsymbol{a},\tau} : C^{0,0}_{2\pi} \to C^{0,0}_{2\pi}$ is invertible, with a continuous inverse.

Lemma 5.23. *Grant Assumptions 4.1, 4.2 and let $\alpha_0 > 0$ such that Assumption 4.6 holds. Let $\tau_0 > 0$. There exists $\eta_0, \epsilon_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the following function is $\mathcal{C}^2$ Fréchet differentiable*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & \mathcal{L}(C^0_{2\pi}; C^0_{2\pi}) \\ (\boldsymbol{a}, \tau) & \mapsto & H^{2\pi}_{\boldsymbol{a},\tau}. \end{array}$$

The same result holds for $K^{2\pi}_{\boldsymbol{a},\tau}$.

Proof. We only prove the result for H, the proof for K being similar. Let $\epsilon_0 > 0$ be chosen arbitrary such that $\epsilon_0 < \tau_0$.
Step 1. We introduce relevant Banach spaces: E denotes the set of continuous functions

$$E := \mathcal{C}([0,2\pi]^2;\mathbb{R}), \qquad \text{equipped with } ||w||_E := \sup_{t,s} |w(t,s)|$$

$$E_0 := \{w \in E,\ \forall s \in [0,2\pi],\ w(2\pi,s) = w(0,s)\}, \qquad \text{equipped with } ||\cdot||_E.$$

We define the following application Φ,

$$\begin{array}{rcl} E_0 & \to & \mathcal{L}(C^0_{2\pi};C^0_{2\pi}) \\ w & \mapsto & \Phi(w) := \left[h \mapsto \left(\int_0^{2\pi} w(t,s)h(s)ds\right)_{t\in[0,2\pi]}\right]. \end{array}$$

Note that Φ is linear and continuous, so in particular $\mathcal{C}^2$. So, to prove the result, it suffices to show that

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & E_0 \\ (\boldsymbol{a}, \tau) & \mapsto & (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s\in[0,2\pi]^2} \end{array}$$

is $\mathcal{C}^2$, where $H^{2\pi}_{\boldsymbol{a},\tau}(t,s)$ is explicitly given by the series (5.27).
Step 2. Let $k \in \mathbb{N}$ be fixed. We prove that the function

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & E \\ (\boldsymbol{a}, \tau) & \mapsto & (H_{\boldsymbol{a},\tau}(t,s-2\pi k))_{t,s\in[0,2\pi]^2} \end{array}$$

is $\mathcal{C}^2$. To proceed, we use the explicit expression of $H_{\boldsymbol{a},\tau}(t,s)$, given by (5.25). Note that we have first to show that the function $(\boldsymbol{a},\tau) \mapsto \varphi^{\boldsymbol{a},\tau}_{t,s}(0) \in \mathbb{R}$ is $\mathcal{C}^2$. This follows (see [Fle80, Th. 3.10.2]) from the fact that $b : \mathbb{R}_+ \to \mathbb{R}$ is $\mathcal{C}^2$ and so the solution of the ODE (5.25) is $\mathcal{C}^2$ with respect to $\boldsymbol{a}$ and τ. Moreover, we have for all $h \in C^0_{2\pi}$,

$$D_{\boldsymbol{a}}\varphi^{\boldsymbol{a},\tau}_{t,s}(0)\cdot h = \int_s^t \tau h(u) \exp\left(\tau \int_u^t b'(\varphi^{\boldsymbol{a},\tau}_{\theta,s}(0))d\theta\right)du.$$

A similar expression holds for $\frac{d}{d\tau}\varphi^{\boldsymbol{a},\tau}_{t,s}(0)$. Using that f is $\mathcal{C}^2$, we deduce that the function

$$(\boldsymbol{a},\tau) \mapsto (H_{\boldsymbol{a},\tau}(t,s-2\pi k))_{t,s\in[0,2\pi]^2} \in E$$

is $\mathcal{C}^2$ Furthermore, we have for instance

$$D_{\boldsymbol{a}}H_{\boldsymbol{a},\tau}(t,s)\cdot h = -H_{\boldsymbol{a},\tau}(t,s)\int_s^t \tau f'(\varphi^{\boldsymbol{a},\tau}_{u,s}(0))\left[D_{\boldsymbol{a}}\varphi^{\boldsymbol{a},\tau}_{u,s}\cdot h\right]du.$$

So, proceeding as in the proof of Lemma 5.13, we deduce the existence of $\eta_0, \lambda_0, A_0 > 0$ (only depending on b, f, α_0, τ_0 and ϵ_0) such that for all $h \in C^0_{2\pi}$ and for all $\tau \in (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$, it holds that

$$\sup_{t,s\in[0,2\pi]^2}\ \sup_{\boldsymbol{a}\in B^{2\pi}_{\eta_0}(\alpha_0)} |D_{\boldsymbol{a}}H_{\boldsymbol{a},\tau}(t,s-2\pi k)\cdot h| \leq A_0||h||_\infty e^{-2\pi k\lambda_0}.$$

Similar estimates hold for the second derivative with respect to $\boldsymbol{a}$ and for the first and second derivative with respect to τ.
Step 3. We have

$$\sum_{k\geq 0}\ \sup_{t,s\in[0,2\pi]^2}\ \sup_{\boldsymbol{a}\in B^{2\pi}_{\eta_0}(\alpha_0)}\ \sup_{h\in C^0_{2\pi},||h||_\infty\leq 1} |D_{\boldsymbol{a}}H_{\boldsymbol{a},\tau}(t,s-2\pi k)\cdot h| \leq \sum_{k\geq 0} A_0 e^{-2\pi k\lambda_0} < \infty.$$

Using [Car67, Th. 3.6.1], we deduce that $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s} \in E$ is Fréchet differentiable, with for all $h \in C^0_{2\pi}$

$$D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau}(t,s) \cdot h = \sum_{k \geq 0} D_{\boldsymbol{a}} H_{\boldsymbol{a},\tau}(t, s - 2\pi k) \cdot h.$$

Note that this last series converges again normally, and so $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s\in[0,2\pi]^2}$ is in fact $\mathcal{C}^1$. Applying again [Car67, Th. 3.6.1], we prove similarly that $\boldsymbol{a} \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s\in[0,2\pi]^2}$ is $\mathcal{C}^2$. The same arguments show that $\tau \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s\in[0,2\pi]^2}$ is $\mathcal{C}^2$.

Step 4. It remains to prove that $(\boldsymbol{a},\tau) \mapsto (H^{2\pi}_{\boldsymbol{a},\tau}(t,s))_{t,s\in[0,2\pi]^2} \in E_0$ is $\mathcal{C}^2$ (we have proved the result for E, not E_0, in the previous step). Let $t,s \in [0,2\pi]$ be fixed, define

$$w \in E, \quad \mathcal{E}^t_s(w) := w(t,s) \in \mathbb{R}.$$

The application $\mathcal{E}^t_s$ is linear and continuous. Moreover, we have seen that $H^{2\pi}_{\boldsymbol{a},\tau} \in E_0$, so

$$\forall s \in [0,2\pi], \quad \mathcal{E}^{2\pi}_s(H^{2\pi}_{\boldsymbol{a},\tau}) = \mathcal{E}^0_s(H^{2\pi}_{\boldsymbol{a},\tau}).$$

Differentiating with respect to $\boldsymbol{a}$, we deduce that for all $h \in C^0_{2\pi}$,

$$\forall s \in [0,2\pi], \quad \mathcal{E}^{2\pi}_s(D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \cdot h) = \mathcal{E}^0_s(D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \cdot h),$$

and so $D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau} \in \mathcal{L}(C^0_{2\pi}, E_0)$. The same results holds for the second derivative with respect to $\boldsymbol{a}$ and the two derivatives with respect to τ. This ends the proof. □

Lemma 5.24. *Grant Assumptions 4.1, 4.2 and let $\alpha_0 > 0$ such that Assumption 4.6 holds. Let $\tau_0 > 0$. There are $\epsilon_0, \eta_0 > 0$ small enough (only depending on b, f, α_0 and τ_0) such that the function*

$$\begin{array}{rcl} B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & C^0_{2\pi} \\ (\boldsymbol{a}, \tau) & \mapsto & \pi_{\boldsymbol{a},\tau} \end{array}$$

is $\mathcal{C}^2$ Fréchet differentiable.

Remark 5.25. *Recall that $\pi_{\boldsymbol{a},\tau}$ is the unique invariant measure of the Markov Chain having $K^{2\pi}_{\boldsymbol{a},\tau}$ has kernel transition probability. So, we study the smoothness of the invariant measure with respect to the parameters $(\boldsymbol{a},\tau)$, knowing the smoothness of the transition probability kernel $(\boldsymbol{a},\tau) \mapsto K^{2\pi}_{\boldsymbol{a},\tau}$. We refer to [GM86] for such sensibility result in the setting of finite discrete-time Markov Chains. Our approach is different and based on the implicit function theorem. In this proof, we consider independent functions $\boldsymbol{a}$ and h (that is we do not have $\boldsymbol{a} = \alpha_0 + h$).*

Proof. Let α_0 and τ_0 be fixed. Let $\delta_0, \epsilon_0 > 0$ be given by Lemma 5.23. Consider the following $\mathcal{C}^2$-Fréchet differentiable function:

$$\begin{array}{rrcl} F: & C^{0,0}_{2\pi} \times B^{2\pi}_{\eta_0}(\alpha_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) & \to & C^{0,0}_{2\pi} \\ & (h, \boldsymbol{a}, \tau) & \mapsto & (\alpha_0 + h) - K^{2\pi}_{\boldsymbol{a},\tau}(\alpha_0 + h). \end{array}$$

It holds that $F(0, \alpha_0, \tau_0) = 0$. Moreover

$$D_h F(0, \alpha_0, \tau_0) = I - K^{2\pi}_{\alpha_0,\tau_0} \in \mathcal{L}(C^{0,0}_{2\pi}, C^{0,0}_{2\pi}),$$

which is invertible with continuous inverse by Lemma 5.22. So the implicit function theorem applies: there exists $(V^{0,0}_{2\pi}, V^0_{2\pi}, V_{\tau_0})$ open neighborhoods of $(0, \alpha_0, \tau_0)$ in $C^{0,0}_{2\pi} \times C^0_{2\pi} \times \mathbb{R}^*_+$ and a $\mathcal{C}^2$-Fréchet differentiable function $U : V^0_{2\pi} \times V_{\tau_0} \to V^{0,0}_{2\pi}$ such that

$$\forall h, \boldsymbol{a}, \tau \in V^{0,0}_{2\pi} \times V^0_{2\pi} \times V_{\tau_0}, \quad F(h, \boldsymbol{a}, \tau) = 0 \Longleftrightarrow h = U(\boldsymbol{a}, \tau).$$

By uniqueness of the invariant measure of the Markov chain with transition kernel $K^{2\pi}_{\boldsymbol{a},\tau}$, we deduce that

$$\pi_{\boldsymbol{a},\tau} = \alpha_0 + U(\boldsymbol{a}, \tau),$$

which is a $\mathcal{C}^2$-Fréchet differentiable function of $(\boldsymbol{a}, \tau)$. □

Proof of Proposition 5.21. Recall that $\rho_{\boldsymbol{a},\tau} = \frac{\pi_{\boldsymbol{a},\tau}}{c_{\boldsymbol{a},\tau}}$, where the constant $c_{\boldsymbol{a},\tau}$ is given by

$$c_{\boldsymbol{a},\tau} = \tau H^{2\pi}_{\boldsymbol{a},\tau}(\pi_{\boldsymbol{a},\tau}).$$

Furthermore, it holds that $\pi_{\alpha_0,\tau_0} = \frac{1}{2\pi}$ and $\rho_{\alpha_0,\tau_0} = \gamma(\alpha_0)$ (see (5.24)). So $c_{\alpha_0,\tau_0} = \frac{1}{2\pi\gamma(\alpha_0)} > 0$. Hence for ϵ_0, η_0 small enough, it holds that

$$\forall \boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0), \forall \tau \in (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0), \quad c_{\boldsymbol{a},\tau} > 0.$$

So, using Lemmas 5.23 and 5.24, it holds that c and ρ are $\mathcal{C}^2$, which ends the proof. □

As a first application of this result, we prove that the mean number of spikes of a neuron driven by a periodic input only depends on the mean of the input current.

Proposition 5.26. *Grant Assumptions 4.1, 4.2 and let $\alpha_0 > 0$ such that Assumption 4.6 holds. Let $\tau_0 > 0$ and consider η_0 be given by Proposition 5.21. Let $h \in C^{0,0}_{2\pi}$ such that $\alpha_0 + h \in B^{2\pi}_{\eta_0}(\alpha_0)$. It holds that*

$$c_{\alpha_0+h,\tau_0} = c_{\alpha_0,\tau_0} = \frac{1}{2\pi\gamma(\alpha_0)}.$$

We denote by c_{α_0} this last quantity. In particular, the mean number of spikes per period

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha_0+h,\tau_0}(u)du = \gamma(\alpha_0)$$

only depends on α_0 (which is the mean of the external current $(\alpha_0 + h(t))_{t\in[0,2\pi]}$).

Proof. Let $\boldsymbol{a} \in B^{2\pi}_{\eta_0}(\alpha_0)$. We prove that

$$\forall h \in C^{0,0}_{2\pi}, \quad D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = 0.$$

We have $c_{\boldsymbol{a},\tau_0} = \tau_0 H^{2\pi}_{\boldsymbol{a},\tau_0}(\pi_{\boldsymbol{a},\tau_0})$. Differentiating with respect to $\boldsymbol{a}$, one gets

$$D_{\boldsymbol{a}} c_{\boldsymbol{a},\tau_0} \cdot h = \tau_0 \left[D_{\boldsymbol{a}} H^{2\pi}_{\boldsymbol{a},\tau_0} \cdot h\right](\pi_{\boldsymbol{a},\tau_0}) + \tau_0 H^{2\pi}_{\boldsymbol{a},\tau_0} D_{\boldsymbol{a}} \pi_{\boldsymbol{a},\tau_0} \cdot h.$$

Recall that $\pi_{\boldsymbol{a},\tau_0} = K^{2\pi}_{\boldsymbol{a},\tau_0}\pi_{\boldsymbol{a},\tau_0}$ so

$$D_{\boldsymbol{a}}\pi_{\boldsymbol{a},\tau_0}\cdot h = \left[D_{\boldsymbol{a}}K^{2\pi}_{\boldsymbol{a},\tau_0}\cdot h\right]\pi_{\boldsymbol{a},\tau_0} + K^{2\pi}_{\boldsymbol{a},\tau_0}\left[D_{\boldsymbol{a}}\pi_{\boldsymbol{a},\tau_0}\cdot h\right].$$

Using Lemma 5.22, one has

$$D_{\boldsymbol{a}}\pi_{\boldsymbol{a},\tau_0}\cdot h = \left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]^{-1}\left[D_{\boldsymbol{a}}K^{2\pi}_{\boldsymbol{a},\tau_0}\cdot h\right]\pi_{\boldsymbol{a},\tau_0}. \tag{5.29}$$

Define on $C^{0,0}_{2\pi}$ the linear operator

$$\forall h \in C^{0,0}_{2\pi}, \quad \mathbb{1}^{2\pi}(h)(t) := \int_0^{2\pi}\mathbb{1}_{\{t\geq s\}}h(s)ds = \int_0^t h(s)ds.$$

We have

$$1 * K_{\boldsymbol{a},\tau_0} = 1 - H_{\boldsymbol{a},\tau_0}, \tag{5.30}$$

so on $C^{0,0}_{2\pi}$,

$$H^{2\pi}_{\boldsymbol{a},\tau_0} = \mathbb{1}^{2\pi}\left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]. \tag{5.31}$$

So

$$H^{2\pi}_{\boldsymbol{a},\tau_0}\left[I - K^{2\pi}_{\boldsymbol{a},\tau_0}\right]^{-1} = \mathbb{1}^{2\pi}.$$

Consequently, we have

$$D_{\boldsymbol{a}}c_{\boldsymbol{a},\tau_0}\cdot h = \tau_0\left[D_{\boldsymbol{a}}H^{2\pi}_{\boldsymbol{a},\tau_0}\cdot h\right](\pi_{\boldsymbol{a},\tau_0}) + \tau_0\mathbb{1}^{2\pi}\left[D_{\boldsymbol{a}}K^{2\pi}_{\boldsymbol{a},\tau_0}\cdot h\right]\pi_{\boldsymbol{a},\tau_0}$$

Differentiating (5.31), one has

$$D_{\boldsymbol{a}}H^{2\pi}_{\boldsymbol{a},\tau_0}\cdot h = -\mathbb{1}^{2\pi}\left[D_{\boldsymbol{a}}K^{2\pi}_{\boldsymbol{a},\tau_0}\cdot h\right],$$

and so for all $h \in C^{0,0}_{2\pi}$, $D_{\boldsymbol{a}}c_{\boldsymbol{a},\tau_0}\cdot h = 0$. Then for all $h \in C^{0,0}_{2\pi}$ such that $\alpha_0 + h \in B^{2\pi}_{\eta_0}(\alpha_0)$, one has

$$c_{\alpha_0+h,\tau_0} - c_{\alpha_0,\tau_0} = \int_0^1\left[D_{\boldsymbol{a}}c_{\alpha_0+th,\tau_0}\cdot h\right]dt = 0.$$

Finally we have $\pi_{\alpha_0,\tau_0} = \frac{1}{2\pi}$ and, by (5.24), $\rho_{\alpha_0,\tau_0} = \gamma(\alpha_0)$. By definition (5.28), we have $c_{\alpha_0,\tau_0} = \frac{\pi_{\alpha_0,\tau_0}}{\rho_{\alpha_0,\tau_0}}$. It ends the proof. □

5.3.6 Strategy to handle the nonlinear equation (1.2)

Grant Assumptions 4.1, 4.2 and let $\alpha_0 > 0$ such that Assumption 4.6 holds. Let $\tau_0 > 0$ be given by Assumption 5.3. For $\eta_0, \epsilon_0 > 0$, define $G : B^{2\pi}_{\eta_0}(\alpha_0)\cap C^{0,0}_{2\pi}\times(\alpha_0-\eta_0,\alpha_0+\eta_0)\times(\tau_0-\epsilon_0,\tau_0+\epsilon_0)\to C^{0,0}_{2\pi}$ such that

$$G(h,\alpha,\tau) := (\alpha+h) - J(\alpha)\rho_{\alpha+h,\tau}. \tag{5.32}$$

Using Propositions 5.21 and 5.26, we choose η_0, ϵ_0 small enough such that G is $\mathcal{C}^2$-Fréchet differentiable and indeed takes values in $C^{0,0}_{2\pi}$. For any constant $\alpha, \tau > 0$, we have, by (5.24), $\rho_{\alpha,\tau} = \gamma(\alpha)$. Recalling that $J(\alpha)\gamma(\alpha) = \alpha$, we have

$$\forall(\alpha,\tau)\in(\alpha_0-\eta_0,\alpha_0+\eta_0)\times(\tau_0-\epsilon_0,\tau_0+\epsilon_0), \quad G(0,\alpha,\tau) = 0. \tag{5.33}$$

Those are the trivial roots of G. To construct the periodic solutions to (1.2), we find the non-trivial roots of G. In fact, Theorem 5.9 is deduced from the following proposition.

Proposition 5.27. *Consider b, f and $\alpha_0, \tau_0 > 0$ such that Assumptions 4.1, 4.2, 4.6, 5.3, 5.4 and 5.7 hold. Let G be defined by (5.32). There exists $X \times V_{\alpha_0} \times V_{\tau_0}$ an open neighborhood of $(0, \alpha_0, \tau_0)$ in $(C^{0,0}_{2\pi}, ||\cdot||_\infty) \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that:*

1. *There exists a continuous curve $\{(h_v, \alpha_v, \tau_v),\ v \in (-v_0, v_0)\}$ of real 2π-periodic solutions of (5.3) passing through $(0, \alpha_0, \tau_0)$ at $v = 0$ and such that for all $v \in (-v_0, v_0)$*
$$(h_v, \alpha_v, \tau_v) \in X \times V_{\alpha_0} \times V_{\tau_0} \quad \text{and} \quad G(h_v, \alpha_v, \tau_v) = 0.$$
Moreover, it holds that
$$\forall v \in (-v_0, v_0), \quad \frac{1}{2\pi}\int_0^{2\pi} h_v(t)\cos(t)dt = v \quad \text{and} \quad \frac{1}{2\pi}\int_0^{2\pi} h_v(t)\sin(t)dt = 0.$$
In particular, $h_v \not\equiv 0$ for $v \neq 0$.
2. *For all $(h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0}$, with $h \not\equiv 0$, it holds that*
$$G(h, \alpha, \tau) = 0 \iff [\exists v \in (-v_0, v_0), \exists \theta \in [0, 2\pi), \quad (h, \alpha, \tau) \equiv (S_\theta(h_v), \alpha_v, \tau_v)].$$

We here prove that our main result is a consequence of this proposition.

Proof that Proposition 5.27 implies Theorem 5.9. Let (h_v, α_v, τ_v) be the continuous curve given by Proposition 5.27. Define $\boldsymbol{a}_v$
$$\forall t \in \mathbb{R}, \quad a_v(t) := \alpha_v + h_v(t/\tau_v).$$
The function $\boldsymbol{a}_v$ is $2\pi\tau_v$-periodic and continuous. From $G(h_v, \alpha_v, \tau_v) = 0$, we deduce that
$$\boldsymbol{a}_v = J(\alpha_v)\rho_{\boldsymbol{a}_v}.$$
Consider $\tilde{\nu}_{\boldsymbol{a}_v}$ defined by (5.22). By Proposition 5.19, $(\tilde{\nu}_{\boldsymbol{a}_v}(t))$ is a $2\pi\tau_v$-periodic solution of (1.2) and $(\tilde{\nu}_{\boldsymbol{a}_v}, \alpha_v, \tau_v)$ satisfies all the properties stated in Theorem 5.9: this gives the existence part of the proof. We now prove uniqueness.

Let $\epsilon_0 > 0$ small enough such that $(\tau_0 - \epsilon_0, \tau_0 + \epsilon_0) \subset V_{\tau_0}$, V_{τ_0} being given by Proposition 5.27. Let $J, \tau > 0$ be fixed, consider $\nu(t)$ a $2\pi\tau$-periodic solution of (1.2) such that
$$|\tau - \tau_0| < \epsilon_0 \quad \text{and} \quad \sup_{t \in [0, 2\pi\tau]} \left| J \int_{\mathbb{R}_+} f(x)\nu(t, dx) - \alpha_0 \right| < \epsilon_1,$$
for some constant $\epsilon_1 > 0$ to be specified later. Define $\boldsymbol{a}$
$$\forall t \in \mathbb{R}, \quad a(t) := J\int_{\mathbb{R}_+} f(x)\nu(t, dx).$$
The function $\boldsymbol{a}$ is $2\pi\tau$-periodic. Let $(X_t)_{t \geq 0}$ be the solution of the nonlinear equation (1.2), starting with the initial condition $\nu(0) \in \mathcal{M}(f^2)$. Under Assumptions 4.1 and 4.2, Theorem 2.8 applies and so the function $t \mapsto \mathbb{E}\, f(X_t)$ is continuous. So $\boldsymbol{a} \in C^0_{2\pi\tau}$. We write
$$a(t) =: \alpha + h(t/\tau),$$

for some constant α and some $h \in C^{0,0}_{2\pi}$. Because $\nu(t)$ is a periodic solution of (1.2), it holds that

$$\boldsymbol{a} = J\rho_{\boldsymbol{a}},$$

or equivalently,

$$\alpha + h = J\rho_{\alpha+h,\tau}. \tag{5.34}$$

We have by assumption

$$|\alpha - \alpha_0| = \left| \frac{1}{2\pi}\int_0^{2\pi} J \int_{\mathbb{R}_+} f(x)\nu(\tau u, dx)du - \frac{1}{2\pi}\int_0^{2\pi} J(\alpha_0)\int_{\mathbb{R}_+} f(x)\nu^\infty_{\alpha_0}(dx)du \right| < \epsilon_1.$$

Recall that α_0 satisfies Assumption 4.6. By Lemma 5.12 and using the continuity of b', we can assume that ϵ_1 is small enough such that Assumption 4.6 is also satisfied by α. Let η_0 be given by Proposition 5.21 (η_0 only depends on b, f, α_0 and τ_0). Provided that $\epsilon_1 \leq \eta_0$, we can apply Proposition 5.26 at (α, τ). It holds that

$$\frac{1}{2\pi}\int_0^{2\pi} \rho_{\alpha+h,\tau}(u)du = \gamma(\alpha),$$

so

$$\alpha = J\gamma(\alpha).$$

This proves that $J = J(\alpha)$. So (5.34) implies that $G(h, \alpha, \tau) = 0$. By the uniqueness part of Proposition 5.27, there exists $\theta \in [0, 2\pi)$ and $v \in (-v_0, v_0)$ such that

$$\forall t, \quad h(t) = h_v(t+\theta), \quad \alpha = \alpha_v, \quad \tau = \tau_v.$$

So, we deduce that $a(t) = \alpha_v + h_v\left(\frac{t+\theta}{\tau_v}\right)$ and $J = J(\alpha_v)$. This ends the proof. $\square$

It remains to prove Proposition 5.27.

5.3.7 Linearization of G.

Define:

$$\forall t \in \mathbb{R}, \quad \Theta_{\alpha,\tau}(t) := \tau\Theta_\alpha(\tau t)\mathbb{1}_{\{t \geq 0\}}, \tag{5.35}$$

where Θ_α is given by (4.8). The main result of this section is the following.

Proposition 5.28. *Let $h \in C^{0,0}_{2\pi}$. It holds that*

$$[D_h G(0, \alpha, \tau) \cdot h](t) = h(t) - J(\alpha)\int_{\mathbb{R}} \Theta_{\alpha,\tau}(t-s)h(s)ds.$$

The proof of this proposition relies on Lemmas 5.29 and 5.30 below. Let $h \in C^{0,0}_{2\pi}$, it holds that

$$D_h G(0, \alpha, \tau) \cdot h = h - J(\alpha) D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h.$$

By eq. (5.28) and Proposition 5.26, one has

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau}\cdot h = \frac{1}{c_\alpha} D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h.$$

To compute $D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h$, we use (5.29) with $\boldsymbol{a}\equiv\alpha$:

$$D_{\boldsymbol{a}}\pi_{\alpha,\tau}\cdot h = (I - K^{2\pi}_{\alpha,\tau})^{-1}\left[D_{\boldsymbol{a}}K^{2\pi}_{\alpha,\tau}\cdot h\right](\tfrac{1}{2\pi}). \tag{5.36}$$

The next lemma is devoted to the computation of $(I - K^{2\pi}_{\alpha,\tau})^{-1}$. Consider $t\mapsto r_\alpha(t)$ the solution of the convolution Volterra integral equation (1.14) (with $\nu=\delta_0$ and $\boldsymbol{a}=\alpha$). That is, r_α solves $r_\alpha = K_\alpha + K_\alpha * r_\alpha$. By Proposition 3.25 there exists a function $\xi_\alpha\in L^1(\mathbb{R}_+)$ such that for all $t\geq 0$,

$$r_\alpha(t) = \gamma(\alpha) + \xi_\alpha(t).$$

Define for all $t\geq 0$, $r_{\alpha,\tau}(t) := \tau r_\alpha(\tau t)$. It solves

$$r_{\alpha,\tau} = K_{\alpha,\tau} + K_{\alpha,\tau} * r_{\alpha,\tau}, \tag{5.37}$$

where $K_{\alpha,\tau}$ is given by (5.25). Similarly, let $\xi_{\alpha,\tau}(t) := \tau\xi_\alpha(\tau t)$. We have

$$r_{\alpha,\tau}(t) = \tau\gamma(\alpha) + \xi_{\alpha,\tau}(t).$$

Recall that by definition, we have

$$K^{2\pi}_{\alpha,\tau}(h)(t) = \int_0^{2\pi} K^{2\pi}_{\alpha,\tau}(t,s)h(s)ds = \int_{-\infty}^t K_{\alpha,\tau}(t-s)h(s)ds.$$

It holds that

Lemma 5.29. *The inverse of the linear operator $I - K^{2\pi}_{\alpha,\tau} : C^{0,0}_{2\pi}\to C^{0,0}_{2\pi}$ is given by $I + r^{2\pi}_{\alpha,\tau}$ where for all $h\in C^{0,0}_{2\pi}$ and $t\in[0,2\pi]$*

$$\begin{aligned} r^{2\pi}_{\alpha,\tau}(h) &:= \tau\gamma(\alpha)\Gamma(h) + \xi^{2\pi}_{\alpha,\tau}(h),\\ \Gamma(h)(t) &:= \int_0^t h(s)ds - \frac{1}{2\pi}\int_0^{2\pi}\int_0^s h(u)duds,\\ \xi^{2\pi}_{\alpha,\tau}(h)(t) &:= \int_{-\infty}^t \xi_{\alpha,\tau}(t-s)h(s)ds. \end{aligned}$$

Proof. Note that $\Gamma(h)$ is the only primitive of h which belongs to $C^{0,0}_{2\pi}$. Moreover, because $t\mapsto\xi_{\alpha,\tau}(t)\in L^1(\mathbb{R}_+)$, we have for $h\in C^{0,0}_{2\pi}$:

$$\int_0^{2\pi}\int_{-\infty}^t \xi_{\alpha,\tau}(t-s)h(s)dsdt = \int_0^{2\pi}\int_0^\infty \xi_{\alpha,\tau}(u)h(t-u)dudt = \int_0^\infty\int_0^{2\pi}\xi_{\alpha,\tau}(u)h(t-u)dtdu = 0.$$

So, $\xi^{2\pi}_{\alpha,\tau}(h)\in C^{0,0}_{2\pi}$ and $r^{2\pi}_{\alpha,\tau}$ is well-defined. To conclude, we have to show that on $C^{0,0}_{2\pi}$

$$K^{2\pi}_{\alpha,\tau}\circ r^{2\pi}_{\alpha,\tau} = r^{2\pi}_{\alpha,\tau}\circ K^{2\pi}_{\alpha,\tau} = r^{2\pi}_{\alpha,\tau} - K^{2\pi}_{\alpha,\tau}.$$

Note that for all $t \in [0, 2\pi]$,

$$\frac{d}{dt}\left[\Gamma(h)(t) - H^{2\pi}_{\alpha,\tau}(h)(t)\right] = K^{2\pi}_{\alpha,\tau}(h)(t).$$

Because $\Gamma(h), H^{2\pi}_{\alpha,\tau}(h) \in C^{0,0}_{2\pi}$, we deduce that

$$\Gamma(K^{2\pi}_{\alpha,\tau}(h)) = \Gamma(h) - H^{2\pi}_{\alpha,\tau}.$$

Moreover, we have (using that $\xi_{\alpha,\tau}, K_{\alpha,\tau} \in L^1(\mathbb{R}_+)$)

$$\begin{aligned}
\xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h))(t) &= \int_{-\infty}^{t} \xi_{\alpha,\tau}(t-s) \int_{-\infty}^{s} K_{\alpha,\tau}(s-u)h(u)duds \\
&= \int_{-\infty}^{t} h(u) \int_{u}^{t} \xi_{\alpha,\tau}(t-s)K_{\alpha,\tau}(s-u)dsdu \\
&= \int_{-\infty}^{t} h(u)(\xi_{\alpha,\tau} * K_{\alpha,\tau})(t-u)du.
\end{aligned}$$

Using (5.30) and (5.37), we deduce the identity

$$K_{\alpha,\tau} * \xi_{\alpha,\tau} = \xi_{\alpha,\tau} * K_{\alpha,\tau} = \xi_{\alpha,\tau} - K_{\alpha,\tau} + \tau\gamma(\alpha)H_{\alpha,\tau}. \tag{5.38}$$

So

$$\xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h)) = \xi^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h) + \tau\gamma(\alpha)H^{2\pi}_{\alpha,\tau}(h).$$

Altogether,

$$r^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h)) = r^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h).$$

We now prove that $K^{2\pi}_{\alpha,\tau}(r^{2\pi}_{\alpha,\tau}(h)) = r^{2\pi}_{\alpha,\tau}(h) - K^{2\pi}_{\alpha,\tau}(h)$. Using (5.38), we have $K^{2\pi}_{\alpha,\tau}(\xi^{2\pi}_{\alpha,\tau}(h)) = \xi^{2\pi}_{\alpha,\tau}(K^{2\pi}_{\alpha,\tau}(h))$. Moreover, because $K^{2\pi}_{\alpha,\tau}(1) = 1$, we have

$$\begin{aligned}
K^{2\pi}_{\alpha,\tau}(\Gamma(h))(t) &= \int_{-\infty}^{t} K_{\alpha,\tau}(t-s) \int_{0}^{s} h(u)duds - \frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{s} h(u)duds \\
&= \left[H_{\alpha,\tau}(t-s)\int_{0}^{s} h(u)du\right]_{-\infty}^{t} - \int_{-\infty}^{t} H_{\alpha,\tau}(t-s)h(s)ds - \frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{s} h(u)duds \\
&= \Gamma(h)(t) - H^{2\pi}_{\alpha,\tau}(h)(t) = \Gamma(K^{2\pi}_{\alpha,\tau}(h))(t).
\end{aligned}$$

It ends the proof. □

So for all $\alpha, \tau \in (\alpha_0 - \eta_0, \alpha_0 + \eta_0) \times (\tau_0 - \epsilon_0, \tau_0 + \epsilon_0)$ and $h \in C^{0,0}_{2\pi}$, it holds that

$$D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h = \frac{1}{c_{\alpha,\tau}} D_{\boldsymbol{a}}\pi_{\alpha,\tau} \cdot h \overset{(5.36)}{=} (I + r^{2\pi}_{\alpha,\tau})\left[D_{\boldsymbol{a}}K^{2\pi}_{\alpha,\tau} \cdot h\right](\gamma(\alpha)). \tag{5.39}$$

Define for all $t \geq 0$, $\Xi_{\alpha,\tau}(t) := \tau\Xi_\alpha(\tau t)$ and denote by $\Xi^{2\pi}_{\alpha,\tau}$ the linear operator

$$\forall h \in C^0_{2\pi}, \forall t \in [0, 2\pi], \quad \Xi^{2\pi}_{\alpha,\tau}(h)(t) := \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u)h(u)du.$$

Lemma 5.30. *For all* $h \in C^0_{2\pi}$ *we have* $\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau} \cdot h\right](\gamma(\alpha)) = \Xi^{2\pi}_{\alpha,\tau}(h)$.

Proof. Given $h \in C^0_{2\pi}$, we have

$$\left[D_{\boldsymbol{a}} K^{2\pi}_{\alpha,\tau} \cdot h\right](\gamma(\alpha))(t) = \gamma(\alpha) \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} K_{\alpha,\tau} \cdot h\right](t,s) ds$$

So we have to prove that

$$\forall h \in C^0_{2\pi}, \quad \gamma(\alpha) \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} K_{\alpha,\tau} \cdot h\right](t,s) ds = \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-s) h(s) ds.$$

When $\tau = 1$, this computation is done in Lemma 4.53. It is first proved that

$$\gamma(\alpha) \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} H_{\alpha} \cdot h\right](t,s) ds = -\int_{\mathbb{R}} \Psi_{\alpha}(t-s) h(s) ds,$$

where $\Psi_\alpha(t)$ is given by (4.30). This computation relies on the explicit expression satisfied by $\left[D_{\boldsymbol{a}} H_{\alpha} \cdot h\right](t,s)$, namely

$$\left[D_{\boldsymbol{a}} H_{\alpha} \cdot h\right](t,s) = H_\alpha(t-s) \int_s^t f'(\varphi^\alpha_{u-s}(0)) \int_s^u h(\theta) \exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0)) dv\right) d\theta du.$$

Using Fubini's Theorem and the identity

$$\int_\theta^t f'(\varphi^\alpha_{u-s}(0)) \exp\left(\int_\theta^u b'(\varphi^\alpha_{v-s}(0)) dv\right) du = \frac{f(\varphi^\alpha_{t-s}(0)) - f(\varphi^\alpha_{\theta-s}(0))}{b(\varphi^\alpha_{\theta-s}(0)) + \alpha}$$

lead to the convolution between Ψ_α and h. We refer to Lemma 4.53 for more details. Then one uses that

$$\int_{-\infty}^{t} \left[D_{\boldsymbol{a}} K_{\alpha} \cdot h\right](t,s) ds = -\frac{d}{dt} \int_{-\infty}^{t} \left[D_{\boldsymbol{a}} H_{\alpha} \cdot h\right](t,s) ds$$

and that $\Xi_\alpha(t) = \frac{d}{dt}\Psi_\alpha(t)$ to obtain the stated identity with $\tau = 1$. The result for $\tau \neq 1$ can be deduced from the case $\tau = 1$. Indeed, given $\alpha > 0$ and $h \in C^0_{2\pi}$, define $\tilde{f} := \tau f$, $\tilde{b} := \tau b$, $\tilde{\alpha} := \tau\alpha$, and $\tilde{h} := \tau h$. By applying the result for $\tilde{\tau} := 1$, $\tilde{b}$, $\tilde{f}$, $\tilde{\alpha}$ and $\tilde{h}$, we obtain exactly the stated equality. □

Proof of Proposition 5.28. We use Lemma 5.30 together with (5.39). For all $h \in C^{0,0}_{2\pi}$, one obtains

$$D_{\boldsymbol{a}} \rho_{\alpha,\tau} \cdot h = \Xi^{2\pi}_{\alpha,\tau}(h) + r^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)).$$

The definition of $r^{2\pi}_{\alpha,\tau}$ yields

$$r^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)) = \tau\gamma(\alpha)\Gamma(\Xi^{2\pi}_{\alpha,\tau}(h)) + \xi^{2\pi}_{\alpha,\tau}(\Xi^{2\pi}_{\alpha,\tau}(h)).$$

Let $\Psi_{\alpha,\tau}(t) := \Psi_\alpha(\tau t)$, such that $\frac{d}{dt}\Psi_{\alpha,\tau}(t) = \Xi_{\alpha,\tau}(t)$. From the identity

$$\frac{d}{dt} \int_{-\infty}^{t} \Psi_{\alpha,\tau}(t-u) h(u) du = \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u) h(u) du,$$

we find that

$$\Gamma(\Xi_{\alpha,\tau}^{2\pi}(h))(t) = \int_{-\infty}^{t} \Psi_{\alpha,\tau}(t-u)h(u)du = \int_{-\infty}^{t} (1 * \Xi_{\alpha,\tau})(t-u)h(u)du.$$

So

$$\begin{aligned}\left[D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h\right](t) = & \int_{-\infty}^{t} \Xi_{\alpha,\tau}(t-u)h(u)du + \tau\gamma(\alpha)\int_{-\infty}^{t} (1 * \Xi_{\alpha,\tau})(t-u)h(u)du \\ & + \int_{-\infty}^{t} \xi_{\alpha,\tau}(t-u)\int_{-\infty}^{u} \Xi_{\alpha,\tau}(u-\theta)h(\theta)d\theta du.\end{aligned}$$

Fubini's Theorem yields

$$\int_{-\infty}^{t} \xi_{\alpha,\tau}(t-u)\int_{-\infty}^{u} \Xi_{\alpha,\tau}(u-\theta)h(\theta)d\theta du = \int_{-\infty}^{t} (\xi_{\alpha,\tau} * \Xi_{\alpha,\tau})(t-\theta)h(\theta)d\theta.$$

Finally, we have

$$\begin{aligned}\Xi_{\alpha,\tau} + \tau\gamma(\alpha)(1 * \Xi_{\alpha,\tau}) + \xi_{\alpha,\tau} * \Xi_{\alpha,\tau} &= \Xi_{\alpha,\tau} + r_{\alpha,\tau} * \Xi_{\alpha,\tau} \qquad (\text{because } r_{\alpha,\tau} = \tau\gamma(\alpha) + \xi_{\alpha,\tau}) \\ &\overset{(4.33)}{=} \Theta_{\alpha,\tau},\end{aligned}$$

so

$$\left[D_{\boldsymbol{a}}\rho_{\alpha,\tau} \cdot h\right](t) = \int_{-\infty}^{t} \Theta_{\alpha,\tau}(t-u)h(u)du.$$

It ends the proof. □

5.3.8 The linearization of G at $(0, \alpha_0, \tau_0)$ is a Fredholm operator

For notational convenience we now write

$$B_0 := D_h G(0, \alpha_0, \tau_0).$$

Proposition 5.31. *We have $N(B_0) = R(Q)$, $R(B_0) = N(Q)$, where Q is the following projector on $C_{2\pi}^{0,0}$:*

$$\forall z \in C_{2\pi}^{0,0}, \quad Q(z)(t) := \left[\frac{1}{2\pi}\int_0^{2\pi} z(s)e^{-is}ds\right]e^{it} + \left[\frac{1}{2\pi}\int_0^{2\pi} z(s)e^{is}ds\right]e^{-it}. \tag{5.40}$$

Remark 5.32. *In particular, $B_0 \in \mathcal{L}(C_{2\pi}^{0,0}, C_{2\pi}^{0,0})$ is a Fredholm operator of index 0, with dim $N(B_0) = 2$.*

Proof. First, let $h \in N(B_0)$. One has for all $t \in \mathbb{R}$

$$h(t) = J(\alpha_0)\int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds.$$

Consider for all $n \in \mathbb{Z}$

$$\tilde{h}_n := \frac{1}{2\pi}\int_0^{2\pi} h(s)e^{-ins}ds$$

the n-th Fourier coefficient of h. We have

$$\forall n \in \mathbb{Z}, \quad \tilde{h}_n = J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)\tilde{h}_n.$$

Assumption 5.4 ensures that

$$\forall n \in \mathbb{Z}\backslash\{-1,1\}, \quad J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in) \neq 1,$$

and so

$$\forall n \in \mathbb{Z}\backslash\{-1,1\}, \quad \tilde{h}_n = 0.$$

We deduce that $h \in R(Q)$. Conversely, if $h \in R(Q)$, there exists $c \in \mathbb{C}$ such that

$$h(t) = ce^{it} + \bar{c}e^{-it}$$

and so

$$\begin{aligned} J(\alpha_0)\int_{\mathbb{R}} \Theta_{\alpha_0,\tau_0}(t-s)h(s)ds &= ce^{it}J(\alpha_0)\int_{\mathbb{R}}\Theta_{\alpha_0,\tau_0}(s)e^{-is}ds + \bar{c}e^{-it}J(\alpha_0)\int_{\mathbb{R}}\Theta_{\alpha_0,\tau_0}(s)e^{is}ds \\ &= ce^{it}J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) + \bar{c}e^{-it}J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(-i) \\ &= h(t). \end{aligned}$$

We used here that $J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) = J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(-i) = 1$ (Assumption 5.3). This proves that $N(B_0) = R(Q)$. Consider now $k \in R(B_0)$, there exists $h \in C^0_{2\pi}$ such that $B_0(h) = k$. We have for all $t \in \mathbb{R}$

$$h(t) - J(\alpha_0)\int_{\mathbb{R}}\Theta_{\alpha_0,\tau_0}(t-s)h(s)ds = k(t).$$

Using that $J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i) = 1$, we deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} k(s)e^{-is}ds = \left[\frac{1}{2\pi}\int_0^{2\pi} h(s)e^{-is}ds\right](1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(i)) = 0.$$

Similarly, $\frac{1}{2\pi}\int_0^{2\pi} k(s)e^{is}ds = 0$ and so $k \in N(Q)$. It remains to show that $N(Q) \subset R(B_0)$. Consider $h \in N(Q)$ and let

$$\tilde{h}_n := \frac{1}{2\pi}\int_0^{2\pi} h(s)e^{-ins}ds$$

be its n-th Fourier coefficient. We have $\tilde{h}_1 = \tilde{h}_{-1} = 0$. Define

$$\forall n \in \mathbb{Z}\backslash\{-1,1\}, \quad \epsilon_n := \frac{J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}{1 - J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}.$$

The function h is continuous, and so h belongs to $L^2([0,2\pi])$. We deduce that

$$\sum_{n \in \mathbb{Z}\backslash\{-1,1\}} |\tilde{h}_n|^2 < \infty.$$

Moreover, because $\Theta_{\alpha_0,\tau_0} \in L^1(\mathbb{R}_+)$, the Riemann-Lebesgue lemma yields the existence of a constant C such that for $n \in \mathbb{Z}$,

$$|n| > 1 \implies |\epsilon_n| \leq \frac{C}{|n|}.$$

We deduce that

$$\sum_{n\in\mathbb{Z}\setminus\{-1,1\}} |n\epsilon_n \tilde{h}_n|^2 < \infty.$$

Consequently, defining

$$\forall t \in \mathbb{R}, \quad w(t) := \sum_{n\in\mathbb{Z}\setminus\{-1,1\}} \epsilon_n \tilde{h}_n e^{int},$$

it holds that $w \in H^1([0, 2\pi])$, and so w is continuous (see for instance [Bre11, Th. 8.2]). Finally, let $k := h + w$. It holds that $k \in C^0_{2\pi}$ and the n-th Fourier coefficient of k is equals to $\frac{\tilde{h}_n}{1-J(\alpha_0)\widehat{\Theta}_{\alpha_0,\tau_0}(in)}$. We deduce that $B_0(k) = h$. This ends the proof. □

5.3.9 The Lyapunov-Schmidt reduction method

The problem of finding the roots of G defined by (5.32) is an infinite dimensional problem. We use the method of Lyapunov-Schmidt to obtain an equivalent problem of finite-dimension - here of dimension 2. The equation $G = 0$ is equivalent to

$$\begin{aligned} QG(Qh + (I-Q)h, \alpha, \tau) &= 0 \\ (I-Q)G(Qh + (I-Q)h, \alpha, \tau) &= 0, \end{aligned}$$

where the projector Q is defined by (5.40). Define the following function W:

$$\begin{array}{rrcl} W: & U_2 \times W_2 \times V_{\alpha_0} \times V_{\tau_0} & \to & R(B_0) \\ & (v, w, \alpha, \tau) & \mapsto & (I-Q)G(v + w, \alpha, \tau), \end{array}$$

where $U_2 \times W_2$ are open neighborhood of $(0, 0)$ in $N(B_0) \times R(B_0)$.

We have $W(0, 0, \alpha_0, \tau_0) = 0$ and $D_w W(0, 0, \alpha_0, \tau_0) = (I-Q)D_h G(0, \alpha_0, \tau_0) = (I-Q)B_0 \in \mathcal{L}(R(B_0), R(B_0))$ which is bijective with continuous inverse. The implicit function theorem applies: there exists a $\mathcal{C}^1$ function $\psi : N(B_0) \times V_{\alpha_0} \times V_{\tau_0} \mapsto R(B_0)$ such that

$$\begin{aligned} &W(v, w, \alpha, \tau) = 0 \text{ for } (v, w, \alpha, \tau) \in U_2 \times W_2 \times V_{\alpha_0} \times V_{\tau_0} \text{ is equivalent to} \\ &w = \psi(v, \alpha, \tau). \end{aligned}$$

Again, the neighborhoods $U_2, W_2, V_{\tau_0}, V_{\alpha_0}$ may be shrunk in this construction. We deduce that

$$G(h, \alpha, \tau) = 0 \text{ for } (h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0} \text{ is equivalent to} \tag{5.41}$$
$$QG(Qh + \psi(Qh, \alpha, \tau), \alpha, \tau) = 0. \tag{5.42}$$

Note that for all $\theta \in \mathbb{R}$, we have for all $\tau > 0$ and $\boldsymbol{a} \in C^0_{2\pi}$, $\rho_{S_\theta(\boldsymbol{a}),\tau} = S_\theta(\rho_{\boldsymbol{a},\tau})$. It follows that

$$G(S_\theta(h), \alpha, \tau) = S_\theta(G(h, \alpha, \tau)).$$

Moreover, it is clear that the projection Q commutes with S_θ (for all $\theta \in \mathbb{R}$, $S_\theta Q = QS_\theta$) and by the local uniqueness of the implicit function theorem, we deduce that

$$\psi(S_\theta(v), \alpha, \tau) = S_\theta(\psi(v, \alpha, \tau)).$$

Using that any element $Qh \in N(B_0)$ can be written

$$Qh = t \mapsto ce^{it} + \bar{c}e^{-it} := ce_0 + \bar{c}\bar{e}_0$$

for some $c \in \mathbb{C}$ and using the definition of Q, we deduce that (5.41) is equivalent to the complex equation:

$$\hat{\Phi}(c, \alpha, \tau) = 0 \text{ for } (c, \alpha, \tau) \in V_0 \times V_{\alpha_0} \times V_{\tau_0}, \text{ where}$$

$$\hat{\Phi}(c, \alpha, \tau) := \frac{1}{2\pi} \int_0^{2\pi} G(ce_0 + \bar{c}\bar{e}_0 + \psi(ce_0 + \bar{c}\bar{e}_0, \alpha, \tau), \alpha, \tau)_t e^{-it} dt$$

and V_0 is an open neighborhood of 0 in $\mathbb{C}$. We have moreover

$$\forall \theta \in \mathbb{R}, \quad \hat{\Phi}(ce^{i\theta}, \alpha, \tau) = e^{i\theta} \hat{\Phi}(c, \alpha, \tau),$$

and so (5.41) is equivalent to

$$\hat{\Phi}(v, \alpha, \tau) = 0 \text{ for } v \in (-v_0, v_0).$$

Note that $\hat{\Phi}(-v, \alpha, \tau) = -\hat{\Phi}(v, \alpha, \tau)$ and in particular

$$\forall \alpha, \tau \in V_{\alpha_0} \times V_{\tau_0}, \quad \hat{\Phi}(0, \alpha, \tau) = 0.$$

This is coherent with (5.33). In order to eliminate these trivial solutions, following [Kie12], we set for $v \in (-v_0, v_0) \setminus \{0\}$:

$$\begin{aligned}\tilde{\Phi}(v, \alpha, \tau) :=& \frac{\hat{\Phi}(v, \alpha, \tau)}{v} \\ =& \int_0^1 D_v \hat{\Phi}(\theta v, \alpha, \tau) d\theta.\end{aligned}$$

To summarize, we have proved that

Lemma 5.33. *There exists $v_0 > 0$ and open neighborhoods $X \times V_{\alpha_0} \times V_{\tau_0}$ of $(0, \alpha_0, \tau_0)$ in $C^{0,0}_{2\pi} \times \mathbb{R}^*_+ \times \mathbb{R}^*_+$ such that the problem*

$$G(h, \alpha, \tau) = 0 \textit{ for } (h, \alpha, \tau) \in X \times V_{\alpha_0} \times V_{\tau_0} \textit{ with } h \neq 0$$

is equivalent to

$$\tilde{\Phi}(v, \alpha, \tau) = 0 \textit{ for } (v, \alpha, \tau) \in (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0}.$$

The next section is devoted to the study of this reduced problem.

5.3.10 Study of the reduced 2D-problem

We denote by cos the cosinus function, such that $ve_0 + v\bar{e_0} = 2v\cos$.

Lemma 5.34. *We have:*

1. $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$.
2. $D_\tau\tilde{\Phi}(0, \alpha_0, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi}\left[D^2_{h\tau}G(0, \alpha_0, \tau_0)\cdot 2\cos\right]_t e^{-it}dt$.
3. $D_\alpha\tilde{\Phi}(0, \alpha_0, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi}\left[D^2_{h\alpha}G(0, \alpha_0, \tau_0)\cdot 2\cos\right]_t e^{-it}dt$.

Proof. We have $\tilde{\Phi}(0, \alpha_0, \tau_0) = D_v\hat{\Phi}(0, \alpha_0, \tau_0)$ and

$$D_v\hat{\Phi}(0, \alpha_0, \tau_0) = \frac{1}{2\pi}\int_0^{2\pi} D_hG(0, \alpha_0, \tau_0)\cdot[2\cos + D_v\psi(0, \alpha_0, \tau_0)\cdot 2\cos]_t\, e^{-it}dt.$$

Moreover, it holds that [see Kie12, Coroll. 1.2.4]

$$D_v\psi(0, \alpha_0, \tau_0)\cdot\cos = 0$$

and $\cos \in N(D_hG(0, \alpha_0, \tau_0))$, so $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$. To prove the second point (the third point is proved similarly), we have $D_\tau\tilde{\Phi}(0, \alpha_0, \tau_0) = D^2_{v\tau}\hat{\Phi}(0, \alpha_0, \tau_0)$. Moreover,

$$\begin{aligned} D_\tau\hat{\Phi}(v, \alpha, \tau) =& \frac{1}{2\pi}\int_0^{2\pi} D_\tau G(2v\cos + \psi(2r\cos, \alpha, \tau), \alpha, \tau)_t e^{-it}dt \\ &+ \frac{1}{2\pi}\int_0^{2\pi}[D_hG(2r\cos + \psi(2v\cos, \alpha, \tau), \alpha, \tau)\cdot D_\tau\psi(2v\cos, \alpha, \tau)]_t\, e^{-it}dt. \end{aligned}$$

So

$$\begin{aligned} D^2_{v\tau}\hat{\Phi}(0, \alpha_0, \tau_0) = & \frac{1}{2\pi}\int_0^{2\pi}\left[D^2_{h\tau}G(0, \alpha_0, \tau_0)\cdot(2\cos + D_v\psi(0, \alpha_0, \tau_0)\cdot 2\cos)\right]_t e^{-it}dt \\ &+ \frac{1}{2\pi}\int_0^{2\pi}\left[D_hG(0, \alpha_0, \tau_0)\cdot D^2_{v\tau}\psi(0, \alpha_0, \tau_0)\cdot 2\cos\right]_t e^{-it}dt \\ &+ \frac{1}{2\pi}\int_0^{2\pi} D^2_{hh}G(0, \alpha_0, \tau_0)\cdot[2\cos + D_v\psi(0, \alpha_0, \tau_0)\cdot 2\cos, D_\tau\psi(0, \alpha_0, \tau_0)]_t e^{-it}dt. \end{aligned}$$

Note that for all α, τ in the neighborhood of α_0, τ_0, one has

$$\psi(0, \alpha, \tau) = 0,$$

so $D_\tau\psi(0, \alpha_0, \tau_0) = 0$. Consequently the third term is null. Recall now that $B_0 := D_hG(0, \alpha_0, \tau_0)$ and by Proposition 5.31, it holds that $QB_0 = 0$. So the second term is also null. Finally, using again that $D_v\psi(0, \alpha_0, \tau_0)\cdot\cos = 0$ we obtain the stated formula. $\square$

By Proposition 5.28, we have for all $h \in C^{0,0}_{2\pi}$

$$D_h G(0, \alpha, \tau) \cdot h = h - J(\alpha)\Theta_{\alpha,\tau} * h,$$

where the function $\Theta_{\alpha,\tau}$ is given by eq. (5.35). It follows that

$$D^2_{h\tau} G(0, \alpha_0, \tau_0) \cdot 2\cos = -2J(\alpha_0)\frac{\partial}{\partial \tau} \left(\Theta_{\alpha_0,\tau} * \cos\right)\big|_{\tau=\tau_0},$$

and so we have

$$D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) = -J(\alpha_0)\frac{\partial}{\partial \tau} \widehat{\Theta}_{\alpha_0,\tau}(i)\Big|_{\tau=\tau_0}.$$

Similarly,

$$D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) = -\frac{\partial}{\partial \alpha} \left(J(\alpha)\widehat{\Theta}_{\alpha,\tau_0}(i)\right)\Big|_{\alpha=\alpha_0}.$$

Lemma 5.35. *Write $J(\alpha_0)\frac{\partial}{\partial z}\widehat{\Theta}_{\alpha_0}(\frac{i}{\tau_0}) =: x_0 + iy_0$. It holds that*

1. $D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) = (ix_0 - y_0)/\tau_0^2$.
2. $D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) = \mathfrak{z}_0'(\alpha_0)(x_0 + iy_0)$, *where $\mathfrak{z}_0'(\alpha_0)$ is defined in Lemma 5.6.*

Proof. From $\Theta_{\alpha,\tau}(t) = \tau\Theta_\alpha(\tau t)$, we have

$$\frac{\partial}{\partial \tau}\Theta_{\alpha,\tau}(t) = \tfrac{1}{\tau}\left[\tau\Theta_\alpha(\tau t) + \tau\Pi_\alpha(\tau t)\right], \quad \text{with} \quad \Pi_\alpha(t) := t\frac{\partial}{\partial t}\Theta_\alpha(t).$$

So

$$\left[\widehat{\frac{\partial}{\partial \tau}\Theta_{\alpha,\tau}}\right](z) = \tfrac{1}{\tau}\left[\widehat{\Theta}_\alpha(\tfrac{z}{\tau}) + \widehat{\Pi}_\alpha(\tfrac{z}{\tau})\right].$$

Moreover, an integration by parts shows that

$$\begin{aligned}\widehat{\Pi}_\alpha(z) &= \int_0^\infty e^{-zt} t\frac{\partial}{\partial t}\Theta_\alpha(t)dt \\ &= -\widehat{\Theta}_\alpha(z) + z\int_0^\infty e^{-zt} t\Theta_\alpha(t)dt. \\ &= -\widehat{\Theta}_\alpha(z) - z\frac{\partial}{\partial z}\widehat{\Theta}_\alpha(z).\end{aligned}$$

Choosing $z = i$ ends the proof of the first point. Define now

$$\Delta(z, \alpha) := J(\alpha)\widehat{\Theta}_\alpha(z) - 1.$$

By the definition of $\mathfrak{z}_0(\alpha)$ (see Lemma 5.6), we have

$$\forall \alpha \in V_{\alpha_0}, \quad \Delta(\mathfrak{z}_0(\alpha), \alpha) = 0.$$

We differentiate with respect to α and obtain

$$\frac{\partial}{\partial z}\Delta(\mathfrak{z}_0(\alpha), \alpha)\mathfrak{z}_0'(\alpha) + \frac{\partial}{\partial \alpha}\Delta(\mathfrak{z}_0(\alpha), \alpha) = 0.$$

Evaluating this expression at $\alpha = \alpha_0$ gives

$$\frac{\partial}{\partial \alpha}\left(J(\alpha)\widehat{\Theta}_\alpha\right)\Big|_{\alpha=\alpha_0}(\tfrac{i}{\tau_0}) = -\mathfrak{Z}_0'(\alpha_0)(x_0 + iy_0),$$

which concludes the proof. □

Lemma 5.36. *There exists $v_0 > 0$, $V_{\alpha_0} \times V_{\tau_0}$ an open neighborhood of (α_0, τ_0) in $(\mathbb{R}_+^*)^2$ and two functions $v \mapsto \tau_v, \alpha_v \in \mathcal{C}^1((-v_0, v_0))$ such that for all $(v, \alpha, \tau) \in (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0}$ we have*

$$\tilde{\Phi}(v, \alpha, \tau) = 0 \iff \tau = \tau_v \text{ and } \alpha = \alpha_v.$$

Proof. We decompose $\tilde{\Phi}$ into real part and imaginary part (without changing the notations), such that now

$$\tilde{\Phi} : (-v_0, v_0) \times V_{\alpha_0} \times V_{\tau_0} \to \mathbb{R}^2.$$

We have $\tilde{\Phi}(0, \alpha_0, \tau_0) = 0$ and

$$\begin{aligned} D_{(\alpha,\tau)}\tilde{\Phi}(0, \alpha_0, \tau_0) &= \begin{pmatrix} \Re D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) & \Re D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) \\ \Im D_\alpha \tilde{\Phi}(0, \alpha_0, \tau_0) & \Im D_\tau \tilde{\Phi}(0, \alpha_0, \tau_0) \end{pmatrix} \\ &= \begin{pmatrix} x_0 \Re\mathfrak{Z}_0'(\alpha_0) - y_0 \Im\mathfrak{Z}_0'(\alpha_0) & -\frac{y_0}{\tau_0^2} \\ x_0 \Im\mathfrak{Z}_0'(\alpha_0) + y_0 \Re\mathfrak{Z}_0'(\alpha_0) & \frac{x_0}{\tau_0^2} \end{pmatrix}. \end{aligned}$$

The determinant of this matrix is $\frac{\Re\mathfrak{Z}_0'(\alpha_0)}{\tau_0^2}(x_0^2 + y_0^2)$ and this quantity is non-null by Assumptions 5.3 and 5.7. Consequently, the implicit function theorem applies and gives the result. □

The proof of Proposition 5.27 then follows immediately from this result and Lemma 5.33. This ends the proof of Theorem 5.9.

5.4 Conclusion and perspectives

We study the existence of periodic solutions through Hopf bifurcations. Our main assumptions are of spectral type. We assume that the complex zeros of $z \mapsto J(\alpha)\widehat{\Theta_\alpha}(z) - 1$ "crosses" the purely imaginary axis at some "bifurcation point" α_0. At this point, the invariant measure $\nu_{\alpha_0}^\infty$ losses its stability and we give sufficient condition ensuring the existence of periodic solutions, for α in the neighborhood of α_0. We study in Chapter 6 an example of functions b and f for which all the spectral assumptions can be analytically verified (see Section 6.2). It would be particularly interesting to study the stability of the periodic solutions. That is, if $(\nu(t))_{t\in[0,T]}$ is a periodic solution of (1.2), at which condition this orbit is locally attractive? In Chapter 4, we have seen that stability of an invariant probability measure is given by the location of the roots of an explicit holomorphic function. It would be interesting to study the existence of a similar criteria, giving the stability of such periodic orbit. Another development would be to reduce the dynamics, near an invariant probability measure, to a finite dimensional manifold by applying idea from the center manifold theory (see [HI11]).

Chapter 6

Explicit examples and numerical methods

We study analytically and numerically several examples of functions b and f which exhibit either multi-stability or oscillations. In particular, we give an explicit example where all the spectral assumptions of Chapter 5 can be analytically verified. In addition, we describe and compare two numerical methods to simulate the mean-field equation: an Euler scheme to approximate the particle system (1.1) and a finite volume method to approximate the solution of the Fokker-Planck equation (1.3). Finally we give an algorithm to determine numerically if an invariant measure of (1.2) is locally stable and to numerically predict the Hopf bifurcations.

6.1 Explicit examples with bistability

We give explicit examples of drift b and rate function f such that the nonlinear equation (1.2) admits multiple invariant measures. Given $m \geq 0$, we choose:

$$\forall x \geq 0, \quad b(x) := m - x \quad \text{and} \quad f(x) := x^2.$$

We denote by δ_0 the Dirac measure at 0 and for all $\alpha > 0$, let ν_α^∞ be given by (3.4):

$$\nu_\alpha^\infty(dx) := \frac{\gamma(\alpha)}{m + \alpha - x} \exp\left(-\int_0^x \frac{y^2}{m + \alpha - y} dy\right) \mathbb{1}_{[0, m+\alpha)}(x) dx.$$

We used that with this choice of b, we have $\sigma_\alpha = m + \alpha$. We first study analytically the case $m = 0$.

6.1.1 Case $m = 0$.

The following proposition gives the number of invariant measures of the nonlinear equation (1.2) when $m = 0$. This result was conjectured in [RT16, Section 7.2.3].

Proposition 6.1. *Let $f(x) = x^2$ and $b(x) = -x$. There exists $\alpha_* > 0$ such that the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is decreasing on $(0, \alpha_*]$ and increasing on $[\alpha_*, \infty)$. Moreover, one has*

$$\lim_{\alpha \downarrow 0} \frac{\alpha}{\gamma(\alpha)} = +\infty, \quad \textit{and} \quad \lim_{\alpha \to \infty} \frac{\alpha}{\gamma(\alpha)} = +\infty.$$

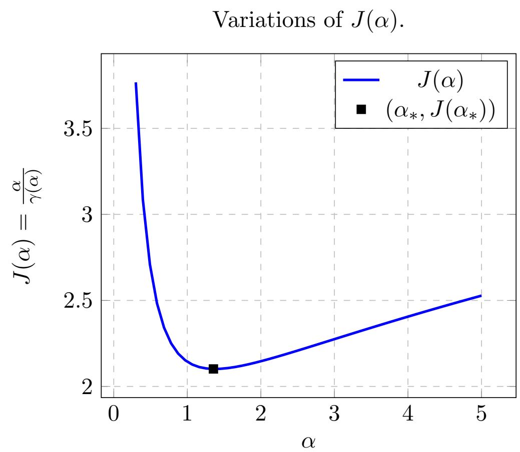


Figure 6.1: Plot of the function $\alpha \mapsto J(\alpha) = \frac{\alpha}{\gamma(\alpha)}$, for $b(x) = -x$ and $f(x) = x^2$. We proved in Proposition 6.1 that this function is decreasing on $(0, \alpha_*]$ and increasing on $[\alpha_*, \infty)$.

Let $J_* := \frac{\alpha_*}{\gamma(\alpha_*)}$. *We deduce that*

1. *For* $J \in [0, J_*)$, δ_0 *is the unique invariant measure of* (1.2).
2. *For* $J \in (J_*, \infty)$, (1.2) *has three invariant measures:* $\{\delta_0,\ \nu^\infty_{\alpha_1},\ \nu^\infty_{\alpha_2}\}$. *with* $\alpha_1 < \alpha_* < \alpha_2$.
3. *For* $J = J_*$, (1.2) *has two invariant measures:* δ_0 *and* $\nu^\infty_{\alpha_*}$.

Proof. The graph of the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is plotted Figure 6.1. Define

$$\forall x \in [0,1), \quad w(x) := x + x^2/2 + \log(1-x) = -\sum_{k \geq 3} \frac{x^k}{k}$$

and

$$\forall \alpha \geq 0, \quad V(\alpha) := \alpha \int_0^1 (1+x) e^{\alpha^2 w(x)} dx. \tag{6.1}$$

Claim It holds that for all $\alpha > 0$

$$\frac{\alpha}{\gamma(\alpha)} = \frac{1}{\alpha} + V(\alpha), \tag{6.2}$$

Proof. First note that with $b(x) = -x$ we have $\varphi^\alpha_t = \alpha(1 - e^{-t})$. So, (3.5) yields

$$\begin{aligned} \frac{\alpha}{\gamma(\alpha)} &= \alpha \int_0^\alpha \frac{1}{\alpha - x} \exp\left(- \int_0^x \frac{y^2}{\alpha - y} dy \right) dx \\ &= \alpha \int_0^1 \frac{1}{1-v} \exp\left(- \int_0^v \frac{(\alpha u)^2}{1-u} du \right) dv \quad \text{with the changes of variables } x = \alpha v \text{ and } y = \alpha u. \end{aligned}$$

Using that $-\frac{u^2}{1-u} = 1 + u - \frac{1}{1-u}$ we deduce that

$$\frac{\alpha}{\gamma(\alpha)} = \alpha \int_0^1 \frac{1}{1-v} e^{\alpha^2 w(v)} dv.$$

We have for all $x \in (0,1)$

$$-\frac{d}{dx} e^{\alpha^2 w(x)} = \alpha^2 e^{\alpha^2 w(x)} \left[\frac{1}{1-x} - (1+x) \right],$$

and so

$$1 = \left[-e^{\alpha^2 w(x)} \right]_0^1 = \alpha^2 \int_0^1 \frac{1}{1-x} e^{\alpha^2 w(x)} dx - \alpha^2 \int_0^1 (1+x) e^{\alpha^2 w(x)} dx.$$

We deduce the claim. Define for all $x \in [0,1)$

$$A(x) := \frac{-4w(x)}{x^3} - (1+x) = \frac{1}{3} + 4x^2 \sum_{k \geq 0} \frac{x^k}{k+5}.$$

Claim It holds that

$$V'(\alpha) = \int_0^1 A(x) e^{\alpha^2 w(x)} dx.$$

In particular V is strictly increasing on $\mathbb{R}_+$.
Proof. We have

$$V'(\alpha) = \int_0^1 (1+x) e^{\alpha^2 w(x)} dx + 2\alpha^2 \int_0^1 (1+x) w(x) e^{\alpha^2 w(x)} dx.$$

Let

$$\forall x \in (0,1), \quad \theta(x) := \frac{(1+x) w(x)}{w'(x)}.$$

We have $\frac{w(x)}{w'(x)} = -\frac{(1-x)w(x)}{x^2}$ and so $\theta(x) = -\frac{1-x^2}{x^2} w(x)$. In particular, θ can be extended to a $\mathcal{C}^1([0,1])$ function with $\theta(0) = \theta(1) = 0$. So

$$\begin{aligned} 2\alpha^2 \int_0^1 (1+x) w(x) e^{\alpha^2 w(x)} dx &= 2\alpha^2 \int_0^1 \theta(x) w'(x) e^{\alpha^2 w(x)} dx \\ &= -2 \int_0^1 \theta'(x) e^{\alpha^2 w(x)} dx. \end{aligned}$$

Moreover, we have $\theta'(x) = \frac{2}{x^3} w(x) + (1+x)$ and so $(1+x) - 2\theta'(x) = A(x)$. This ends the proof of the Claim.
For all $\alpha \geq 1$, we have

$$V'(\alpha) \geq \frac{1}{3} \int_0^1 e^{\alpha^2 w(x)} dx \geq \frac{1}{6\alpha} \alpha \int_0^1 (1+x) e^{\alpha^2 w(x)} dx = \frac{1}{6\alpha} V(\alpha).$$

Consequently, we have $\forall \alpha \geq 1,\ V(\alpha) \geq V(1) \alpha^{1/6}$. Using (6.2), we deduce that

$$\lim_{\alpha \downarrow 0} \frac{\alpha}{\gamma(\alpha)} = +\infty, \quad \text{and} \quad \lim_{\alpha \to \infty} \frac{\alpha}{\gamma(\alpha)} = +\infty.$$

It remains to study the variations of $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$. Using (6.2), we have

$$\frac{d}{d\alpha} \frac{\alpha}{\gamma(\alpha)} = \frac{\alpha^2 V'(\alpha) - 1}{\alpha^2} = \frac{W(\alpha^2) - 1}{\alpha^2},$$

with

$$W(\alpha) := \alpha \int_0^1 A(x) e^{\alpha w(x)} dx.$$

Claim The function W is increasing on $\mathbb{R}_+$.
Proof. Let $D(x) := \frac{A(x)w(x)}{w'(x)}$. We have

$$\begin{aligned} W'(\alpha) &= \int_0^1 A(x) e^{\alpha w(x)} dx + \int_0^1 D(x) \alpha w'(x) e^{\alpha w(x)} dx \\ &= \int_0^1 [A(x) - D'(x)] e^{\alpha w(x)} dx. \end{aligned}$$

To conclude it suffices to show that for all $x \in [0,1)$, $A(x) - D'(x) \geq 0$, which follows from the explicit formula satisfied by A and D. To end the proof it remains to show that
Claim: $\lim_{\alpha \to \infty} W(\alpha) = +\infty$.
This follows from $W(\alpha^2) = \alpha^2 V'(\alpha) \geq \alpha^2 \frac{1}{6\alpha} V(1) \alpha^{1/6}$. □

6.1.2 Case $m > 0$

When $m > 0$, the Dirac measure δ_0 is not anymore an invariant probability measure of (1.2): in that case all the invariant probability measures have the form ν_α^∞, for some $\alpha > 0$ satisfying $\alpha = J\gamma(\alpha)$. Consider V the function given by (6.1). Recall that V is strictly increasing. The following Proposition shows uniqueness of the invariant probability for m large enough.

Proposition 6.2. *Let $b(x) = m - x$ and $f(x) = x^2$. Assume $m > 0$ is large enough such that $mV(m) \geq 1$. Then, the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is strictly increasing on $\mathbb{R}_+$ and*

$$\lim_{\alpha \to \infty} \frac{\alpha}{\gamma(\alpha)} = \infty.$$

So, for all $J \geq 0$, (1.2) has exactly one invariant measure.

Proof. We have, using the first Claim of the proof of Proposition 6.1:

$$\begin{aligned} \frac{\alpha}{\gamma(\alpha)} &= \alpha \int_0^{m+\alpha} \frac{1}{m+\alpha-x} \exp\left(-\int_0^x \frac{y^2}{m+\alpha-y} dy\right) dx \\ &= \frac{\alpha}{m+\alpha}\left[\frac{1}{m+\alpha} + V(m+\alpha)\right]. \end{aligned}$$

So

$$\frac{d}{d\alpha}\frac{\alpha}{\gamma(\alpha)} = \frac{(m-\alpha) + m(m+\alpha)V(m+\alpha)}{(m+\alpha)^3} + \frac{\alpha}{m+\alpha} V'(m+\alpha).$$

The second term is non-negative because V is increasing. Assume $mV(m) \geq 1$. We have $V(m+\alpha) \geq V(m)$ and so the first term is also non-negative. It ends the proof. Numerically, the equation $mV(m) = 1$ yields $m \approx 0.92$. □

For m small enough however, we find numerically that the situation is similar to the case $m = 0$ studied in Proposition 6.1. Depending on the value of J, there is 1,2 or 3 invariant probability measures. Consider for instance $m = 0.1$. We observe numerically that there exists $0 < \alpha_*^1 < \alpha_*^2$ such that the function $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$ is non-decreasing on $[0, \alpha_*^1]$, non-increasing on $[\alpha_*^1, \alpha_*^2]$ and finally non-decreasing on $[\alpha_*^2, \infty)$. Let $J_*^1 := \frac{\alpha_*^1}{\gamma(\alpha_*^1)}$ and $J_*^2 := \frac{\alpha_*^2}{\gamma(\alpha_*^2)}$. We have (see Figure 6.2(a)):

1. For $J < J_*^2$, there is a unique invariant measure, locally stable.

2. For $J \in (J_*^2, J_*^1)$, there are three invariant measures: $\nu_{\alpha_1}^\infty, \nu_{\alpha_2}^\infty$ and $\nu_{\alpha_3}^\infty$ for some $\alpha_1 < \alpha_2 < \alpha_3$. We find numerically that $\nu_{\alpha_1}^\infty$ and $\nu_{\alpha_3}^\infty$ are locally stable, while $\nu_{\alpha_2}^\infty$ is not.

3. For $J > J_*^1$, there is one invariant measure, locally stable.

4. Finally, for the edge cases $J \in \{J_*^1, J_*^2\}$, there are two invariant measures.

We report in the plane (m, J) the number of invariant measures (see Figure 6.2(b)). For m large enough, Proposition 6.2 ensures that for all $J \geq 0$, there is exactly one invariant measure. We find that the transition occurs for $m_*^{\text{cusp}} \approx 0.18$, through a cusp bifurcation.

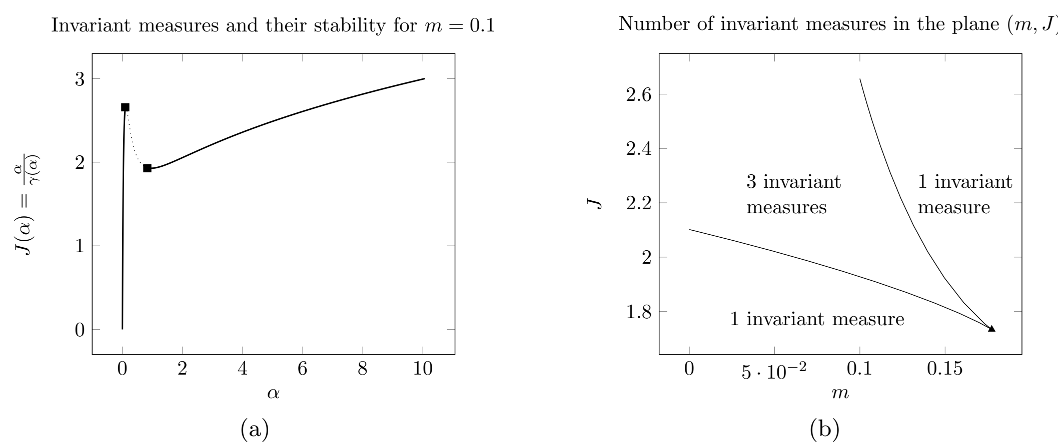


Figure 6.2: Let $b(x) = m - x$ and $f(x) = x^2$. (a) For $m = 0.1$, we report the graph of $\alpha \mapsto \frac{\alpha}{\gamma(\alpha)}$. The function is non-decreasing on $[0, \alpha_*^1]$, non-increasing on $[\alpha_*^1, \alpha_*^2]$ and finally non-decreasing on $[\alpha_*^2, \infty)$. Let $J_*^1 := \frac{\alpha_*^1}{\gamma(\alpha_*^1)}$ and $J_*^2 := \frac{\alpha_*^2}{\gamma(\alpha_*^2)}$. The coordinates of the two black squares are (α_*^1, J_*^1) and (α_*^2, J_*^2). The stability of the invariant measures is determined using the algorithm described in Section 6.4. (b) For each point (m, J) of the plane, we compute the number of invariant measures of (1.2). We find a cusp bifurcation at $(m_*^{\text{cusp}}, J_*^{\text{cusp}}) \approx (0.18, 1.73)$. In particular, for $m > m_*^{\text{cusp}}$ there is always one unique invariant probability measure. The two figures are computed using the Julia package BifurcationKit.jl [Vel20].

6.2 An explicit example with a Hopf bifurcation

We now give a simple example of functions f and b such that Hopf bifurcations occurs and such that the spectral assumptions of Theorem 5.9 can be analytically verified. Our minimal example satisfies all the assumptions of Theorem 5.9, except Assumption 4.2, because the function f we consider is not continuous. Indeed, to simplify the computation, we consider the step function

$$\forall x \in \mathbb{R}_+, \quad f(x) := \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1/\beta & \text{for } x \ge 1, \end{cases} \tag{6.3}$$

where $\beta > 0$ is a (small) parameter of the model.

6.2.1 Some generalities when f is a step function

We shall specify later the exact shape of b, for now we only assume that

$$\inf_{x \in [0,1]} b(x) > 0.$$

This ensures in particular that the Dirac mass at 0 is not an invariant measure. We now consider some fixed constant $\alpha \ge 0$. Let, for all $x \in [0, 1]$

$$t^*_\alpha(x) := \inf\{t \ge 0,\ \varphi^\alpha_t(x) = 1\},$$

the time required for the deterministic flow to hit 1, starting from x. A simple computation shows that

$$t^*_\alpha(x) = \int_x^1 \frac{dy}{b(y) + \alpha}.$$

Let $H^x_\alpha(t)$ be defined by (1.13) (with $\nu = \delta_x$, $\boldsymbol{a} \equiv \alpha$ and $s = 0$). Using the explicit shape of f, we find for all $x \in [0, 1]$,

$$H^x_\alpha(t) := \begin{cases} 1 & \text{for } 0 \le t < t^*_\alpha(x), \\ e^{-\frac{t - t^*_\alpha(x)}{\beta}} & \text{for } t \ge t^*_\alpha(x). \end{cases}$$

Moreover,

$$\forall x > 1, \quad H^x_\alpha(t) = e^{-t/\beta}. \tag{6.4}$$

Altogether,

$$\forall z \in \mathbb{C} \text{ with } \Re(z) > -1/\beta, \quad \widehat{H}_\alpha(z) = \frac{1 - e^{-z t^*_\alpha(0)}}{z} + \frac{e^{-z t^*_\alpha(0)}}{z + \beta^{-1}}.$$

Note that in particular (using that $1/\gamma(\alpha) = \widehat{H}_\alpha(0)$)

$$1/\gamma(\alpha) = t^*_\alpha(0) + \beta.$$

So

$$J(\alpha) := \frac{\alpha}{\gamma(\alpha)} = \int_0^1 \frac{dy}{1 + b(y)/\alpha} + \alpha\beta$$

is a strictly increasing function of α: for a fixed value of $J > 0$, there is a unique $\alpha > 0$ solution of $\alpha = J\gamma(\alpha)$ and the corresponding ν_α^∞ is the unique invariant measure of (1.2). Let $\sigma_\alpha = \lim_{t\to\infty} \varphi_t^\alpha(0)$. This invariant measure is given by

$$\nu_\alpha^\infty(x) = \begin{cases} \frac{\gamma(\alpha)}{b(x)+\alpha} & \text{for } x \in [0,1), \\ \frac{\gamma(\alpha)}{b(x)+\alpha} \exp\left(-\frac{1}{\beta}\int_1^x \frac{dy}{b(y)+\alpha}\right) & \text{for } x \in [1,\sigma_\alpha), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, for $x \in [0,1]$ and $t > t_\alpha^*(x)$,

$$\frac{d}{dx} H_\alpha^x(t) = -\frac{1}{\beta} \frac{e^{-\frac{t - t_\alpha^*(x)}{\beta}}}{b(x) + \alpha}.$$

So the Laplace transform of $\frac{d}{dx} H_\alpha^x(t)$ is, for all $z \in \mathbb{C}$ with $\Re(z) > -1/\beta$

$$\forall x \in [0,1], \quad \int_0^\infty e^{-zt} \frac{d}{dx} H_\alpha^x(t) dt = -\frac{e^{-t_\alpha^*(x) z}}{b(x) + \alpha} \frac{1}{1 + \beta z}.$$

Consider Ψ_α given by (4.29). For all $z \in \mathbb{C}$ with $\Re(z) > -1/\beta$

$$\begin{aligned} J(\alpha)\widehat{\Psi}_\alpha(z) &= -\frac{\alpha}{\gamma(\alpha)} \int_0^{\sigma_\alpha} \int_0^\infty e^{-zt} \frac{d}{dx} H_\alpha^x(t) dt \; \nu_\alpha^\infty(x) dx \\ &= \frac{\alpha}{1 + \beta z} \int_0^1 \frac{e^{-t_\alpha^*(x) z}}{(b(x) + \alpha)^2} dx. \end{aligned}$$

Indeed, using (6.4), it holds that $\frac{d}{dx} H_\alpha^x(t) = 0$ for $x > 1$. Finally, the change of variable

$$x = \varphi_u^\alpha(0), \quad u \in [0, t_\alpha^*(0)),$$

such that $t_\alpha^*(x) = t_\alpha^*(0) - u$, shows that

$$J(\alpha)\widehat{\Psi}_\alpha(z) = \frac{\alpha e^{-z t_\alpha^*(0)}}{1 + \beta z} \int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0)) + \alpha} du.$$

So by Remark 4.28, the (local) stability of the invariant measure ν_α^∞ is given by the location of the roots of the following holomorphic function, defined for all $\Re(z) > -1/\beta$:

$$\boxed{J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = \frac{\alpha e^{-z t_\alpha^*(0)}}{1 + \beta z} \int_0^{t_\alpha^*(0)} \frac{e^{uz}}{b(\varphi_u^\alpha(0)) + \alpha} du - \frac{1 - e^{-z t_\alpha^*(0)}}{z} - \frac{\beta e^{-z t_\alpha^*(0)}}{1 + \beta z}}.$$

6.2.2 A linear drift b.

We now specify the shape of b. We choose:

$$\forall x \geq 0, \quad b(x) = m - x,$$

for some parameter $m > 1$, such that $b(x) + \alpha = \sigma_\alpha - x$, with $\sigma_\alpha = m + \alpha$. We then have $\varphi^\alpha_u(0) = \sigma_\alpha(1 - e^{-u})$ and so

$$t^*_\alpha(0) = \log\left(\frac{\sigma_\alpha}{\sigma_\alpha - 1}\right).$$

Finally

$$\int_0^{t^*_\alpha(0)} \frac{e^{uz}}{b(\varphi^\alpha_u(0)) + \alpha} du = \frac{1}{\sigma_\alpha} \int_0^{t^*_\alpha(0)} e^{(z+1)u} du = \frac{1}{\sigma_\alpha} \frac{e^{(z+1)t^*_\alpha(0)} - 1}{z+1},$$

so

$$J(\alpha)\widehat{\Psi}_\alpha(z) - \widehat{H}_\alpha(z) = \frac{\alpha}{\sigma_\alpha} \frac{e^{t^*_\alpha(0)} - e^{-zt^*_\alpha(0)}}{(1+\beta z)(z+1)} - \frac{1 - e^{-zt^*_\alpha(0)}}{z} - \frac{\beta e^{-zt^*_\alpha(0)}}{1 + \beta z}.$$

Consequently, we have to study the complex solutions of

$$\Re(z) > -\beta^{-1}, \quad \frac{\alpha}{m+\alpha-1} \frac{1 - \left(\frac{m+\alpha}{m+\alpha-1}\right)^{-(z+1)}}{(1+\beta z)(z+1)} - \frac{1 - \left(\frac{m+\alpha}{m+\alpha-1}\right)^{-z}}{z} - \frac{\beta\left(\frac{m+\alpha}{m+\alpha-1}\right)^{-z}}{1+\beta z} = 0. \tag{6.5}$$

Remark 6.3. *In fact this analysis can be easily extended to any linear drift*

$$b(x) = \kappa(m - x),$$

with $\kappa, m \in \mathbb{R}$. Indeed, adapting slightly the proof of Theorem 4.14, when $\kappa \leq 0$ it holds that $f + b' \geq 0$ and so the unique non trivial invariant measure is locally stable: there is no Hopf bifurcation. If on the other hand $\kappa > 0$, by setting

$$\tilde{\kappa} = 1, \quad \tilde{\alpha} = \frac{\alpha}{\kappa}, \quad \tilde{m} = m \quad \tilde{\beta} = \kappa\beta,$$

we can easily reduce the problem to $\kappa = 1$.

We now make the following change of variable

$$\omega := \log\left(\frac{m+\alpha}{m+\alpha-1}\right) \quad \text{and} \quad \delta := \frac{\alpha}{m+\alpha-1},$$

with $\omega > 0$ et $\delta \in (0,1)$. That is, we have

$$\alpha = \frac{\delta}{e^\omega - 1}, \quad m = 1 + \frac{1-\delta}{e^\omega - 1}. \tag{6.6}$$

With this change of variable, (6.5) becomes

$$\Re(z) > -\beta^{-1}, \quad \delta \frac{1}{1+\beta z} \frac{1 - e^{-\omega(z+1)}}{1+z} - \frac{1 - e^{-\omega z}}{z} - \frac{\beta e^{-\omega z}}{1+\beta z} = 0. \tag{6.7}$$

Assume now that

$$\beta + \omega - \delta(1 - e^{-\omega}) \neq 0, \tag{6.8}$$

such that $z = 0$ is not a solution of the equation. Multiplying by $(1 + \beta z)z$ on both side, we finally find that we have to study the zeros of

$$\Re(z) > -\beta^{-1}, \quad U(\beta, \delta, \omega, z) = 0,$$

with

$$\boxed{U(\beta, \delta, \omega, z) := \delta \frac{z}{z+1}(1 - e^{-\omega(z+1)}) + e^{-\omega z} - (1 + \beta z).} \tag{6.9}$$

6.2.3 On the roots of U

An explicit parametrization of the purely imaginary roots

We now describe all the imaginary roots of U. If $z = iy,\ y \geq 0$, the equation $U(\beta, \delta, \omega, z) = 0$ yields

$$\begin{cases} \cos(\omega y) + \sin(\omega y) y(1 - \delta e^{-\omega}) &= 1 - \beta y^2 \\ -\sin(\omega y) + \cos(\omega y) y(1 - \delta e^{-\omega}) &= y(1 + \beta - \delta). \end{cases} \tag{6.10}$$

For $\omega > 0$ et $y \geq 0$ fixed, (6.10) admits a unique solution in (β, δ), given by

$$\begin{aligned} \beta^0_\omega(y) &:= \frac{(1 + e^\omega)(1 - \cos(\omega y)) - (e^\omega - 1) y \sin(\omega y)}{y^2 e^\omega - y^2 \cos(\omega y) - y \sin(\omega y)}, \\ \delta^0_\omega(y) &:= \frac{e^\omega (1 + y^2)(1 - \cos(\omega y))}{y^2 e^\omega - y^2 \cos(\omega y) - y \sin(\omega y)}. \end{aligned} \tag{6.11}$$

Proposition 6.4. *The parametric curve $(\beta^0_\omega(y), \delta^0_\omega(y))_{y>0}$ admits exactly two multiple points given by*

$$(0, 0) \quad \text{and} \quad (0, \frac{2}{1 + e^{-\omega}}).$$

Apart from those two points, the curve does not intersect itself.

Proof. Squaring the two equations of (6.10) and summing the result, one gets

$$1 + y^2(1 - \delta e^{-\omega})^2 = (1 - \beta y^2)^2 + y^2(1 + \beta - \delta)^2,$$

that is

$$(1 - \delta e^{-\omega})^2 = -2\beta + \beta^2 y^2 + (1 + \beta - \delta)^2. \tag{6.12}$$

Note that if $\beta \neq 0$, for fixed values of δ, β, there is a unique y satisfying this equation. This proves that all the multiple points are located on the axis $\beta = 0$. When $\beta = 0$, the equation becomes

$$(1 - \delta e^{-\omega})^2 = (1 - \delta)^2,$$

whose solutions are

$$\delta = 0 \quad \text{and} \quad \delta = \frac{2}{1 + e^{-\omega}}.$$

Those are indeed multiple points. For $(0, 0)$ for instance, it suffices to consider $y = \frac{2\pi k}{\omega}, k \in \mathbb{N}^*$. This ends the proof. □

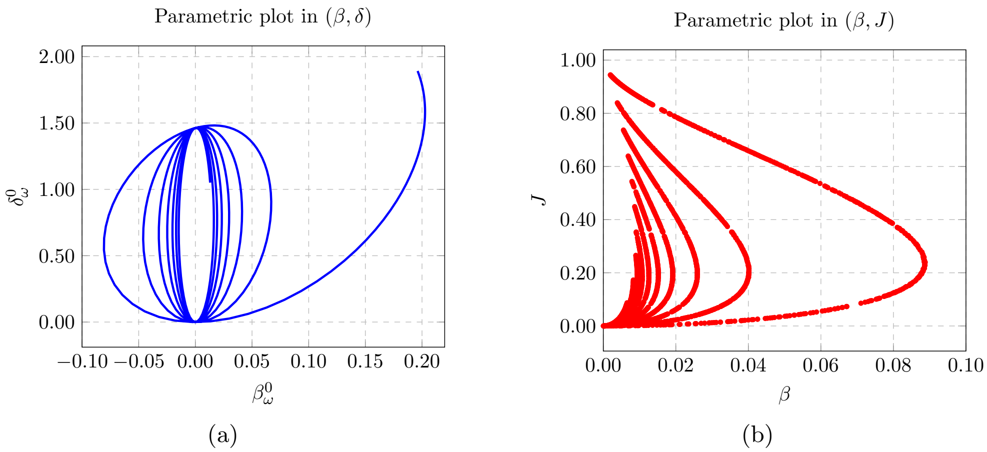


Figure 6.3: Description of the purely imaginary roots of U. (a) The parametric curve $(\beta^0_\omega(y), \delta^0_\omega(y))$, plotted with $\omega = 1$ and $y \in [0, 15.5\pi]$. Each point of the curve corresponds to a purely imaginary roots of U. (b) Purely imaginary solutions of U plotted in the plane (β, J), the value of m being fixed ($m = 3/2$).

6.2.4 Construction of the bifurcation point satisfying all the spectral assumptions.

Let $\omega_0 > 0$ being fixed, chosen arbitrarily. Let $y_0 := \frac{2\pi}{\omega_0}(1 - \frac{\epsilon_0}{\omega_0})$ with $\epsilon_0 > 0$ (small) to be chosen later. Let $\beta_0 := \beta^0_{\omega_0}(y_0)$ and $d_0 := \delta^0_{\omega_0}(y_0)$. We have

$$\beta_0 = \epsilon_0 + \mathcal{O}(\epsilon_0^2) \qquad \text{as } \epsilon_0 \to 0.$$

and

$$d_0 = \frac{e^{\omega_0}}{2(e^{\omega_0} - 1)} \left(1 + \frac{(2\pi)^2}{\omega_0^2}\right) \epsilon_0^2 + \mathcal{O}(\epsilon_0^2) \qquad \text{as } \epsilon_0 \to 0.$$

We then have from (6.9)

$$\frac{\partial U}{\partial z}(\beta_0, d_0, \omega_0, iy_0) = -\omega_0 - (1 + 2i\pi)\epsilon_0 + \mathcal{O}(\epsilon_0^2) \qquad \text{as } \epsilon_0 \to 0.$$

This quantity is non-null provided that ϵ_0 is sufficiently small. The implicit function theorem applies and gives the existence of a $\mathcal{C}^1$ function

$$(\beta, \delta, \omega) \mapsto z_0(\beta, \delta, \omega),$$

defined in the neighborhood of (β_0, d_0, ω_0) and such that

$$U(\beta, \delta, \omega, z_0(\beta, \delta, \omega)) = 0, \qquad \text{with} \qquad z_0(\beta_0, d_0, \omega_0) = iy_0.$$

Furthermore, one has

$$\frac{\partial}{\partial \delta} z_0(\beta_0, d_0, \omega_0) = -\frac{\frac{\partial U}{\partial \delta}(\beta_0, d_0, \omega_0, iy_0)}{\frac{\partial U}{\partial z}(\beta_0, d_0, \omega_0, iy_0)} \overset{(6.9)}{=} 2\pi \frac{1 - e^{-\omega_0}}{\omega_0} \frac{2\pi + i\omega_0}{(2\pi)^2 + \omega_0^2} + \mathcal{O}(\epsilon_0) \qquad \text{as } \epsilon_0 \to 0$$

and

$$\frac{\partial}{\partial \omega} z_0(\beta_0, d_0, \omega_0) = -\frac{\frac{\partial U}{\partial \omega}(\beta_0, d_0, \omega_0, iy_0)}{\frac{\partial U}{\partial z}(\beta_0, d_0, \omega_0, iy_0)} \stackrel{(6.9)}{=} -\frac{2i\pi}{\omega_0} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0.$$

We finally set

$$\alpha_0 := \frac{d_0}{e^{\omega_0} - 1}, \quad m_0 := 1 + \frac{1 - d_0}{e^{\omega_0} - 1},$$

and

$$\mathfrak{z}_0(\alpha) := z_0(\beta_0, \frac{\alpha}{m_0 + \alpha - 1}, \log\left(\frac{m_0 + \alpha}{m_0 + \alpha - 1}\right)),$$

such that

$$\begin{aligned} \frac{d}{d\alpha}\mathfrak{z}_0(\alpha_0) =& 2\pi \frac{1 - e^{-\omega_0}}{\omega_0} \frac{2\pi + i\omega_0}{(2\pi)^2 + \omega_0^2} \frac{m_0 - 1}{(m_0 - 1 - \alpha_0)^2} \\ &+ \frac{2i\pi}{\omega_0} \frac{1}{(m_0 - 1 + \alpha_0)(m_0 + \alpha_0)} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0. \end{aligned}$$

The second term on the right hand side is purely imaginary. So

$$\Re \frac{d}{d\alpha}\mathfrak{z}_0(\alpha_0) = \frac{1 - e^{-\omega_0}}{\omega_0} \frac{(2\pi)^2}{(2\pi)^2 + \omega_0^2} \frac{m_0 - 1}{(m_0 - 1 - \alpha_0)^2} + \mathcal{O}(\epsilon_0) \quad \text{as } \epsilon_0 \to 0.$$

This quantity is strictly positive provided that ϵ_0 is small enough. By choosing the parameters of the model to be $\beta = \beta_0$ and $m = m_0$, the Assumptions 5.3, 5.4 and 5.7 are satisfied at the point $\alpha = \alpha_0$. In particular, Assumption 5.4 follows from Proposition 6.4.

6.3 Numerical methods

6.3.1 Monte Carlo simulation of the particle system

We first give a straightforward Monte Carlo Euler scheme with constant time step $\Delta_t > 0$ to simulate the solution of the particle system (1.1). We consider $N \geq 1$ particles and we compute their membrane potentials $(X^{i,N}_n)_{i \in \{1, \cdots, N\}}$ at the discrete times $\{n\Delta_t,\ n \geq 0\}$. Consider $(U^i_n)_{n \geq 1, i \in \{1, \cdots, N\}}$ a sequence of i.i.d. random variables uniformly distributed on $[0, 1]$, independent of the initial conditions $(X^{i,N}_0)_{i \in \{1, \cdots, N\}}$. Given $n \geq 1$ and $(X^{i,N}_{n-1})_{i \in \{1, \cdots, N\}}$, the update rules are

Step 1. $S^{i,N}_n := \mathbb{1}_{\{U^i_n \leq \Delta_t f(X^{i,N}_{n-1})\}}$

Step 2. $\tilde{X}^{i,N}_n := X^{i,N}_{n-1} + \Delta_t b(X^{i,N}_{n-1}) + \frac{J}{N} \sum_{j \neq i} S^{j,N}_n$.

Step 3. $X^{i,N}_n := \tilde{X}^{i,N}_n (1 - S^{i,N}_n)$.

A variant is the following: replace Step 2. with

$$\tilde{X}_n^{i,N} = X_{n-1}^{i,N} + \Delta_t b(X_{n-1}^{i,N}) + \frac{J\Delta_t}{N}\sum_{j=1}^{N} f(X_{n-1}^{j,N}).$$

Both schemes give similar results for N large enough. This method is easily implemented on GPU. Using a simple Julia implementation, we can simulate up to $N = 10^9$ particles.

6.3.2 Simulation of the nonlinear Fokker-Planck equation

We give an explicit finite volume scheme to compute the solution of (1.3).The method is adapted from the (implicit) 2D scheme given in [ACV19]. We work on the compact domain $[0, v_{\max}]$: we discretize it using a regular grid with N_v subdivisions. That is, given $N_v \in \mathbb{N}^*$, let $\Delta_v := \frac{v_{\max}}{N_v}$ such that

$$[0, v_{\max}] = \bigcup_{i=1}^{N_v} \Omega^i \quad \text{with} \quad \Omega^i := [v_i - \frac{\Delta_v}{2}, v_i + \frac{\Delta_v}{2}] \quad \text{and} \quad v_i := \frac{\Delta_v}{2} + \Delta_v(i-1). \tag{6.13}$$

We denote by Δ_t the time step of the scheme. Let $t_n := n\Delta_t$ and consider $\nu(t_n)$ the solution of (1.3) at time t_n. For $i \in \{1, \cdots, N_v\}$, we compute the finite volume approximations $\nu_n := (\nu_n^i)_{i\in\{1,\cdots,N_v\}}$ of $\nu(t_n)$, that is

$$\nu_n^i \approx \frac{1}{\Delta_v}\langle \nu(t^n), \mathbb{1}_{\Omega^i}\rangle.$$

We split the PDE (1.3) by writing

$$\partial_t \nu(t) = \mathcal{L}^*_{\text{transport}}(\nu(t)) + \mathcal{L}^*_{\text{jump}}(\nu(t)),$$

with

$$\begin{aligned}\mathcal{L}^*_{\text{transport}}(\nu) &:= -\partial_x\left[(b + J\langle \nu, f\rangle)\,\nu\right]\\ \mathcal{L}^*_{\text{jump}}(\nu) &:= -f\nu + \langle \nu, f\rangle \delta_0.\end{aligned}$$

Discretization of the transport operator

Consider a measure μ and write $\mu^i := \frac{1}{\Delta_v}\langle \mu, \mathbb{1}_{\Omega^i}\rangle$. Hence,

$$\langle \mu, f\rangle \approx \Delta_v \sum_{j=1}^{N_v} f(v_j)\mu^j. \tag{6.14}$$

To discretize the transport operator, we use the following explicit upwind scheme (see [CIR52]):

$$\frac{1}{\Delta_v}\langle \mathcal{L}^*_{\text{transport}}(\mu), \mathbb{1}_{\Omega^i}\rangle \approx -\frac{1}{\Delta_v}\left(F^{i+1/2}(\mu) - F^{i-1/2}(\mu)\right), \tag{6.15}$$

with:

$$F^{1/2}(\mu) = F^{N_v+1/2}(\mu) := 0,$$

$$\forall i \in \{1, \cdots, N_v - 1\}, \quad F^{i+1/2}(\mu) := \begin{cases} V^{i+1/2}(\mu)\ \mu^i, & \text{if } V^{i+1/2}(\mu) > 0 \\ V^{i+1/2}(\mu)\ \mu^{i+1}, & \text{otherwise} \end{cases}$$

$$\text{and} \quad V^{i+1/2}(\mu) := b(v_i + \frac{\Delta_v}{2}) + \Delta_v J \sum_{j=1}^{N_v} f(v_j)\mu^j.$$

So the vector $V(\mu)$ is a spatial discretization of $b + J\langle \mu, f\rangle$. We used (6.14) to approximate the interactions part $J\langle \mu, f\rangle$. Note moreover that

$$\sum_{j=1}^{N_v} \left(F^{j+1/2}(\mu) - F^{j-1/2}(\mu)\right) = F^{N_v+1/2}(\mu) - F^{1/2}(\mu) = 0,$$

so the scheme preserves the mass.

Discretization of the jump operator

To discretize the jump operator, consider a measure $\mu(0)$ and write again $\mu^i(0) := \frac{1}{\Delta_v}\langle \mu(0), \mathbb{1}_{\Omega^i}\rangle$. The PDE

$$\partial_t \mu(t) = \mathcal{L}^*_{\text{jump}}(\mu(t))$$

translates to the following system of ODEs:

$$\begin{cases} \dot{\mu}^i(t) = -f(v_i)\mu^i(t), & \forall i \in \{2, \cdots, N_v\} \\ \dot{\mu}^1(t) = \sum_{j=2}^{N_v} f(v_j)\mu^j(t), & \text{for } i = 1. \end{cases}$$

We solve explicitly this system and find:

$$\begin{cases} \mu^i(t) = \mu^i(0)e^{-f(v_i)t}, & \forall i \in \{2, \cdots, N_v\} \\ \mu^1(t) = \mu^1(0) + \sum_{j=2}^{N_v} \mu^j(0)(1 - e^{-f(v_j)t}), & \text{for } i = 1. \end{cases}$$

Note that this scheme preserves the mass and the positivity:

$$\forall t > 0, \quad \sum_{i=1}^{N_v} \mu^i(t) = \sum_{i=1}^{N_v} \mu^i(0) \quad \text{and} \quad \forall i \in \{1, \cdots, N_v\}, \quad \left[\mu^i(0) \geq 0\right] \implies \left[\mu^i(t) \geq 0\right].$$

Update rules

Overall, the update rules are the following. Given $\nu_n = (\nu_n^i)_{i\in\{1,\cdots,N_v\}}$ an approximation of the solution of (1.3) at time t_n, we set for all $i \in \{1, \cdots, N_v\}$:

$$\tilde{\nu}^i_{n+1} := \nu^i_n - \frac{\Delta_t}{\Delta_v}\left(F^{i+1/2}(\nu_n) - F^{i-1/2}(\nu_n)\right).$$

$$\nu^i_{n+1} := \begin{cases} \tilde{\nu}^i_{n+1} e^{-f(v_i)\Delta_t}, & \forall i \in \{2, \cdots, N_v\} \\ \tilde{\nu}^1_{n+1} + \sum_{j=2}^{N_v} \tilde{\nu}^j_{n+1}(1 - e^{-f(v_j)\Delta_v}), & \text{for } i = 1. \end{cases}$$

That is, we first apply the transport operator, and then the jump operator.

Remark 6.5. *Let $S_A(t)$ be a semi-group with generator A. The semi-group $S_{A+B}(t)$ with generator $A+B$ is obtained by the following Dyson-Phillips series (see [EN00, Th. 1.10])*

$$S_{A+B}(t) = \sum_{n=0}^{\infty} S^n_{A+B}(t),$$

with $S^0_{A+B}(t) := S_A(t)$ and $S^{n+1}_{A+B}(t) := \int_0^t S_A(t-s) B S^n_{A+B}(s) ds$. We only keep the first two terms of the series and use the following approximation, valid for Δ_t small enough

$$\begin{aligned} S_{A+B}(\Delta_t) \approx S_A(\Delta_t) + \int_0^{\Delta_t} S_A(\Delta_t - s) B S_A(s) ds &\approx S_A(\Delta_t) + S_A(\Delta_t)\Delta_t B S_A(0) \\ &= S_A(\Delta_t)\left(I + \Delta_t B\right), \end{aligned}$$

using that $S_A(0) = I$, I denoting the identity operator. Our updates rules are finding by choosing A to be the jump operator and B to be the transport operator.

Adaptive time steps

An important refinement of this method is to use **adaptive** time steps. The length of the time step is chosen such that the Courant–Friedrichs–Lewy stability condition holds, namely:

$$\frac{V^{\max}_n \Delta^n_t}{\Delta_v} \leq C_{\max} \leq 1. \tag{6.16}$$

with

$$V^{\max}_n := \max_{i \in \{1,\cdots,N_v\}} |V^{i+1/2}(\nu_n)|.$$

That is, we chose the n-th step size to be

$$\Delta^n_t = \frac{C_{\max}\Delta_v}{V^{\max}_n}.$$

The parameter $C_{\max} \leq 1$ is the **Courant number** of the scheme.

6.3.3 Comparison of the two schemes

We compare numerically the order of convergence in time and space of the two schemes. To do so, we choose $b(x) = 1-x$, $f(x) = x^2$, $J = 1/2$ and the law of the initial condition X_0 is the uniform probability measure on $[0, 3/2]$. Using both schemes, we then estimate the value of $\mathbb{E} f(X_T)$ with $T = 2$. We compare the results to a reference value. In Figure 6.4, we plotted few approximations of the jump rate $t \mapsto \mathbb{E} f(X_t)$ using the two schemes with different parameters.

The time complexity of the finite volume scheme is $\mathcal{O}(\frac{N_v}{\Delta_t}) \overset{(6.16)}{=} \mathcal{O}(N_v^2)$, while the time complexity of the Monte Carlo Euler scheme is $\mathcal{O}(\frac{N^2}{\Delta_t})$. To fairly compare the two schemes, given $\delta > 0$ (small), we choose the parameters $N_v(\delta)$ (for the finite volume scheme) and $N(\delta), \Delta_t(\delta)$ (for the Monte Carlo Euler scheme) such that

$$\delta \approx \frac{1}{N_v^2(\delta)},$$

and

$$\delta \approx \frac{\Delta_t(\delta)}{N^2(\delta)}.$$

With this choice, the computation time of the two schemes is, in both cases, of order $\mathcal{O}(1/\delta)$. For the finite volume scheme, we choose

$$N_v(\delta) := \left\lfloor \frac{1}{\sqrt{\delta}} \right\rfloor.$$

For the Monte Carlo Euler scheme, we choose

$$\Delta_t(\delta) := \delta^{1/3}, \quad \text{and} \quad N(\delta) = \lfloor \delta^{-2/3} \rfloor.$$

We compute the reference value $\mathbb{E} f(X_T) \approx 0.62427$. To do so, we used the finite volume scheme with $N_v = 40000$. We report, for different values of δ, the errors of the two schemes with respect to this reference value. We find that the error of the Monte Carlo Euler scheme is proportional to $\delta^{1/3}$ (see Figure 6.6), while the error of the finite volume scheme has an error proportional to $\delta^{1/2}$ (see Figure 6.5). This suggests that the two schemes are not asymptotically of the same order. To estimate $\mathbb{E} f(X_T)$ with an accuracy of ϵ, the Monte Carlo Euler scheme needs a computation time of order $\mathcal{O}(\epsilon^{-3})$ while the finite volume scheme only needs a computation time of order $\mathcal{O}(\epsilon^{-2})$.

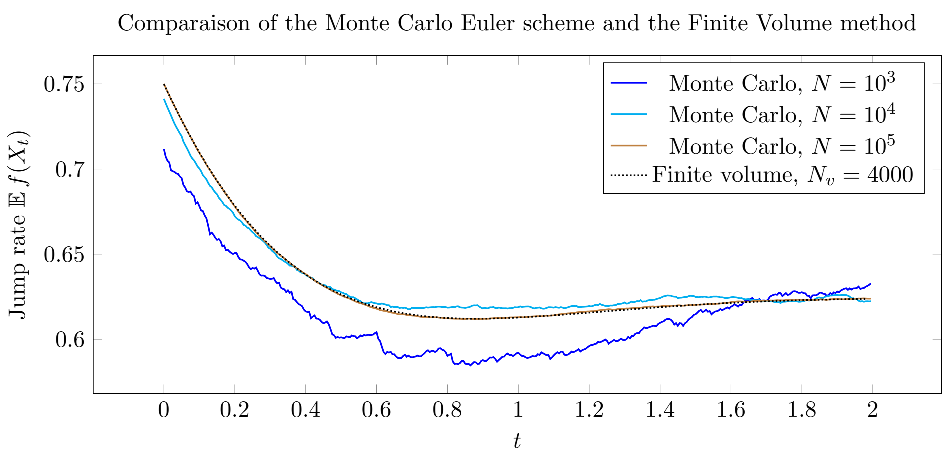


Figure 6.4: We compare the Monte Carlo Euler scheme, described in Section 6.3.1, to the finite volume scheme of Section 6.3.2. We choose $f(x) = x^2, b(x) = 1 - x$ and $J = 0.5$ and simulate up to $T = 2$. The initial condition is the uniform probability measure on $[0, 3/2]$. For the Monte Carlo Euler scheme, we choose a time step of $\Delta_t := 10^{-4}$ and the number of particles $N \in \{10^3, 10^4, 10^5\}$. For the Volume finite method, we choose $C_{\max} := 0.5$, $N_v = 4000$, $v_{\max} = 3.0$.

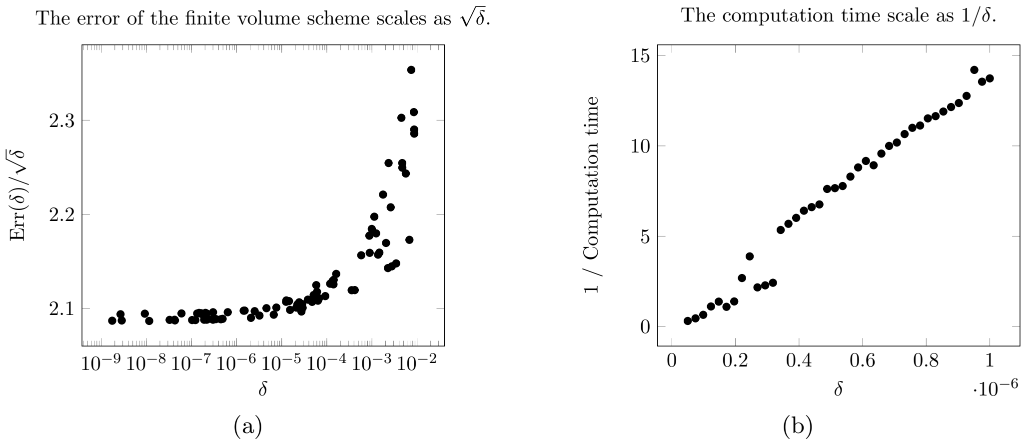


Figure 6.5: For 100 values of δ randomly chosen in $[10^{-9}, 10^{-2}]$, let $N_v(\delta) := \lceil 1/\sqrt{\delta} \rceil$. This parameter is used to compute an approximation of $\mathbb{E}\, f(X_T)$ using the finite volume scheme. We report the normalized error $\frac{\mathrm{Err}(\delta)}{\sqrt{\delta}}$ as a function of δ. This suggests that the error rate of the scheme is of order $\sqrt{\delta}$, while the computation time is of order $1/\delta$. Parameters $f(x) = x^2$, $b(x) = 1 - x$, $C_{\max} = 0.1$, $J = 0.5$, $v_{\max} = 3.0$.

6.4 Numerical stability of an invariant probability measure.

Given some interaction parameter $J_0 > 0$, consider $\nu^\infty_{\alpha_0}$ an invariant measure of (1.2). The constant $\alpha_0 > 0$ satisfies $\alpha_0 = J_0 \gamma(\alpha_0)$. We have seen in Chapter 4 that the (local) stability of $\nu^\infty_{\alpha_0}$ can be determined by inspecting the location of the roots of the holomorphic function $z \mapsto J_0 \widehat{\Theta}_{\alpha_0}(z) - 1$, where $\Theta_{\alpha_0}(t)$ is given by (4.8). By Theorem 4.13, local stability holds when all the roots have negative real parts. The goal of this Section is to provide an effective algorithm to compute these roots, in order to decide numerically if $\nu^\infty_{\alpha_0}$ is locally stable or not.

The idea is the compute numerically the eigenvalues and the eigenvectors of the generator of the linearized Fokker-Planck equation. The nonlinear Fokker-Planck equation (1.3) writes

$$\dot{\nu} = F(\nu),$$

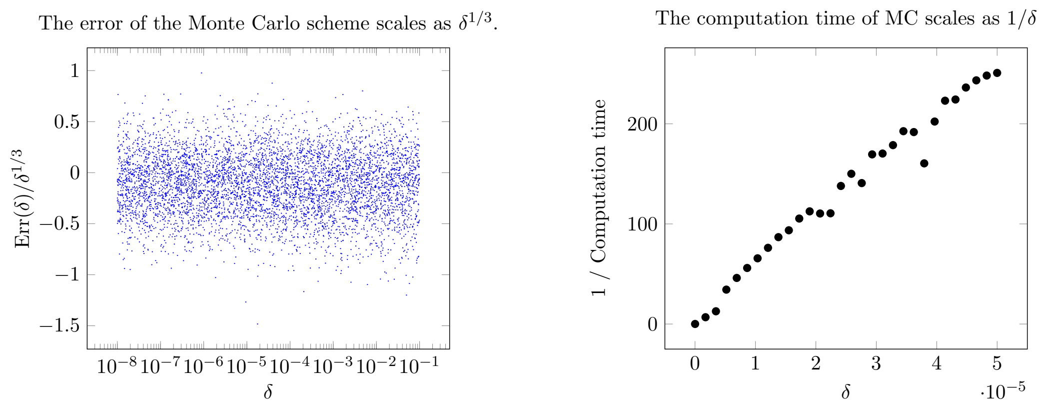
with

$$F(\nu) := -\partial_x \left[(b + J_0 \langle \nu, f \rangle) \nu \right] - f\nu + \langle \nu, f \rangle \delta_0.$$

We have $F(\nu^\infty_{\alpha_0}) = 0$ and so for $\phi = \nu - \nu^\infty_{\alpha_0}$, the linearized Fokker-Planck equation informally writes

$$\begin{aligned} \dot{\phi} &= D_\nu F(\nu^\infty_{\alpha_0}) \cdot \phi \\ &\overset{(4.12)}{=} \mathcal{L}^*_{\alpha_0}(\phi) + \mathcal{B}_{\alpha_0}(\phi). \end{aligned}$$

We study the eigenvalues and the associated eigenvectors of $\mathcal{L}^*_{\alpha_0}(\phi)$ and of $D_\nu F(\nu^\infty_{\alpha_0})$.



(a) (b)

Figure 6.6: For 5000 values of δ randomly chosen in $[10^{-9}, 10^{-1}]$, we let $N(\delta) := \delta^{-2/3}$ and $\Delta_t(\delta) := \delta^{1/3}$. Those parameters are used to compute an approximation of $\mathbb{E}\, f(X_T)$, using the Monte Carlo Euler scheme described in Section 6.3.1. We report the normalized error $\frac{\text{Err}(\delta)}{\delta^{1/3}}$ as a function of δ. This suggests that the error rate of the scheme is of order $\delta^{1/3}$, while the computation time is in $1/\delta$. Parameters: $f(x) = x^2, b(x) = 1 - x, J = 0.5, T = 2$.

***Claim* 6.6.** *Let z_0 be an eigenvalue of the linear operator $\mathcal{L}^*_{\alpha_0}$, with $\Re(z_0) > -f(\sigma_{\alpha_0})$. Then, it holds that $\widehat{K}_{\alpha_0}(z_0) = 1$. Conversely, if $\widehat{K}_{\alpha_0}(z_0) = 1$, then z_0 is an eigenvalue of $\mathcal{L}^*_{\alpha_0}$ and an associated eigenvector is*

$$\nu^{z_0}_{\alpha_0}(dx) := \frac{1}{b(x) + \alpha_0} \exp\left(- \int_0^x \frac{f(y) + z_0}{b(y) + \alpha_0} dy \right) \mathbb{1}_{[0, \sigma_{\alpha_0})}(x). \tag{6.17}$$

Remark 6.7. *In particular, 0 is an eigenvalue of $\mathcal{L}^*_{\alpha_0}$ and $\nu^\infty_{\alpha_0}$ is the associated eigenvector. Recall that for $z \neq 0$, $\widehat{K}_{\alpha_0}(z) = 1 \overset{(3.13)}{\iff} \widehat{H}_{\alpha_0}(z) = 0$. So the other eigenvalues of $\mathcal{L}^*_{\alpha_0}(\phi)$ are the zeros of $\widehat{H}_\alpha$. Therefore, to compute the complex zeros of $\widehat{H}_\alpha$, it suffices the compute the eigenvalues (of a discrete approximation) of $\mathcal{L}^*_{\alpha_0}$.*

Proof. Using that $\nu^\infty_{\alpha_0}$ is the invariant probability measure, we have $\mathcal{L}^*_{\alpha_0} \nu^\infty_{\alpha_0} = 0$, and so 0 is an eigenvalue. Consider now $z_0 \neq 0$ another eigenvalue and let $\phi(0)$ be the associated eigenvector. Consider $\phi(t)$ defined by

$$\phi(t) = e^{z_0 t} \phi(0). \tag{6.18}$$

Using that $\mathcal{L}^*_{\alpha_0} \phi(0) = z_0 \phi(0)$, we deduce that $\phi(t)$ solves

$$\dot{\phi}(t) = \mathcal{L}^*_{\alpha_0} \phi(t).$$

Note that the semi-group associated to $\mathcal{L}^*_{\alpha_0}$ preserves the mass, so

$$\int_{\mathbb{R}_+} \phi(t)(dx) = e^{z_0 t} \int_{\mathbb{R}_+} \phi(0)(dx) = \int_{\mathbb{R}_+} \phi(0)(dx).$$

Because $z_0 \neq 0$, it necessarily holds that

$$\int_{\mathbb{R}_+} \phi(0)(dx) = 0.$$

Let $r^{\phi(0)}_{\alpha_0}(t) = \langle \phi(t), f \rangle$. It solves the Volterra integral equation (1.14):

$$r^{\phi(0)}_{\alpha_0}(t) = K^{\phi(0)}_{\alpha_0}(t) + \int_0^t K_{\alpha_0}(t-u) r^{\phi(0)}_{\alpha_0}(u) du.$$

Taking the Laplace transform and using (6.18), one has

$$\langle \phi(0), f \rangle = (z - z_0) \widehat{K}^{\phi(0)}_{\alpha_0}(z) + \langle \phi(0), f \rangle \widehat{K}_{\alpha_0}(z).$$

Moreover, we necessarily have $\langle \phi(0), f \rangle \neq 0$, otherwise $\nu(0) = 0$. One deduces that $\widehat{K}_{\alpha_0}(z_0) = 1$. Reciprocally, if $\widehat{K}_{\alpha_0}(z_0) = 1$ with $z_0 \neq 0$, one can check that $\nu^{z_0}_{\alpha_0}$, given by (6.17), is an eigenvector of $\mathcal{L}^*_{\alpha_0}$ associated to z_0. In particular the changes of variable $y = \varphi^{\alpha_0}_u(0)$ and $x = \varphi^{\alpha_0}_t(0)$ in (6.17) yields

$$\int_{\mathbb{R}_+} \nu^{z_0}_{\alpha_0}(dx) = \int_{\mathbb{R}_+} e^{-z_0 t} \exp\left(- \int_0^t f(\varphi^{\alpha_0}_u(0)) du \right) dt = \widehat{H}_{\alpha_0}(z_0) = 0.$$

□

Define

$$\kappa(\alpha) := \frac{\alpha}{\gamma(\alpha) J_0}.$$

We have similarly:

Claim 6.8. *The linear operator $D_\nu F(\nu^\infty_{\alpha_0})$ has 0 as eigenvalue, with an associated eigenvector given by*

$$\left. \frac{d}{d\alpha} \kappa(\alpha) \nu^\infty_\alpha \right|_{\alpha = \alpha_0}$$

Consider $z_0 \in \mathbb{C}^*$, with $\Re(z_0) > -\lambda^*_{\alpha_0}$ another eigenvalue. Then it holds that $J_0 \widehat{\Theta}_{\alpha_0}(z_0) = 1$.

Proof. First, we have for all $\alpha > 0$, $F(\kappa(\alpha)\nu^\infty_\alpha) = 0$. We differentiate with respect to α this equality and find:

$$D_\nu F(\nu^\infty_{\alpha_0}) \cdot \left. \frac{d}{d\alpha} \kappa(\alpha) \nu^\infty_\alpha \right|_{\alpha = \alpha_0} = 0.$$

This shows that 0 is an eigenvalue. Second, the nonlinear Fokker-Planck equation conserves the mass. So for all ν

$$\int_{\mathbb{R}_+} F(\nu)(dx) = 0.$$

Differentiating at $\nu = \nu^\infty_{\alpha_0}$ with respect to ν, we have for $\phi = \nu - \nu^\infty_\alpha$

$$\int_{\mathbb{R}_+} \left[D_\nu F(\nu^\infty_{\alpha_0}) \cdot \phi \right] (dx) = 0.$$

In other words, $D_\nu F(\nu_{\alpha_0}^\infty)$ is also mass conservative. Consider $z_0 \in \mathbb{C}^*$ with $\Re(z_0) > -\lambda_{\alpha_0}^*$ an eigenvalue of $D_\nu F(\nu_{\alpha_0}^\infty)$, associated to the eigenvector $\phi(0)$. Then $\phi(t) := e^{z_0 t}\phi(0)$ solves

$$\dot{\phi(t)} = D_\nu F(\nu_{\alpha_0}^\infty) \cdot \phi(t).$$

Because $D_\nu F(\nu_{\alpha_0}^\infty)$ preserves the mass and because $z_0 \neq 0$, one deduces that

$$\int_{\mathbb{R}_+} \phi(0)(dx) = 0.$$

Moreover, we have

$$\int_0^\infty f(x)\phi(0)(dx) \neq 0.$$

Indeed, proceed by contradiction, if $\langle \phi(0), f\rangle = 0$, we have $\mathcal{B}_{\alpha_0}\phi(0) = 0$ and so z_0 is an eigenvalue of $\mathcal{L}_{\alpha_0}^*$:

$$D_\nu F(\nu_{\alpha_0}^\infty)\phi(0) = \mathcal{L}_{\alpha_0}^*\phi(0) = z_0\phi(0).$$

But we have seen that 0 is the only eigenvalue of $\mathcal{L}_{\alpha_0}^*$ satisfying $\Re(z) > -\lambda_{\alpha_0}^*$. We now use (4.14). One has

$$e^{z_0 t}\langle \phi(0), f\rangle = \int_0^\infty r_{\alpha_0}^x(t)\phi(0)dx + J_0\int_0^t \Theta_{\alpha_0}(t-s)e^{z_0 s}\langle \phi(0), f\rangle ds.$$

We take the Laplace transform and obtain for all $z \in \mathbb{C}$ with $\Re(z) > \Re(z_0)$:

$$1 = G_{z_0}(z) + J_0\widehat{\Theta}_{\alpha_0}(z),$$

with

$$G_{z_0}(z) := \frac{z - z_0}{\langle \phi(0), f\rangle}\int_0^\infty\int_0^\infty e^{-zt} r_\alpha^x(t)\phi(0)(dx)dt.$$

We deduce that $J_0\widehat{\Theta}_{\alpha_0}(z_0) = 1$. This ends the proof. □

To approximate numerically the eigenvalues of $D_v F(\nu_{\alpha_0}^\infty)$, we approximate $D_v F(\nu_{\alpha_0}^\infty)$ by a matrix of size $N_v \times N_v$, using the Finite Volume method described in Section 6.3.2.

Discretization of the transport operator

Given a measure ν and $i \in \{1, \cdots, N_v\}$, we set $\nu^i := \frac{1}{\Delta_v}\langle \mu, \mathbb{1}_{\Omega^i}\rangle$, where Δ_v, Ω_i are given by (6.13).

Consider $g : \mathbb{R}_+ \to \mathbb{R}$. We approximate the transport operator $-\partial_x(g\nu)$ using the same upwind-scheme than in (6.15). Because here $-\partial_x(g\nu)$ is a linear operator, this approximation can be written as product between a matrix of size $N_v \times N_v$ and the vector $(\nu^j)_{j\in\{1,\cdots,N_v\}}$:

$$\frac{1}{\Delta_v}\langle -\partial_x(g(x)\nu), \mathbb{1}_{\Omega^i}\rangle \approx \left[L_{[g]}^{\text{trans}} \cdot (\nu^j)\right]_i.$$

Here, $L^{\text{trans}}_{[g]}$ is the following tridiagonal matrix:

$$L^{\text{trans}}_{[g]} := -\frac{1}{\Delta_v}\left(M^{+1/2}_{[g]} - M^{-1/2}_{[g]}\right), \tag{6.19}$$

where the matrices $M^{+1/2}_{[g]}$ and $M^{+1/2}_{[g]}$ are given by, for all $i, j \in \{1, \cdots, N_v\}$:

$$\left(M^{+1/2}_{[g]}\right)_{i,j} := \begin{cases} g(v_i + 1/2), & \text{if } j = i,\ i < N_v \text{ and } g(v_i + 1/2) > 0 \\ g(v_i + 1/2), & \text{if } j = i + 1 \text{ and } g(v_i + 1/2) \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\left(M^{-1/2}_{[g]}\right)_{i,j} := \begin{cases} g(v_i - 1/2), & \text{if } j = i - 1 \text{ and } g(v_i - 1/2) > 0 \\ g(v_i - 1/2), & \text{if } j = i,\ i > 1 \text{ and } g(v_i - 1/2) \leq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Discretization of $\mathcal{L}^*_{\alpha_0}$

Recall that $\mathcal{L}^*_{\alpha_0}\nu = -\partial_x[(b + \alpha_0)\nu] - f\nu + \langle \nu, f\rangle\delta_0$. We approximate the transport part $-\partial_x[(b + \alpha_0)\nu]$ using the matrix $L^{\text{trans}}_{[b+\alpha_0]}$ (given by (6.19) with $g \equiv b + \alpha_0$). The jump part $-f\nu + \langle \nu, f\rangle\delta_0$ is approximated by the following matrix L^{jump}, defined by:

$$\forall i, j \in \{1, \cdots, N_v\}, \quad (L^{\text{jump}})_{i,j} := \begin{cases} -f(v_i) & \text{if } i = j \text{ and } i > 1 \\ f(v_j) & \text{if } i = 1 \text{ and } j > 1 \\ 0 & \text{otherwise.} \end{cases} \tag{6.20}$$

Overall, $\mathcal{L}^*_{\alpha_0}$ is approximated by the matrix $L^{\text{trans}}_{[b+\alpha_0]} + L^{\text{jump}}$.

Discretization of $D_\nu F(\nu^\infty_{\alpha_0})$

Recall that $D_\nu F(\nu^\infty_{\alpha_0}) = \mathcal{L}^*_{\alpha_0} + \mathcal{B}_{\alpha_0}$. It remains to explain how to approximate $\mathcal{B}_{\alpha_0}\phi = -J_0\langle \phi, f\rangle \partial_x \nu^\infty_{\alpha_0}$. First, to approximate $\partial_x \nu^\infty_{\alpha_0}$, we compute the eigenvectors/eigenvalues of the matrix $L^{\text{trans}}_{[b+\alpha_0]} + L^{\text{jump}}$. This matrix has 0 as eigenvalue. Let $\nu^{\infty, N_v}_{\alpha_0}$ be the associated eigenvector, normalized such that

$$\sum_{j=1}^{N_v} \nu^{\infty, N_v}_{\alpha_0}(j) = 1.$$

Hence $\nu^{\infty, N_v}_{\alpha_0}$ is an approximation of the invariant measure $\nu^\infty_{\alpha_0}$. We then approximate $\partial_x \nu^\infty_{\alpha_0}$ by the vector

$$L^{\text{trans}}_{[1]}\nu^{\infty, N_v}_{\alpha_0},$$

where $L^{\text{trans}}_{[1]}$ is given by (6.19) with $g \equiv 1$. Finally, we approximate the operator $\mathcal{B}_{\alpha_0}$ by the matrix B_{α_0}, given by

$$\forall i, j \in \{1, \cdots, N_v\}, \quad (B_{\alpha_0})_{i,j} := -J_0 f(v_j)\left[L^{\text{trans}}_{[1]}\nu^{\infty, N_v}_{\alpha_0}\right](i). \tag{6.21}$$

6.4.1 Overall algorithm

We summarize the algorithm to compute the zeros of $z \mapsto \widehat{H}_{\alpha_0}(z)$ and $z \mapsto J(\alpha_0)\widehat{\Theta}_{\alpha_0}(z) - 1$.

Input: b, f and $\alpha_0 > 0$.
Discretization parameters: $N_v \in \mathbb{N}^*$ and $v_{\max} > 0$ (see (6.13)).
Step 1. Compute the matrix $L_{\alpha_0} := L^{\text{trans}}_{[b+\alpha_0]} + L^{\text{jump}}$, using (6.19) and (6.20).
Step 2. Compute the eigenvalues, eigenvectors of L_{α_0}. Let $\nu^{\infty,N_v}_{\alpha_0}$ be the eigenvector associated to the eigenvalue 0. Normalize it such that

$$\sum_{j=1}^{N} \nu^{\infty,N_v}_{\alpha_0}(j) = 1.$$

Return the other eigenvalues, which are approximations of the zeros of $z \mapsto \widehat{H}_{\alpha_0}(z)$.
Step 3. Compute J_0 by

$$J_0 := \frac{\alpha_0}{\sum_{j=1}^{N_v} f(v_j)\nu^{\infty,N_v}_{\alpha_0}(j)}.$$

Step 4. Compute the matrix B_{α_0} using (6.21). Compute the eigenvalues of $L_{\alpha_0} + B_{\alpha_0}$, remove the value 0. Return the other eigenvalues, which are approximations of the zeros of $z \mapsto J(\alpha_0)\widehat{\Theta}_{\alpha_0}(z) - 1$.

6.4.2 Numerical validation of the algorithm

We validate numerically the algorithm using the analytical example of Section 6.2.

We choose f to be the step function given by (6.3) (with $\beta = \beta_0$ to be specify) and $b(x) = m_0 - x$. Let $\omega_0 := 1$ and $y_0 := 2\pi(1 - 0.1)$. We choose:

$$\beta_0 := \beta^0_{\omega_0}(y_0), \quad d_0 := \delta^0_{\omega_0}(y_0),$$

where the functions $\beta^0_{\omega_0}$ and $\delta^0_{\omega_0}$ are given by (6.11). We then define α_0 and m_0 using (6.6):

$$\alpha_0 := \frac{d_0}{e^{\omega_0} - 1}, \quad \text{and} \quad m_0 := 1 + \frac{1 - d_0}{e^{\omega_0} - 1}.$$

With this choice of parameters, the zeros of $z \mapsto J(\alpha_0)\widehat{\Theta}_{\alpha_0}(z) - 1$ are given by the solutions of (6.7). Moreover, it holds that the pair of imaginary numbers $\pm iy_0$ are solutions. We report in Figure 6.7 a computation of these zeros using the algorithm of Section 6.4.1 and compare the results to the "ground-truth" given by (6.7). We choose $v_{\max} = 2$ and $N_v \in \{4000, 8000\}$. We find a good match between the ground-truth and the zero computed with the algorithm of Section 6.4.2 (see Figure 6.7).

6.4.3 A complete illustration

We finally illustrate the algorithm with a numerical analysis of the following example:

$$\forall x \geq 0, \quad f(x) := x^{10}, \quad b(x) := 2 - 2x.$$

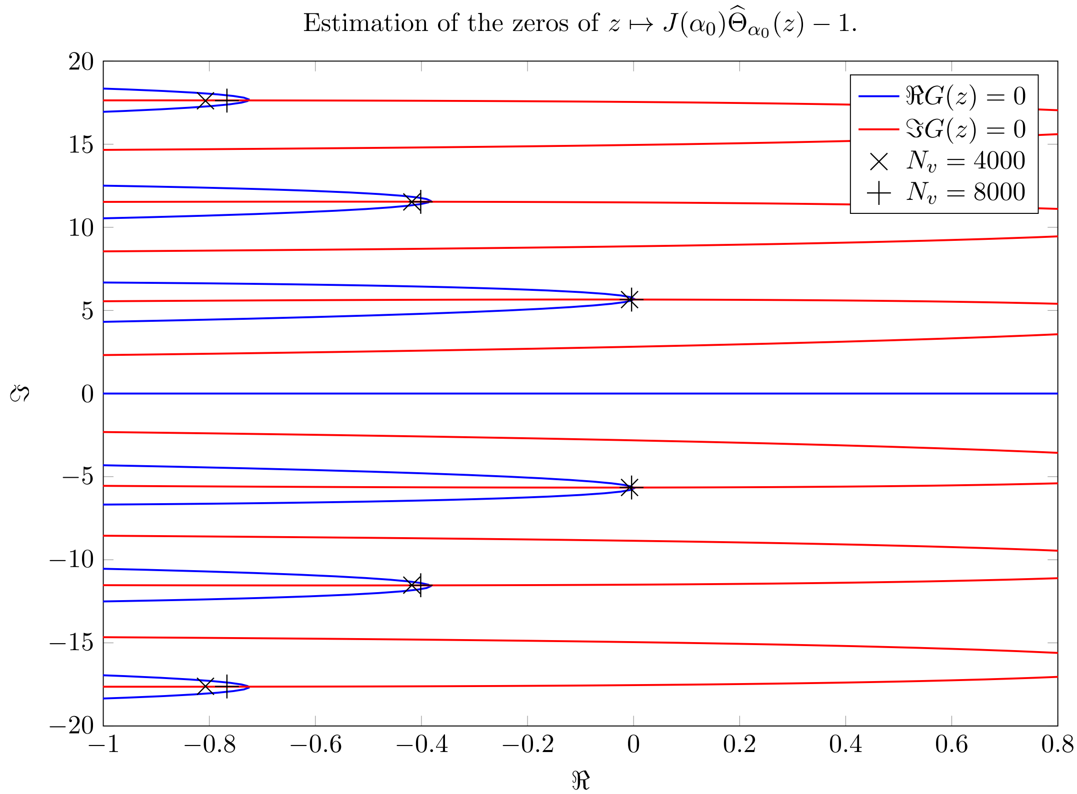


Figure 6.7: Estimations of the zeros of $z \mapsto J(\alpha_0)\widehat{\Theta}_{\alpha_0}(z) - 1$ using the algorithm described in Section 6.4.2. The exact setting is described Section 6.4.2. We used $v_{\max} = 2$ and $N_v \in \{4000, 8000\}$. Let $G(z)$ be the function defined by (6.7). The zeros of G match with the zeros of $z \mapsto J(\alpha_0)\widehat{\Theta}_{\alpha_0}(z) - 1$. The blue curve corresponds to the solution of the equation $\Re G(z) = 0$, while the red curve corresponds to $\Im G(z) = 0$. Hence, the points where the two curves intersect are the "true" zeros of G. With our choice of parameters, it holds that $\pm i 2\pi(1 - 0.1) \approx \pm i 5.655$ are a pair of zeros of G. With $N_v = 8000$, the algorithm computes the pair of zeros $-0.004 \pm i5.653$. Overall the error is or order $4 \cdot 10^{-3}$.

We have seen in Figure 5.1 a simulation with $J = 0.8$ featuring stable oscillations. We now give the bifurcation diagram. First we find numerically that the function $\alpha \mapsto J(\alpha) := \frac{\alpha}{\gamma(\alpha)}$ is strictly increasing. So, for all J, (1.2) has a unique invariant measure. We compute its stability using the algorithm of Section 6.4.1. We find two Hopf bifurcations: one for $J \approx 0.71$ and the other for $J \approx 1.06$ (see Figure 6.8(a)). Overall the invariant probability measure is stable if $J < 0.71$ or $J > 1.06$, and unstable for $J \in [0.71, 1.06]$. For $J \in [0.71, 1.06]$, we compute the stable periodic orbits and report the period T as a function of J (see Figure 6.8(b)) and the following minimum and maximum

$$\min_{t \in [0,T]} J\,\mathbb{E}\, f(X_t) \quad \text{and} \quad \max_{t \in [0,T]} J\,\mathbb{E}\, f(X_t)$$

as a function of J (see Figure 6.8(a), purple curves). Finally, we plot the periodic orbit for $J = 0.72$ (Figure 6.8(c)) and $J = 0.80$ (Figure 6.8(d)).

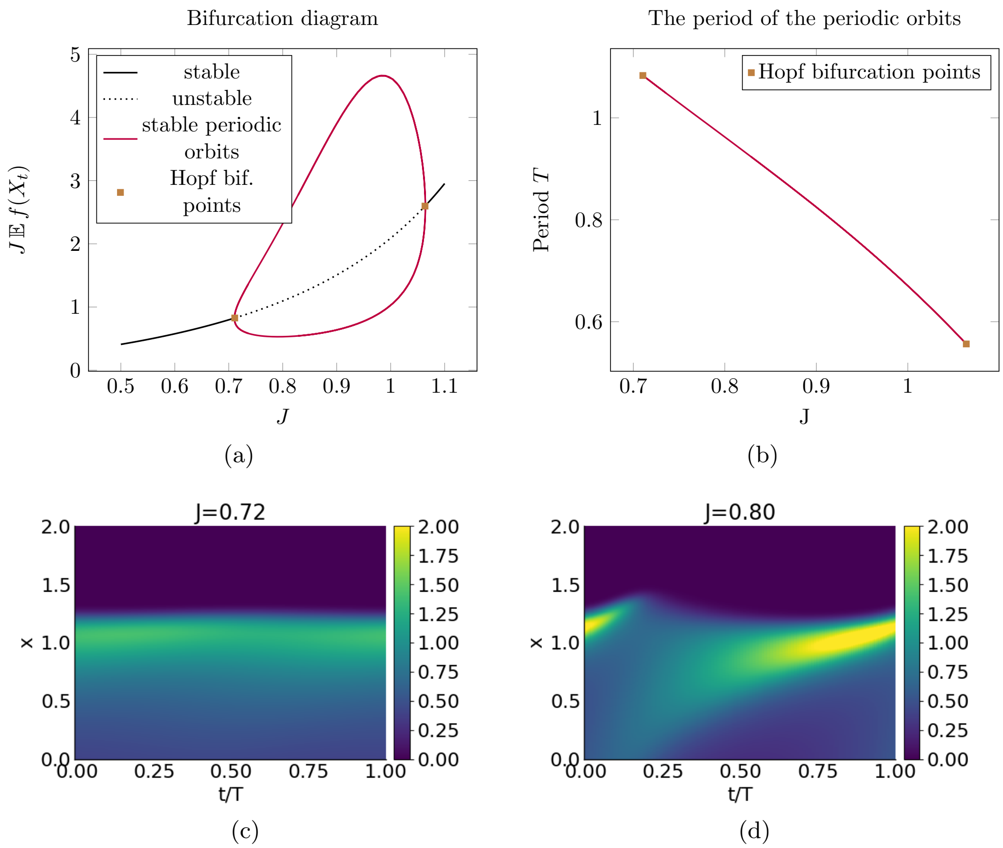


Figure 6.8: (a) Bifurcation diagram for $f(x) = x^{10}$ and $b(x) = 2 - 2x$. The black curve is computed using the algorithm described in Section 6.4.1, with $N_v = 1000$ and $v_{\max} = 2$. We find that the unique invariant measure is unstable for $J \in [0.71, 1.06]$: for this example, (1.2) exhibits two Hopf bifurcations, the first one at $J \approx 0.71$ and the second one at $J \approx 1.06$. For $J \in [0.71, 1.06]$, we compute the stable periodic orbit and report the minimum and the maximum value of $t \mapsto J\,\mathbb{E}\,f(X_t)$ over a period (the two purple curves), as well as the period of the solution as a function of J (see (b)). (c) We plot the periodic orbit for $J = 0.72$, that is close to the first Hopf bifurcation. Note that the solution is close to the invariant probability measure. (d) For $J = 0.8$ however, the periodic orbit is far from the invariant probability measure.

Bibliography

[AD20] M. Arnaudon and P. Del Moral. “A second order analysis of McKean-Vlasov semigroups”. In: *Ann. Appl. Probab.* (2020). (to appear). arXiv: 1906.05140 [math.PR].

[ACV19] B. Aymard, F. Campillo, and R. Veltz. *Mean-field limit of interacting 2D nonlinear stochastic spiking neurons*. 2019. arXiv: 1906.10232 [math.NA].

[BCGMZ13] J.-B. Bardet, A. Christen, A. Guillin, F. Malrieu, and P.-A. Zitt. “Total variation estimates for the TCP process”. In: *Electron. J. Probab.* 18 (2013), no. 10, 21. DOI: 10.1214/EJP.v18-1720.

[BRTV98] S. Benachour, B. Roynette, D. Talay, and P. Vallois. “Nonlinear self-stabilizing processes. I. Existence, invariant probability, propagation of chaos”. In: *Stochastic Process. Appl.* 75 (2) (1998), pp. 173–201. DOI: 10.1016/S0304-4149(98)00018-0.

[BRV98] S. Benachour, B. Roynette, and P. Vallois. “Nonlinear self-stabilizing processes. II. Convergence to invariant probability”. In: *Stochastic Process. Appl.* 75 (2) (1998), pp. 203–224. DOI: 10.1016/S0304-4149(98)00019-2.

[BGM10] F. Bolley, A. Guillin, and F. Malrieu. “Trend to equilibrium and particle approximation for a weakly selfconsistent Vlasov-Fokker-Planck equation”. In: *M2AN Math. Model. Numer. Anal.* 44 (5) (2010), pp. 867–884. DOI: 10.1051/m2an/2010045.

[Bre11] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011. DOI: 10.1007/978-0-387-70914-7.

[Bri88] D. R. Brillinger. “Maximum likelihood analysis of spike trains of interacting nerve cells.” In: *Biol. Cybern.* 59 (1988), pp. 189–200. DOI: 10.1007/BF00318010.

[But14] O. A. Butkovsky. “On ergodic properties of nonlinear Markov chains and stochastic McKean-Vlasov equations”. In: *Theory Probab. Appl.* 58 (4) (2014), pp. 661–674. DOI: 10.1137/S0040585X97986825.

[CCP11a] M. J. Cáceres, J. A. Carrillo, and B. Perthame. “Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states”. In: *J. Math. Neurosci.* 1 (2011), Art. 7, 33. DOI: 10.1186/2190-8567-1-7.

[CCP11b] M. J. Cáceres, J. A. Carrillo, and B. Perthame. “Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states”. In: *J. Math. Neurosci.* 1 (2011), Art. 7, 33. DOI: 10.1186/2190-8567-1-7.

[CD18] R. Carmona and F. Delarue. *Probabilistic theory of mean field games with applications. I.* Probability Theory and Stochastic Modelling. Mean field FBSDEs, control, and games. Springer, Cham, 2018, pp. xxv+713.

[CPSS15] J. A. Carrillo, B. Perthame, D. Salort, and D. Smets. “Qualitative properties of solutions for the noisy integrate and fire model in computational neuroscience”. In: *Nonlinearity* 28 (9) (2015), pp. 3365–3388. DOI: 10.1088/0951-7715/28/9/3365.

[Car67] H. Cartan. *Calcul différentiel.* Hermann, Paris, 1967.

[Che17a] J. Chevallier. “Fluctuations for mean-field interacting age-dependent Hawkes processes”. In: *Electron. J. Probab.* 22 (2017), Paper No. 42, 49. DOI: 10.1214/17-EJP63.

[Che17b] J. Chevallier. “Mean-field limit of generalized Hawkes processes”. In: *Stochastic Process. Appl.* 127 (12) (2017), pp. 3870–3912. DOI: 10.1016/j.spa.2017.02.012.

[CCDR15] J. Chevallier, M. J. Cáceres, M. Doumic, and P. Reynaud-Bouret. “Microscopic approach of a time elapsed neural model”. In: *Math. Models Methods Appl. Sci.* 25 (14) (2015), pp. 2669–2719. DOI: 10.1142/S021820251550058X.

[Cor20] Q. Cormier. *A mean-field model of Integrate-and-Fire neurons: non-linear stability of the stationary solutions.* 2020. arXiv: 2002.08649 [math.PR].

[CTV20a] Q. Cormier, E. Tanré, and R. Veltz. “Long time behavior of a mean-field model of interacting neurons”. In: *Stochastic Process. Appl.* 130 (5) (2020), pp. 2553–2595. DOI: 10.1016/j.spa.2019.07.010.

[CTV20b] Q. Cormier, E. Tanré, and R. Veltz. *Hopf bifurcation in a Mean-Field model of spiking neurons.* 2020. arXiv: 2008.11116 [math.PR].

[Cos90] O. L. V. Costa. “Stationary distributions for piecewise-deterministic Markov processes”. In: *J. Appl. Probab.* 27 (1) (1990), pp. 60–73. DOI: 10.2307/3214595.

[CIR52] R. Courant, E. Isaacson, and M. Rees. “On the solution of nonlinear hyperbolic differential equations by finite differences”. In: *Comm. Pure Appl. Math.* 5 (1952), pp. 243–255. DOI: 10.1002/cpa.3160050303.

[Dav84] M. H. A. Davis. “Piecewise-deterministic Markov processes: a general class of nondiffusion stochastic models”. In: *J. Roy. Statist. Soc. Ser. B* 46 (3) (1984). With discussion, pp. 353–388.

[DGLP15] A. De Masi, A. Galves, E. Löcherbach, and E. Presutti. “Hydrodynamic limit for interacting neurons”. In: *J. Stat. Phys.* 158 (4) (2015), pp. 866–902. DOI: 10.1007/s10955-014-1145-1.

[DIRT15a] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré. “Global solvability of a networked integrate-and-fire model of McKean-Vlasov type”. In: *Ann. Appl. Probab.* 25 (4) (2015), pp. 2096–2133. DOI: 10.1214/14-AAP1044.

[DIRT15b] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré. "Particle systems with a singular mean-field self-excitation. Application to neuronal networks". In: *Stochastic Process. Appl.* 125 (6) (2015), pp. 2451–2492. DOI: 10.1016/j.spa.2015.01.007.

[DL17] S. Ditlevsen and E. Löcherbach. "Multi-class oscillating systems of interacting neurons". In: *Stochastic Process. Appl.* 127 (6) (2017), pp. 1840–1869. DOI: 10.1016/j.spa.2016.09.013.

[DV17] A. Drogoul and R. Veltz. "Hopf bifurcation in a nonlocal nonlinear transport equation stemming from stochastic neural dynamics". In: *Chaos* 27 (2) (2017), pp. 021101, 6. DOI: 10.1063/1.4976510.

[DV21] A. Drogoul and R. Veltz. "Exponential stability of the stationary distribution of a mean field of spiking neural network". In: *J. Differential Equations* 270 (2021), pp. 809–842. DOI: 10.1016/j.jde.2020.08.001.

[DO16] A. Duarte and G. Ost. "A model for neural activity in the absence of external stimuli". In: *Markov Process. Related Fields* 22 (1) (2016), pp. 37–52.

[DOR15] A. Duarte, G. Ost, and A. A. Rodríguez. "Hydrodynamic limit for spatially structured interacting neurons". In: *J. Stat. Phys.* 161 (5) (2015), pp. 1163–1202. DOI: 10.1007/s10955-015-1366-y.

[EN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Springer-Verlag, New York, 2000.

[ELL19] X. Erny, E. Löcherbach, and D. Loukianova. "Conditional propagation of chaos for mean field systems of interacting neurons". In: (2019). arXiv: 1909.02925 [math.PR].

[FST19] O. Faugeras, É. Soret, and E. Tanré. "Asymptotic behaviour of a network of neurons with random linear interactions". In: (2019). HAL: https://hal.inria.fr/hal-01986927.

[Fel41] W. Feller. "On the integral equation of renewal theory". In: *Ann. Math. Statistics* 12 (1941), pp. 243–267. DOI: 10.1214/aoms/1177731708.

[Fle80] T. M. Flett. *Differential analysis*. Differentiation, differential equations and differential inequalities. Cambridge University Press, Cambridge-New York, 1980.

[FL16] N. Fournier and E. Löcherbach. "On a toy model of interacting neurons". In: *Ann. Inst. Henri Poincaré Probab. Stat.* 52 (4) (2016), pp. 1844–1876. DOI: 10.1214/15-AIHP701.

[Gab18] P. Gabriel. "Measure solutions to the conservative renewal equation". In: *CIMPA School on Mathematical Models in Biology and Medicine*. Vol. 62. ESAIM Proc. Surveys. EDP Sci., Les Ulis, 2018, pp. 68–78. DOI: 10.1051/proc/201862186206.

[Ger95] W. Gerstner. "Time structure of the activity in neural network models". In: *Phys. Rev. E* 51 (1 1995), pp. 738–758. DOI: 10.1103/PhysRevE.51.738.

[GKNP14] W. Gerstner, W. M. Kistler, R. Naud, and L. Paninski. *Neuronal Dynamics: From single neurons to networks and models of cognition*. Cambridge University Press, 2014.

[GPPP12] G. Giacomin, K. Pakdaman, X. Pellegrin, and C. Poquet. "Transitions in active rotator systems: invariant hyperbolic manifold approach". In: *SIAM J. Math. Anal.* 44 (6) (2012), pp. 4165–4194. DOI: 10.1137/110846452.

[GM86] G. H. Golub and C. D. Meyer Jr. "Using the QR factorization and group inversion to compute, differentiate, and estimate the sensitivity of stationary probabilities for Markov chains". In: *SIAM J. Algebraic Discrete Methods* 7 (2) (1986), pp. 273–281. DOI: 10.1137/0607031.

[GLS90] G. Gripenberg, S.-O. Londen, and O. Staffans. *Volterra integral and functional equations*. Vol. 34. Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1990. DOI: 10.1017/CBO9780511662805.

[GM74] M. E. Gurtin and R. C. MacCamy. "Non-linear age-dependent population dynamics". In: *Arch. Rational Mech. Anal.* 54 (1974), pp. 281–300. DOI: 10.1007/BF00250793.

[Hal69] J. K. Hale. *Ordinary differential equations*. Pure and Applied Mathematics, Vol. XXI. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1969.

[HI11] M. Haragus and G. Iooss. *Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems*. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, Les Ulis, 2011. DOI: 10.1007/978-0-85729-112-7.

[HS19] S. Heesen and W. Stannat. "Fluctuation limits for mean-field interacting nonlinear Hawkes processes". In: (2019). arXiv: 1912.10854 [math.PR].

[HT12] S. Herrmann and J. Tugaut. "Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small-noise limit". In: *ESAIM Probab. Stat.* 16 (2012), pp. 277–305. DOI: 10.1051/ps/2011152.

[HKL18] P. Hodara, N. Krell, and E. Löcherbach. "Non-parametric estimation of the spiking rate in systems of interacting neurons". In: *Stat. Inference Stoch. Process.* 21 (1) (2018), pp. 81–111. DOI: 10.1007/s11203-016-9150-4.

[HP19] P. Hodara and I. Papageorgiou. "Poincaré-Type Inequalities for Compact Degenerate Pure Jump Markov Processes". In: *Mathematics* 7 (6) (2019). DOI: 10.3390/math7060518.

[HH52] A. L. Hodgkin and A. F. Huxley. "A quantitative description of membrane current and its application to conduction and excitation in nerve". In: *The Journal of Physiology* 117 (4) (1952), pp. 500–544. DOI: 10.1113/jphysiol.1952.sp004764. eprint: https://physoc.onlinelibrary.wiley.com/doi/pdf/10.1113/jphysiol.1952.sp004764.

[Hör90] L. Hörmander. *The analysis of linear partial differential operators. I.* Second. Springer Study Edition. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1990, pp. xii+440. DOI: 10.1007/978-3-642-61497-2.

[JBHD11] P. Jahn, R. W. Berg, J. Hounsgaard, and S. Ditlevsen. "Motoneuron membrane potentials follow a time inhomogeneous jump diffusion process". In: *J. Comput. Neurosci.* 31 (3) (2011), pp. 563–579. DOI: 10.1007/s10827-011-0326-z.

[Kie12] H. Kielhöfer. *Bifurcation theory.* Second. Vol. 156. Applied Mathematical Sciences. An introduction with applications to partial differential equations. Springer, New York, 2012. DOI: 10.1007/978-1-4614-0502-3.

[Lap07] L. Lapicque. "Recherches quantitatives sur l'excitation électrique des nerfs traitée comme une polarisation". In: *J Physiol Pathol Générale* (1907), 9:567–578.

[LM20] E. Löcherbach and P. Monmarché. "Metastability for systems of interacting neurons". In: (2020). arXiv: 2004.13353 [math.PR].

[LP20a] E. Luçon and C. Poquet. "Emergence of Oscillatory Behaviors for Excitable Systems with Noise and Mean-Field Interaction: A Slow-Fast Dynamics Approach". In: *Comm. Math. Phys.* 373 (3) (2020), pp. 907–969. DOI: 10.1007/s00220-019-03641-y.

[LP20b] E. Luçon and C. Poquet. "Periodicity induced by noise and interaction in the kinetic mean-field FitzHugh-Nagumo model". In: *Ann. Appl. Probab.* (2020). (to appear). arXiv: 1811.00305 [math.PR].

[MKJSSH13] S. Mack, E. Kandel, T. Jessell, J. Schwartz, S. Siegelbaum, and A. Hudspeth. *Principles of Neural Science, Fifth Edition.* Principles of Neural Science. McGraw-Hill Education, 2013.

[MQW18] S. Mischler, C. Quiñinao, and Q. Weng. "Weak and strong connectivity regimes for a general time elapsed neuron network model". In: *J. Stat. Phys.* 173 (1) (2018), pp. 77–98. DOI: 10.1007/s10955-018-2122-x.

[MS16] S. Mischler and J. Scher. "Spectral analysis of semigroups and growth-fragmentation equations". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 33 (3) (2016), pp. 849–898. DOI: 10.1016/j.anihpc.2015.01.007.

[MW18] S. Mischler and Q. Weng. "Relaxation in Time Elapsed Neuron Network Models in the Weak Connectivity Regime". In: *Acta Appl. Math.* 157 (2018), pp. 45–74. DOI: 10.1007/s10440-018-0163-4.

[ODBBF15] M. E. J. Obien, K. Deligkaris, T. Bullmann, D. J. Bakkum, and U. Frey. "Revealing neuronal function through microelectrode array recordings". In: *Frontiers in Neuroscience* 8 (2015), p. 423. DOI: 10.3389/fnins.2014.00423.

[PPS10] K. Pakdaman, B. Perthame, and D. Salort. "Dynamics of a structured neuron population". In: *Nonlinearity* 23 (1) (2010), pp. 55–75. DOI: 10.1088/0951-7715/23/1/003.

[PPS13] K. Pakdaman, B. Perthame, and D. Salort. "Relaxation and self-sustained oscillations in the time elapsed neuron network model". In: *SIAM J. Appl. Math.* 73 (3) (2013), pp. 1260–1279. DOI: 10.1137/110847962.

[Per07] B. Perthame. *Transport equations in biology*. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.

[PDRDM19] B. Pietras, F. Devalle, A. Roxin, A. Daffertshofer, and E. Montbrió. "Exact firing rate model reveals the differential effects of chemical versus electrical synapses in spiking networks". In: *Phys. Rev. E* 100 (4) (2019), p. 042412. DOI: 10.1103/PhysRevE.100.042412.

[Pro05] P. E. Protter. *Stochastic integration and differential equations*. Vol. 21. Stochastic Modelling and Applied Probability. Second edition. Version 2.1, Corrected third printing. Springer-Verlag, Berlin, 2005. DOI: 10.1007/978-3-662-10061-5.

[Prü83] J. Prüss. "Stability analysis for equilibria in age-specific population dynamics". In: *Nonlinear Anal.* 7 (12) (1983), pp. 1291–1313. DOI: 10.1016/0362-546X(83)90002-0.

[RT16] P. Robert and J. Touboul. "On the dynamics of random neuronal networks". In: *J. Stat. Phys.* 165 (3) (2016), pp. 545–584. DOI: 10.1007/s10955-016-1622-9.

[Sch85a] M. Scheutzow. "Noise can create periodic behavior and stabilize nonlinear diffusions". In: *Stochastic Process. Appl.* 20 (2) (1985), pp. 323–331. DOI: 10.1016/0304-4149(85)90219-4.

[Sch85b] M. Scheutzow. "Some examples of nonlinear diffusion processes having a time-periodic law". In: *Ann. Probab.* 13 (2) (1985), pp. 379–384.

[SDG17] T. Schwalger, M. Deger, and W. Gerstner. "Towards a theory of cortical columns: From spiking neurons to interacting neural populations of finite size". In: *PLOS Computational Biology* 13 (4) (2017), pp. 1–63. DOI: 10.1371/journal.pcbi.1005507.

[Sta87] O. J. Staffans. "Hopf bifurcation for functional and functional-differential equations with infinite delay". In: *J. Differential Equations* 70 (1) (1987), pp. 114–151. DOI: 10.1016/0022-0396(87)90171-9.

[SHMHSWHN06] K. Sugino, C. Hempel, M. Miller, A. Hattox, P. Shapiro, C. Wu, Z. J. Huang, and S. B. Nelson. "Molecular taxonomy of major neuronal classes in the adult mouse forebrain". In: *Nature Neuroscience* 9 (1) (2006), pp. 99–107. DOI: 10.1038/nn1618.

[Tug13] J. Tugaut. “Convergence to the equilibria for self-stabilizing processes in double-well landscape”. In: *Ann. Probab.* 41 (3A) (2013), pp. 1427–1460. DOI: 10.1214/12-AOP749.

[Tug14] J. Tugaut. “Phase transitions of McKean-Vlasov processes in double-wells landscape”. In: *Stochastics* 86 (2) (2014), pp. 257–284. DOI: 10.1080/17442508.2013.775287.

[Vel20] R. Veltz. *BifurcationKit.jl*. 2020. HAL: https://hal.inria.fr/hal-02902346.

[Ver06] A. Y. Veretennikov. “On ergodic measures for McKean-Vlasov stochastic equations”. In: *Monte Carlo and quasi-Monte Carlo methods 2004*. Springer, Berlin, 2006, pp. 471–486. DOI: 10.1007/3-540-31186-6_29.

[WTGWSLM04] Y. Wang, M. Toledo-Rodriguez, A. Gupta, C. Wu, G. Silberberg, J. Luo, and H. Markram. “Anatomical, physiological and molecular properties of Martinotti cells in the somatosensory cortex of the juvenile rat”. In: *The Journal of Physiology* 561 (1) (2004), pp. 65–90. DOI: 10.1113/jphysiol.2004.073353.

[Web85] G. F. Webb. *Theory of nonlinear age-dependent population dynamics*. Vol. 89. Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 1985.

[Wid41] D. V. Widder. *The Laplace Transform*. Princeton Mathematical Series, v. 6. Princeton University Press, Princeton, N. J., 1941, pp. x+406.

[WC72] H. R. Wilson and J. D. Cowan. “Excitatory and Inhibitory Interactions in Localized Populations of Model Neurons”. In: *Biophys J* 12 (1) (1972), pp. 1–24. DOI: 10.1016/S0006-3495(72)86068-5.