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CAN THE NONLOCAL CHARACTERIZATION OF SOBOLEV SPACES BY BOURGAIN ET AL. BE USEFUL TO SOLVE VARIATIONAL PROBLEMS?*

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Abstract. We question whether the recent characterization of Sobolev spaces by Bourgain, Brezis, and Mironescu (2001) could be useful to solve variational problems on $W^{1,p}(\Omega)$. To answer this, we introduce a sequence of functionals so that the seminorm is approximated by an integral operator involving a differential quotient and a radial mollifier. Then, for the approximated formulation, we prove existence, uniqueness, and convergence of the solution to the unique solution of the initial formulation. We show that these results can also be extended in the BV-case. Interestingly, this approximation leads to a unified implementation, for Sobolev spaces (including with high p-values) and for the BV space. Finally, we show how this theoretical study can indeed lead to a numerically tractable implementation, and we give some image diffusion results as an illustration.

Key words. calculus of variation, functional analysis, Sobolev spaces, BV, variational approach, integral approximations, nonlocal formulations

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1. Introduction. The goal of this work is to propose a new unifying method for solving variational problems defined on the Sobolev spaces $W^{1,p}(\Omega)$ or on the space of functions of bounded variations $BV(\Omega)$ of the form

$$\inf_{u \in W^{1,p}(\Omega)} F(u), \tag{1.1}$$

with

$$F(u) = \int_\Omega |\nabla u(x)|^p dx + \int_\Omega h(x, u(x)) dx.$$

To solve this problem numerically, particularly in the case when $p = 1$, several methods have been proposed; see, e.g., [8, 13, 14, 7, 18]. These methods mainly rely on regularization or duality results.

In this article we propose an alternative method based on a recent new characterization of the Sobolev spaces by Bourgain, Breszis, and Mironescu [5], and further extended by Ponce [16] in the BV-case. In [5] the authors showed that the Sobolev seminorm of a function f can be approximated by a sequence of integral operators involving a differential quotient of f and a suitable sequence of radial mollifiers:

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy = K_{N,p} \int_\Omega |\nabla u|^p dx.$$

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In this paper, our main contribution is to show how this characterization can be used to approximate the variational formulation (1.1) by defining the sequence of functionals

$$F_n(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy + \int_\Omega h(x, u(x))\ dx.$$

To do this, we prove that the sequence of minimizers of F_n converges to the solution of the original variational formulation. We prove this result for any $p \geq 1$, so that the BV-case is also covered (thanks to results by Ponce [16]). Note that approximation is not constrained by the fidelity attach term (see [7]). Numerically, we propose a unified subgradient approach for all $p \geq 1$, and we show how to discretize the nonlocal singular term with a finite element–type method.

Interestingly, the nonlocal term in F_n has some similarities to recent contributions by Gilboa and Osher [12] and Gilboa et al. [11], who propose to minimize nonlocal functionals of the type

$$\int_\Omega \int_\Omega \phi(|u(x) - u(y)|) w(|x, y|) dx dy,$$

where ϕ is a convex positive function and w is a weighting function. The authors propose a general formalism for nonlocal smoothing terms but define them heuristically for their applications in image processing (see also the link to neighborhood filters [6]). In our contribution, the nonlocal term that we propose comes from the approximation of a seminorm, so that we will show some regularity results on the solution. Notice that one related major difference is the weighting function, which is in our case singular.

This paper is organized as follows. In section 2, we recall the main results from [5] that we will use herein and define the sequence of the approximating functional F_n. In section 3, we present the most significant results of the paper, considering the case $p > 1$: we prove existence and uniqueness of a minimizer u_n of F_n, characterize its regularity, derive the optimality condition, and finally show that u_n converges to the unique solution of the initial formulation. In section 4, we describe how those results can be extended to the case $p = 1$, which corresponds to the BV-case. Finally, we show in section 5 how this theoretical study can indeed lead to a numerically tractable implementation, and we give some image diffusion results as an illustration.

2. The Bourgain–Brezis–Mironescu result. Let us first recall the result of Bourgain, Brezis, and Mironescu [5].

PROPOSITION 2.1. *Assume $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$, and let $\rho \in L^1(\mathbb{R}), \rho \geq 0$. Then*

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho(|x-y|) dx dy \leq C \|u\|^p_{W^{1,p}} \|\rho\|_{L^1(\mathbb{R})}, \tag{2.1}$$

where $\|u\|^p_{W^{1,p}}$ denotes the (semi)norm defined by $\|u\|^p_{W^{1,p}} = \int_\Omega |\nabla u|^p dx$ and C depends only on p and Ω.

Now let us suppose that (ρ_n) is a sequence of radial mollifiers, i.e.,

$$\rho_n \geq 0, \quad \int_{\mathbb{R}^N} \rho_n(|x|) dx = 1, \tag{2.2}$$

and for every $\delta > 0$, we assume that

$$\lim_{n \to \infty} \int_\delta^\infty \rho_n(r) r^{N-1} dr = 0. \tag{2.3}$$

With conditions (2.2) and (2.3), which we will assume throughout this article, we have the following proposition.

PROPOSITION 2.2. *If $1 < p < \infty$ and $u \in W^{1,p}(\Omega)$, then*

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy = K_{N,p} \|u\|^p_{W^{1,p}}, \tag{2.4}$$

where $K_{N,p}$ depends only on p and N.

In this paper, we propose to apply Propositions 2.1 and 2.2 for solving general variational problems of the form

$$\inf_{u \in W^{1,p}(\Omega)} F(u), \tag{2.5}$$

with

$$F(u) = \int_\Omega |\nabla u(x)|^p dx + \int_\Omega h(x, u(x)) dx, u \in W^{1,p}(\Omega). \tag{2.6}$$

To do this, following [5], we introduce the nonlocal formulation

$$\inf_{u \in L^p(\Omega)} F_n(u), \tag{2.7}$$

with

$$F_n(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy + \int_\Omega h(x, u(x))\ dx. \tag{2.8}$$

Our goal is to establish in which sense formulation (2.7)–(2.8) approximates the initial formulation (2.5)–(2.6).

3. Approximation of variational problems on $W^{1,p}(\Omega)$, $p > 1$. Thanks to Proposition 2.1, functional $F_n(u)$ is well-defined on $W^{1,p}(\Omega)$. However, one cannot prove directly that F_n admits a unique minimizer on $W^{1,p}(\Omega)$, since minimizing sequences cannot be bounded in that space. Thus we need to consider the minimization over the larger space $L^p(\Omega)$, and problem (2.7) is in fact an unbounded problem in $L^p(\Omega)$.

In this section, we prove the following results:

- For n fixed, we show in section 3.1 that problem (2.7) admits a unique solution $u_n \in L^p(\Omega)$.
- Then we show in section 3.2 that u_n is more regular and belongs to the Sobolev space $W^{s,p}(\Omega)$ with $1/2 < s < 1$. Moreover, we show that all minimizing sequences are bounded on $W^{s,p}(\Omega)$. The main consequence is that minimizing sequences $(u_n^l)_l$ indeed converge strongly to u_n. This additional regularity will also enable us to consider problems with Dirichlet boundary conditions, since one can give a meaning to the trace operator on that space.
- The previous regularity result will be fundamental in section 3.3 when we consider that n tends to infinity. Applying some results by Ponce [16], we will show that u_n converges to the unique solution u of the original formulation (2.5).
- In section 3.4 we establish the expression of the Euler–Lagrange equation.

Remark. Note that throughout this section and in the proofs, we will denote by C a universal constant that may be different from one line to the other. If the constant depends on n, for example, it will be denoted by $C(n)$.

3.1. Existence and uniqueness of a solution u_n in $L^p(\Omega)$. Now, let us show that functional (2.8) admits a unique minimizer. It is clear by using again Proposition 2.1 and the fact that $\|\rho_n\|_{L^1(\mathbb{R})} = 1$ that we have for all v in $W^{1,p}(\Omega)$

$$\inf_{u \in L^p(\Omega)} F_n(u) \leq \inf_{u \in W^{1,p}(\Omega)} F_n(u) \leq F_n(v) \leq C\|v\|^p_{W^{1,p}} + \int_\Omega h(x, v(x))\ dx,$$

from which we deduce that $\inf_{u\in L^p(\Omega)} F_n(u)$ is bounded by a finite constant (independent of n).

PROPOSITION 3.1. *Assume that $h \geq 0$, the function $x \mapsto h(x, u(x))$ is in $L^1(\Omega)$ for all u in $L^p(\Omega)$, h is convex with respect to its second argument, and, for each n, the function $t \mapsto \rho_n(t)$ is nonincreasing. Then functional* (2.8) *admits a unique minimizer in $L^p(\Omega)$.*

Before proving this proposition, let us recall a technical lemma from Bourgain, Brezis, and Mironescu (Lemma 2 in [5]) that we will use in the proof of Proposition 3.1.

LEMMA 3.2. *Let $g, k : (0, \delta) \to \mathbb{R}_+$. Assume $g(t) \leq g(t/2)$ for $t \in (0, \delta)$, and that k is nonincreasing. Then for all $M > 0$, there exists a constant $C(M) > 0$ such that*

$$\int_0^\delta t^{M-1} g(t)k(t)dt \geq C(M)\delta^{-M} \int_0^\delta t^{M-1} g(t)dt \int_0^\delta t^{M-1}k(t)dt. \tag{3.1}$$

Proof of Proposition 3.1. Let us consider a minimizing sequence u_n^l of $F_n(u)$ with $n > 0$ fixed. Since $h \geq 0$ and $\inf_{u\in L^p(\Omega)} F_n(u)$ is bounded, then there exists a constant C such that

$$\int_\Omega \int_\Omega \frac{|u_n^l(x) - u_n^l(y)|^p}{|x-y|^p} \rho_n(|x-y|)dxdy \leq C. \tag{3.2}$$

We are going to apply techniques borrowed from Brezis, Bourgain, and Mironescu [5, Theorem 4]. Without loss of generality, we may assume that $\Omega = \mathbb{R}^N$ and that the support of u_n^l is included in a ball B of diameter 1. This can be achieved by extending each function u_n^l by reflection across the boundary in a neighborhood of $\partial\Omega$. We may also assume the normalization condition $\int_\Omega u_n^l(x)dx = 0$ for all n and l. Let us define for each n, l, $t > 0$

$$E_n^l(t) = \int_{S^{N-1}} \int_{\mathbb{R}^N} |u_n^l(x + tw) - u_n^l(x)|^p dxdw, \tag{3.3}$$

where S^{N-1} denotes the unit sphere of $\mathbb{R}^N$. Straightforward changes of variables show that

$$\int_\Omega \int_\Omega \frac{|u_n^l(x) - u_n^l(y)|^p}{|x-y|^p} \rho_n(|x-y|)dxdy = \int_0^1 t^{N-1} \frac{E_n^l(t)}{t^p} \rho_n(t)dt,$$

and thus (3.2) can be equivalently expressed as

$$\int_0^1 t^{N-1} \frac{E_n^l(t)}{t^p} \rho_n(t)dt \leq C. \tag{3.4}$$

Now since we have supposed that u_n^l is of zero mean, we can write

$$u_n^l(x) = u_n^l(x) - \frac{1}{|B|} \int_B u_n^l(y)dy.$$

Thus

$$\int |u_n^l(x)|^p dx = \int \Big| u_n^l(x) - \frac{1}{|B|}\int_B u_n^l(y)dy\Big|^p dx = \frac{1}{|B|^p}\int \Big| \int_B u_n^l(x) - u_n^l(y)dy\Big|^p\, dx,$$

and, thanks to the Hölder inequality, there exists a constant C such that

$$\int |u_n^l(x)|^p dx \le C\int_{|h|\le 1}\Big(\int |u_n^l(x+h)-u_n^l(x)|^p\, dx\Big)dh = C\int_0^1 t^{N-1}E_n^l(t)dt. \tag{3.5}$$

Now, an interesting property of E_n^l is that

$$E_n^l(2t) \le 2^p E_n^l(t). \tag{3.6}$$

Inequality (3.6) follows from the triangle inequality $|a+b|^p \le 2^{p-1}(|a|^p + |b|^p)$:

$$\begin{aligned} E_n^l(2t) &= \int_{S^{N-1}}\int_{\mathbb{R}^N} |u_n^l(x+2tw) - u_n^l(x)|^p dxdw \\ &= \int_{S^{N-1}}\int_{\mathbb{R}^N} |u_n^l(x+2tw) - u_n^l(x+tw) + u_n^l(x+tw) - u_n^l(x)|^p dxdw \\ &\le 2^{p-1}\Big(\int_{S^{N-1}}\int_{\mathbb{R}^N} |u_n^l(x+2tw) - u_n^l(x+tw)|^p dxdw \\ &\qquad + \int_{S^{N-1}}\int_{\mathbb{R}^N} |u_n^l(x+tw) - u_n^l(x)|^p dxdw\Big) \qquad (3.7) \\ &\le 2^p E_n^l(t), \end{aligned}$$

since both integrals in (3.7) are equal (up to a change of variable).

To conclude we apply Lemma 3.2 with $M = N$, $\delta = 1$, $k(t) = \rho_n(t)$, and $g(t) = \frac{E_n^l(t)}{t^p}$ (this choice is valid thanks to the hypotheses on ρ_n and property (3.6)). We obtain

$$\begin{aligned} \int_0^1 t^{N-1}\rho_n(t)\frac{E_n^l(t)}{t^p}dt &\ge C\int_0^1 t^{N-1}\rho_n(t)dt\int_0^1 t^{N-1}\frac{E_n^l(t)}{t^p}dt \\ &\ge C\int_0^1 t^{N-1}\rho_n(t)dt\int_0^1 t^{N-1}E_n^l(t)dt, \qquad (3.8) \end{aligned}$$

where we have used in the last inequality the fact that $0 < t < 1$. Let us denote $d(n) = \int_0^1 t^{N-1}\rho_n(t)dt > 0$; we obtain, thanks to (3.4), (3.5), and (3.8), that there exists a constant $C(n) > 0$ (but which is independent of l) such that

$$\big|u_n^l\big|_{L^p(\Omega)} \le C(n). \tag{3.9}$$

From (3.9), we deduce that, up to a subsequence, u_n^l tends weakly in $L^p(\Omega)$ to some $u_n \in L^p(\Omega)$ as $l \to +\infty$. Then we deduce that the sequence $w_n^l(x,y) = u_n^l(x) - u_n^l(y)$ tends weakly in $L^p(\Omega\times\Omega)$ to $w_n(x,y) = u_n(x) - u_n(y)$. Since the functional

$$w \to \int_\Omega\int_\Omega |w(x,y)|^p\frac{\rho_n(|x-y|)}{|x-y|^p}dxdy$$

is nonnegative, convex, and lower semicontinuous from $L^p(\Omega\times\Omega) \to \bar{R}$, we easily get

$$F_n(u_n) \le \varliminf_{l\to\infty} F_n(u_n^l) = \inf_{u\in L^p(\Omega)} F_n(u),$$

where the symbol $\varliminf$ denotes the lower limit. Therefore u_n is a minimizer of F_n. Moreover it is unique since the function $t \mapsto |t|^p$ is strictly convex for $p > 1$. $\square$

3.2. Regularity result for u_n. We have obtained the existence of a minimizer in $L^p(\Omega)$. Let us show that the solution is in fact more regular than just L^p.

As for $W^{1,p}(\Omega)$, the space $W^{s,p}(\Omega)$ can be characterized by a differential quotient. For $0 < s < 1$ and $1 \le p < \infty$, we define

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega); \frac{|u(x)-u(y)|}{|x-y|^{s+N/p}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the norm

$$|u|^p_{W^{s,p}(\Omega)} = \int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}} dxdy.$$

Let us consider n fixed and let us denote by $C(n)$ a universal positive constant depending on n (i.e., $C(n)$ may be different from one line to the next). Let $(u_n^l)_l$ be a minimizing sequence of (2.7) so that

$$\int_\Omega \int_\Omega \frac{|u_n^l(x)-u_n^l(y)|^p}{|x-y|^p} \rho_n(|x-y|) dxdy \le C(n). \tag{3.10}$$

Then we would like to prove that (3.10) implies

$$\int_\Omega \int_\Omega \frac{|u_n^l(x)-u_n^l(y)|^p}{|x-y|^{sp+N}} dxdy \le C(n) \tag{3.11}$$

for some $1/2 < s < 1$ and some constant other constant $C(n)$, thus showing that u_n^l belongs to $W^{s,p}(\Omega)$.

Proposition 3.3. *Let q be a real number such that $\frac{p}{2} < q < p$ and $(p-1) \le q$, and let us assume that ρ_n verifies* (2.2)–(2.3) *and also that conditions of Proposition* 3.1 *are fulfilled. Moreover let us suppose that the functions $t \to \rho_n(t)$ and $t \to t^{q+2-p}\rho_n(t)$ are nonincreasing for $t \ge 0$. Then $u_n^l \in W^{q/p,p}(\Omega)$ for all l.*

Proof. Without loss of generality, let us prove Proposition 3.3 for the case $N = 2$. Equivalently, thanks to (3.3) of E_n^l, we can rewrite (3.10) and (3.11) so that one needs to prove that

$$\int_0^1 t \frac{E_n^l(t)}{t^p} \rho_n(t) dt \le C(n) \tag{3.12}$$

implies

$$\int_0^1 t \frac{E_n^l(t)}{t^{sp+2}} dt \le C(n).$$

Let us apply Lemma 3.2 with $M = \delta = 1$, $g(t) = \frac{E_n^l(t)}{t^{q+1}}$, $k(t) = t^{q+2-p}\rho_n(t)$. Assuming the hypothesis on $g(t)$ is true, Lemma 3.2 gives

$$\int_0^1 \frac{E_n^l(t)\rho_n(t)}{t^{p-1}} dt \ge C(M) \int_0^1 \frac{E_n^l(t)}{t^{q+1}} dt \int_0^1 t^{q+2-p} \rho_n(t) dt. \tag{3.13}$$

Therefore

$$\int_0^1 \frac{E_n^l(t)}{t^{q+1}} dt \le \frac{1}{C(M) \int_0^1 t^{q+2-p} \rho_n(t) dt} \int_0^1 \frac{E_n^l(t)\rho_n(t)}{t^{p-1}} dt,$$

and according to (3.12), we get

$$\int_0^1 \frac{E_n^l(t)}{t^{q+1}}dt \leq \frac{C(n)/C(M)}{\int_0^1 t^{q+2-p}\rho_n(t)dt},$$

where the right-hand term is bounded independently of l. Thus $u_n^l \in W^{s,p}(\Omega)$ with $s = \frac{q}{p}$, and since we have supposed $\frac{p}{2} < q < p$ we have $\frac{1}{2} < s < 1$.

So it remains to show that function $g(t)$ verifies the hypothesis of Lemma 3.2. We have to check $g(t) \leq g(t/2)$. Since $g(t) = \frac{E_n^l(t)}{t^{q+1}}$ then $g(t/2) = \frac{E_n^l(t/2)}{t^{q+1}}2^{q+1} \geq 2^{q+1-p}\frac{E_n^l(t)}{t^{q+1}} = 2^{q+1-p}g(t)$ (thanks to (3.3)). Thus we get $g(t/2) \geq g(t)$ if $q+1-p \geq 0$, i.e., if $q \geq (p-1)$. ∎

Depending on p, one needs to find a function $\rho_n(t)$ so that $\rho_n(t)$ and $t^{q+2-p}\rho_n(t)$ are decreasing, and verify (2.2) and (2.3). Let us show that such a ρ_n function exists. We define

$$\rho_n(t) = Cn^2\rho(nt) \quad \text{with} \quad C = \frac{1}{\int_{\mathbb{R}^2}\rho(|x|)dx} \tag{3.14}$$

and, depending on the values of p, we propose the following functions:

$$\rho(t) = \begin{cases} \exp(-t)/t^{q+1} & \text{if } p=1, \text{with } 0.5 < q < 1, \\ \exp(-t)/t^q & \text{if } p=2, \text{with } 1 < q < 2, \\ \exp(-t)/t & \text{if } p>2, \text{with } q = p-1. \end{cases} \tag{3.15}$$

As a consequence, we have the following proposition.

PROPOSITION 3.4. *Let* $(u_n^l)_l$ *be a minimizing sequence of* (2.7). *Let us suppose that* h *verifies the conditions of Proposition* 3.1 *and the coercivity condition* $h(x,u) \geq a|u|^p + b$, *with* $a > 0$. *Then the sequence* $(u_n^l)_l$ *is bounded in* $W^{q/p,p}(\Omega)$ *uniformly with respect to* l. *Therefore, up to a subsequence,* u_n^l *tends weakly to* u_n *in* $W^{q/p,p}(\Omega)$ *(and strongly in* $L^p(\Omega)$*).*

Another direct consequence of Proposition 3.3 is the following.

LEMMA 3.5. *We have* $\inf_{u\in L^p(\Omega)} F_n(u) = \inf_{u\in W^{s,p}(\Omega)} F_n(u)$, *and the solution of the problem posed on* $L^p(\Omega)$ *is also the solution of the problem posed in* $W^{s,p}(\Omega)$.

Proof. Since $W^{s,p}(\Omega) \subset L^p(\Omega)$, then

$$\inf_{u\in L^p(\Omega)} F_n(u) \leq \inf_{u\in W^{s,p}(\Omega)} F_n(u).$$

By definition, since u_n is the minimizer of F_n in $L^p(\Omega)$, we have

$$F_n(u_n) = \inf_{u\in L^p(\Omega)} F_n(u) \leq \inf_{u\in W^{s,p}(\Omega)} F_n(u),$$

but as $u_n \in W^{s,p}(\Omega)$, we have finally

$$\inf_{u\in W^{s,p}(\Omega)} F_n(u) \leq F_n(u_n) = \inf_{u\in L^p(\Omega)} F_n(u) \leq \inf_{u\in W^{s,p}(\Omega)} F_n(u),$$

which concludes the proof. ∎

Remark. Yet another consequence of Proposition 3.3 is that one can also consider problems with Dirichlet boundary conditions if necessary: If one needs to solve problem (2.5) with a Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$, then one can impose the minimizing sequence of (2.7) to verify $u_n^l = \varphi$ on $\partial\Omega$ (which has a meaning thanks to this regularity result), so that, by continuity of the trace operator, we have $u_n = \varphi$ on $\partial\Omega$. Thus u_n is the unique minimizer in $W^{q/p,p}(\Omega)$ of problem (2.7), also verifying the Dirichlet boundary condition.

3.3. Study of the $\lim_{n\to\infty} u_n$. In section 3 we proved the existence of a unique solution u_n for problem (2.7), with n fixed, which is in fact in $W^{s,p}(\Omega)$. Now, we are going to examine the asymptotic behavior of (2.7) as $n \to \infty$. Throughout this section we will suppose the hypotheses stated in Proposition 3.3 and 3.4 hold. By definition of a minimizer, we have, for all $v \in W^{q/p,p}(\Omega)$,

$$F_n(u_n) \le F_n(v) = \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^p} \rho_n(|x-y|)dxdy + \int_\Omega h(x,v(x))dx. \tag{3.16}$$

Thus by using (2.1) and the fact that $|\rho_n|_{L^1} = 1$ we deduce from (3.16) that $F_n(u_n)$ is bounded uniformly with respect to n. In particular, we get for some constant $C > 0$

$$\int_\Omega \int_\Omega \frac{|u_n(x)-u_n(y)|^p}{|x-y|^p} \rho_n(|x-y|)dxdy \le C.$$

By using the same technique as in Proposition 3.3, we still have that (u_n) is bounded in $W^{q/p,p}(\Omega)$. Therefore there exists u such that (up to a subsequence) $u_n \to u$ in $L^p(\Omega)$-strong. Moreover, by applying Theorem 4 from [5], we obtain that $u \in W^{1,p}(\Omega)$. We claim that u is the unique solution of problem (2.5), i.e., for all $v \in W^{1,p}(\Omega)$,

$$\int_\Omega |\nabla u(x)|^p dx + \int_\Omega h(x,u(x))dx \le \int_\Omega |\nabla v(x)|^p dx + \int_\Omega h(x,v(x))dx. \tag{3.17}$$

To prove (3.17) we refer the reader to the paper by Ponce [16]. In this paper the author studies in the same spirit as [5] new characterizations of Sobolev spaces and also of the space $BV(\Omega)$ of functions of bounded variations (see also section 4). The author considers more general differential quotients than the ones in [5], namely, functionals of the form

$$E_n(u) = \int_\Omega \int_\Omega w\left(\frac{|u(x)-u(y)|}{|x-y|}\right) \rho_n(|x-y|)dxdy.$$

By studying the asymptotic behavior, Ponce [16] obtained new characterizations of $W^{1,p}(\Omega)$ but also of $BV(\Omega)$. In particular, for $w(t) = |t|^p$ the author proved that $E_n(u)$ Γ-converge (up to a multiplicative constant) to $E(u) = \int_\Omega |\nabla u|^p dx$.

We have the following proposition.

Proposition 3.6.

(i) *The sequence of functionals*

$$F_n(u) = E_n(u) + \int_\Omega h(x,u(x))dx$$

Γ-*converges (up to a multiplicative constant) to*

$$F(u) = E(u) + \int_\Omega h(x,u(x))dx.$$

(ii) *The sequence u_n of minimizers of $F_n(u)$, which is precompact in $L^p(\Omega)$, converges to the unique minimizer of $F(u)$.*

Proof. Item (i) is the Γ-convergence result shown by Ponce [16]. Item (ii) is a direct consequence of general Γ-convergence properties, since we proved that the sequence (u_n) is bounded in $W^{s,p}(\Omega)$, and thus converges strongly in $L^p(\Omega)$ to u (up to a subsequence). ∎

3.4. Euler–Lagrange equation. Since u_n is a global minimizer of $F_n(u)$ it necessarily verifies $F_n'(u_n) = 0$, i.e., an Euler–Lagrange equation. The Euler–Lagrange equation is given in the following proposition.

PROPOSITION 3.7. *If function h is differentiable, verifies conditions of Propositions 3.1 and 3.4, and verifies for all u and a.e. x an inequality of the form $|\frac{\partial h(x,u)}{\partial u}| \le l(x) + b|u|^{p-1}$ for some function $l(x) \in L^1(\Omega)$, $l(x) > 0$ and some $b > 0$, then the unique minimizer u_n of $F_n(u)$ verifies for a.e. x*

$$2p \int_\Omega \frac{|u_n(x) - u_n(y)|^{p-2}}{|x-y|^p}(u_n(x) - u_n(y))\rho_n(|x-y|)dy + \frac{\partial h(x, u_n(x))}{\partial u} = 0. \tag{3.18}$$

Proof. Let us focus on the smoothing term and denote

$$E_n(u_n) = \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y)|^p}{|x-y|^p}\rho_n(|x-y|)dxdy,$$

and let us consider for all v in $W^{1,p}(\Omega)$ the differential quotient

$$D_v(t) = \frac{E_n(u_n + tv) - E_n(u_n)}{t}.$$

We have

$$D_v(t) = \int_\Omega \int_\Omega \frac{|u_n(x) - u_n(y) + t(v(x) - v(y))|^p - |u_n(x) - u_n(y)|^p}{|x-y|^p}\rho_n(|x-y|)dxdy.$$

Thanks to Taylor's formula, there exists $c(t, x, y)$ with $|c(t, x, y) - (u_n(x) - u_n(y))| < t|v(x) - v(y)|$ such that

$$D_v(t) = p \int_\Omega \int_\Omega \frac{(v(x) - v(y))c(t, x, y)|c(t, x, y)|^{p-2}}{|x-y|^p}\rho_n(|x-y|)dxdy.$$

Moreover, we have, as $t \to 0$

$$\begin{aligned}&\frac{(v(x) - v(y))c(t, x, y)|c(t, x, y)|^{p-2}}{|x-y|^p}\rho_n(|x-y|)\\ &\qquad \to \frac{(v(x) - v(y))(u_n(x) - u_n(y))|u_n(x) - u_n(y)|^{p-2}}{|x-y|^p}\rho_n(|x-y|).\end{aligned}$$

On the other hand

$$|c(t, x, y)|^{p-1} \le 2^p(|u_n(x) - u_n(y)|^{p-1} + |v(x) - v(y)|^{p-1}).$$

Thus

$$\begin{aligned}&\left|\frac{(v(x) - v(y))c(t, x, y)|c(t, x, y)|^{p-2}}{|x-y|^p}\rho_n(|x-y|)\right|\\ &\qquad \le 2^p\left(\frac{|v(x) - v(y)||u_n(x) - u_n(y)|^{p-1}}{|x-y|^p}\rho_n(|x-y|) + \frac{|v(x) - v(y)|^p}{|x-y|^p}\rho_n(|x-y|)\right).\end{aligned} \tag{3.19}$$

Let us discuss the integrability of the right-hand side terms denoted, respectively, by A and B. The second term B is bounded by an integrable function because $v \in W^{1,p}(\Omega)$ and thanks to Proposition 2.1. The first term A gives

$$A = \frac{|v(x)-v(y)|}{|x-y|}\rho_n^{\frac{1}{p}}(x-y)\left|\frac{u_n(x)-u_n(y)}{|x-y|}\right|^{p-1}\rho_n^{\frac{p-1}{p}}(x-y),$$

where

$$\frac{|v(x)-v(y)|}{|x-y|}\rho_n^{\frac{1}{p}}(x-y)$$

is in $L^p(\Omega)$ since $v \in W^{1,p}(\Omega)$ and thanks to Proposition 2.1, and

$$\left|\frac{u_n(x)-u_n(y)}{|x-y|}\right|^{p-1}\rho_n^{\frac{p-1}{p}}(x-y)$$

is in $L^{\frac{p}{p-1}}(\Omega)$ since u_n is a minimizing sequence. So A is also bounded by an integrable function.

Therefore we can apply Lebesgue's dominated convergence theorem (n is fixed) and get

$$\langle E_n'(u_n), v\rangle = p\int_\Omega \frac{|u_n(x)-u_n(y)|^{p-2}}{|x-y|^p}(v(x)-v(y))(u_n(x)-u_n(y))\rho_n(|x-y|)dy.$$

The computation of the derivative of $\int_\Omega h(x,u(x))dx$ is classical. Thus the desired result (3.18) by remarking that the function $(x,y) \mapsto \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^p}$ is antisymmetric with respect to (x,y). □

4. Extension of previous results to the $BV(\Omega)$-case ($p = 1$). A similar result to that of Proposition 2.2 holds if $p = 1$; see [16]. In this case we need to search for a solution for problem (2.5) in $BV(\Omega)$, the space of functions of bounded variations [1, 10]. In fact most results are still valid in this case with some adaptations. We do not reproduce here details of their proofs, which rely upon the work by Ponce [16], who has, as said before, generalized to $BV(\Omega)$ the results of [5] stated in the $W^{1,p}(\Omega)$ case.

Let us recall the main steps and show how the results can be extended.

- The first point is that the proof of Proposition 3.1 does not apply in the case $p = 1$ since we cannot extract from a sequence bounded in $L^1(\Omega)$ a weakly converging subsequence. Thus we have to show that a minimizing sequence u_n^l of $F_n(u)$ is bounded in the Sobolev space $W^{q,1}(\Omega)$, with $0.5 < q < 1$. To do that, we use the same proof as in Proposition 3.3. Then, thanks to the two-dimensional Rellich–Kondrachov theorem $W^{q,1}(\Omega) \subset L^r(\Omega)$ with compact injection for $1 \leq r < \frac{2}{2-q}$ (note that if $0.5 < q < 1$, then $4/3 < \frac{2}{2-q} < 2$). Therefore, up to a subsequence, $u_n^l(x)$ tends, a.e., to some function $u_n(x)$. Then by using Fatou's lemma we get $F_n(u_n) \leq \liminf_{l\to\infty} F_n(u_n^l)$; i.e., u_n is a minimizer of F_n.
- The result when n tends to infinity is again obtained thanks to the Γ-convergence result by Ponce and the compactness of the sequence u_n in $L^r(\Omega)$. As a result, u_n converges strongly in $L^1(\Omega)$ to $u \in BV(\Omega)$.

- Finally, the Euler–Lagrange equation (3.18) is no longer true in the case $p = 1$ since the function $t \to |t|$ is not differentiable. However, it is subdifferentiable. Therefore (3.18) changes into an inclusion

$$0 \in \partial E_n(u_n) + \frac{\partial h}{\partial u}(x, u_n), \tag{4.1}$$

where $E_n(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_n(|x-y|)dxdy$. In (4.1), we can choose any element of the subdifferential, and, for example,

$$2 \int_\Omega \frac{1}{|x-y|} \mathrm{sign}(u_n(x) - u_n(y))\rho_n(|x-y|)dy, \tag{4.2}$$

where

$$\mathrm{sign}(s) = \begin{cases} -1 & \text{if } s < 0, \\ 0 & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases} \tag{4.3}$$

5. Implementation details and results.

5.1. A unified discrete implementation. In this section, we give the implementation details to solve the general variational problem (2.7) in a unified way (for n fixed) for both Sobolev and BV spaces.

The goal is to solve the differential inclusion

$$0 \in \partial F_n(u_n),$$

with a standard subgradient descent approach [17, 4]:

$$\begin{cases} u^{k+1}(x) = u^k(x) - \alpha^k g^k(x), \\ u^0(x) = u_0(x) \ \forall x \in \Omega, \end{cases} \tag{5.1}$$

where α^k is the kth step size and g^k is any subgradient in $\partial F_n(u_n)$.

Taking into account the expression of the gradient or subgradient, we have here

$$u^{k+1}(x) = u^k(x) + \alpha^k \Big(-\frac{\partial h}{\partial u}(x, u^k(x)) - 2p I_{u^k}(x) \Big), \tag{5.2}$$

with

$$I_{u^k}(x) = \int_\Omega \frac{|u^k(x) - u^k(y)|^{p-1}}{|x-y|^p} \mathrm{sign}(u^k(x) - u^k(y))\rho_n(|x-y|)dy \ \forall p. \tag{5.3}$$

Note that (5.3) is a unified expression which corresponds to the gradient when $p > 1$ (see the Euler–Lagrange equation in section 5.1), or a given element of the subdifferential in the BV-case (see section 4). We remind the reader that the definition of ρ_n also depends on p (see (3.15)).

Now the problem is to discretize in space the integral $I_{u^k}(x)$, which has a singular kernel, not defined when $x = y$. Let us introduce the function J_{u^k} such that

$$I_{u^k}(x) = \int_\Omega \frac{J_{u^k}(x, y)}{|x-y|} dy, \tag{5.4}$$

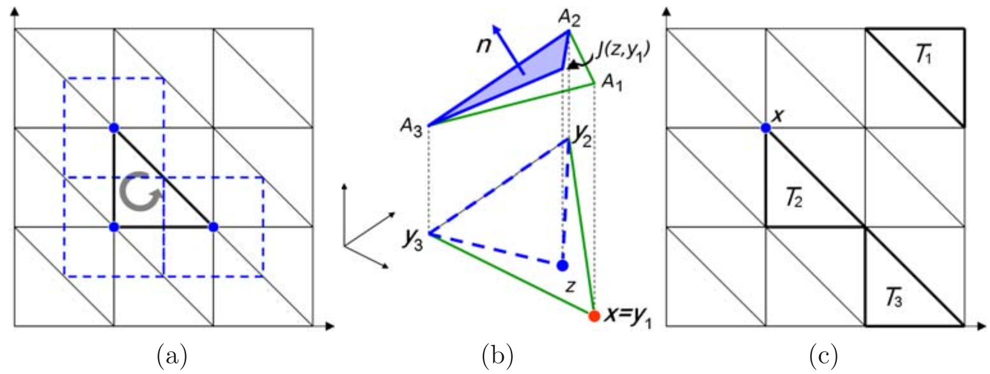

Fig. 1. (a) *Mesh definition. Pixels are represented by the dashed squares. The circles correspond to the centers of the pixels defining the nodes of the mesh. Four nodes define two triangles.* (b) *In the special case when x is a node ($x = y_1$ in the figure), one needs an interpolation to define $J_{u^k}(x, y)$. In that situation, another point z close to the node is introduced and a linear interpolation is estimated.* (c) *Different cases depending on the situation of x with respect to T_i. Triangle T_1 has no edge aligned with x; for triangle T_2, x is one node; for T_3, x is aligned with one edge.*

with

$$J_{u^k}(x, y) = \frac{|u^k(x) - u^k(y)|^{p-1}}{|x - y|^{p-1}} \operatorname{sign}(u^k(x) - u^k(y))\rho_n(|x - y|).$$

Because of the singularity, simple schemes using finite differences and integral approximations, for example, will fail. Here we propose to do the following:

- Discretize the space using a triangulation. We denote by $\mathcal{T}$ the family of triangles covering Ω (see Figure 1).
- Interpolate linearly the function $J_{u_k}(x, y)$ on each triangle (x fixed).
- Find explicit expressions for the integral $J_{u_k}(x, y)/|x - y|$ on each triangle. Note that this kind of estimation also appears, for instance, in electromagnetism problems such as MEG-EEG (see, e.g., [9]), where one needs to estimate such singular integrals on meshed domains (three-dimensional domains here).

Let us now detail each step. First, integral (5.4) becomes

$$I_{u^k}(x) = \sum_{T_i \in \mathcal{T}} \int_{T_i} \frac{J_{u^k}(x, y)}{|x - y|} dy. \tag{5.5}$$

Then let us approximate $J_{u^k}(x, y)$ on each triangle by a linear interpolation. We assume that x is given and fixed. Given one triangle $T \in \mathcal{T}$, let us denote the three nodes of T by $\{y_i = (y_i^1, y_i^2)^T\}_{i=1..3}$, where the subscript indicates the component. Then we define $\{A_i\}_{i=1..3}$ to be the three-dimensional points

$$A_i = (y_i^1, y_i^2, J_{u^k}(x, y_i))^T.$$

Note that as soon as $x \neq y_i$, $J_{u^k}(x, y_i)$ is well-defined. Otherwise, if x is in fact a node of T, for example, y_1 (see Figure 1(b)), then we use a linear interpolation algorithm: We introduce one point $z \in T$ close to y_1, estimate the value of $J_{u^k}(z, y_1)$ at this point, and deduce the value of $J_{u^k}(x, y_1)$ by interpolation.

So, given $\{A_i\}_{i=1..3}$, we can in fact choose any node y_j and write

$$J_{u^k}(x, y) = J_{u^k}(x, y_j) - \frac{1}{n^3} \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} (y - y_j), \tag{5.6}$$

where n is the normal to the triangle $A_1A_2A_3$ (see Figure 1(b)). With (5.6) we obtain

$$\begin{aligned} \int_T \frac{J_{u^k}(x,y)}{|x-y|}dy &= J_{u^k}(x,y_j)\int_T \frac{1}{|x-y|}dy - \frac{1}{n^3}\begin{pmatrix} n^1 \\ n^2 \end{pmatrix}\int_T \frac{(y-y_j)}{|x-y|}dy \\ &= J_{u^k}(x,y_j)\int_T \frac{1}{|x-y|}dy \\ &\quad - \frac{1}{n^3}\begin{pmatrix} n^1 \\ n^2 \end{pmatrix}\left[\int_T \frac{(y-x)}{|x-y|}dy + (x-y_j)\int_T \frac{1}{|x-y|}dy\right]. \end{aligned} \tag{5.7}$$

So, in order to estimate the integral over triangle T, one need only estimate

$$\int_T \frac{1}{|x-y|}dy \quad \text{and} \quad \int_T \frac{(y-x)}{|x-y|}dy. \tag{5.8}$$

If we introduce the distance function

$$\mathrm{Dist}(x,y) = |x-y| = \sqrt{(x^1-y^1)^2+(x^2-y^2)^2},$$

so that

$$\nabla_y \mathrm{Dist}(x,y) = \frac{y-x}{|x-y|},$$

$$\triangle_y \mathrm{Dist}(x,y) = \frac{1}{\mathrm{Dist}(x,y)},$$

then we have the following relations:

$$\int_T \frac{1}{|x-y|}dy = \int_T \triangle_y \mathrm{Dist}(x,y)\,dy = \sum_{i=1,2}\int_{\partial T}\frac{\partial \mathrm{Dist}}{\partial y^i}(x,y)N^i\,ds, \tag{5.9}$$

$$\int_T \frac{(y-x)}{|x-y|}dy = \int_T \nabla_y \mathrm{Dist}(x,y)\,dy = \int_{\partial T} \mathrm{Dist}(x,y)N\,ds, \tag{5.10}$$

where N is the normal to the edges of the triangle T. So we need to estimate the two kinds of integrals defined on the boundaries of the triangles. This can be done explicitly, as follows.

LEMMA 5.1. *Let us consider a segment $S=(\alpha,\beta)$ of extremities $\alpha=(\alpha^1,\alpha^2)$, $\beta=(\beta^1,\beta^2)$, N the normal to this segment, and x a fixed given point. Let us define*

$$\begin{aligned} &a = |\alpha\beta|, && \delta = a^2b^2 - c^2, && l_1 = c/\sqrt{\delta}, \\ &b = |x\alpha|, && d = \vec{x\alpha}\cdot N, && l_2 = (a^2+c)/\sqrt{\delta}, \\ &c = \vec{x\alpha}\cdot\vec{\alpha\beta}. \end{aligned}$$

Then we have

$$\sum_{i=1,2}\int_S \frac{\partial \mathrm{Dist}}{\partial y^i}(x,y)N^i\,ds = \begin{cases} 0 \text{ if } x \text{ is aligned with } S, \\ d(\operatorname{asinh}(l_2)-\operatorname{asinh}(l_1)) \text{ otherwise,} \end{cases} \tag{5.11}$$

and

$$\int_S \mathrm{Dist}(x,y) N\,ds = \begin{cases} a^2/2 \ \textit{if } x=\alpha \textit{ or } x=\beta, \\ a^2/2 + c \ \textit{if } c = ab \ (x \textit{ aligned with } \vec{\alpha\beta} \textit{ and } c>0, \\ -a^2/2 - c \ \textit{if } c=-ab \ (x \textit{ aligned with } \vec{\alpha\beta}) \textit{ and } c<0, \\ \delta/a^2 \left(l_2\sqrt{1+l_2^2} + \operatorname{asinh}(l_2) - l_1\sqrt{1+l_1^2} - \operatorname{asinh}(l_1)\right) \ \textit{otherwise.} \end{cases} \tag{5.12}$$

Proof. Let us show how to obtain (5.11) when x, α, and β are not aligned. To do this, let us parametrize the segment $S=[\alpha,\beta]$ so that

$$S = \left\{ y(t) = t\begin{pmatrix} \beta^1 \\ \beta^2 \end{pmatrix} + (1-t)\begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix}; \quad t\in(0,1)\right\}.$$

The unitary normal vector of the segment S is given by

$$N = \begin{pmatrix} -(\beta^2-\alpha^2) \\ \beta^1-\alpha^1 \end{pmatrix} \frac{1}{\sqrt{(\beta^1-\alpha^1)^2+(\beta^2-\alpha^2)^2}}.$$

So we have

$$I = \sum_{i=1,2}\int_S \frac{\partial \mathrm{Dist}}{\partial y^i}(x,y)N^i\,ds = \sum_{i=1,2}\int_0^1 \frac{y^i(t)-x^i}{|x-y(t)|}N^i|\alpha\beta|\,ds.$$

After some algebraic computations, we get

$$I = \alpha\beta\cdot x\alpha^{\perp}\int_0^1 \frac{dt}{\sqrt{t^2|\alpha\beta|^2+|x\alpha|^2+2\ t\ x\alpha\cdot\alpha\beta}},$$

with $x\alpha^{\perp} = \left(\begin{smallmatrix} -(\alpha^2-x^2) \\ \alpha^1-x^1 \end{smallmatrix}\right)$. Using the notation defined in Lemma 5.1, and since $\delta>0$ (x, α, and β are not aligned), we have

$$I = \alpha\beta\cdot x\alpha^{\perp}\frac{a}{\sqrt{\delta}}\int_0^1 \frac{dt}{\sqrt{\frac{a^4}{\delta}\left(t+\frac{c}{a^2}\right)^2+1}}.$$

We can explicitly compute the integral with the change of variable

$$z = \frac{a^2}{\sqrt{\delta}}\left(t+\frac{c}{a^2}\right),$$

so that we obtain

$$I = \frac{\alpha\beta\cdot x\alpha^{\perp}}{|\alpha\beta|}(\operatorname{asinh}(l_2)-\operatorname{asinh}(l_1)),$$

which concludes the proof. Other cases follow from similar arguments. □

With Lemma 5.1, one can estimate (5.9) and (5.10) and thus (5.7). By summing over all the squares and for a given x, we obtain the estimation of the integral $I_{u^k}(x)$ (5.5), and then we can iterate (5.2).

5.2. Experiments on image restoration. Let $u : \Omega \subset R^2 \to R$ be an original image describing a real scene, and let u_0 be the observed image of the same scene (i.e., a degradation of u). We assume that

$$u_0 = R\,u + \eta, \tag{5.13}$$

where η stands for a white additive Gaussian noise and where R is a linear operator representing the blur (usually a convolution). Given u_0, the problem is then to reconstruct u knowing (5.13). Supposing that η is a white Gaussian noise, and according to the maximum likelihood principle, we can find an approximation of u by solving the least-squares problem

$$\inf_u \int_\Omega |u_0 - Ru|^2 \; dx,$$

where Ω is the domain of the image. However, this is well known to yield to an ill-posed problem [15, 3].

A classical way to overcome ill-posed minimization problems is to add a regularization term to the energy so that the problem is to minimize

$$F(u) = \int_\Omega |u_0 - Ru|^2 \; dx + \lambda \int_\Omega |\nabla u|^p \; dx. \tag{5.14}$$

The first term in $F(u)$ measures the fidelity to the data. The second is a smoothing term. In other words, we search for a u that best fits the data so that its gradient is low (so that noise will be removed). The parameter λ is a positive weighting constant. For $p = 1$ we have in fact a BV-norm which leads to discontinuous solutions (see [2] for a review).

Remark that (5.14) is of the form (2.5), with $h(x, u(x)) = |u_0(x) - Ru(x)|^2$. Without loss of generality, we will assume that the operator R is the identity operator. So, in this section, we show some numerical results considering the minimization of the nonlocal functional

$$F_n(u) = \int_\Omega |u_0 - u|^2 \; dx + \lambda \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(|x-y|) dx dy \tag{5.15}$$

for a given n.

The first result, shown in Figure 2, illustrates an image restoration result on a real noisy image for $p = 1$. The result is as expected, which is very close to classical TV results. We recall that this approximation of the BV regularization problem is indeed independent of the fidelity attach term.

The second result, shown in Figure 3, is another image restoration result on a simple synthetic step image, which illustrates the effect of the parameter p on the edges. For example, we recover the classical observation for $p = 1$ or $p = 2$. More importantly, we show that our approximation can be successfully used to handle variational problems posed on $W^{1,p}(\Omega)$ with high values of p which, to our knowledge, generally leads to numerically unstable schemes.

6. Conclusion. Our main contribution was to show that the characterization result due to Bourgain, Brezis, and Mironescu [5] for the Sobolev seminorm can indeed be successfully applied to solve variational problems. It was not a priori straightforward that this characterization of $W^{1,p}$ could be useful in the theoretical and numerical analysis of problems of calculus of variations.

original noisy restored ($p = 1$)

FIG. 2. *Example of image restoration.*

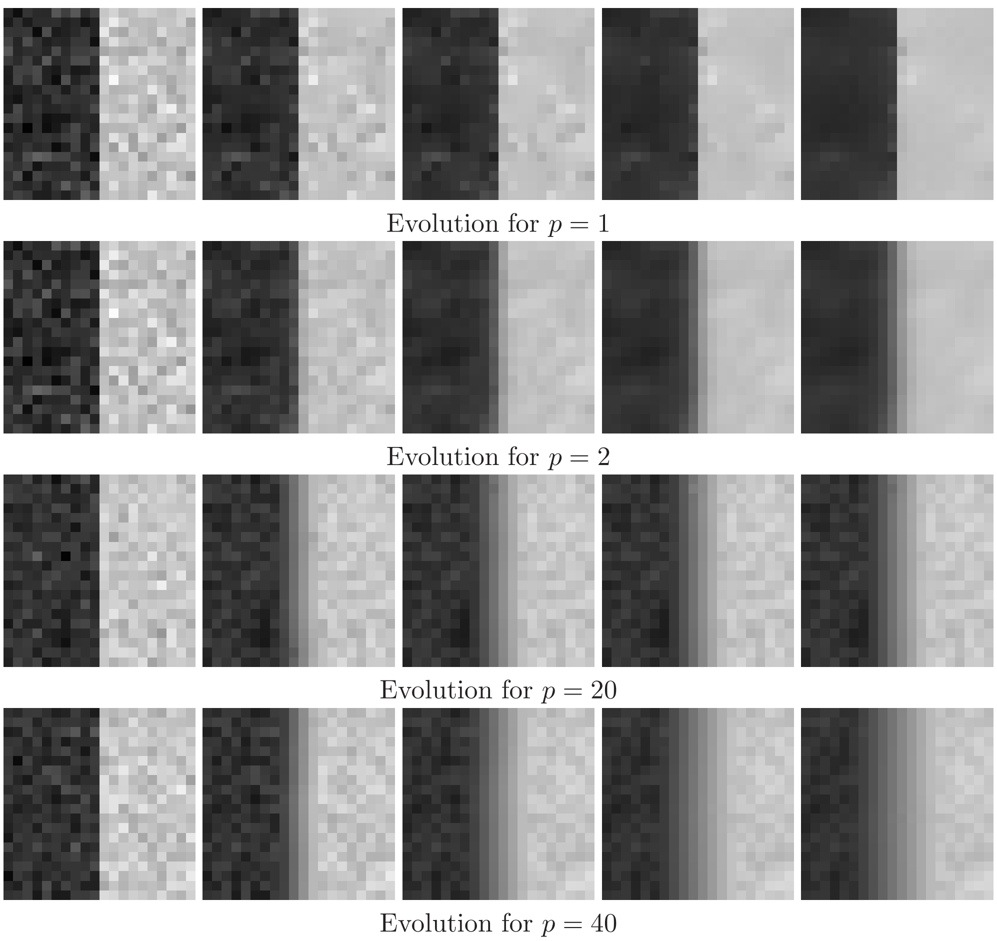

FIG. 3. *Example of evolutions with various values of p applied to a synthetic noisy image.*

A step further, we proved that our results can be extended also in the BV-case, thanks to Ponce's results [16]. Note that the BV-case is not a simple extension from the $W^{1,p}$-case, and it requires some adaptations.

Interestingly, we show that this approach allows us to treat problems posed in $W^{1,p}$ with high values of p, which is a challenging problem as far as we know.

Finally, our contribution does not target a particular field of application, and image restoration was proposed here as an illustration: We wanted also to show that this alternative formulation, which leads to nonlocal terms with singular kernels, can be implemented.

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