



Synchronization in a Kuramoto mean field game

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ABSTRACT

The classical Kuramoto model is studied in the setting of an infinite horizon mean field game. The system is shown to exhibit both synchronization and phase transition. Incoherence below a critical value of the interaction parameter is demonstrated by the stability of the uniform distribution. Above this value, the game bifurcates and develops self-organizing time homogeneous Nash equilibria. As interactions get stronger, these stationary solutions become fully synchronized. Results are proved by an amalgam of techniques from nonlinear partial differential equations, viscosity solutions, stochastic optimal control and stochastic processes.

ARTICLE HISTORY

Received 23 October 2022
Accepted 17 June 2023

KEYWORDS

Mean field games; Kuramoto model; synchronization; viscosity solutions

2020 MATHEMATICS

SUBJECT CLASSIFICATION

35Q89; 35D40; 39N80;
91A16; 92B25

1. Introduction

Originally motivated by systems of chemical and biological oscillators, the classical Kuramoto model [1] has found an amazing range of applications from neuroscience to Josephson junctions in superconductors, and has become a key mathematical model to describe self organization in complex systems. These autonomous oscillators are coupled through a nonlinear interaction term which plays a central role in the long time behavior of the system. While the system is unsynchronized when this term is not sufficiently strong, fascinatingly they exhibit an abrupt transition to self organization above a critical value of the interaction parameter. Synchronization is an emergent property that occurs in a broad range of complex systems such as neural signals, heart beats, fire-fly lights and circadian rhythms, and the Kuramoto dynamical system is widely used as the main phenomenological model. Expository papers [2, 3] and the references therein provide an excellent introduction to the model and its applications.

The analysis of the coupled Kuramoto oscillators through a mean field game formalism is first explored by [4, 5] proving bifurcation from incoherence to coordination by a formal linearization and a spectral argument. [6] further develops this analysis in their application to a jet-lag recovery model. We follow these pioneering studies and analyze the Kuramoto model as a discounted infinite horizon stochastic game in the limit when the number of oscillators goes to infinity. We treat the system of oscillators as an infinite particle system, but instead of positing the dynamics of the particles, we let the individual particles endogenously determine their behaviors by minimizing a cost functional and hopefully, settling in a Nash equilibrium. Once the search for equilibrium is recast in this way, equilibria are given by solutions of nonlinear systems. Analytically, they are characterized by a backward dynamic

programming equation coupled to a forward Fokker-Planck-Kolmogorov equation (see equation (1.3) below), and in the probabilistic approach, by forward-backward stochastic differential equations. Stability analysis of the solutions is delicate because of this forward-backward nature of the solution, and to the best of our knowledge, it remains a challenging problem. Except possibly in the finite horizon potential case (cf. [7] and the references therein) it has not been fully addressed in the existing literature on the subject. For the stability results of the Kuramoto model in the classical setting, the interested reader could consult [8, 9] and the references therein.

With finitely many oscillators, we consider the following version of the model already introduced in [5, 6]. We fix a large integer N and for $i \in \{1, \dots, N\}$ let θ_t^i be the phase of the i -th oscillator at time $t \geq 0$. We assume the phases θ_t^i are controlled Ito diffusion processes satisfying, $d\theta_t^i = \alpha_t^i dt + \sigma dB_t^i$, where B^i 's are independent Brownian motions, and the control processes α^i are exerted by the individual oscillators so as to simultaneously minimize their costs given by

$$\alpha^i \mapsto J^i(\alpha) := \mathbb{E} \int_0^\infty e^{-\beta t} \left[\kappa L(\theta_t^i, \boldsymbol{\theta}_t) + \frac{1}{2} (\alpha_t^i)^2 \right] dt,$$

where $\alpha = (\alpha^1, \dots, \alpha^N)$ and $\boldsymbol{\theta}_t = (\theta_t^1, \dots, \theta_t^N)$. The positive constants σ, β are respectively, the common standard deviations of the random shocks affecting the dynamics of the phases, and the common discounting factor used to compute the present value of the cost. The centrally important positive constant κ models the strength of the interactions between the oscillators.

In line with the classical literature on Kuramoto's synchronization theory, we assume that the running cost function L is given by

$$L(\theta^i, \boldsymbol{\theta}) = \frac{1}{N} \sum_{j \neq i} 2 (\sin((\theta^i - \theta^j)/2))^2 = \frac{1}{N} \sum_{j=1}^N 2 (\sin((\theta^i - \theta^j)/2))^2.$$

The cost L accounts for the cooperation between the N oscillators by incentivizing them to align their frequencies, while the term $(\alpha_t^i)^2$ represents a form of kinetic energy which is also to be minimized. It is convenient to express the above cost functional by using the empirical distribution measure of the oscillators as follows,

$$L(\theta_t^i, \boldsymbol{\theta}_t) = c(\theta_t^i, \bar{\mu}_t^N), \quad \text{where} \quad c(\theta, \mu) := \int 2 (\sin((\theta - \theta')/2))^2 \mu(d\theta'), \quad (1.1)$$

and the empirical measure $\bar{\mu}_t^N$ is given by

$$\bar{\mu}_t^N = \bar{\mu}(\boldsymbol{\theta}_t) := \frac{1}{N} \sum_{j=1}^N \delta_{\theta_t^j}.$$

As the finite particle system is essentially intractable, especially for large values of N , we follow the approach of [10–15] that is now considered standard, and approximate the Nash equilibria for the above system of oscillators by letting their number N go to infinity. Then, for a given flow of probability measures $\boldsymbol{\mu} = (\mu_t)_{t \geq 0}$, the stochastic optimal control problem for the representative oscillator is to minimize

$$\alpha \in \mathcal{A} \mapsto \mathbb{E} \int_0^\infty e^{-\beta t} \left(\ell(t, X_t) + \frac{1}{2} \alpha_t^2 \right) dt, \quad (1.2)$$

where \mathcal{A} is the set of all progressively measurable processes, the *running cost* $\ell(t, x)$ is equal to $\kappa c(x, \mu_t)$ with c as in (1.1), and X_t is the controlled phase of the representative oscillator given by $X_t = X_0 + \int_0^t \alpha_u du + \sigma B_t$, for a Brownian motion B_t . The Nash equilibrium, as defined in Definition 3.1, is achieved when the flow $\mu = (\mu_t)_{t \geq 0}$ is given by the marginal laws of the optimal process X_t^* . By direct methods, Lemma 4.5 proves the existence of such equilibrium flows starting from any initial distribution.

The model is equivalently described by the following two coupled partial differential equations for the value function v and the density $\mu(t, x)$

$$\begin{aligned} \beta v(t, x) - \partial_t v(t, x) - \frac{\sigma^2}{2} \partial_{xx}^2 v(t, x) + \frac{1}{2} (\partial_x v(t, x))^2 &= \kappa c(x, \mu_t) \\ \partial_t \mu(t, x) - \frac{\sigma^2}{2} \partial_{xx}^2 \mu(t, x) - \partial_x (\partial_x v(t, x) \mu(t, x)) &= 0. \end{aligned} \quad (1.3)$$

The first equation is the backward Hamilton-Jacobi-Bellman equation, and the second equation is the forward Fokker-Planck equation. It is immediate that the uniform distribution $U(dx) = dx/(2\pi)$ on the torus gives a stationary equilibrium flow. Indeed, $c(x, U) \equiv 1$ and therefore, the optimal control for the above problem with the constant flow U is identically equal to zero. As the uniform distribution has no special structure, it represents incoherence among the oscillators, and when the interaction parameter is small, we show that all the solutions of the Kuramoto mean field game converge to this incoherent state. This global attraction is proved in Lemma 4.3 for $\kappa < \beta\sigma^2/4$ which is analogous to the long time behavior studied in [16] with small running cost, proving not only stability but also uniqueness. Theorem 4.4 considers all κ less than the critical value

$$\kappa_c := \beta\sigma^2 + \sigma^4/2, \quad (1.4)$$

and proves that there are solutions that start “close” to the uniform distribution and converge to it as time tends to infinity. Thus, Lemma 4.3 and Theorem 4.4 reveal that incoherence is the main paradigm in the sub-critical regime $\kappa < \kappa_c$. Theorem 4.1 analyzes the case $\kappa > \kappa_c$, and proves that there are infinitely many non-uniform stationary solutions for these interaction parameter values. In particular, these solutions do not converge to the incoherent uniform distribution and numerically they are stable. Furthermore, Theorem 4.2 shows convergence to full synchronization as κ gets larger. As in the earlier works [4, 5], we interpret non-uniform distributions as self-organizing states, and formally conclude that synchronization is the main paradigm for $\kappa > \kappa_c$.

Hence, these results suggest that κ_c is the sharp threshold for the stability of incoherence, and that there is a phase transition from total disorder to self organization exactly at this critical interaction parameter κ_c . However, further analysis is needed to strengthen this statement. Indeed, the phase transition in the classical Kuramoto model is a pitchfork bifurcation. For the mean-field game version, it is also possible that for $\kappa > \kappa_c$, the uniform measure could become unstable and the non-uniform invariant probability measure become stable. We leave these interesting questions and the uniqueness of non-uniform states to future studies.

These questions also arise in a similar one-dimensional model for congestion with a local interaction studied in [17, 18]. In particular, they consider deterministic optimal control problems with ergodic cost. Additionally, a class of problems with trigonometric running cost are studied and some explicit solutions are constructed in [19].

The classical Kuramoto model with noise has been the object of many studies, and the mean field version is the following McKean-Vlasov stochastic differential equation

$$dX_t = -\kappa \int_{\mathbb{T}} \sin(X_t - y) \mathcal{L}(X_t)(dy) dt + \sigma dB_t,$$

where $\mathcal{L}(X_t)$ is the law of the random variable X_t . The uniform distribution is shown in [20] to be both locally and globally stable when $\kappa < \sigma^2$. The corresponding finite particle system is studied in [21, 22]. There, it is proven that the solutions of the finite model remain close to the solution of the above equation for a very long time, on the order of $o(\exp(N))$. Similar results are also proved for the Kuramoto mean field game with an ergodic cost in [5], by using bifurcation theory techniques including the Lyapunov-Schmidt reduction method to show the existence of non-uniform stationary solutions near the critical value $\kappa_c^* = \sigma^4/2$. Rabinowitz bifurcation theorem and other global techniques are used in [23] for similar results.

The classical Kuramoto model and its mean field game versions provide a mechanism for the analysis of self organization. However, they cannot model synchronization with external drivers, thus requiring additional terms. Indeed, the jet-lag recovery model of [6] introduce a cost for misalignment with the exogenously given sunlight frequency, providing an incentive to be in synch with the environment as well. These studies are clear evidences of the modeling potential of the mean field game formalism in all models when self organization is the salient feature.

The paper is organized as follows. After a short section on notation, the Kuramoto mean field game is introduced in Section 3, and the main results are stated in Section 4. Section 5 briefly summarizes all control problems used in the paper. Stationary solutions are defined and a fixed-point characterization is proved in Section 6. The super-critical case is studied in Section 7 and full synchronization in Section 8. Incoherence is demonstrated in Section 9 by proving the convergence of all solutions to the uniform distribution when the interaction parameter is small, and local stability of the uniform distribution is established in Section 10 for all $\kappa < \kappa_c$. For completeness, solutions starting from any distribution are constructed in the Appendix A, and we provide the expected comparison result for a degenerate Eikonal equation in the Appendix B.

2. Notation

The *state-space* is the one-dimensional torus $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, $\mathcal{P}(\mathbb{T})$ is the space of all probability measures on \mathbb{T} . For $\nu \in \mathcal{P}(\mathbb{T})$, $f \in C(\mathbb{T})$, we use the standard notation $\nu(f) := \int_{\mathbb{T}} f(x) \nu(dx)$. We say that a probability measure $\nu \in \mathcal{P}(\mathbb{T})$ is the *law* $\mathcal{L}(X)$ of X , if $\mathbb{E}[f(X)] = \nu(f)$ for every $f \in C(\mathbb{T})$. We also use the following space of continuous functions,

$$\mathcal{C} := \{ \xi = (\gamma, \eta) : [0, \infty) \mapsto \mathbb{R}^2 : \text{continuous and bounded} \}.$$

We fix a filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ supporting an \mathbb{F} -adapted Brownian motion $(B_t)_{t \geq 0}$. We assume that the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ satisfies the usual conditions, i.e. \mathcal{F}_0 is complete and \mathcal{F}_t is right-continuous. The initial filtration is nontrivial so that for any probability measure $\mu_0 \in \mathcal{P}(\mathbb{T})$, one can construct an \mathcal{F}_0 measurable, \mathbb{T} valued random variable X_0 with distribution μ_0 . For $t \geq 0$, the set of *admissible controls* \mathcal{A}_t is the collection all progressively measurable processes $\alpha : [t, \infty) \rightarrow \mathbb{R}$, and we set $\mathcal{A} := \mathcal{A}_0$.

For $\mu \in \mathcal{P}(\mathbb{T})$, $z \in \mathbb{T}$ and a Borel subset $B \subset \mathbb{T}$, we define the *translation of μ* by,

$$\mu(B; x) := \mu(\{z \in \mathbb{T} : x + z \in B\}). \tag{2.1}$$

Finally, we record several elementary trigonometric identities that are used repeatedly. For $\mu \in \mathcal{P}(\mathbb{T})$, let $c(x, \mu)$ be as in the Introduction. As $2(\sin(x/2))^2 = 1 - \cos(x)$,

$$c(x, \mu) = \int_{\mathbb{T}} 2(\sin((x - y)/2))^2 \mu(dy) = 1 - a(\mu) \cos(x) - b(\mu) \sin(x), \tag{2.2}$$

where $a(\mu) := \mu(\cos)$, and $b(\mu) := \mu(\sin)$. In particular, there is $z^* \in \mathbb{T}$ such that

$$a(\mu(\cdot; z^*)) = g(\mu) := \sqrt{(a(\mu))^2 + (b(\mu))^2}, \quad \text{and} \quad b(\mu(\cdot; z^*)) = 0. \tag{2.3}$$

3. Kuramoto mean-field game

Given a flow of probability measures $\mu = (\mu_t)_{t \geq 0}$, set

$$\ell_\mu(t, x) := \kappa[c(x, \mu_t) - 1] = -\kappa \mu_t(\cos) \cos(x) - \kappa \mu_t(\sin) \sin(x), \quad x \in \mathbb{T}, t \geq 0.$$

Consider the optimal control problem (1.2) with this running cost. Then, the problem is

$$v_\mu := \inf_{\alpha \in \mathcal{A}} J_{\mu, \kappa}(\alpha) := \inf_{\alpha \in \mathcal{A}} \mathbb{E} \int_0^\infty e^{-\beta t} \left(\ell_\mu(t, X_t^\alpha) + \frac{1}{2} \alpha_t^2 \right) dt, \tag{3.1}$$

where as in the introduction, $X_t^\alpha := X_0 + \int_0^t \alpha_u du + \sigma B_t$, with a Brownian motion $(B_t)_{t \geq 0}$ and an initial condition X_0 satisfying $\mathcal{L}(X_0) = \mu_0$.

Definition 3.1. We say that $\mu = (\mu_t)_{t \geq 0}$ is a solution to the Kuramoto mean-field game with interaction parameter κ starting from initial distribution μ_0 , if there exists $\alpha^* \in \mathcal{A}$ such that $J_{\mu, \kappa}(\alpha^*) = \inf_{\alpha \in \mathcal{A}} J_{\mu, \kappa}(\alpha)$ and $\mu_t = \mathcal{L}(X_t^{\alpha^*})$ for all $t \geq 0$.

Example 3.2. Consider an initial condition X_0 satisfying $\mathbb{E} \cos(X_0) = \mathbb{E} \sin(X_0) = 0$, and the flow of probability measures $\mu = (\mu_t)_{t \geq 0}$ with $\mu_t := \mathcal{L}(X_0 + \sigma B_t)$. Then, for every $t \geq 0$,

$$\begin{aligned} \ell_\mu(t, x) &= -\kappa(\mu_t(\cos) \cos(x) + \mu_t(\sin) \sin(x)) \\ &= -\kappa(\cos(x) \mathbb{E}[\cos(X_0 + \sigma B_t)] + \sin(x) \mathbb{E}[\sin(X_0 + \sigma B_t)]) \\ &= -\kappa(\cos(x) e^{-\frac{\sigma^2}{2} t} \mathbb{E}[\cos(X_0)] + \sin(x) e^{-\frac{\sigma^2}{2} t} \mathbb{E}[\sin(X_0)]) = 0. \end{aligned}$$

Therefore, for any $\alpha \in \mathcal{A}$, $J_{\mu, \kappa}(\alpha) = \mathbb{E} \int_0^\infty e^{-\beta t} \frac{\alpha_t^2}{2} dt \geq 0 = J_{\mu, \kappa}(0)$, implying that $\alpha^* \equiv 0$ is the minimizer of $J_{\mu, \kappa}(\alpha)$, and μ is the law of the dynamics controlled by α^* . Hence, μ is a solution of the Kuramoto mean-field game for every κ .

Now suppose that $\mathcal{L}(X_0)$ is the uniform probability measure on the torus $U(dx) = dx/(2\pi)$. As any translation of U is equal to itself, $\mu_t = \mathcal{L}(X_0 + \sigma B_t) = U$ for all $t \geq 0$. Thus, U is a stationary solution. □

The uniform distribution represents complete incoherence, and we refer to it as the *incoherent* (or *uniform*) solution. We next introduce the stationary solutions of the Kuramoto mean-field game.

Definition 3.3. We call a probability measure $\mu \in \mathcal{P}(\mathbb{T})$ a stationary solution if the constant flow $\mu = (\mu_t)_{t \geq 0}$ with $\mu_t = \mu$ for all $t \geq 0$ is a solution of the Kuramoto mean-field game. We say that μ is self-organizing or non-uniform if it is not equal to the uniform measure U .

We record the following simple result for future reference.

Lemma 3.4. *The uniform probability measure U on the torus is the incoherent stationary solution of the Kuramoto mean-field game. Moreover, a stationary solution μ is the uniform probability measure if and only if $\mu(\cos) = \mu(\sin) = 0$.*

Proof In [Example 3.2](#), we have shown that U is a stationary solution and that $c(\cdot, U) \equiv 1$. Now suppose that μ is a stationary solution with $\mu(\cos) = \mu(\sin) = 0$. Then, as in [Example 3.2](#), we conclude that the optimal solution of the control problem (1.2) is $\alpha^* \equiv 0$, and the optimal state process satisfies $dX_t^* = \sigma dB_t$. As by stationarity $\mathcal{L}(X_t^*) = \mu$ for every $t \geq 0$, the density f of μ solves the Fokker-Plank equation $f_{xx}(x) = 0$ on the torus. Hence, f is equal to a constant, and $\mu = U$. \square

Remark 3.5 (Invariance by translation). Assume that μ is a stationary solution. The symmetry of the problem implies that the translated measure $\mu(\cdot; z)$ is also a stationary solution for every z .

4. Main results

In this section, we state all the main results of the paper. Recall the critical interaction parameter κ_c of (1.4). In [Section 7](#), we study the super-critical case $\kappa > \kappa_c$, and prove the following result.

Theorem 4.1 (Super-critical interaction: synchronization). *For all interaction parameters $\kappa > \kappa_c$, there are non-uniform stationary solutions of the Kuramoto mean field game.*

Suppose μ is one of the non-uniform stationary solutions given by the above result. Then, any translation $\mu(\cdot; z)$ is also a stationary solution. We conjecture that up to these translations, there exists a unique non-uniform stationary solution of the Kuramoto mean-field game for every interaction parameter $\kappa > \kappa_c$, see [Remark 7.4](#).

We interpret these non-uniform stationary solutions as partially organized states of the Kuramoto mean-field game, and conclude that for interaction parameters κ larger than the critical value κ_c , there is self organization. As κ gets larger the stationary measure become more localized and [Theorem 4.2](#), proved in [Section 8](#), shows convergence to the fully synchronized regime corresponding to stationary Dirac measures.

Theorem 4.2 (Strong interaction: full synchronization). *Let μ_n be a sequence of non-uniform stationary solutions of the Kuramoto mean-field game with interaction parameters κ_n tending to infinity. Then, there exists a sequence $z_n \in \mathbb{T}$ such that the translated stationary solutions $\mu_n(\cdot; z_n)$ converge in law to the Dirac measure $\delta_{\{0\}}$.*

We already argued in [Example 3.2](#) that the uniform measure is always a stationary solution for all interaction parameters. In [Section 9](#), we consider small interaction parameters and prove that all solutions converge to this incoherent state.

Lemma 4.3 (Weak interaction: incoherence). *If $\kappa < \beta\sigma^2/4$, then any solution $\boldsymbol{\mu} = (\mu_t)_{t \geq 0}$ of the Kuramoto mean field game with interaction parameter κ converges to the incoherent state, i.e., as t tends to infinity, μ_t converges in law to U .*

The next result, proved in [Section 10](#), addresses the local stability for all $\kappa < \kappa_c$, showing that a phase transition occurs *exactly* at κ_c . This result requires the initial distribution μ_0 to be sufficiently close to the uniform distribution. To quantify the distance of any measure $\mu_0 \in \mathcal{P}(\mathbb{T})$ to the uniform measure, we set

$$d(\mu_0) := \max \left\{ |\mu_0(\cos)|, |\mu_0(\sin)|, |\mu_0(\sin \cos)|, \left| \mu_0(\cos^2) - \frac{1}{2} \right| \right\}. \quad (4.1)$$

Theorem 4.4 (Sub-critical interaction: desynchronization). *For $\kappa < \kappa_c$, there is c_κ such that for every $\mu_0 \in \mathcal{P}(\mathbb{T})$ satisfying $d(\mu_0) \leq c_\kappa$, there exists a solution $\boldsymbol{\mu}^* = (\mu_t^*)_{t \geq 0}$ of the Kuramoto mean field game with interaction parameter κ and initial distribution μ_0 , such that μ_t^* converges in law to the uniform distribution as t tends to infinity. Moreover, this convergence is exponential in the sense that for some $\lambda_\kappa^* > 0$,*

$$\sup_{t \geq 0} e^{\lambda_\kappa^* t} d(\mu_t^*) < \infty. \quad (4.2)$$

The existence of solutions to mean field games is well known for problems with ergodic cost [[13–15](#)]. However, for discounted infinite horizon problems it follows directly from our general approach. Thus, we provide this proof for completeness in [Appendix A](#).

Lemma 4.5 (Existence of solutions). *For any probability measure $\mu_0 \in \mathcal{P}(\mathbb{T})$ and $\kappa \geq 0$, there exists a solution $\boldsymbol{\mu} = (\mu_t)_{t \geq 0}$ of the Kuramoto mean field game with interaction parameter κ starting from initial distribution μ_0 .*

4.1. Illustration of the results

We illustrate our main results by computing numerically the solutions of the Kuramoto mean field game for different parameters. To do so, we follow the method of [[24](#)]. We fix a large time horizon, and then solve iteratively the HJB and the Fokker-Planck equations, until the flow of probability measures converges to a fixed point. We use finite difference schemes: an explicit scheme for the Fokker-Planck equation, and an implicit scheme for the HJB equation, in order to handle the non-linearity. We consider the problem with parameters $\beta = 1/2$, $\sigma = 1$ with critical value $\kappa_c = 1$ and examine two interaction parameters.

The first case $\kappa = 0.8$ is below the threshold, and we are in the regime considered in [Theorem 4.4](#). We compute a solution with initial condition $\nu(dx) = C \exp(-\sin(x)) dx$. Left panel in [Figure 1](#) illustrates the convergence of the solution to the uniform distribution.

The case $\kappa = 2$ is above the critical value and [Theorem 4.1](#) implies that there are non-uniform stationary solutions. Indeed, we compute a solution of the Kuramoto mean field game with initial distribution that has two clusters around $\pi/2$ and $3\pi/2$,

$$\nu(dx) = C \chi_{[\pi/4, \pi/4 + \pi/10] \cup [\pi, \pi + \pi/10]}(x) dx.$$

As seen in the right panel of [Figure 1](#), the two clusters quickly merge and the solution converges towards a non-uniform invariant probability measure, whose shape is reported [Figure 2](#).

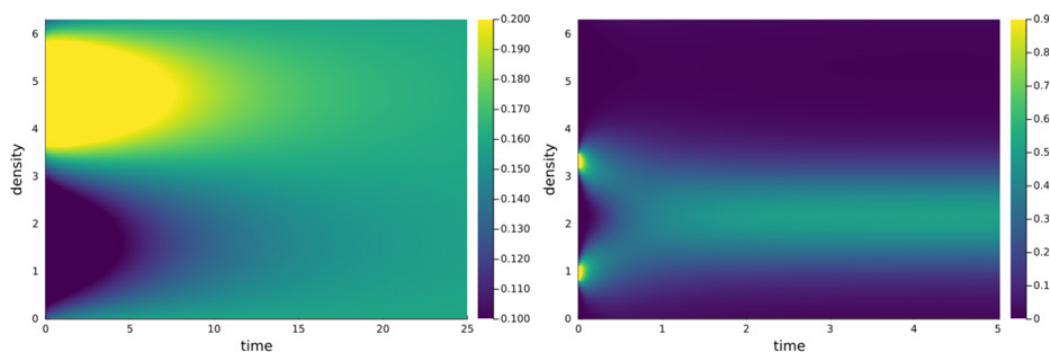


Figure 1. Left panel sub critical interaction, right panel super critical.

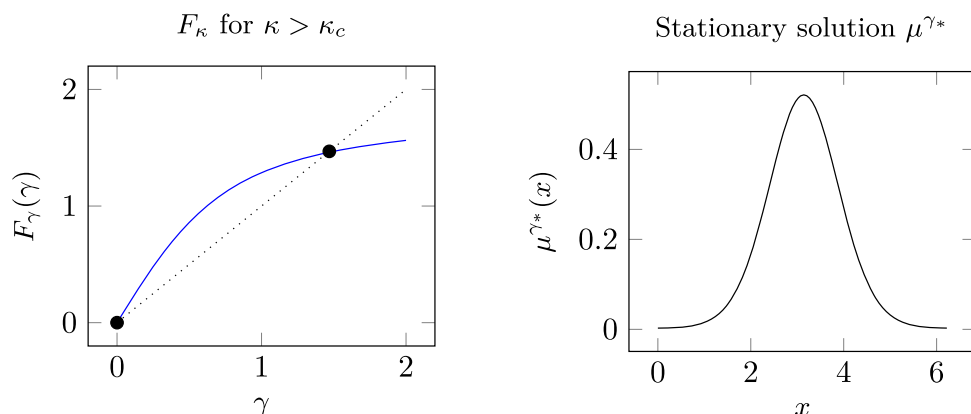


Figure 2. For parameters $\beta = 1/2, \sigma = 1, \kappa = 2\kappa_c = 2$, the function F_κ has two fixed points at $\gamma = 0$ and $\gamma_* \approx 1.47$.

In all our numerical experiments with $\kappa > \kappa_c$, the solutions converge to shifts of the solutions constructed in the proof of [Theorem 4.1](#). The exact translation is determined by the initial distribution. We do not provide a study of this interesting phenomenon.

5. Control problems

The original and central stochastic optimal control problem is defined in (3.1). However, in the sequel, we use several other closely related problems in our analysis. So to highlight the subtle differences among them and to provide a general overview of the notation, we define all of them in this section. It is also clear that adding a constant to the running cost of any control problem does not alter the minimizing control. As we are only interested in the optimal behavior, we use this flexibility and appropriately modify the problem whenever it is convenient.

5.1. Inhomogeneous problems

For $\xi = (\gamma, \eta) \in \mathcal{C}$, we consider the stochastic control problem

$$v_\xi(\mu_0) := \inf_{\alpha \in \mathcal{A}} J_\xi(\mu_0, \alpha) := \inf_{\alpha \in \mathcal{A}} \mathbb{E} \int_0^\infty e^{-\beta t} \left(\ell_\xi(t, X_t^\alpha) + \frac{1}{2} \alpha_t^2 \right) dt, \quad (5.1)$$

where $X_t^\alpha := X_0 + \int_0^t \alpha_u du + \sigma B_t$ is as before with initial data satisfying $\mathcal{L}(X_0) = \mu_0$, and the running cost is given by,

$$\ell_\xi(t, x) = -\gamma(t) \cos(x) - \eta(t) \sin(x), \quad x \in \mathbb{T}, t \geq 0. \quad (5.2)$$

We let X^ξ be the optimal state process. The dependence on μ_0 through the condition $\mathcal{L}(X_0^\xi) = \mu_0$ is omitted in the notation for simplicity. To characterize the dynamics of X^ξ , we also need to introduce a family of control problems starting from any pair $(t, x) \in [0, \infty) \times \mathbb{T}$. Recall that \mathcal{A}_t is the set of all progressively measurable control process $\alpha : [t, \infty) \mapsto \mathbb{R}$. We set

$$v^\xi(t, x) := \inf_{\alpha \in \mathcal{A}_t} J_\xi(t, x, \alpha) := \inf_{\alpha \in \mathcal{A}_t} \mathbb{E} \int_t^\infty e^{-\beta(u-t)} [\ell_\xi(u, X_u^{\alpha, (t,x)}) + \frac{1}{2} \alpha_u^2] du, \quad (5.3)$$

where

$$X_u^{\alpha, (t,x)} = x + \int_t^u \alpha_s ds + \sigma [B_u - B_t], \quad u \geq t. \quad (5.4)$$

We use the notation $X^{\alpha, x} = X^{\alpha, (0,x)}$. For a given ξ , let α^* be the optimal control with initial data (t, x) and let $X^{\xi, (t,x)} = X^{\alpha^*, (t,x)}$ be the optimal state process making the dependence on ξ explicit.

5.2. Stationary problem

When the flow μ is given by one probability measure $\mu \in \mathcal{P}(\mathbb{T})$, we obtain a stationary problem. The corresponding value function is given by,

$$v_{\mu, \kappa}(x) := \inf_{\alpha \in \mathcal{A}} J_{\mu, \kappa}(x, \alpha) := \inf_{\alpha \in \mathcal{A}} \mathbb{E} \int_0^\infty e^{-\beta t} [\ell_{\mu, \kappa}(X_t^{\alpha, x}) + \frac{1}{2} \alpha_t^2] dt, \quad (5.5)$$

where as before $\ell_{\mu, \kappa}(x) := \kappa [c(x, \mu) - 1] = -\kappa [\mu(\cos) \cos(x) + \mu(\sin) \sin(x)]$.

5.3. Parametrized problems

Similarly, we may consider functions $\xi \in \mathcal{C}$ that are time-homogeneous. Additionally, in this case we can translate the corresponding measure appropriately so that the second component is zero. So we only use the first component $\gamma \in \mathbb{R}$ and let $\ell_\gamma(x) := -\gamma \cos(x)$. We then set

$$v^\gamma(x) := \inf_{\alpha \in \mathcal{A}} J_\gamma(x, \alpha) := \inf_{\alpha \in \mathcal{A}} \mathbb{E} \int_0^\infty e^{-\beta t} [\ell_\gamma(X_t^{\alpha, x}) + \frac{1}{2} \alpha_t^2] dt. \quad (5.6)$$

We further elaborate on this problem in Section 6.

6. Stationary solutions

In this section, we establish a one-to-one correspondence between the stationary solutions and fixed points of a scalar function of one-variable that we construct.

6.1. System of partial differential equations

It is well-known that the solutions of mean-field games can be obtained by solving a system of coupled partial differential equations, (6.1) and (6.2) in the present situation. Indeed, the dynamic programming (Hamilton-Jacobi-Bellman) equation related to the stochastic optimal control problem (1.2) with any time-homogeneous running cost ℓ is given by,

$$\beta v(x) - \frac{\sigma^2}{2} v_{xx}(x) + \frac{1}{2} (v_x(x))^2 = \ell(x). \quad (6.1)$$

For smooth ℓ , the above equation has classical solutions (cf. [Lemma 6.3](#)) and the solution is the value function given by (5.6) with running cost ℓ . Moreover, the optimal feedback control is $\alpha^*(x) = -v_x(x)$, and the optimal state process solves $dX_t^* = -v_x(X_t^*)dt + \sigma dB_t$. The stationary law of X_t^* has a density f that solves the following stationary Fokker-Plank equation,

$$\partial_x \left(v_x(x)f(x) + \frac{\sigma^2}{2}f_x(x) \right) = 0. \quad (6.2)$$

The unique solution f_v of the above equation is explicitly available, cf. (6.3).

Remark 6.1. We emphasize that the initial condition X_0^* of the optimal process is random and its density is given by f_v . In particular, the density of X_0^* is also a part of the solution. This is in contrast with the time-varying problems (3.1) and (5.1), in which the initial distribution μ_0 is given and the solutions depend on μ_0 .

Recall the value function $v_{\mu,\kappa}$ of (5.5), and let $f_{v_{\mu,\kappa}}$ be the solution of (6.2) with this value function. The following characterization follows directly from these definitions.

Lemma 6.2. *A probability measure μ is a stationary solution of the Kuramoto mean field game with interaction parameter κ , if and only if its density is equal to $f_{v_{\kappa,\mu}}$.*

We close this subsection with another simple result reported for completeness.

Lemma 6.3. *For $\beta > 0$ and $\ell \in C^1(\mathbb{T})$, there exists a unique solution $v \in C^2(\mathbb{T})$ of (6.1). Moreover, when ℓ is even so is v .*

Proof As the equation (6.1) is one-dimensional and uniformly elliptic, a unique smooth solution v of it can be constructed by classical and direct arguments. Now suppose that ℓ is even, and set $\widehat{v}(x) := v(-x)$. It is clear that \widehat{v} also solves (6.1). Thus, by uniqueness $v = \widehat{v}$. \square

6.2. Characterization

Using the system of differential equations (6.1) and (6.2), we establish a one-to-one correspondence between the stationary solutions and fixed points of a scalar-valued function of one-variable. For $\gamma \geq 0$, let v^γ be as in (5.6) and set $\mu^\gamma(dx) = f_{v^\gamma}(x)dx$. Then, the solution is explicitly given by,

$$\mu^\gamma(dx) = \frac{1}{Z^\gamma} \exp\left(-\frac{2}{\sigma^2}v^\gamma(x)\right) dx, \quad Z^\gamma = \int_{\mathbb{T}} \exp\left(-\frac{2}{\sigma^2}v^\gamma(y)\right) dy. \quad (6.3)$$

For $\kappa \geq 0$, set

$$F_\kappa(\gamma) := \kappa \mu^\gamma(\cos), \quad \gamma \geq 0, \quad (6.4)$$

Note that for $\gamma = 0$, the measure μ^0 is the uniform measure as $v^\gamma(x) \equiv 0$, and therefore, $\gamma = 0$ is a fixed point of the function F_κ for every κ . The case $\gamma > 0$ is treated next. For the following discussion, recall $a(\mu)$, $b(\mu)$ of (2.2), $g(\mu)$ of (2.3), and $\mu(\cdot; z)$ of (2.1).

Proposition 6.4. *A probability measure $\mu \in \mathcal{P}(\mathbb{T})$ is a non-uniform stationary solution of the Kuramoto mean-field game with an interaction parameter κ , if and only if $\kappa g(\mu)$ is a strictly*

positive fixed point of F_κ and $\mu^{\kappa g(\mu)} = \mu(\cdot; z)$ for some $z \in \mathbb{T}$. Moreover, if $\gamma \in (0, \kappa]$ is a fixed point of F_κ , then μ^γ is a non-uniform stationary solution.

The above result also implies that the existence of non-uniform stationary solutions is equivalent to the existence of positive fixed points of F_κ .

Proof Suppose that μ is a stationary solution. By Remark 3.5, any translation $\mu(\cdot; z)$ is again a stationary solution. Choose $z \in \mathbb{T}$ as in (2.3) so that $b(\mu(\cdot; z)) = 0$, and $a(\mu(\cdot; z)) = g(\mu) \geq 0$. Set, $\gamma := \kappa a(\mu(\cdot; z)) = \kappa g(\mu)$. By (2.2),

$$\ell_{\mu(\cdot; z), \kappa}(x) := \kappa [c(x, \mu(\cdot; z)) - 1] = -\gamma \cos(x).$$

Then, the value function $v_{\kappa, \mu(\cdot; z)}$ of (5.5), and v^γ of (5.6) are equal. Moreover, as $\mu(\cdot; z)$ is a stationary solution, it is equal to the law of the optimal state process of this problem. Therefore, its density is equal to the solution f_{v^γ} of the Fokker-Plank equation (6.2). Hence, $\mu(\cdot; z) = \mu^\gamma$. By its definition $F_\kappa(\gamma) = \kappa a(\mu^\gamma)$, and by our choice $\gamma := \kappa a(\mu(\cdot; z))$. Hence, γ is a fixed point of F_κ .

To prove the opposite implication, assume that $\gamma := \kappa g(\mu)$ is a fixed point of F_κ . By Lemma 6.3, v^γ and therefore the density of μ^γ are even. This implies that $b(\mu^\gamma) = 0$. Also $\gamma = F_\kappa(\gamma) = \kappa a(\mu^\gamma)$. Hence by (2.2), $\ell_{\mu^\gamma, \kappa} = -\gamma \cos(x)$ and v_{κ, μ^γ} of (5.5) is equal to v^γ of (5.6). Hence, by Lemma 6.2, μ^γ is a stationary solution.

Moreover, choose $z \in \mathbb{T}$ as in (2.3) so that $\kappa c(x, \mu(\cdot; z)) = \kappa - \kappa g(\mu) \cos(x)$. Since by definition $\gamma = \kappa g(\mu)$, the control problems (1.2) for the stationary flows $\mu(\cdot; z)$ and μ^γ are the same. Hence, $\mu(\cdot; z)$ is equal μ^γ with $\gamma = \kappa g(\mu)$.

Finally suppose that $\gamma \in (0, \kappa]$ is a fixed point of F_κ . In the above we have shown that μ^γ is a stationary solution. Moreover, $\kappa \mu^\gamma(\cos) = F_\kappa(\gamma) = \gamma \neq 0$. Hence, by Lemma 3.4, μ^γ is non-uniform. \square

7. Partial self-organization

In this section, we use the characterization obtained in the previous section to prove the existence of non-uniform (or self-organizing) stationary solutions for super-critical parameters, proving Theorem 4.1. Towards this goal, we analyze the function F_κ defined by (6.4) near the origin and at infinity. A numerical example of F_κ with $\kappa > \kappa_c$ is given in Figure 2.

Set $A(\gamma) := a(\mu^\gamma)$ so that by (6.4), $F_\kappa(\gamma) = \kappa A(\gamma)$.

Lemma 7.1. *The function A defined above is differentiable at the origin and $A'(0) = 1/\kappa_c$. In particular, $F'_\kappa(0) > 1$ for all $\kappa > \kappa_c$ and there is $\gamma_0 > 0$ such that*

$$F_\kappa(\gamma) > \gamma, \quad \forall \kappa \geq 2\kappa_c, \gamma \in (0, \gamma_0). \quad (7.1)$$

Proof As v^γ solves (6.1) with $\ell(x) = -\gamma \cos(x)$, $k(x) := (v^\gamma)_x(x)$ solves

$$\beta k(x) - \frac{\sigma^2}{2} k_{xx}(x) + (v^\gamma)_{xx}(x) k_x(x) = \gamma \sin(x).$$

Since $|\sin(x)| \leq 1$, by maximum principle, we conclude that $|k(x)| \leq (\gamma/\beta)$. Next consider $h(x) := v^\gamma(x) - u(x)$ where $u(x) = -(2\gamma \cos(x))/(2\beta + \sigma^2)$. Since u solves the linear

equation $\beta u(x) - \frac{\sigma^2}{2} u_{xx}(x) = -\gamma \cos(x)$, h satisfies

$$\beta h(x) - \frac{\sigma^2}{2} h_{xx}(x) = -\frac{1}{2} k^2(x).$$

By Feynman–Kac,

$$|h(x)| = \frac{1}{2} \left| \int_0^\infty e^{-\beta t} \mathbb{E} [k^2(x + \sigma B_t)] dt \right| \leq \frac{\gamma^2}{2\beta^3}.$$

Summarizing, we have shown that

$$v^\gamma(x) = -\frac{2\gamma \cos(x)}{2\beta + \sigma^2} + h(x), \quad \text{and} \quad |h(x)| \leq \frac{\gamma^2}{2\beta^3}.$$

As a result, $\lim_{\gamma \downarrow 0} v^\gamma(x) = 0$ and by dominated convergence, $\lim_{\gamma \downarrow 0} Z^\gamma = 2\pi$. Therefore, for any $y \in \mathbb{T}$,

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \frac{1}{\gamma} [\exp(-\frac{2}{\sigma^2} v^\gamma(y)) - 1] &= \lim_{\gamma \rightarrow 0} \frac{1}{\gamma} \left[\exp(-\frac{2}{\sigma^2} [-\frac{2\gamma \cos(y)}{2\beta + \sigma^2} + h(y)]) - 1 \right] \\ &= \lim_{\gamma \rightarrow 0} \frac{1}{\gamma} [\exp(-\frac{2}{\sigma^2} [-\frac{2\gamma \cos(y)}{2\beta + \sigma^2}]) - 1] = \frac{4}{\sigma^2(2\beta + \sigma^2)} \cos(y). \end{aligned}$$

Moreover, there exists a constant c such that $|\exp(-\frac{2}{\sigma^2} v^\gamma(y)) - 1| \leq c\gamma$ for every $y \in \mathbb{T}$. We also directly calculate that

$$Z^0 = 2\pi, \quad \lim_{\gamma \rightarrow 0} F_\kappa(0) = 0.$$

Hence, by the dominated convergence theorem and the above calculations,

$$\begin{aligned} A'(0) &= \lim_{\gamma \rightarrow 0} \frac{A(\gamma)}{\gamma} = \lim_{\gamma \rightarrow 0} \frac{1}{Z^\gamma} \int_{-\pi}^\pi \cos(y) \frac{1}{\gamma} \exp(-\frac{2}{\sigma^2} v^\gamma(y)) dy \\ &= \frac{1}{2\pi} \lim_{\gamma \rightarrow 0} \int_{-\pi}^\pi \cos(y) \frac{1}{\gamma} [\exp(-\frac{2}{\sigma^2} v^\gamma(x)) - 1] dy \\ &= \frac{1}{2\pi} \frac{4}{\sigma^2(2\beta + \sigma^2)} \int_{-\pi}^\pi \cos^2(y) dy = \frac{2}{\sigma^2(2\beta + \sigma^2)}. \end{aligned}$$

To prove the final statement, we choose $\gamma_0 > 0$ so that $A(\gamma) \geq 2\gamma/(3\kappa_c)$ for all $\gamma \in [0, \gamma_0]$. Then, for $\kappa \geq 2\kappa_c$, $F_\kappa(\gamma) = \kappa A(\gamma) \geq 4\gamma/3 > \gamma$ for all $\gamma \in [0, \gamma_0]$. □

We continue with an easy upper bound.

Lemma 7.2 (Upper bound).

$$v^\gamma(x) \leq -\frac{\gamma}{\beta} + \sqrt{\gamma} \left[\frac{x^2}{2} + \frac{\sigma^2}{2\beta} \right], \quad \forall x \in [-\pi, \pi].$$

Proof Since $-\cos(x) \leq -1 + x^2/2$, $v^\gamma \leq \bar{v}^\gamma$, where

$$\bar{v}^\gamma(x) := \inf_{\alpha \in \mathcal{A}} \mathbb{E} \int_0^\infty e^{-\beta t} (-\gamma + \frac{1}{2} [\alpha_t^2 + \gamma (X_t^{x,\alpha})^2]) dt.$$

This linear quadratic stochastic optimization problem has an explicit solution given by

$$\bar{v}^\gamma(x) = -\frac{\gamma}{\beta} + \sqrt{\gamma} [ax^2 + b],$$

where $a = -\frac{\beta}{4\sqrt{\gamma}} + \frac{1}{4}\sqrt{\frac{\beta^2}{\gamma} + 4} \leq \frac{1}{2}$, and $b = a\sigma^2/\beta \leq \sigma^2/(2\beta)$. □

Lemma 7.3. F_κ is continuous on \mathbb{R}_+ .

Proof Fix $\gamma, \delta \in \mathbb{R}$ and define $u(x) := v^{\gamma+\delta}(x) - v^\gamma(x)$. In view of (6.1),

$$\beta u(x) - \frac{\sigma^2}{2} u_{xx}(x) + \frac{1}{2} u_x(x) (v_x^{\gamma+\delta}(x) + v_x^\gamma(x)) = -\delta \cos(x).$$

By maximum principle, we conclude that $\|u\|_\infty \leq |\delta|/\beta$. In particular, $\|v^\gamma - v^0\|_\infty \leq |\gamma|/\delta$, so $\gamma \mapsto \|v^\gamma\|_\infty$ is bounded on bounded sets. Hence, by an application of the dominated convergence theorem, we conclude that F_κ is continuous. \square

Proof of Theorem 4.1 By its definition $|F_\kappa(\gamma)| \leq \kappa$. Moreover, F_κ is continuous on \mathbb{R}_+ , and is differentiable at $\gamma = 0$ with $F'_\kappa(0) = \kappa/\kappa_c$. Therefore, for $\kappa > \kappa_c$, $F'_\kappa(0) > 1$ and consequently, F_κ has a second fixed point $\gamma_* > 0$. Since F_κ is bounded by κ , $\gamma_* \leq \kappa$. By Proposition 6.4, μ_{γ_*} is a non-uniform stationary solution of the Kuramoto mean-field game with interaction parameter κ . \square

Remark 7.4. In our numerical experiments, we always find that the function $\gamma \mapsto F_\kappa(\gamma)$ is concave on \mathbb{R}_+ , as depicted in the Figure 2, and observed that in the super-critical case, self-organizing stationary solutions are unique up to translations. Moreover, time inhomogeneous solutions converge to a stationary solution. We thus conjecture that the function F_κ is concave for \mathbb{R}_+ for every interaction parameter and that non-uniform solutions are unique. Concavity would also imply that this unique stationary measure converges to the uniform measure as $\kappa \downarrow \kappa_c$. A complete analysis of these observations and the conjecture would be highly interesting.

8. Full synchronization: $\kappa \rightarrow \infty$

In this section, we prove Theorem 4.2. An important step is the following result.

Proposition 8.1. As γ tend to infinity, μ^γ converges in law to the Dirac measure $\delta_{\{0\}}$.

The above result follows from several lemmas proved in the next Section 8.1.

Proof of Theorem 4.2 Let μ_n and κ_n be as in the statement of the theorem. Choose z_n as in (2.3). Then by Proposition 6.4, $\gamma_n = \kappa_n g(\mu_n) = \kappa_n a(\mu_n(\cdot; z_n))$ is a fixed point of F_{κ_n} and $\mu_n(\cdot; z_n) = \mu^{\gamma_n}$. We claim that $\lim_{n \rightarrow \infty} \gamma_n = \infty$. If this claim holds, then by Proposition 8.1, we conclude that $\mu_n(\cdot; z_n)$ converges in law to $\delta_{\{0\}}$, completing the proof of Theorem 4.2.

We continue by proving our claim that $\lim_{n \rightarrow \infty} \gamma_n = \infty$, by a counter argument. So we assume that on a subsequence γ_n remains bounded. Without loss of generality, we take the subsequence to be the whole sequence. Then, on a subsequence, denoted by n again, γ_n converges to γ^* . It is clear that v^{γ_n} converges to v^{γ^*} and consequently, μ^{γ_n} converges in law to $\mu^* := \mu^{\gamma^*}$. Also, for sufficiently large n , $\kappa_n \geq 2\kappa_c$ and in view of (7.1), the fixed point γ_n of F_{κ_n} is larger than γ_0 . So we conclude that the limit point $\gamma^* \geq \gamma_0 > 0$.

Summarizing $v^* := v^{\gamma^*}$ is the value function of (1.2) with running cost $\ell(x) = -\gamma^* \cos(x)$. The stationary law of the optimal state process is $\mu^* = \mu^{\gamma^*}$. Furthermore,

$$a(\mu^*) = \lim_{n \rightarrow \infty} a(\mu^{\gamma_n}) = \lim_{n \rightarrow \infty} \frac{F_{\kappa_n}(\gamma_n)}{\kappa_n} = \lim_{n \rightarrow \infty} \frac{\gamma_n}{\kappa_n} = 0.$$

Also by (6.3),

$$\begin{aligned}
 a(\mu^*) &= \int \cos(x) \mu^*(dx) = \frac{1}{Z_{\gamma^*}} \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(x) \exp\left(-\frac{2}{\sigma} v^*(x)\right) dx \\
 &= \frac{1}{Z_{\gamma^*}} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) \exp\left(-\frac{2}{\sigma} v^*(x)\right) dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(x) \exp\left(-\frac{2}{\sigma} v^*(x)\right) dx \right] \\
 &= \frac{1}{Z_{\gamma^*}} \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) \exp\left(-\frac{2}{\sigma} v^*(x)\right) dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x + \pi) \exp\left(-\frac{2}{\sigma} v^*(x + \pi)\right) dx \right] \\
 &= \frac{1}{Z_{\gamma^*}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x) \exp\left(-\frac{2}{\sigma} v^*(x)\right) \left[1 - \exp\left(\frac{2}{\sigma} [v^*(x) - v^*(x + \pi)]\right) \right] dx.
 \end{aligned}$$

By Lemma 8.2, we conclude that the above integral is strictly positive, which is in contradiction with the fact that $a(\mu^*) = 0$. \square

The following lemma is used in the above proof. Set

$$\widehat{w}(x) := v^\gamma(x) - v^\gamma(x + \pi), \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Lemma 8.2. For $x, y \in \mathbb{R}$, $\gamma > 0$, if $\cos(x) \geq \cos(y)$, then $v^\gamma(x) \leq v^\gamma(y)$. In particular, $\widehat{w}(x) \leq 0$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and it is not identically equal to 0.

Proof Let v^γ be as in (5.6). For any stopping time τ , dynamic programming implies that

$$v^\gamma(x) = \inf_{\alpha \in \mathcal{A}} J_\gamma(x, \tau, \alpha),$$

where

$$J_\gamma(x, \tau, \alpha) := \mathbb{E} \int_0^\tau e^{-\beta t} [-\gamma \cos(X_t^{\alpha, x}) + \frac{1}{2} \alpha_t^2] dt + e^{-\beta \tau} v^\gamma(X_\tau^{\alpha, x}),$$

as in Section 5, $X_t^{\alpha, x} = x + \int_0^t \alpha_u du + \sigma B_t$. In particular,

$$v^\gamma(x) - v^\gamma(y) \leq \sup_{\alpha \in \mathcal{A}} [J_\gamma(x, \tau, \alpha) - J_\gamma(y, \tau, \alpha)]. \quad (8.1)$$

First suppose that $\cos(x) = \cos(y)$. Then, either $x = y$ or $x = -y$. As v^γ is even, in either case $v^\gamma(x) = v^\gamma(y)$.

We now fix x, y such that $\cos(x) > \cos(y)$ and consider the stopping time

$$\tau = \inf\{t > 0 : \cos(X_t^{\alpha, x}) = \cos(X_t^{\alpha, y})\}.$$

Then, $\cos(X_\tau^{\alpha, x}) = \cos(X_\tau^{\alpha, y})$, and consequently, $v^\gamma(X_\tau^{\alpha, x}) = v^\gamma(X_\tau^{\alpha, y})$. Since for every $t \in [0, \tau)$, $\cos(X_t^{\alpha, x}) > \cos(X_t^{\alpha, y})$ we have

$$J_\gamma(x, \tau, \alpha) - J_\gamma(y, \tau, \alpha) \leq 0.$$

Hence, by (8.1) we conclude that $v^\gamma(x) \leq v^\gamma(y)$.

Since for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos(x) \geq \cos(x + \pi)$, this implies that $\widehat{w}(x) \leq 0$. Suppose that $v^\gamma(0) = v^\gamma(\pi)$. As $\cos(0) \geq \cos(x) \geq \cos(\pi)$ for every $x \in [0, \pi]$, this implies that $v^\gamma \equiv v^\gamma(0)$. However, v^γ is a classical solution of (6.1) with $\ell(x) = -\gamma \cos(x)$ and a constant function is not a solution of this equation. So we conclude that $\widehat{w}(0) < 0$. \square

8.1. Proof of Proposition 8.1

The central analytical object in this proof is the following scaled function

$$w^\gamma(x) := \frac{1}{\sqrt{\gamma}} \left[\gamma^\gamma(x) + \frac{\gamma}{\beta} \right], \tag{8.2}$$

which solves the equation

$$\beta_\gamma w^\gamma(x) - \frac{\sigma_\gamma^2}{2} w_{xx}^\gamma(x) + \frac{1}{2} (w_x^\gamma(x))^2 = 1 - \cos(x) = 2(\sin(x/2))^2, \tag{8.3}$$

where $\sigma_\gamma := \sigma \gamma^{-1/4}$, and $\beta_\gamma := \beta \gamma^{-1/2}$. Then, w^γ has the following stochastic optimal control representation,

$$w^\gamma(x) = \inf_{\alpha \in \mathcal{A}} L_\gamma(x, \alpha) := \mathbb{E} \int_0^\infty e^{-\beta_\gamma t} \left[\frac{\alpha_t^2}{2} + 1 - \cos(X_t^{\alpha, x}) \right] dt,$$

with $X_t^{\alpha, x} := x + \int_0^t \alpha_u du + \sigma_\gamma B_t$. Therefore, $w^\gamma \geq 0$. Using Lemma 7.2, we deduce that

$$|w^\gamma(x)| \leq \frac{x^2}{2} + \frac{\sigma^2}{2\beta} \leq \frac{\pi^2}{2} + \frac{\sigma^2}{2\beta}, \quad \forall x \in [-\pi, \pi].$$

We start with a uniform Lipschitz estimate.

Lemma 8.3. *For all $\gamma > 0$,*

$$|w_x^\gamma(x)| \leq \frac{1}{2} \left[3 + \pi + \pi^2 + \frac{\sigma^2}{\beta} \right], \quad \forall x \in [0, 2\pi].$$

Proof Let $L_\gamma(z, \alpha)$ be as above. Fix $x, x', \epsilon > 0$ and choose an ϵ -optimal control $\alpha^\epsilon \in \mathcal{A}$ satisfying $L_\gamma(x, \alpha^\epsilon) \leq w^\gamma(x) + \epsilon$. Define a new control α' by,

$$\alpha'_t := \begin{cases} \alpha_t^\epsilon + (x - x'), & \text{if } t \leq 1, \\ \alpha_t^\epsilon, & \text{if } t > 1. \end{cases}$$

It is clear that $\alpha' \in \mathcal{A}$ and also we have the following,

$$X_t^{\alpha', x'} - X_t^{\alpha^\epsilon, x} = \begin{cases} (x' - x)(1 - t), & \text{if } t \leq 1, \\ 0, & \text{if } t > 1. \end{cases}$$

This implies that

$$L_\gamma(x', \alpha') - L_\gamma(x, \alpha^\epsilon) = \mathbb{E} \int_0^1 e^{-\beta_\gamma t} (A_t + B_t) dt,$$

where $A_t := \cos(X_t^{\alpha^\epsilon, x}) - \cos(X_t^{\alpha', x'})$ and $B_t := \frac{1}{2} [\alpha_t^\epsilon + (x - x')]^2 - \frac{1}{2} (\alpha_t^\epsilon)^2$. Therefore, $|A_t| \leq |x - x'|$, and

$$|B_t| \leq |\alpha_t^\epsilon| |x - x'| + \frac{(x - x')^2}{2} \leq \frac{1}{2} [|x - x'| (\alpha_t^\epsilon)^2 + |x - x'| + (x - x')^2].$$

These imply that

$$\begin{aligned}
 w^\gamma(x') - w^\gamma(x) - \epsilon &\leq L_\gamma(x', \alpha') - L_\gamma(x, \alpha^\epsilon) \\
 &\leq \mathbb{E} \int_0^1 e^{-\beta_\gamma t} (A_t + B_t) dt \\
 &\leq \frac{1}{2} [3|x - x'| + |x - x'|^2] + |x - x'| \mathbb{E} \int_0^1 e^{-\beta_\gamma t} \frac{1}{2} (\alpha_t^\epsilon)^2 dt \\
 &\leq \frac{1}{2} [3 + |x - x'|] |x - x'| + |x - x'| L_\gamma(x, \alpha^\epsilon) \\
 &\leq \frac{1}{2} [3 + \pi] |x - x'| + |x - x'| [w^\gamma(x) + \epsilon] \\
 &\leq \frac{1}{2} [3 + \pi + \pi^2 + \frac{\sigma^2}{\beta} + 2\epsilon] |x - x'|.
 \end{aligned}$$

As the argument is symmetric in x, x' and ϵ is arbitrary, the proof of the lemma is complete. \square

Above estimates imply that $(w^\gamma)_{\gamma>0}$ is equicontinuous and uniformly bounded. Hence, by Arzelà–Ascoli, it converges uniformly on subsequences. We continue by identifying the limit of $w^\gamma(\cdot) - w^\gamma(0)$ which is sufficient for our purposes. We achieve this by using standard tools from the theory of viscosity solution [25–27].

Proposition 8.4. *As γ tends to infinity, $w^\gamma(\cdot) - w^\gamma(0)$ converges uniformly to w given by,*

$$w(x) := 4(1 - |\cos(x/2)|). \quad (8.4)$$

In particular, the function w is the unique viscosity solution of the Eikonal equation

$$\frac{1}{2}(w_x)^2 = 2(\sin(x/2))^2, \quad x \in (0, 2\pi), \quad \text{with} \quad w(0) = w(2\pi) = 0. \quad (8.5)$$

Proof Observe that w^γ is a classical and hence, a viscosity solution of (8.3). As $\beta_\gamma, \sigma_\gamma$ converge to zero, the equation (8.3) formally converges to the Eikonal equation (8.5). Then, the classical stability results for viscosity solutions (cf. Theorem 1.4 in [26] or Lemma II.6.2 in [27]), any uniform limit w of $w^\gamma(\cdot) - w^\gamma(0)$ is a viscosity solution of (8.5). We also directly verify that w defined above is a viscosity solution of (8.5). By the standard comparison result for this equation (proved in Lemma B.1 for completeness), we conclude that, any uniform limit of $w^\gamma(\cdot) - w^\gamma(0)$ is equal to w . \square

Proof of Proposition 8.1. Set $\tilde{w}_\gamma := \frac{2}{\sigma^2}(w^\gamma(x) - w^\gamma(0))$, so that by (8.2),

$$v^\gamma(x) = \frac{\sqrt{\gamma}\sigma^2}{2}\tilde{w}_\gamma - \frac{\gamma}{\beta} + \sqrt{\gamma}w^\gamma(0).$$

The definition of μ^γ implies that

$$\mu^\gamma(dx) = \frac{\exp(-\sqrt{\gamma}\tilde{w}_\gamma) dx}{\int_{-\pi}^{\pi} \exp(-\sqrt{\gamma}\tilde{w}_\gamma(y)) dy} =: \frac{1}{\tilde{Z}_\gamma} \exp(-\sqrt{\gamma}\tilde{w}_\gamma) dx.$$

By Proposition 8.4, \tilde{w}_γ converges to $\tilde{w}(x) := \frac{8}{\sigma^2}(1 - |\cos(x/2)|)$. Hence, $\tilde{w}_\gamma(x) = \tilde{w}(x) + \epsilon_\gamma(x)$, for some function ϵ_γ converging uniformly to zero as γ tends to infinity. It is clear that

on $x \in [-\pi, \pi]$, \tilde{w} is strictly convex and has a unique global minimum $\tilde{w}(0) = 0$. Therefore, there exists a constant c_1 such that

$$\int_{-\pi}^{\pi} \chi_{\{\tilde{w}(y) \leq \frac{1}{\sqrt{\gamma}}\}} dy \geq c_1 \gamma^{-1/4}.$$

Fix $\epsilon > 0$. There is γ_ϵ such that for all $\gamma \geq \gamma_\epsilon$, we have $\|\epsilon_\gamma\|_\infty \leq \epsilon$. Therefore, for $\gamma \geq \gamma_\epsilon$,

$$\begin{aligned} \tilde{Z}_\gamma &= \int_{-\pi}^{\pi} \exp(-\sqrt{\gamma}[\tilde{w}(y) + \epsilon_\gamma(y)]) dy \\ &\geq \exp(-\epsilon\sqrt{\gamma}) \int_{\{\tilde{w} \leq \frac{1}{\sqrt{\gamma}}\}} \exp(-\sqrt{\gamma}\tilde{w}(y)) dy \\ &\geq \exp(-[\epsilon\sqrt{\gamma} + 1]) c_1 \gamma^{-1/4} =: (c_0 e^{\epsilon\sqrt{\gamma}} \gamma^{1/4})^{-1}. \end{aligned}$$

As $\|\epsilon_\gamma\|_\infty \leq \epsilon$ for all $\gamma \geq \gamma_\epsilon$, for these values of γ the following estimate holds

$$\begin{aligned} \int_{-\pi}^{\pi} \chi_{\{\tilde{w} \geq 3\epsilon\}} \mu^\gamma(dx) &= \frac{1}{\tilde{Z}_\gamma} \int_{-\pi}^{\pi} \chi_{\{\tilde{w} \geq 3\epsilon\}} e^{-\sqrt{\gamma}\tilde{w}_\gamma} dx \\ &\leq c_0 e^{\epsilon\sqrt{\gamma}} \gamma^{1/4} e^{\sqrt{\gamma}\|\epsilon_\gamma\|_\infty} \int_{-\pi}^{\pi} \chi_{\{\tilde{w} \geq 3\epsilon\}} e^{-\sqrt{\gamma}\tilde{w}(x)} dx \\ &\leq 2\pi c_0 \gamma^{1/4} e^{-\epsilon\sqrt{\gamma}}. \end{aligned}$$

Since the above quantity converges to zero as γ tends to infinity, we conclude that any limit point of μ^γ does not have any mass in the set $\{\tilde{w} \geq 3\epsilon\}$ for every ϵ . This set shrinks to the singleton $\{0\}$ as ϵ tends to zero. Hence, μ^γ converges in law to $\delta_{\{0\}}$. \square

9. Weak interaction and incoherence

In this section we consider small κ values, and prove the convergence of all solutions to the Kuramoto mean field game to the uniform solution as time gets larger. In the next section, we consider all $\kappa < \kappa_c$, and prove the existence of convergent solutions provided that the initial distribution is sufficiently close to the uniform distribution.

9.1. Setting

For a continuous function $\xi = (\gamma, \eta) \in \mathcal{C}$ and a probability measure $\mu_0 \in \mathcal{P}(\mathbb{T})$, recall the value function $v^\xi(t, x)$ of (5.3), running cost ℓ_ξ of (5.2), the state processes $X^{\alpha, (t, x)}$ of (5.4), and the optimal state process $X^{\xi, (t, x)}$ of (5.6), X^ξ of the problem (5.1) with initial distribution $\mathcal{L}(X_0^\xi) = \mu_0$ (the dependence on μ_0 is omitted in the notation for simplicity).

It is well-known [27] that the value function $v^\xi(t, x)$ of (5.3) is a classical solution of the time inhomogeneous dynamic programming equation,

$$\beta v^\xi(t, x) - v_t^\xi(t, x) - \frac{\sigma^2}{2} v_{xx}^\xi(t, x) + \frac{1}{2} (v_x^\xi(t, x))^2 = \ell_\xi(t, x), \quad t > 0, x \in \mathbb{T}. \tag{9.1}$$

Then, the optimal state process X^ξ is given by

$$dX_t^\xi = -v_x^\xi(t, X_t^\xi) dt + \sigma dB_t, \tag{9.2}$$

with initial data $\mathcal{L}(X_0) = \mu_0$.

We now define a map

$$\mathcal{T}(\cdot; \mu_0) : \xi \in \mathcal{C} \mapsto \mathcal{T}(\xi; \mu_0) := (\mathbb{E}[\cos(X_t^\xi)], \mathbb{E}[\sin(X_t^\xi)])_{t \geq 0}. \quad (9.3)$$

For a given probability flow $\mu = (\mu_t)_{t \geq 0}$, define

$$\xi(\mu) = (\mu_t(\cos), \mu_t(\sin))_{t \geq 0}. \quad (9.4)$$

The following is an immediate consequence of the definitions.

Lemma 9.1. *A probability flow $\mu = (\mu_t)_{t \geq 0}$ is a solution of the Kuramoto mean field game if and only if $\xi(\mu)$ defined in (9.4) is a fixed point of $\kappa \mathcal{T}(\cdot; \mu_0)$. Moreover, if ξ is a fixed point of $\kappa \mathcal{T}(\cdot; \mu_0)$, then the probability flow $(\mathcal{L}(X_t^\xi))_{t \geq 0}$ is a solution of the Kuramoto mean field game starting from the distribution μ_0 .*

9.2. Estimates

For any function $k : [0, \infty) \times \mathbb{T} \mapsto \mathbb{R}^d$ and $t \geq 0$, we set

$$\|k\|_{t, \infty} := \sup_{u \geq t} \|k(u, \cdot)\|_\infty.$$

Lemma 9.2. *For any $\xi \in \mathcal{C}$, $\kappa > 0$ and $t \geq 0$, $\|v_x^\xi\|_{t, \infty} \leq \frac{1}{\beta} \|\xi\|_{t, \infty}$.*

Proof For any $x, y \in \mathbb{T}$ and $t \geq 0$,

$$\begin{aligned} v^\xi(t, x) - v^\xi(t, y) &\leq \sup_{\alpha \in \mathcal{A}_t} [J_\xi(t, x, \alpha) - J_\xi(t, y, \alpha)] \\ &\leq \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \int_t^\infty e^{-\beta(u-t)} [\ell_\xi(u, X_u^{\alpha, (t, x)}) - \ell_\xi(u, X_u^{\alpha, (t, y)})] du, \end{aligned}$$

where ℓ_ξ is as in (5.2). Then,

$$\left| \ell_\xi(u, X_u^{\alpha, (t, x)}) - \ell_\xi(u, X_u^{\alpha, (t, y)}) \right| \leq \|\xi\|_{t, \infty} |x - y|, \quad \forall u \geq 0, x, y \in \mathbb{T},$$

and therefore,

$$v^\xi(t, x) - v^\xi(t, y) \leq \sup_{\alpha \in \mathcal{A}_t} \mathbb{E} \int_t^\infty e^{-\beta(u-t)} \|\xi\|_{t, \infty} |x - y| du = \frac{\|\xi\|_{t, \infty}}{\beta} |x - y|. \quad \square$$

The following estimate follows directly from Ito's formula.

Lemma 9.3. *For any $\xi \in \mathcal{C}$, $n \geq 1$, and $0 \leq t \leq s$,*

$$\left| \mathbb{E}[\cos(nX_s^\xi)] \right| + \left| \mathbb{E}[\sin(nX_s^\xi)] \right| \leq 2e^{-\frac{n^2 \sigma^2}{2}(s-t)} + \frac{4\|\xi\|_{t, \infty}}{n\beta \sigma^2}.$$

Proof Set $A_t := \mathbb{E}[\cos(nX_t^\xi)]$. Ito formula implies that

$$A_s - A_t = n \mathbb{E} \left[\int_t^s v_x^\xi(u, X_u^\xi) \sin(nX_u^\xi) du \right] - \frac{n^2 \sigma^2}{2} \int_t^s A_u du.$$

By Duhamel's principle, we have

$$A_s = e^{-\frac{n^2 \sigma^2}{2}(s-t)} A_t + \int_t^s e^{-\frac{n^2 \sigma^2}{2}(s-u)} n \mathbb{E} [v_x^\xi(u, X_u^\xi) \sin(nX_u^\xi)] du.$$

Therefore, by the previous lemma,

$$\begin{aligned} |A_s| &\leq e^{-\frac{n^2\sigma^2}{2}(s-t)} |A_t| + \int_t^s \frac{n\|\xi\|_{t,\infty}}{\beta} \exp\left(-\frac{n^2\sigma^2}{2}(s-u)\right) du \\ &\leq e^{-\frac{n^2\sigma^2}{2}(s-t)} + \frac{2\|\xi\|_{t,\infty}}{n\beta\sigma^2} [1 - e^{-\frac{n^2\sigma^2}{2}(s-t)}]. \end{aligned}$$

The inequality for the sin is proved exactly the same way. □

9.3. Proof of Lemma 4.3

Let $\mu = (\mu_t)_{t \geq 0}$ be a solution of the Kuramoto mean field game. By Lemma 9.1, $\xi_t := \xi_t(\mu)$ given by (9.4) is a fixed-point of $\kappa\mathcal{T}(\cdot; \mu_0)$. Hence, $\xi_t = \kappa \left(\mathbb{E}[\cos(X_t^\xi)], \mathbb{E}[\sin(X_t^\xi)] \right)$. By Lemma 9.3, for any $0 \leq t \leq \tau$,

$$\begin{aligned} \|\xi\|_{\tau,\infty} &= \kappa \sup_{s \geq \tau} \left[\left| \mathbb{E}[\cos(X_s^\xi)] \right| + \left| \mathbb{E}[\sin(X_s^\xi)] \right| \right] \leq \kappa \sup_{s \geq \tau} \left[2e^{-\frac{\sigma^2}{2}(s-t)} + \frac{4\|\xi\|_{t,\infty}}{\beta\sigma^2} \right] \\ &= 2\kappa e^{-\frac{\sigma^2}{2}(\tau-t)} + \frac{4\kappa\|\xi\|_{t,\infty}}{\beta\sigma^2}. \end{aligned}$$

As $\|\xi\|_{\tau,\infty}$ is non-increasing in τ , it has a limit, and the above inequality implies that

$$\lim_{\tau \rightarrow \infty} \|\xi\|_{\tau,\infty} \leq \lim_{\tau \rightarrow \infty} 2\kappa e^{-\frac{\sigma^2}{2}(\tau-t)} + \frac{4\kappa\|\xi\|_{t,\infty}}{\beta\sigma^2} = \frac{4\kappa\|\xi\|_{t,\infty}}{\beta\sigma^2}, \quad \forall t \geq 0.$$

We now take the limit as t tends to infinity. The result is the following,

$$\lim_{\tau \rightarrow \infty} \|\xi\|_{\tau,\infty} \leq \frac{4\kappa}{\beta\sigma^2} \lim_{t \rightarrow \infty} \|\xi\|_{t,\infty}.$$

Thus for $\kappa < \beta\sigma^2/4$, $\lim_{t \rightarrow \infty} \|\xi\|_{t,\infty} = 0$. By Lemma 9.3,

$$\lim_{t \rightarrow \infty} \left[\left| \mathbb{E}[\cos(nX_t^\xi)] \right| + \left| \mathbb{E}[\sin(nX_t^\xi)] \right| \right] = 0,$$

for every n . Let f be a twice continuously differentiable function on \mathbb{T} . Then,

$$f(x) = c_0 + \sum_{n=1}^{\infty} [c_n \cos(nx) + e_n \sin(nx)],$$

for some constants c_n, e_n . One may directly show that they satisfy $\sum_{n=1}^{\infty} [|c_n| + |e_n|] < \infty$. Hence, by dominated convergence,

$$\lim_{t \rightarrow \infty} \mu_t(f) = c_0 + \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} (c_n \mathbb{E}[\cos(nX_t^\xi)] + e_n \mathbb{E}[\sin(nX_t^\xi)]) = c_0 = U(f).$$

This implies the convergence of μ_t to the uniform distribution U . □

10. Sub-critical case: $\kappa < \kappa_c$

For a positive constant $\lambda > 0$, consider the subspace of \mathcal{C} given by,

$$\mathcal{C}_\lambda := \{ \xi = (\gamma, \eta) \in \mathcal{C} : \|\xi\|_\lambda < \infty \}, \quad \text{where} \quad \|\xi\|_\lambda := \sup_{t \geq 0} e^{\lambda t} |\xi_t| = \sup_{t \geq 0} e^{\lambda t} (|\gamma(t)| + |\eta(t)|).$$

Then, $(\mathcal{C}_\lambda, \|\cdot\|_\lambda)$ is a Banach space.

10.1. Preliminaries

Let $v_x^\xi(t, x)$ be as in Section 9.1, and recall that the optimal state processes X^ξ and $X^{\xi, (t, x)}$ solve the same stochastic differential equation (9.2) but with different initial conditions. Namely, $X_0^\xi = X_0$ satisfies $\mathcal{L}(X_0) = \mu_0$, and $X_t^{\xi, (t, x)} = x$. The drift term in the stochastic differential equation (9.2) is v_x^ξ . We differentiate the equation (9.1), to show that it solves the following equation,

$$\beta v_x^\xi - v_{xt}^\xi(t, x) - \mathcal{M}(v_x^\xi)(t, x) = (\ell_\xi)_x(t, x) = \gamma(t) \sin(x) - \eta(t) \cos(x),$$

where \mathcal{M} is the infinitesimal generator of the stochastic differential equation (9.2), i.e., for a smooth function φ of (t, x) ,

$$\mathcal{M}(\varphi)(t, x) = \frac{\sigma^2}{2} \varphi_{xx}(t, x) - \varphi_x(t, x) v_x^\xi(t, x).$$

Hence, we have the following representation of v_x^ξ ,

$$v_x^\xi(t, x) = \int_t^\infty e^{-\beta(u-t)} \left[\gamma(u) B_u^\xi(t, x) - \eta(u) A_u^\xi(t, x) \right] du, \quad (10.1)$$

where

$$A_u^\xi(t, x) := \mathbb{E} \left[\cos(X_u^{\xi, (t, x)}) \right], \quad B_u^\xi(t, x) := \mathbb{E} \left[\sin(X_u^{\xi, (t, x)}) \right].$$

Lemma 10.1. For every $t \geq 0, x \in \mathbb{T}$,

$$\|v_x^\xi\|_\lambda^* := \sup_{t \geq 0, x \in \mathbb{T}} |v_x^\xi(t, x)| e^{\lambda t} \leq \frac{\|\xi\|_\lambda}{\beta}.$$

Proof We use the representation (10.1) with the estimate

$$|\gamma(u) B_u^\xi(t, x) - \eta(u) A_u^\xi(t, x)| \leq |\gamma(u)| + |\eta(u)| \leq \|\xi\|_\lambda e^{-\lambda u} \leq \|\xi\|_\lambda e^{-\lambda t}, \quad \forall u \geq t,$$

to obtain

$$|v_x^\xi(t, x)| \leq \int_t^\infty e^{-\beta(u-t)} \|\xi\|_\lambda e^{-\lambda t} du = \frac{\|\xi\|_\lambda}{\beta} e^{-\lambda t}. \quad \square$$

We close this subsection with a simple application of the Ito's rule.

Lemma 10.2. Let X be a solution of the stochastic differential equation (9.2) on (t, ∞) . Then, for $f(x) = \cos(x)$ or $f(x) = \sin(x)$,

$$\left| \mathbb{E}[f(X_u)] - e^{-\frac{\sigma^2}{2}(u-t)} \mathbb{E}[f(X_t)] \right| \leq \frac{2 \|v_x^\xi\|_\lambda^*}{\sigma^2} e^{-\lambda t}, \quad \forall u \geq t.$$

Proof We only consider $f(x) = \cos(x)$, the other case is proved in exactly the same way. By Ito's rule,

$$d \cos(X_u) = -\sin(X_u) dX_u - \frac{\sigma^2}{2} \cos(X_u) du.$$

Therefore,

$$\mathbb{E}[\cos(X_u)] = e^{-\frac{\sigma^2}{2}(u-t)} \mathbb{E}[\cos(X_t)] + \int_t^u e^{-\frac{\sigma^2}{2}(u-s)} \mathbb{E}[v_x^\xi(s, X_s) \sin(X_s)] ds.$$

Moreover, $\mathbb{E}[v_x^\xi(s, X_s) \sin(X_s)] \leq \|v_x^\xi\|_\lambda^* e^{-\lambda t}$ for every $s \geq t$. We substitute this into the above equation to complete the proof of the claimed inequality. \square

Set

$$\rho := \frac{\kappa_c}{\sigma^2} = \beta + \frac{\sigma^2}{2}.$$

Lemma 10.3. For $\xi = (\gamma, \eta) \in \mathcal{C}_\lambda$, the negative drift v_x^ξ has the representation,

$$v_x^\xi(t, x) = w^\xi(t, x) + r^\xi(t, x) = M_t(\xi) \sin(x) - N_t(\xi) \cos(x) + r^\xi(t, x),$$

where

$$M_t(\xi) = \int_t^\infty e^{-\rho(s-t)} \gamma(s) ds, \quad N_t(\xi) = \int_t^\infty e^{-\rho(s-t)} \eta(s) ds.$$

In addition,

$$\|(M_t(\xi), N_t(\xi))\|_\lambda \leq \frac{\|\xi\|_\lambda}{\rho} \quad |r^\xi(t, x)| \leq \frac{4}{\beta^2 \sigma^2} \|\xi\|_\lambda^2 e^{-2\lambda t}. \tag{10.2}$$

Proof We first use Lemma 10.2 with $X^{\xi, (t, x)}$ to obtain

$$A_u^\xi(t, x) = e^{-\frac{\sigma^2}{2}(u-t)} \cos(x) + a_u^\xi(t, x), \quad \text{and} \quad |a_u^\xi(t, x)| \leq \frac{2 \|v_x^\xi\|_\lambda^*}{\sigma^2} e^{-\lambda t},$$

$$B_u^\xi(t, x) = e^{-\frac{\sigma^2}{2}(u-t)} \sin(x) + b_u^\xi(t, x), \quad \text{and} \quad |b_u^\xi(t, x)| \leq \frac{2 \|v_x^\xi\|_\lambda^*}{\sigma^2} e^{-\lambda t}.$$

Let w^ξ, M_t, N_t be as in the statement of the lemma. Then, formula (10.1) gives,

$$r^\xi(t, x) := v_x^\xi(t, x) - w^\xi(t, x) = \int_t^\infty e^{-\beta(u-t)} \left[\gamma(u) b_u^\xi(t, x) - \eta(u) a_u^\xi(t, x) \right] du.$$

The claimed estimates of N, M, r^ξ now follows directly from Lemma 10.1. \square

10.2. Linearization

For small ξ , one expects the value function and its derivatives to be small. We exploit this formal observation and obtain the following representation of the map $\mathcal{T}(\cdot; \mu_0)$. Recall, $w^\xi, N(\xi), M(\xi)$ of Lemma 10.3, and $d(\mu_0)$ of (4.1).

Proposition 10.4. For any $\xi \in \mathcal{C}_\lambda, t \geq 0, \mu_0 \in \mathcal{P}(\mathbb{T})$, and $\lambda \leq \sigma^2/8$,

$$\mathcal{T}_t(\xi; \mu_0) = e^{-\frac{\sigma^2}{2}t} (\mu_0(\cos), \mu_0(\sin)) + \Xi_t(\xi) + R_t(\xi, \mu_0),$$

where the linear operator Ξ is given by

$$\Xi_t(\xi) := \frac{1}{2} \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} (M_u(\xi), N_u(\xi)) du.$$

Moreover, there is a constant $c_2(\beta, \sigma) > 0$ satisfying

$$|R_t(\xi, \mu_0)| \leq \left[\frac{8d(\mu_0)}{\kappa_c} \|\xi\|_\lambda + c_2(\beta, \sigma) \|\xi\|_\lambda^2 \right] e^{-2\lambda t}. \quad (10.3)$$

Proof Recall that for $\xi = (\gamma, \eta) \in \mathcal{C}_\lambda$ and $t \geq 0$,

$$\mathcal{T}_t(\xi; \mu_0) = (\mathbb{E}[\cos(X_t^\xi), \mathbb{E}[\sin(X_t^\xi)]], \quad t \geq 0,$$

where X^ξ solves (9.2) with initial condition $\mathcal{L}(X_0) = \mu_0$. We continue in several steps.

Step 1. By Ito's rule,

$$\mathbb{E}[\cos(X_t^\xi)] = e^{-\frac{\sigma^2}{2}t} \mu_0(\cos) + \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} \mathbb{E}[v_x^\xi(u, X_u^\xi) \sin(X_u^\xi)] du. \quad (10.4)$$

Moreover, by Lemma 10.3,

$$v_x^\xi(u, X_u^\xi) \sin(X_u^\xi) = M_u(\xi) \sin^2(X_u^\xi) - N_u(\xi) \sin(X_u^\xi) \cos(X_u^\xi) + r^\xi(u, X_u^\xi) \sin(X_u^\xi).$$

Step 2. Set $Y_u := \sin^2(X_u^\xi) - \frac{1}{2}$. By Ito's rule,

$$dY_u = 2v_x^\xi(u, X_u^\xi) \sin(X_u^\xi) \cos(X_u^\xi) du - 2\sigma^2 Y_u du + (\dots) dB_u.$$

This implies that

$$\mathbb{E}[Y_u] = e^{-2\sigma^2 u} \mathbb{E}[Y_0] + \int_0^u 2e^{-2\sigma^2(u-s)} \mathbb{E}[v_x^\xi(s, X_s^\xi) \sin(X_s^\xi) \cos(X_s^\xi)] ds.$$

Then, by Lemma 10.1 and (4.1),

$$\begin{aligned} |\mathbb{E}[\sin^2(X_u^\xi)] - \frac{1}{2}| &\leq e^{-2\sigma^2 u} |\mathbb{E}[\sin^2(X_0^\xi)] - \frac{1}{2}| + \frac{\|\xi\|_\lambda}{\beta\sigma^2} e^{-\lambda u} \\ &\leq e^{-2\sigma^2 u} d(\mu_0) + \frac{\|\xi\|_\lambda}{\beta\sigma^2} e^{-\lambda u}. \end{aligned} \quad (10.5)$$

A similar calculation implies that

$$|\mathbb{E}[\sin(X_u^\xi) \cos(X_u^\xi)]| \leq e^{-2\sigma^2 u} d(\mu_0) + \frac{\|\xi\|_\lambda}{\beta\sigma^2} e^{-\lambda u}. \quad (10.6)$$

Step 3. Set

$$R_u^1(\xi; \mu_0) := \mathbb{E}[v_x^\xi(u, X_u^\xi) \sin(X_u^\xi)] - \frac{1}{2} M_u(\xi).$$

We directly estimate R^1 by using the previous steps and (10.2). The result is the following,

$$\begin{aligned} |R_u^1(\xi; \mu_0)| &\leq (|N_u(\xi)| + |M_u(\xi)|) [e^{-2\sigma^2 u} d(\mu_0) + \frac{\|\xi\|_\lambda}{\beta\sigma^2} e^{-\lambda u}] + \mathbb{E}[|r^\xi(u, X_u^\xi)|] \\ &\leq \frac{2\|\xi\|_\lambda}{\rho} e^{-\lambda u} [e^{-2\sigma^2 u} d(\mu_0) + \frac{\|\xi\|_\lambda}{\beta\sigma^2} e^{-\lambda u}] + \frac{4}{\beta^2\sigma^2} \|\xi\|_\lambda^2 e^{-2\lambda u} \\ &\leq \left[\frac{2d(\mu_0)}{\rho} \|\xi\|_\lambda + c(\beta, \sigma) \|\xi\|_\lambda^2 \right] e^{-2\lambda u}, \end{aligned}$$

where

$$c(\beta, \sigma) := \frac{(2\beta + 4\rho)}{\rho\beta^2\sigma^2} = \frac{(12\beta + 4\sigma^2)}{(2\beta + \sigma^2)\beta^2\sigma^2}.$$

Step 4. Using Step 3 in (10.4), we obtain

$$\mathbb{E}[\cos(X_t^\xi)] = e^{-\frac{\sigma^2}{2}t} \mu_0(\cos) + \frac{1}{2} \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} M_u(\xi) \, du + R_t^2(\xi, \mu_0),$$

where

$$R_t^2(\xi, \mu_0) = \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} R_u^1(\xi, \mu_0) \, du.$$

By the estimate obtained in Step 3, and as $\lambda < \sigma^2/8$,

$$\begin{aligned} |R_t^2(\xi, \mu_0)| &\leq \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} \left[\frac{2d(\mu_0)}{\rho} \|\xi\|_\lambda + c(\beta, \sigma) \|\xi\|_\lambda^2 \right] e^{-2\lambda u} \, du \\ &\leq \frac{4}{\sigma^2} \left[\frac{2d(\mu_0)}{\rho} \|\xi\|_\lambda + c(\beta, \sigma) \|\xi\|_\lambda^2 \right] e^{-2\lambda t}. \end{aligned}$$

Step 5. Proceeding exactly as in the previous steps, we also obtain,

$$\mathbb{E}[\sin(X_t^\xi)] = e^{-\frac{\sigma^2}{2}t} \mu_0(\sin) + \frac{1}{2} \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} N_u(\xi) \, du + R_t^3(\xi, \mu_0),$$

where

$$|R_t^3(\xi, \mu_0)| \leq \frac{4}{\sigma^2} \left[\frac{2d(\mu_0)}{\rho} \|\xi\|_\lambda + c(\beta, \sigma) \|\xi\|_\lambda^2 \right] e^{-2\lambda t}.$$

As $\rho\sigma^2 = \kappa_c$, above estimates implies (10.3) with $c_2(\beta, \sigma) = 4c(\beta, \sigma)/\sigma^2$. □

Corollary 10.5. *For every λ small, Ξ is bounded on \mathcal{C}_λ . In particular, for all $\xi \in \mathcal{C}_\lambda$, $\mathcal{T}(\xi; \mu_0) \in \mathcal{C}_\lambda$ and*

$$\frac{\|\Xi(\xi)\|_\lambda}{\|\xi\|_\lambda} \leq \frac{1}{(\kappa_c - 2\lambda\rho)}, \quad \forall \xi \in \mathcal{C}_\lambda.$$

Proof By the definitions of M , and N ,

$$|(M_u, N_u)| \leq \int_u^\infty e^{-\rho(s-u)} e^{-\lambda u} \|\xi\|_\lambda \, ds = \frac{\|\xi\|_\lambda}{\rho + \lambda} e^{-\lambda u} \leq \frac{\|\xi\|_\lambda}{\rho} e^{-\lambda u}.$$

Then,

$$|\Xi_t(\xi)| \leq \frac{1}{2} \int_0^t e^{-\frac{\sigma^2}{2}(t-u)} \frac{\|\xi\|_\lambda}{\rho} e^{-\lambda u} \, du = \frac{\|\xi\|_\lambda}{(\sigma^2 - 2\lambda)\rho} e^{-\lambda t} = \frac{\|\xi\|_\lambda}{(\kappa_c - 2\lambda\rho)} e^{-\lambda t},$$

where in the last calculation we used the identity $\kappa_c = \sigma^2\rho$. As all terms in the representation of $\mathcal{T}(\xi; \mu_0)$ are in \mathcal{C}_λ , consequently, so is $\mathcal{T}(\xi; \mu_0)$. □

Note that for all $\kappa < \kappa_c$, and all sufficiently small λ , the map $\xi \in \mathcal{C}_\lambda \mapsto \kappa \Xi(\xi) \in \mathcal{C}_\lambda$ is a contraction. Therefore, in view of Proposition 10.4, $\kappa\mathcal{T}$ is equal to a contraction perturbed by a quadratic nonlinearity. This formal observation drives the subsequent analysis.

10.3. Proof of Theorem 4.4

We start with a uniform bound. Recall $d(\mu_0)$ of (4.1).

Lemma 10.6. For every $\kappa < \kappa_c$, there are $C_\kappa, c_\kappa, \lambda_\kappa > 0$ depending only on κ, β, σ , such that if $d(\mu_0) \leq c_\kappa$, then

$$\|\xi\|_{\lambda_\kappa} \leq C_\kappa \quad \Rightarrow \quad \|\kappa \mathcal{T}(\xi; \mu_0)\|_{\lambda_\kappa} \leq C_\kappa.$$

Proof We fix $\mu_0, \kappa < \kappa_c$ and set

$$\delta := \frac{\kappa_c - \kappa}{4\kappa_c}, \quad \Rightarrow \quad \kappa = \kappa_c(1 - 4\delta).$$

Choose $\lambda_\kappa < \delta\sigma^2/2$. Then,

$$(\kappa_c - 2\lambda_\kappa\rho)(1 - 3\delta) = \kappa_c(1 - \frac{2\lambda_\kappa}{\sigma^2})(1 - 3\delta) \geq \kappa_c(1 - \delta)(1 - 3\delta) \geq \kappa_c(1 - 4\delta) = \kappa.$$

By definition $\kappa \mathcal{T}_0(\xi; \mu_0)_0 = \kappa(\mu_0(\cos), \mu_0(\sin))$. Therefore, $|\kappa \mathcal{T}_0(\xi; \mu_0)_0| \leq 2\kappa d(\mu_0)$. Then, by Proposition 10.4 and Corollary 10.5,

$$\begin{aligned} \|\kappa \mathcal{T}_0(\xi; \mu_0)\|_{\lambda_\kappa} &\leq |\kappa \mathcal{T}_0(\xi; \mu_0)_0| + \left[\frac{\kappa}{\kappa_c - 2\lambda_\kappa\rho} + \frac{8\kappa d(\mu_0)}{\kappa_c} \right] \|\xi\|_{\lambda_\kappa} + \kappa c_2(\beta, \sigma) \|\xi\|_{\lambda_\kappa}^2 \\ &\leq 2\kappa d(\mu_0) + \left[(1 - 3\delta) + \frac{8\kappa d(\mu_0)}{\kappa_c} \right] \|\xi\|_{\lambda_\kappa} + \kappa c_2(\beta, \sigma) \|\xi\|_{\lambda_\kappa}^2. \end{aligned}$$

Set

$$C_\kappa = C(\kappa, \beta, \sigma) := \frac{\delta}{\kappa c_2(\beta, \sigma)}, \quad c_\kappa = c(\kappa, \beta, \sigma) := \min\left\{ \frac{\delta C_\kappa}{2\kappa}, \frac{\delta}{8} \right\}.$$

Then, if $d(\mu_0) \leq c_\kappa$ and $\|\xi\|_{\lambda_\kappa} \leq C_\kappa$,

$$\begin{aligned} \|\kappa \mathcal{T}(\xi; \mu_0)\|_{\lambda_\kappa} &\leq 2\kappa c_\kappa + [(1 - 3\delta) + 8d(\mu_0)]\|\xi\|_{\lambda_\kappa} + [\kappa c_2(\beta, \sigma)\|\xi\|_{\lambda_\kappa}] \|\xi\|_{\lambda_\kappa} \\ &\leq \delta C_\kappa + (1 - 2\delta)C_\kappa + [\kappa c_2(\beta, \sigma)C_\kappa] C_\kappa \\ &= C_\kappa [\delta + (1 - 2\delta) + \delta] = C_\kappa. \end{aligned}$$

This completes the proof of this lemma. □

Let $\lambda_\kappa, C_\kappa, c_\kappa > 0$ be as in the above lemma and set

$$\mathcal{B}_\kappa := \{ \xi \in \mathcal{C}_{\lambda_\kappa} : \|\xi\|_{\lambda_\kappa} \leq C_\kappa \} \subset \mathcal{C}_\lambda, \quad \forall \lambda \in (0, \lambda_\kappa].$$

We have shown above that $\kappa \mathcal{T}(\cdot; \mu_0)$ maps \mathcal{B}_κ into itself provided that $d(\mu_0) \leq c_\kappa$.

Lemma 10.7. For every $0 < \lambda < \lambda_\kappa$ and $d(\mu_0) \leq c_\kappa$, $\kappa \mathcal{T}(\mathcal{B}_\kappa; \mu_0)$ is pre-compact in \mathcal{C}_λ .

Proof Fix $0 < \lambda < \lambda_\kappa$ and μ_0 with $d(\mu_0) \leq c_\kappa$. Let ξ^n be a sequence \mathcal{B}_κ , and set $\zeta^n := \kappa \mathcal{T}(\xi^n; \mu_0)$. By the previous lemma, $\zeta^n \in \mathcal{B}_\kappa$. As in the proof of Lemma 4.5 given in Appendix A.4, we use Lemma A.3 with Arzelà–Ascoli in a diagonal argument to construct a subsequence, denoted by n again, and $\zeta^* \in \mathcal{B}_\kappa$ such that ζ^n converges to ζ^* uniformly on every compact set $[0, T]$. Since $\zeta^n, \zeta^* \in \mathcal{B}_\kappa$, for every n, T ,

$$\sup_{t \geq T} |\zeta_t^n - \zeta_t^*| e^{\lambda t} \leq \sup_{t \geq T} |\zeta_t^n - \zeta_t^*| e^{(\lambda - \lambda_\kappa)t} e^{\lambda_\kappa t} \leq \|\zeta^n - \zeta^*\|_{\lambda_\kappa} e^{-(\lambda_\kappa - \lambda)T} \leq 2C_\kappa e^{-(\lambda_\kappa - \lambda)T}.$$

Given $\epsilon > 0$, we choose $T_\epsilon > 0$ such that $2C_\kappa e^{-(\lambda_\kappa - \lambda)T_\epsilon} \leq \epsilon$. As ζ^n converges to ζ^* uniformly on every bounded set, there exists n_ϵ satisfying

$$\sup_{t \in [0, T_\epsilon]} |\zeta_t^n - \zeta_t^*| e^{\lambda t} \leq \epsilon, \quad \forall n \geq n_\epsilon.$$

Hence, for every $n \geq n_\epsilon$, $\|\zeta^n - \zeta^*\|_\lambda \leq \epsilon$. So we conclude that ζ^n converges to ζ^* in $\|\cdot\|_\lambda$, proving that $\kappa \mathcal{T}(\mathcal{B}_\kappa; \mu_0)$ is pre-compact in \mathcal{C}_λ . \square

Proof of Theorem 4.4 Let $\lambda_\kappa, c_\kappa, C_\kappa$ be as in the Lemma 10.6. Fix $\lambda \in (0, \lambda_\kappa)$. Suppose that the initial distribution μ_0 satisfies $d(\mu_0) \leq c_\kappa$. We have shown that $\kappa \mathcal{T}(\cdot; \mu_0)$ is pre-compact on the convex set \mathcal{B}_κ and it maps \mathcal{B}_κ onto itself. The continuity of $\kappa \mathcal{T}(\cdot; \mu_0)$ can be proved as in Lemma A.4. Therefore, we can use the Schauder fixed point theorem to conclude that there exists $\xi^* \in \mathcal{C}_\lambda$ so that $\xi^* = \kappa \mathcal{T}(\xi^*; \mu_0)$. Let X^* be the optimal process for the problem (3.1) with ξ^* . In view of Lemma 9.1, $(\mathcal{L}(X_t^*))_{t \geq 0}$ is a solution of the Kuramoto mean field game with interaction parameter κ and initial distribution μ_0 . We now use Lemma (9.3) as in Section 9.3, to conclude that $\mathcal{L}(X_t^*)$ converges to the uniform distribution.

As $\xi^* \in \mathcal{B}_\kappa$, we have $\|\xi^*\|_\lambda \leq C_\kappa$. Therefore, by (10.5) and (10.6), we conclude that (4.2) holds with $\lambda_\kappa^* = \lambda$. \square

Funding

Research of Carmona was partially supported by AFOSR FA9550-19-1-0291 and ARPA-E DE-AR0001289. Research of Soner was partially supported by the National Science Foundation grant DMS 2106462.

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A. Existence of solutions

We first approximate the infinite horizon problem (3.1) by finite horizon problems and prove the existence of solutions for them. Then, we use a limiting argument to construct solutions to the original problem.

A.1. Finite horizon problem

For a finite horizon T , we modify the control problem (3.1) slightly and consider

$$\text{minimize } \alpha \mapsto J_\mu(\alpha; T) := \mathbb{E} \int_t^T e^{-\beta(u-t)} [\ell_\mu(u, X_u^\alpha) + \frac{1}{2} \alpha_u^2] du, \quad t \in [0, T], x \in \mathbb{T},$$

where ℓ_μ and X are as in (3.1). The solution to the finite horizon Kuramoto mean field game is defined exactly as in Definition 3.1.

Arguments of the Section 9.1 leading to Lemma 9.1 can be followed *mutatis mutandis* to obtain a similar fixed point characterization of the solutions. Indeed, let

$$\mathcal{C}_{T,\kappa} := \{ \xi = (\gamma, \eta) : [0, T] \mapsto \mathbb{R}^2 : \text{continuous and } \|\xi\|_\infty \leq \kappa \},$$

and for $t \in [0, T]$, $x \in \mathbb{T}$ set

$$v^{\xi, T}(t, x) := \inf_{\alpha \in \mathcal{A}_t} J^{\xi, T}(t, x, \alpha) := \mathbb{E} \int_t^T e^{-\beta(u-t)} [\ell_\xi(u, X_u^{\alpha, (t, x)}) + \frac{1}{2} \alpha_u^2] du, \quad (\text{A.1})$$

where ℓ_ξ is as in (5.2) and $X^{\alpha,(t,x)}$ is as in (5.3).

Note that the corresponding dynamic programming equation is exactly (9.1). The only difference is that the equation holds for $t \in (0, T)$ and $v^{\xi,T}$ satisfies the terminal condition $v^{\xi,T}(\cdot, T) \equiv 0$. This equation has a smooth solution, and in particular, the Lipschitz estimate $|v_x^{\xi,T}(x, t)| \leq \|\xi\|_\infty/\beta$ is proved as in the proof of Lemma 9.2. Also the optimal state process $X^{\xi,T}$ starting from any initial condition X_0 is the unique solution of the stochastic differential equation (9.2) with $v_x^{\xi,T}$ replacing v_x^ξ , i.e.,

$$dX_t^{\xi,T} = -v_x^{\xi,T}(t, X_t^{\xi,T}) dt + \sigma dB_t, \tag{A.2}$$

Definition A.1. A flow of probability measures $\mu := (\mu_t)_{t \in [0, T]}$ is a solution of the finite horizon Kuramoto mean field game with initial data μ_0 , if and only if the solution of (A.2) with $\mathcal{L}(X_0) = \mu_0$ satisfies $\mathcal{L}(X_t) = \mu_t$ for all $t \in [0, T]$.

As in Lemma 9.1 to prove the existence of a solution to the finite horizon Kuramoto mean field game, it suffices construct a fixed point of $\kappa\mathcal{T}(\cdot; T, \mu_0)$, where

$$\mathcal{T}(\cdot; T, \mu_0) : \xi \in \mathcal{C}_{T,\kappa} \mapsto \mathcal{T}(\xi; T, \mu_0) := (\mathbb{E}[\cos(X_t^{\xi,T})], \mathbb{E}[\sin(X_t^{\xi,T})])_{t \in [0, T]}.$$

A.2. A convergence result

In this subsection, we consider the stochastic optimal control problem (A.1) with a running cost ℓ_ξ given by (5.2) and with both finite and infinite T . It is classical that the value function $v^{T,\xi}$ of (A.1) or $v^{\infty,\xi} := v^\xi$ of (5.3) are smooth, classical solutions of the dynamic programming equation (9.1). The main result of this subsection is the following convergence result that is used repeatedly in our forthcoming arguments.

Lemma A.2. For $T \leq \infty$, suppose that T_n converges to T and a sequence $\xi^n \in \mathcal{C}_{T_n,\kappa}$ converges locally uniformly to $\xi^* \in \mathcal{C}_{T,\kappa}$. Then, $v^n := v^{\xi^n, T_n}$ and v_x^n converge locally uniformly to $v^* := v^{T,\xi^*}$ and to v_x^* , respectively.

Proof Convergence of the value function follows directly from the definitions. Also we have argued earlier that $|v_x^{\xi^n, T_n}(t, x)| \leq \|\xi^n\|_\infty/\beta$. In particular, v_x^n is uniformly bounded.

Set $w^n := v_{xx}^n$ and $\ell^n := \ell_{\xi^n}$. The dynamic programming equation (9.1) implies that w^n satisfies the linear parabolic equation

$$-w_t^n(t, x) + \beta w^n(t, x) - \frac{\sigma^2}{2} w_{xx}^n(t, x) + w_x^n(t, x) v_x^n(t, x) = (\ell^n)_{xx}(t, x) - (w^n(t, x))^2,$$

with $w^n(T, \cdot) \equiv 0$ when $T < \infty$. Since $|\xi^n| \leq \kappa$, we have $(\ell^n)_{xx}(t, x) \leq \kappa$. Therefore, Feynman-Kac implies that

$$w^n(t, x) = v_{xx}^n(t, x) \leq \frac{\kappa}{\beta}, \quad \forall t \in [0, T], x \in \mathbb{T}.$$

Therefore, $\bar{v}^n(t, x) := v^n(t, x) - \frac{\kappa}{2\beta}x^2$ is concave. Consider a sequence (t_n, x_n) converging to (t_0, x_0) and set $p_n := v_x^n(t_n, x_n) - (\kappa/\beta)x_n$. Then, $p_n = \bar{v}_x^n(t_n, x_n)$ and since \bar{v}^n is concave, we have

$$\bar{v}^n(t_n, y) \leq \bar{v}^n(t_n, x_n) + p_n(y - x_n), \quad \forall y \in \mathbb{R}.$$

Since as argued before v_x^n is uniformly bounded, p_n converges to p^* on a subsequence. Set $\bar{v}^*(t, x) := v^*(t, x) - \frac{\kappa}{2\beta}x^2$ and let n tend to infinity, to conclude that

$$\bar{v}^*(t_0, y) \leq \bar{v}^*(t_0, x_0) + p^*(y - x_0), \quad \forall y \in \mathbb{R}.$$

As \bar{v}^* is concave and differentiable, the above inequality implies that $p^* = v_x^*(t_0, x_0)$, proving the local uniform convergence of v_x^n to v_x^* . □

A.3. Finite horizon solution

Fix $T > 0$ and for $\xi \in \mathcal{C}_{T,\kappa}$, let $X^{\xi,T}$ be as in (A.2), and set

$$A_t^{\xi,T} := \mathbb{E}[\cos(X_t^{\xi,T})], \quad B_t^{\xi,T} := \mathbb{E}[\sin(X_t^{\xi,T})], \quad t \in [0, T].$$

Lemma A.3. *There exists a constant c_* depending only on β, σ, κ such that*

$$|A_s^{\xi,T} - A_t^{\xi,T}| + |B_s^{\xi,T} - B_t^{\xi,T}| \leq c_*(s - t), \quad \forall \xi \in \mathcal{C}_{T,\kappa}, 0 \leq t \leq s \leq T.$$

Proof Using Ito’s formula as in the proof of Lemma 9.3 we arrive at the following estimate:

$$A_s^{\xi,T} = e^{-\frac{\sigma^2}{2}(s-t)} A_t^{\xi,T} + \int_t^s e^{-\frac{\sigma^2}{2}(u-t)} \mathbb{E}[v_x^{\xi,T}(u, X_u^{\xi,T}) \sin(X_u^{\xi,T})] du.$$

Since $|v_x^{\xi,T}(x, t)| \leq \|\xi\|_\infty/\beta$ and $\|\xi\|_\infty \leq \kappa$ for every $\xi \in \mathcal{C}_{T,\kappa}$, we conclude that A is uniformly Lipschitz. The statement for B is proved exactly the same way. □

We have shown that $\kappa\mathcal{T}(\cdot; T, \mu_0)$ maps $\mathcal{C}_{T,\kappa}$ into itself, and by Arzelà–Ascoli, the above uniform Lipschitz estimate implies that it is a compact map.

Lemma A.4. *For every $T < \infty$, interaction parameter κ , and μ_0 , there are solutions to the finite horizon Kuramoto mean field game.*

Proof We first prove that $\mathcal{T}(\cdot; T, \mu_0)$ is continuous on $\mathcal{C}_{T,\kappa}$. Suppose that a sequence $\xi^n \in \mathcal{C}_{T,\kappa}$ converges uniformly to ξ^* . Let $v^n := v^{\xi^n,T}$ be the value function defined in (A.1), and set $v^* := v^{\xi^*,T}$. Since $T < \infty$, by Lemma A.2, v^n, v_x^n converge uniformly to v^*, v_x^* . Let X^n be the solution of (9.2) with v_x^n and X^* be the solution with v_x^* . Since v_x^n converges to v_x^* uniformly, we conclude that X_t^n converges to X_t^* in L^1 for every $t \in [0, T]$. Consequently, $\mathbb{E}[\cos(X_t^n), \sin(X_t^n)]$ converges to $\mathbb{E}[\cos(X_t^*), \sin(X_t^*)]$ for every $t \in [0, T]$, and this implies the continuity of \mathcal{T} .

Summarizing, we have shown that $\kappa\mathcal{T}(\cdot; T, \mu_0)$ is a continuous, compact operator mapping $\mathcal{C}_{T,\kappa}$ into itself. Therefore, we can apply the Schauder fixed point theorem to conclude that $\kappa\mathcal{T}(\cdot; T, \mu_0)$ has a fixed point. Then, the finite horizon version of Lemma 9.1 implies that there are solutions to the finite horizon Kuramoto mean field game. □

A.4. Proof of Lemma 4.5

Let \mathbb{N} be the set of all positive integers. We represent subsequences by strictly increasing functions of \mathbb{N} into itself. Fix κ and for $m \in \mathbb{N}$, set

$$\mathcal{C}_m := \mathcal{C}(m, \kappa) = \{\xi : [0, m] \mapsto \mathbb{R}^2 : \|\xi\|_\infty \leq \kappa\}.$$

Let $\mu^m = (\mu_t^m)_{t \in [0, m]}$ be a solution of the Kuramoto mean field game with horizon m , and $\xi^m := \xi(\mu^m) \in \mathcal{C}_m$ be as in (9.4). By Lemma A.3 they are uniformly Lipschitz continuous, and by their definition, are bounded by κ . We now use the diagonal argument to construct a locally convergent subsequence. Set $M_0(n) = n$ for every $n \in \mathbb{N}$. For $m \in \mathbb{N}$ we recursively construct subsequences $N_m, M_m : \mathbb{N} \mapsto \mathbb{N}$ as follows. Suppose that M_m is constructed so that $M_m(1) \geq m$, and the sequence of functions

$$(\xi^{M_m(n)})_{n \in \mathbb{N}} = (\xi^{M_m(1)}, \xi^{M_m(2)}, \dots) \subset \mathcal{C}_m$$

is uniformly convergent to a function $\xi^{m,*} \in \mathcal{C}_m$. If $n \geq 2$,

$$M_m(n) \geq M_m(2) \geq M_m(1) + 1 \geq m + 1.$$

Hence, $(\xi^{M_m(n)})_{n=2,3,\dots}$ are all in \mathcal{C}_{m+1} . Moreover, they are uniformly Lipschitz continuous on $[0, m + 1]$. By Arzelà–Ascoli there exists an increasing function $N_m : \mathbb{N} \mapsto \{2, 3, \dots\}$ such that with

$$M_{m+1}(n) := M_m(N_m(n)), \quad n \in \mathbb{N},$$

the sequence of functions $(\xi^{M_{m+1}(n)})_{n \in \mathbb{N}} = (\xi^{M_{m+1}(1)}, \xi^{M_{m+1}(2)}, \dots) \subset \mathcal{C}_{m+1}$ is uniformly convergent to a function $\xi^{m+1,*} \in \mathcal{C}_{m+1}$. Moreover, $M_{m+1}(1) = M_m(N_m(1)) \geq M_m(2) \geq M_m(1) + 1 \geq m + 1$. Hence, we can repeat the process to construct M_m, N_m as claimed. It is also clear that the limit functions satisfy the consistency condition

$$m \leq m' \quad \Rightarrow \quad \xi_t^{m,*} = \xi_t^{m',*}, \quad \forall t \in [0, m].$$

Then, the function $\xi_t^* := \xi_t^{m,*}$ when $t \in [0, m]$, is a well-defined and is in \mathcal{C} . Notice that by construction, $\{M_{m'}(n) : n \in \mathbb{N}\} \subset \{M_m(n) : n \in \mathbb{N}\}$, for every $m \leq m'$.

Finally, for $n \in \mathbb{N}$, set $K(n) := M_n(n)$. Then, $(K(n))_{n \geq m} \subset (M_m(n'))_{n' \in \mathbb{N}}$ for every $m \in \mathbb{N}$, i.e., K after the index m is a subsequence of M_m . Therefore,

$$\lim_{n \rightarrow \infty} \xi_t^{K(n)} = \lim_{n \rightarrow \infty} \xi_t^{M_m(n)} = \xi_t^{m,*} = \xi_t^*, \quad \forall t \in [0, m].$$

Moreover, this convergence is uniform. Hence, as n tends to infinity the sequence of functions $\xi^{K(n)}$ converge to ξ^* uniformly on every $[0, m]$. Set $\ell_m(t, x) := \ell^{\xi^{K(m)}}$, $\ell^* := \ell^{\xi^*}$, $v^m := v^{\xi^{K(m)}, m}$, and $v^* := v^{\xi^*}$. Note that v^{ξ^*} is also equal to v^{∞, ξ^*} . Then, by Lemma A.2, v^m, v_x^m, ℓ_m converge locally uniformly to $v^*, v_x^*,$ and respectively ℓ^* . As before, let X^m be given by (9.2) with v_x^m , and X^* be the solution of (9.2) with v_x^* . Then, X^m is the optimal state process for v^m and X^* for v^* . Also, for every $t \geq 0$, X_t^m converges to X_t^* almost surely. As ξ^m is a fixed point of $\mathcal{T}(\cdot; T_m, \mu_0)$, we have

$$\kappa \mathbb{E}[\cos(X_t^*), \sin(X_t^*)] = \lim_{m \rightarrow \infty} \kappa \mathbb{E}[\cos(X_t^m), \sin(X_t^m)] = \lim_{m \rightarrow \infty} \xi_t^m = \xi_t^*.$$

Hence, ξ is a fixed point of the map $\kappa \mathcal{T}(\cdot; \mu_0)$. By Lemma 9.1, the probability flow $(\mathcal{L}(X_t^*))_{t \geq 0}$ is a solution of the Kuramoto mean field game starting from the distribution μ_0 . \square

B. A comparison result

We provide the proof of the comparison result for (8.5) which essentially follows from standard techniques. The fact that the forcing term in the equation vanishes at the boundary does not allow us to find an immediate reference in the literature.

Lemma B.1 (Comparison lemma). *Suppose that continuous functions w and u are a viscosity sub and respectively super-solution of (8.5), and satisfy the boundary conditions:*

$$w(0) = u(0) = w(2\pi) = u(2\pi) = 0.$$

Then, $w \leq u$ on $[0, 2\pi]$.

Proof Towards a counter-position, we assume that $\max_{[0, 2\pi]}[w - u] =: c_1 > 0$. For small constants $\delta, \epsilon > 0$, set

$$\Phi_{\delta, \epsilon}(x, y) := (1 - \delta)w(x) - u(y) - \frac{1}{2\epsilon}|x - y|^2, \quad x, y \in [0, 2\pi].$$

Choose $x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^* \in [0, 2\pi]$ satisfying $\max_{[0, 2\pi]} \Phi_{\delta, \epsilon} = \Phi_{\delta, \epsilon}(x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^*)$. Followings are elementary consequences.

1. Clearly, we have $\Phi_{\delta, \epsilon}(x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^*) \leq \|w\|_\infty + \|u\|_\infty$. Also, using that $\Phi_{\delta, \epsilon}(x_{\delta, \epsilon}^*, x_{\delta, \epsilon}^*) + \Phi_{\delta, \epsilon}(y_{\delta, \epsilon}^*, y_{\delta, \epsilon}^*) \leq 2\Phi_{\delta, \epsilon}(x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^*)$, we have:

$$|x_{\delta, \epsilon}^* - y_{\delta, \epsilon}^*|^2 \leq 2\epsilon(\|w\|_\infty + \|u\|_\infty).$$

2. Let x_δ^*, y_δ^* be any limit point of $x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^*$ as $\epsilon \downarrow 0$. By the above step, $x_\delta^* = y_\delta^*$.
3. By definitions $(1 - \delta)w(x) - u(x) = \Phi_{\delta, \epsilon}(x, x) \leq \Phi_{\delta, \epsilon}(x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^*) \leq w(x_{\delta, \epsilon}^*) - u(y_{\delta, \epsilon}^*)$ for any x . We use a limit argument to conclude that $(1 - \delta)w(x) - u(x) \leq w(x_\delta^*) - u(y_\delta^*)$.
4. Let x^* be any limit point of x_δ^* as $\delta \downarrow 0$. Then, for any x ,

$$\begin{aligned} w(x) - u(x) &= \lim_{\delta \downarrow 0} (1 - \delta)w(x) - u(x) \leq \liminf_{\delta \downarrow 0} (1 - \delta)w(x_\delta^*) - u(y_\delta^*) \\ &= \hat{w}(x^*) - u(x^*). \end{aligned}$$

Thus, $w(x^*) - u(x^*) = \max_{[0, 2\pi]}[w - u] =: c_1 > 0$.

5. Since w, u are continuous and $(w - u)(0) = (w - u)(2\pi) = 0$, there exists $\epsilon_0, a \in (0, \pi/2)$ such that for every $\delta, \epsilon \in (0, \epsilon_0]$,

$$x_{\delta, \epsilon}^*, y_{\delta, \epsilon}^* \in (a, 2\pi - a), \quad \Rightarrow \quad \sin^2(x_{\delta, \epsilon}^*/2), \sin^2(y_{\delta, \epsilon}^*/2) \geq \sin^2(a/2). \quad (\text{B.1})$$

We now proceed as in the usual comparison proof in the theory of viscosity solutions which we provide for completeness. We first observe that $x \in [0, 2\pi] \mapsto \Phi_{\delta, \epsilon}(\cdot, y_{\delta, \epsilon}^*)$ is maximized at $x_{\delta, \epsilon}^*$. Hence,

$$x_{\delta, \epsilon}^* \in \operatorname{argmax}_{[0, 2\pi]} \left[w - \frac{1}{(1 - \delta)} \varphi \right], \quad \text{where} \quad \varphi(x) := \frac{1}{2\epsilon}|x - y_{\delta, \epsilon}^*|^2.$$

Since w is a viscosity subsolution of (8.5), the following inequality holds,

$$\frac{1}{2(1 - \delta)^2} |p_{\delta, \epsilon}|^2 \leq 2 \sin^2(x_{\delta, \epsilon}^*/2), \quad \text{where} \quad p_{\delta, \epsilon} = \nabla \varphi(x_{\delta, \epsilon}^*) = \frac{x_{\delta, \epsilon}^* - y_{\delta, \epsilon}^*}{\epsilon}. \quad (\text{B.2})$$

Proceeding almost exactly as above and using the fact that u is a viscosity supersolution, we arrive at the following inequality,

$$\frac{1}{2} |p_{\delta, \epsilon}|^2 \geq 2 \sin^2(y_{\delta, \epsilon}^*/2) \geq 2 \sin^2(a/2) > 0, \quad (\text{B.3})$$

where the final inequality follows from (B.1). We now subtract the above inequality from (B.2). The result is the following,

$$\frac{1}{2}(p_{\delta,\epsilon})^2((1-\delta)^{-2}-1) \leq 2[\sin^2(x_{\delta,\epsilon}^*/2) - \sin^2(y_{\delta,\epsilon}^*/2)].$$

We let $\epsilon \downarrow 0$ while keeping δ fixed. Then by Step 2, $|x_{\delta,\epsilon}^* - y_{\delta,\epsilon}^*|$ converges to zero. Hence,

$$\limsup_{\epsilon \downarrow 0} \frac{1}{2}(p_{\delta,\epsilon})^2 \leq 0.$$

This is contradiction with (B.3). □