A MATHEMATICAL STUDY OF THE RELAXED OPTICAL FLOW PROBLEM IN THE SPACE $BV(\Omega)$

G. AUBERT AND P. KORNPROBST *

Abstract

This paper describes a variational approach for estimating a discontinuous optical flow from a sequence of images. Defined as the apparent motion of the image brightness pattern, the optical flow is very important in the computer vision community where its accurate estimation is strongly needed. After a fast overview of existing methods, we present a new variational method that we study in the space of Bounded Variations. We first present an integral representation of the optical flow problem which appears to be not lower semicontinuous. The relaxed functional is then calculated. We conclude by challenging questions about the possible numerical analysis of the abstract results.

Keywords

Measure theory, space of bounded variations, convex functions of measures, Γ-convergence, elliptic equations, relaxation of ill-posed problems, optical flow, computer vision.

AMS Subject Classifications. 35J, 49J, 65N

1. Introduction.

This paper deals with the estimation of the movement in a sequence of images. This velocity field will be called the optical flow. In the Computer Vision community, it is well known that the optical flow is a rich source of informations about the geometrical structure of the world. Numerous numerical algorithms on the optical flow estimation and its applications have been performed. They have clearly shown how the optical flow can be used to recover information about slant and tilt of surface elements, egomotion, shape information, time to collision, etc [31, 32, 30, 34, 33, 29, 28, 38, 50, 24, 37, 49, 27, 9, 40, 41]

Almost all these approaches use the classical brightness constancy assumption that relates the gradient of brightness to the components of the local flow to estimate. Because this problem is ill-posed, additional constraints are usually required. The most used one is to add a quadratic smoothness constraint as done originally by Horn and Schunk [29]. However, in order to estimate the optical flow more accurately, other constraints involving high order spatial derivatives have also been used [40]. Nevertheless, several of the proposed methods lacked robustness to the presence of occlusion, and yielded smooth optical flow. The variational approach proposed in this paper is motivated by the need to recover the optical flow while preventing the method from trying to smooth the solution across the flow discontinuities. To cope with discontinuities, we propose in this article a complete mathematical study of the relaxed optical flow problem in the space $BV(\Omega)$. We first present an integral repre-

*Laboratoire J.A. Dieudonné - UNSA - UMR n° 6621 du CNRS - 06108 Nice Cedex 2 - FRANCE, e-mail: *gaubert@math.unice.fr*, Tel: +33 4 92 07 62 82, Fax: +33 4 93 51 79 74

sentation of the optical flow problem which appears to be no lower semicontinuous.

This article is organized as follows :

In section 1, we define the problem and propose a variational approach to solve it. The general idea is based on a conservation law of the intensity along the trajectories. We will have to deal with an ill-posed problem that we will solve by regularizing the unknowns.

Section 3 presents some general recalls about the space of Bounded Variations (noted $BV(\Omega)$). Classically used for problems coming from Computer Vision, this space permits to get discontinuities along curves (in dimension 2).

In section 4, we concentrate on the meaning of the energy we defined. This will permit us to consider some integral representation results of the duality pairing of an integrable function with a measure. Same kind of results have been proved by Anzellotti [3] and we will extend them under weaker assumptions. This will enable us to obtain a fully developed expression for energy that we have to minimize.

Unfortunately, the proposed energy is not lower semi continuous for the weak topology of $BV(\Omega)$. Section 5 is devoted to the computation of the relaxed functional. This part is mainly technical and is based on ideas developed by Bouchitté et al [14, 13].

Finally we prove in section 6 that there exists a solution in $\mathbf{BV}(\Omega)$ for the relaxed formulation.

2. The optical flow problem : definition and modelisation.

2.1. Definition. As shown in figure 2.1, we can modelize a camera as a simple projective model. Consequently, the first idea is to say that the 2D velocity field in the

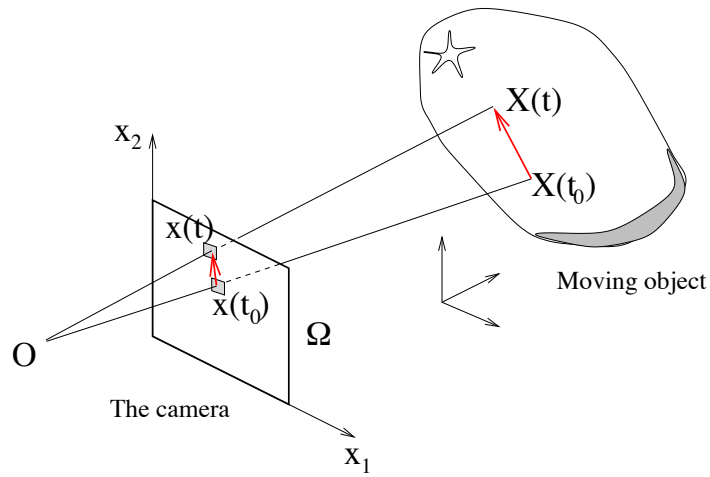

FIG. 2.1. *Pinhole model of camera. Ω is the domain of the image, O is the optical center.*

image corresponds to the projection of the 3D velocity field of the objects. However, variations of intensity due to shadows do not correspond to any real motion. The importance of the light source can be seen towards other phenomena. For instance, if the object is sparkling, the reflected luminosity changes rapidly with the position. This is the case for bodywork, glasses,... Finally, notice the problem of noise in images

which is unavoidable. This intensity variations may be interpreted as false motions which have no physical meaning.

Thanks to these remarks we will define the optical flow as the 2D velocity field describing the changes in intensity between images. In many cases, it can be interpreted as an approximation of the projection of the 3D velocity field which animates physical objects. We will see in the next section how we can traduce it mathematically.

2.2. A short overview. In this last decade, numerous methods have been proposed to compute optical flow. Several ideas have been used : working with regions, curves, lines or points. There is also a wide range of methodologies : wavelets, Markov random fields, Fourier analysis and naturally partial differential equations [29, 28, 38, 50, 24, 37, 49, 27, 18, 40, 41]. We refer the interested reader to two (mainly computational) general surveys :

- Barron, Fleet and Beauchemin [9] explain the main different techniques and perform numerical quantitative experiments to compare them (the database used for tests is also available).
- Orkisz and Clarysse [39] propose an "updated" version of the preceding one.

In this article we will concentrate upon the class of *differential methods* (as named by Barron, Fleet and Beauchemin) which have been proved to be among the best one [9]. Their common point is the consistency intensity hypothesis of a point during its movement. More precisely, we will assume that :

$$\text{"\textit{The intensity of a point keeps constant along its trajectory}"} \tag{2.1}$$

This hypothesis is called the *optical flow constraint* (noted in the sequel OFC). We can consider it as reasonable, almost along short times, for which changes of the brightness.

Let $x(t) = (x_1(t), x_2(t)) \in \Omega \subset R^2$ be the projection of the point $X(t) \in R^3$ at time t (see Figure 2.1,). For $x \in \Omega$, we denote by $u(t, x)$ the reflected intensity (the brightness) of the point x at time t. Let t_0 be fixed. Using these notations, a natural way to express (2.1) is :

$$u(t, x(t)) = u(t_0, x(t_0)) \tag{2.2}$$

By differentiating (2.2) with respect to t, we obtain, for $t = t_0$:

$$\sigma(x) \cdot Du(t_0, x) + u_t(t_0, x) = 0 \qquad x \in \Omega$$

where $\sigma = (\sigma_1, \sigma_2)^T = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}\right)^T$ is the unknown velocity field, $D\cdot$ is the spatial gradient operator, and u_t denotes the temporal derivative of $u(t, x)$. (derivatives are written in the distributional sense). This equation is called the *Optical Flow Constraint* (OFC). Naturally, this scalar equation is insufficient to compute both components of the flow field. This problem is usually called the *aperture problem.* Additional constraints are therefore required to reduce the space of admissible functions. Several possibilities are then possible : use additional constraints, consider special movements (rigid or fluids), regularize the velocity field,... We refer to [6] where we notably propose an overview of these different methods.

Our starting point will be the method proposed by Horn and Schunk in 1981 [29]. The idea is to minimize the following energy :

$$E_{HS}(\sigma) = \underbrace{\int_\Omega ((\sigma \cdot Du) + u_t)^2 dx}_{\mathtt{A}} + \alpha^r \sum_{j=1}^{2} \underbrace{\int_\Omega \|D\sigma_j\|^2 dx}_{\mathtt{B}} \tag{2.3}$$

where α^r is a positive constant. The interpretation of this functional is the following : we would like that the OFC be zero (term A) and that the gradient magnitude be minimum (term B). Notice that term B is the classical Tikhonov-Arsenin [48] relaxation known to smoothing isotropically. With this method, we obtain a smooth optical flow, and flow discontinuities are lost.

2.3. Setting the problem. The purpose of this work is to propose a model able to cope with the discontinuities of the optical flow. Starting from (2.3), we propose to minimize the following energy :

$$E(\sigma) = \underbrace{\int_\Omega |(\sigma \cdot Du) + u_t|}_{\mathtt{A}} + \alpha^r \underbrace{\sum_{j=1}^{2} \int_\Omega \phi(D\sigma_j)}_{\mathtt{B}} + \alpha^h \underbrace{\int_\Omega c(x)\|\sigma\|^2 dx}_{\mathtt{C}} \tag{2.4}$$

where α^r, α^h are positive constants, $\phi(\cdot)$ and $c(\cdot)$ to be precised. We refer the interested reader to [35, 6] for the detailed construction of this model. Let us describe briefly the main differences :

(i) the term A is comparable to term A in (2.3). Here we choose the L^1 norm which must be interpreted in term of measures. As we will see in the sequel, since the data u belongs *a priori* to $BV(\Omega)$, we cannot use the L^2 norm as done in (2.3).

(ii) the term B is again a regularization term. The functions $\phi(\cdot)$ has been chosen so that we can preserve discontinuities. The key idea is to forbid smoothing across discontinuities. Such idea have initially been proposed in the image restoration background [45, 20, 5, 7] and many functions have been proposed. Typically, admissible functions are convex functions with linear growth at infinity. For instance we will choose the minimal hypersurface function :

$$\phi(s) = \sqrt{s^2 + 1}$$

We mention that the term B will be interpretated as convex functions of measures.

(iii) Finally, the term C permits to handle with the homogeneous regions. Typically, $c(x)$ is high for low spatial gradients of u (hence penalizing velocities in poor information zones) and low for high spatial gradients of u (no intervention).

3. General recalls. In this section we only recall main notations and definitions. We refer to [1, 22, 25, 23, 53] for the complete theory.

Let Ω be a bounded open set in R^N, with Lipschitz-regular boundary $\partial\Omega$. We denote by $\mathcal{L}^N$ or dx the N-Lebesgue dimensional measure in R^N and by $\mathcal{H}^\alpha$ the $\alpha-$dimensional Hausdorff measure. We also set $|E| = \mathcal{L}^N(E)$, the Lebesgue measure of a measurable set $E \subset R^N$. $\mathcal{B}(\Omega)$ denotes the family of the Borel subsets of Ω. We will respectively denote the strong, the weak and weak$\star$ convergences in a space $V(\Omega)$ by $\xrightarrow[V(\Omega)]{}$, $\rightharpoonup_{V(\Omega)}$, $\overset{\star}{\rightharpoonup}_{V(\Omega)}$. Spaces of vector valued functions will be noted by bold characters.

Working with images requires that the functions we consider can be discontinuous along curves. This is impossible with classical Sobolev spaces such as $W^{1,1}(\Omega)$. This is why we need to use the space of bounded variations (noted $BV(\Omega)$) defined by :

$$BV(\Omega) = \left\{ u \in L^1(\Omega); sup \int_\Omega u div(\varphi) dx < \infty : \varphi \in \mathcal{C}_0^1(\Omega)^N, |\varphi|_\infty \leq 1 \right\}$$

where $\mathcal{C}_0^1(\Omega)$ is the set of differentiable functions with compact support in Ω. We will note :

$$|Du|(\Omega) = sup \left\{ \int_\Omega u div(\varphi) dx : \varphi \in \mathcal{C}_0^1(\Omega)^2, |\varphi|_\infty \leq 1 \right\}$$

If $u \in BV(\Omega)$ and Du is the gradient in the sense of distributions, then Du is a vector valued Radon measure and $|Du|(\Omega)$ is the total variation of Du on Ω. The set of Radon measure is noted $\mathcal{M}(\Omega)$

The product topology of the strong topology of $L^1(\Omega)$ for u and of the weak$\star$ topology of measures for Du will be called the weak$\star$ topology of BV, and will be denoted by $BV - w\star$.

$$u^n \underset{BV-w\star}{\rightharpoonup} u \iff \left\{ \begin{array}{ccc} u^n & \underset{L^1(\Omega)}{\longrightarrow} & u \\ Du^n & \underset{\mathcal{M}(\Omega)}{\overset{\star}{\rightharpoonup}} & Du \end{array} \right. \tag{3.1}$$

We recall that every bounded sequence in $BV(\Omega)$ admits a subsequence converging in $BV - w\star$.

We define the approximate upper limit $u^+(x)$ and the approximate lower limit $u^-(x)$ by :

$$u^+(x) = inf \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(\{u > t\} \cap B_\rho(x))}{\rho^N} = 0 \right\}$$
$$u^-(x) = sup \left\{ t \in [-\infty, +\infty] : \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(\{u < t\} \cap B_\rho(x))}{\rho^N} = 0 \right\}$$

where $B_\rho(x)$ is the ball of center x and radius ρ. We denote by S_u the jump set, that is to say the complement up to a set of $\mathcal{H}^{N-1}$ measure zero of the set of Lebesgue points, *i.e.* the set of points x where $u^+(x)$ is different $u^-(x)$, namely :

$$S_u = \{x \in \Omega / u^-(x) < u^+(x)\}.$$

After choosing a normal $n_u(x)$ $(x \in S_u)$ pointing toward the largest value of u, we recall the following decompositions ([2] for more details) :

$$Du = \nabla u \cdot \mathcal{L}_N + C_u + (u^+ - u^-) n_u \cdot \mathcal{H}^{N-1}_{|S_u} \tag{3.2}$$

$$|Du|(\Omega) = \int_\Omega \|\nabla u\| dx + \int_{\Omega \setminus S_u} |C_u| + \int_{S_u} (u^+ - u^-) d\mathcal{H}^{N-1} \tag{3.3}$$

where ∇u is the density of the absolutely continuous part of Du with respect to the Lebesgue measure, C_u is the Cantor part and $\mathcal{H}^{N-1}$ is the Hausdorff measure of dimension $N-1$.

We then recall the definition of a convex function of measures. We refer to the works of Goffman-Serrin [26] and Demengel-Temam [19] for more details. Let $\phi(\cdot)$ be convex and finite on R with linear growth at infinity. Let ϕ^∞ be the asymptote (or recession) function defined by $\phi^\infty(z) = \lim_{s\to\infty} \frac{\phi(sz)}{s} \in [0;+\infty)$. Then, for $u \in BV(\Omega)$, using classical notations, we define :

$$\int_\Omega \phi(Du) = \int_\Omega \phi(\|\nabla u\|)dx + \phi^\infty(1)\int_{S_u}(u^+ - u^-)d\mathcal{H}^{N-1} + \phi^\infty(1)\int_{\Omega\setminus S_u}|C_u| \quad (3.4)$$

We finally mention that this function is lower semi-continuous for the $BV - w\star$-topology.

4. The integral representation of the optical flow problem.

4.1. The precise formulation. This section is devoted to the mathematical study of the optical flow model proposed in section 2.3. Let us recall it. Without loss of generality, we will assume that $\alpha^r = \alpha^h = 1$. For $u \in BV(R\times\Omega)$, the problem is to find σ minimizing the following energy :

$$E(\sigma) = \int_\Omega |(\sigma\cdot Du) + u_t| + \sum_{j=1}^N \int_\Omega \phi(D\sigma_j) + \int_\Omega c(x)\|\sigma\|^2 dx \quad (4.1)$$

where the infimum is taken other the space $\mathbf{BV}(\Omega)$. The function $\phi(\cdot)$ verifies :

$$\phi : R \to R^+ \text{ is an even and convex function, nondecreasing on } R^+ \quad (4.2)$$

$$\text{There exist constants } d > 0 \text{ and } b \geq 0 \text{ such that } dx - b \leq \phi(x) \leq dx + b \text{ for all } x \in R \quad (4.3)$$

$$\phi^\infty(1) = 1 \quad (4.4)$$

$$\phi^*(x^*) \leq k \qquad \forall x^* \in dom(\phi^*) \qquad (k \text{ constant}) \quad (4.5)$$

where ϕ^* is the Legendre-Fenchel conjugate function of ϕ. Remark that hypotheses (4.3) and (4.5) permit to assert that (see [43]) :

$$\phi(x) \geq \phi^\infty(x) - k \quad (4.6)$$

Finally, we assume that the function $c(\cdot)$ verifies the following assumptions :

$$c \in \mathcal{C}^\infty(\Omega) \quad (4.7)$$

$$\text{there exists a constant } m_c > 0 \text{ such that } c(x) \in [m_c, 1] \text{ for all } x \text{ in } \Omega \quad (4.8)$$

4.2. The duality pairing $(\sigma\cdot Du)$: an extended integral representation. This part is devoted to better understand the functional to be minimized (4.1), and especially the product $(\sigma\cdot Du)$. As a matter of fact, what can we say about the product of an integrable function and a measure? This question as been treated for special cases, with suitable hypotheses on σ and u (see [3, 47, 12]). For example, Anzellotti [3] supposes :

$$\sigma \in \mathbf{X}(\Omega) \cap \mathcal{C}^0(\Omega; R^N) \text{ and } u(t_0,\cdot) \in BV(\Omega)$$
$$\sigma \in \mathbf{X}(\Omega) \text{ and } u(t_0,\cdot) \in W^{1,1}(\Omega)$$

where $\mathbf{X}(\Omega) = \{\sigma \in \mathbf{L}^\infty(\Omega)\ ;\quad div(\sigma) \in L^N(\Omega)\}$. Our aim is to extend his results for a more general class of product $(\sigma \cdot Du)$. We suppose :

$$\sigma \in \mathbf{BV}(\Omega) \cap \mathbf{X}(\Omega) \tag{4.9}$$

$$u(t_0, \cdot) \in SBV(\Omega) \cap L^\infty(\Omega) \tag{4.10}$$

where $SBV(\Omega)$ is the space of special bounded variations (the Cantor part of Du is zero).

Remark The hypothesis (4.10) is quite general. We mention to the interested reader a more applied work where we only assumed that the data u is Lipschitz [6]. In that case, there is no problem to define the L^1 norm of the optical flow constraint and we proved existence and uniqueness of the minimization problem posed on $\mathbf{BV}(\Omega)$. We also proposed a convergent algorithm to approximate the solution (using Γ-convergence arguments) and showed some numerical results on synthetic and real sequences. ♣

The space $\mathbf{X}(\Omega)$ is a Banach space endowed with the norm

$$||\sigma||_{\mathbf{X}(\Omega)} = ||\sigma||_{\mathbf{L}^\infty(\Omega)} + ||div(\sigma)||_{L^N(\Omega)}$$

and we can define a weak$\star$ topology on $\mathbf{X}(\Omega)$ by

$$\sigma^n \xrightarrow[\mathbf{X}(\Omega)]{} \sigma \quad \Longleftrightarrow \quad \begin{cases} \sigma^n \xrightarrow[\mathbf{L}^\infty(\Omega)]{\star} \sigma \\ div(\sigma^n) \xrightarrow[L^N(\Omega)]{} div(\sigma) \end{cases}$$

To make sense to the pairing $(\sigma \cdot Du)$, the first idea is to define it by duality :

$$\int_\Omega \varphi(\sigma \cdot Du) = -\int_\Omega u\varphi div(\sigma)dx - \int_\Omega u\sigma\nabla\varphi dx \quad \text{for all} \quad \varphi \ \text{ in } \ \mathcal{C}_c^1(\Omega) \tag{4.11}$$

Remark that with hypotheses (4.9) and (4.10), the right-hand side of (4.11) is completely defined. We can prove [3, 12] that $(\sigma \cdot Du)$ is a bounded measure, absolutely continuous with respect to $|Du|$.

Our aim is to find an integral representation of that measure.

To this end, we need to introduce the precise representation of σ, noted $\tilde{\sigma}$ and defined by :

$$\tilde{\sigma}(x) = \lim_{r \to 0} \frac{1}{\mathcal{L}^N(B(x,r))} \int_{B(x,r)} \sigma(y)dy \tag{4.12}$$

If σ is simply in $\mathbf{L^1}(\Omega)$, the right-hand side limit exists $\mathcal{L}^N$ a.e. and is equal to $\sigma(x)$. However, if σ is also in $\mathbf{BV}(\Omega)$, we can explicitely write the limit $\mathcal{H}^{N-1}$ a.e. using σ^+, σ^- We have [52, 22] :

$$\tilde{\sigma}(x) = \frac{\sigma^+(x) + \sigma^-(x)}{2} \qquad \mathcal{H}^{N-1} \text{ a.e. on } S_\sigma \tag{4.13}$$

Another interesting property of $\tilde{\sigma}$ is that we have the following approximation result :

$$\tilde{\sigma}(x) = \lim_{\varepsilon \to 0} \eta_\varepsilon \star \sigma(x) \qquad \mathcal{H}^{N-1} \text{ a.e.} \tag{4.14}$$

where (η_ε) are the usual mollifiers : $\eta_\varepsilon \in \mathcal{C}_c^\infty(R^N)$, $\quad spt(\eta_\varepsilon) \subset B(0,\varepsilon)$, $\quad 0 \leq \eta_\varepsilon \leq 1$, $\displaystyle\int_{R^N} \eta_\varepsilon(x)dx = 1$. The function $\tilde{\sigma}$ is called the *precise representation* of σ since it permits in some way to define $\sigma - \mathcal{H}^{N-1}$ a.e.. Remark that $\tilde{\sigma}$ and σ are in fact the same elements in $\mathbf{BV}(\Omega)$ (they belong to the same equivalent class of $\mathcal{L}^N$a.e. equal functions) so that their distributional derivatives are the same.

From now on, we will consider that $N = 2$. For two measures μ and ν in $\mathcal{M}(\Omega)$, we will denote by $\frac{d\mu}{d\nu}$ the Radon-Nikodym derivative of μ with respect to ν (See [22] for more details).

The main result of this section is the following:

PROPOSITION 4.1. *If $\sigma \in \mathbf{X}(\Omega) \cap \mathbf{BV}(\Omega)$ and $u(t_0,\cdot) \in SBV(\Omega) \cap L^\infty(\Omega)$, then we have :*

$$\int_B (\sigma \cdot Du) = \int_B \tilde{\sigma}(x) \cdot \frac{dDu}{d|Du|}(x)|Du| \quad \textit{for all Borel set } B \textit{ in } \Omega \tag{4.15}$$

Moreover, if $u \in SBV(\Omega)$, we obtain :

$$\int_\Omega (\sigma \cdot Du) = \int_\Omega \sigma \cdot \nabla u dx + \int_{S_u} \tilde{\sigma} \cdot n_u (u^+ - u^-) d\mathcal{H}^{N-1} \tag{4.16}$$

Before proving this result we mention a convergence result which can be demonstrated using arguments from [3].

LEMMA 4.2. *Let $\sigma_\varepsilon = \eta_\varepsilon \star \sigma(x)$. If $\sigma \in \mathbf{BV}(\Omega) \cap \mathbf{X}(\Omega)$, then we have :*

$$\sigma_\varepsilon \xrightarrow[\mathbf{L}^\infty(A)]{\star} \sigma \tag{4.17}$$

$$div(\sigma_\varepsilon) \xrightarrow[L^p(A)]{} div(\sigma) \qquad (p < \infty) \tag{4.18}$$

for all open set $A \subset \Omega$. Moreover, for all $u \in BV_{loc}(\Omega) \cap L^\infty(\Omega)$, one has :

$$(\sigma_\varepsilon \cdot Du) \xrightarrow[\mathcal{M}(\Omega)]{} (\sigma \cdot Du) \tag{4.19}$$

Proof. of Proposition 4.1. If we denote $\sigma_\varepsilon = \eta_\varepsilon \star \sigma$, then for all $\varphi \in \mathcal{C}_c^1(\Omega)$, we have (see lemma 4.2) :

$$< (\tilde{\sigma} \cdot Du), \varphi >= \lim_{\varepsilon \to 0} < (\sigma_\varepsilon \cdot Du), \varphi > \tag{4.20}$$

As $Du \ll |Du|$, by the Radon-Nikodym's theorem, there exists a function $h \in L^1_{|Du|}(\Omega)$, $|h(x)| = 1$, such that $Du = h|Du|$. So we can write, since $\sigma_\varepsilon \in L^\infty_{|Du|}(\Omega)$ $(\sigma_\varepsilon \in \mathcal{C}^\infty(\Omega))$:

$$< (\sigma_\varepsilon \cdot Du), \varphi >= \int_\Omega \varphi \sigma_\varepsilon \cdot h|Du|. \tag{4.21}$$

Equations (4.20) and (4.21) imply :

$$\langle (\sigma \cdot Du), \varphi \rangle \;=\; \lim_{\varepsilon \to 0} \int_\Omega \varphi \sigma_\varepsilon \cdot h|Du|. \tag{4.22}$$

What remains to show is the permutation between the limit and the integral in (4.22). To do this, we use the Lebesgue dominated convergence theorem. Classically, two requirements are necessary :

- the pointwise convergence of $\varphi(x)\sigma_\varepsilon(x)\cdot h(x)$ to $\varphi(x)\tilde{\sigma}(x)\cdot h(x)$. It comes from (4.14). Notice that the pointwise convergence is true $\mathcal{H}^{N-1}$-a.e. and consequently $|Du|$-a.e..

- find a function which "dominates" the sequence. In fact, since Ω is bounded, it is sufficient to prove that the L^∞ norm of $\varphi(x)\sigma_\varepsilon(x)\cdot h(x)$ is bounded uniformly by a constant. Since φ is in $\mathcal{C}^1_c(\Omega)$ and $|h(x)|=1$, it is enough to show that there exists a constant C such that $\|\sigma_\varepsilon\|_{\mathbf{L}^\infty{}_{Du}(\Omega)}\leq C$. In fact we have :

$$\|\sigma_\varepsilon\|_{\mathbf{L}^\infty_{\mathbf{Du}}(\Omega)}\leq \sup_{R^N}|\sigma_\varepsilon| \underset{[*]}{\leq} \|\sigma\|_{\mathbf{L}^\infty_{\mathcal{L}^\mathbf{N}}(\Omega)}\leq C$$

where inequality $[*]$ is shown in [3]. Consequently, we can apply the Lebesgue dominated convergence theorem. This permits to pass to the limit in (4.22) as $\varepsilon\to 0$, and we get (4.15). It is then an easy task to get (4.16) from (4.15), using the decomposition (3.2). □

4.3. Application to the optical flow problem. Now that we have found an expression of the product $(\sigma\cdot Du)$, we give in the next proposition the integral representation of the functional E which will be used in the sequel.

We will assume that :

$$u\in SBV(R\times\Omega)\cap L^\infty(R\times\Omega) \tag{4.23}$$

There exists $h_1\in L^1(\Omega)$ and $h_2\in L^1_{\mathcal{H}^1}(S_u)$ such that :

$$u_t = h_1\mathcal{L}^2 + h_2\mathcal{H}^1|_{S_u} \tag{4.24}$$

Notice that assumption means that the measure u_t is absolutely continuous with respect to $|Du|$. This is physically correct since when there is no texture (no gradient) no intensity variation should be observed.

PROPOSITION 4.3. *We assume that $N=2$. Let u verifying hypotheses (4.23)-(4.24), Then the function E defined on $\mathbf{X}\cap\mathbf{BV}(\Omega)$ by :*

$$E(\sigma)=\int_\Omega|(\sigma\cdot Du)+u_t|+\sum_{j=1}^2\int_\Omega\phi(D\sigma_j)+\int_\Omega c(x)\|\sigma\|^2dx \tag{4.25}$$

with hypotheses (4.2)-(4.3), (4.4)-(4.6), (4.7)-(4.8), can be rewritten as :

$$\begin{aligned}E(\sigma)=\ &\int_\Omega|\sigma\cdot\nabla u+h_1|dx+\int_{S_u}|\tilde{\sigma}\cdot n_u(u^+-u^-)+h_2|d\mathcal{H}^1\\ &+\sum_{j=1}^2\int_\Omega\phi(D\sigma_j)+\int_\Omega c(x)\|\sigma\|^2dx\end{aligned} \tag{4.26}$$

Proof. Thanks to [46] (Theorem 6.13), we know that if ν is a positive measure on $\mathcal{M}(\Omega)$, $g\in L^1_\nu$ and λ the measure defined by :

$$\lambda(E)=\int_E g d\nu$$

Then we have :

$$|\lambda|(E) = \int_E |g| d\nu$$

Moreover, using the decomposition of the measure u_t and the result (4.16), we have :

$$\int_\Omega (\sigma \cdot Du) + u_t = \int_\Omega (\sigma \cdot \nabla u + h_1) dx + \int_{S_u \cap \Omega} (\tilde{\sigma} \cdot n_u (u^+ - u^-) + h_2) d\mathcal{H}^1$$

Using the fact that dx and $d\mathcal{H}^1$ are mutually singular and applying the above theorem permit to conclude the proof. □

Commentary about proposition 4.3. The interesting point in the integral representation (4.26) is that we no longer need that the divergence of σ should be integrable. Consequently, (4.26) can be viewed as an extension of E defined *a priori* for $\sigma \in \mathbf{BV}(\Omega)$. The next section is devoted to the theoretical study of that extension.

5. The relaxed problem. After introducing notations and assumptions in section 5.1, we show in section 5.2 that the functional that we are considering is not lower semi continuous for the $BV - w\star$ topology. As a consequence, the existence of a solution can not be shown for the initial problem. We then search for the relaxed functional for a suitable topology in section 5.4 after proving some preliminary results in section 5.3.

5.1. Notations and assumptions. To simplify proofs and notations, we will assume in that section that $N = 2$ and that S_u is a single $\mathcal{C}^1$ curve as shown in figure 5.1 where main notations are introduced. Notice that the parameter α corresponds to the distance between S_u and $S_u^{e,\alpha}$ (or $S_u^{i,\alpha}$). We will also use the superscript i (resp. e) to mention that we are considering the restriction of the function to $\Omega^i \equiv \Omega^{i,0}$ (resp. $\Omega^e \equiv \Omega^{e,0}$). The Hausdorff measure of dimension 1 is noted ds.

Using these notations we rewrite the integral on S_u of (4.26) which is :

$$\int_{S_u} |\tilde{\sigma} \cdot n_u (u^+ - u^-) + h_2| ds. \tag{5.1}$$

Let $b = \pm 1$ the function such that $n_u = bn$ where n is the normal oriented towards the exterior (Voir Figure 5.1). Let $\tilde{h}_2$ the function defined by $\tilde{h}_2 = bh_2$. It is then easy to check that (5.1) may be rewritten :

$$\int_{S_u} \left| \frac{\sigma^i + \sigma^e}{2} \cdot n(u^+ - u^-) + \tilde{h}_2 \right| ds \tag{5.2}$$

So, changing h_2 in $\tilde{h}_2$ permits to have a normal independent of u. We will use this expression easier to handle. To simplify notations, we will omit the tilda superscript for h_2.

5.2. Statement of the problem. Let us first recall precisely the problem that we are going to study. Let $h_1 \in L^1(\Omega)$ and $h_2 \in L^1_{\mathcal{H}^1}(S_u)$ given. Let $\phi(\cdot)$ a function verifying (4.2)-(4.3), (4.4)-(4.6)and c satisfying (4.7)-(4.8). Let E be the functional

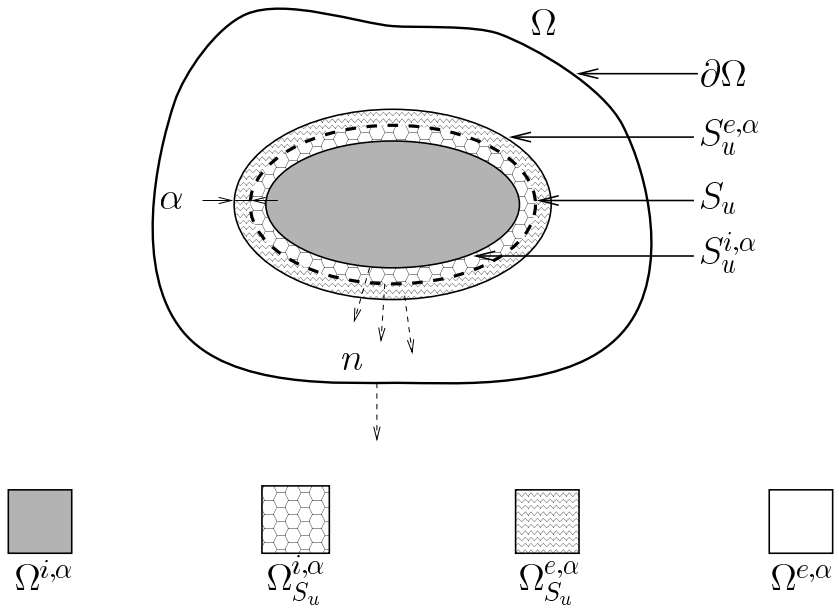

FIG. 5.1. *Notations of the simplified problem. Notice that the normals are all oriented towards the exterior.*

defined over $\mathbf{BV}(\Omega)$ by :

$$\begin{aligned} E(\sigma) = & \int_\Omega |\sigma \cdot \nabla u + h_1| dx + \int_{S_u} |\tilde{\sigma} \cdot n(u^+ - u^-) + h_2| ds \\ & + \sum_{j=1}^{2} \int_\Omega \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2 dx \end{aligned} \tag{5.3}$$

We remark that it is well defined on $\mathbf{BV}(\Omega)$ thanks to the embedding of $\mathbf{BV}(\Omega)$ into $\mathbf{L^2}(\Omega)$ $(N = 2)$ (see for instance [25]). Our aim is to study the existence of a solution to the minimization problem :

$$\inf_{\sigma \in \mathbf{BV}(\Omega)} E(\sigma) \tag{5.4}$$

Following the direct method of the calculus of variations, let (σ^n) be a minimizing sequence of (5.3). Thanks to hypotheses on functions $\phi(\cdot)$ ans $c(\cdot)$, we can obtain a uniform majoration in $\mathbf{BV}(\Omega)$ and in $\mathbf{L^2}(\Omega)$, so we can extract a subsequence converging to some σ for the topology $BV - w\star$ and L^2-weak. The question is : can we deduce an existence result for (5.4)? To answer this question, let us split the functional E in two parts, namely P and L, defined by :

$$P(\sigma) = \int_\Omega |\sigma \cdot \nabla u + h_1| dx + \sum_{j=1}^{2} \int_\Omega \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2 dx \tag{5.5}$$

$$L(\sigma) = \int_{S_u} \left| \frac{\sigma^i + \sigma^e}{2} \cdot n(u^+ - u^-) + h_2 \right| ds \tag{5.6}$$

It is easy to show that we have :

$$\lim_n P(\sigma^n) \geq P(\sigma)$$

but, we cannot say anything about the term L. The reason is that the functional L is defined through traces and the trace application is not continuous for the weak$\star$ topology of $\mathbf{BV}(\Omega)$. Consequently, the functional E is not lower semi continuous for the $BV - w\star$ topology. In such a situation, the idea is to study the relaxed functional.

We recall that for a functional F defined over a topological metrisable space X, the relaxed functional, noted $R(F)$, verifies :

$$\forall u \in X, \forall u^n \to u. \liminf_n F(u^n) \geq R(F)(u) \tag{5.7}$$

$$\forall u \in X, \exists u^n \to u. \limsup_n F(u^n) \leq R(F)(u). \tag{5.8}$$

$R(F)$ is in fact the higher lower semi-continuous functional less than or equal to F. We refer the interested reader to [36, 16] for a complete overview of the relaxation properties and consequences.

5.3. Preliminary results. As it is usual when we have this kind of problems, we need to introduce additional variables and some notations. Let us define the functionals $\tilde{L}$ and E_1 by :

$$\tilde{L} : \mathbf{M}(S_u) \times \mathbf{M}(S_u) \to R \tag{5.9}$$

$$\tilde{L}(\mu^i, \mu^e) = \int_{S_u} d|\nu|$$

$$\text{where} \quad \nu = \frac{\mu^i + \mu^e}{2} \cdot n(u^+ - u^-) + h_2 ds \tag{5.10}$$

and

$$E_1 : \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u) \to R \tag{5.11}$$

$$E_1(\sigma, \mu^i, \mu^e) = \begin{cases} P(\sigma) + \tilde{L}(\mu^i, \mu^e) \text{ if } \mu^i = \sigma^i ds \text{ and } \mu^e = \sigma^e ds \\ +\infty \text{ otherwise} \end{cases}$$

It is straightforward to see that :

$$\inf_{\sigma \in BV(\Omega)} E(\sigma) = \inf_{(\sigma, \mu^i, \mu^e) \in \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)} E_1(\sigma, \mu^i, \mu^e) \tag{5.12}$$

The functionals (5.3) and (5.11) are not weakly lower semi continuous, so it is natural to search for the relaxed functionals of E and E_1, noted $R(E)$ and $R(E_1)$, for a suitable topology.

Thanks to classical results [36, 16], we have using (5.12) :

$$\inf_{\sigma \in BV(\Omega)} E(\sigma) = \inf_{\sigma \in BV(\Omega)} R(E)(\sigma) = \inf_{(\sigma, \mu^i, \mu^e) \in \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)} R(E_1)(\sigma, \mu^i, \mu^e).$$

Moreover, since the relaxed functionals are lower semi continuous, existence results can be proved. Our aim is then to compute these relaxed functionals which is the main result of section 5. To this end, we will use the definitions (5.7) and (5.8). Difficulties are twofold :

- the first idea is that we must "guess" the expression of the functional which is *a priori* unknown. This will be done using the property (5.7) with some care;

- to check that the "guess" is really the relaxed functional, we need to verify (5.8). The main difficulty is that we must find the sequence (u^n) converging to a given u. However, we will see how we can avoid this difficulty.

We mention that the notion of relaxation is classical in many problems occuring in the Calculus of Variations : phase transition, fracture mechanics, plasticity,... For recent advances and bibliography, we refer to [10].

The specificity of this work is that the surface energy is defined over a fixed set independent of the unknown σ. Moreover, we give an explicit representation of the relaxed energy. We are going to establish that the functional $\overline{E}_1$ defined by :

$$\overline{E}_1 : \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u) \to R \tag{5.13}$$

$$\begin{aligned}\overline{E}_1(\sigma, \mu^i, \mu^e) = &\int_\Omega |\sigma \cdot \nabla u + h_1| dx + \sum_{j=1}^{2} \int_{\Omega^i \cup \Omega^e} \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2 \\ &+ \int_{S_u} d|\nu| + \int_{S_u} \|\mu^i - \mu^e\|_1 + \int_{S_u} \|\mu^i - \sigma^i ds\|_1 + \|\mu^e - \sigma^e ds\|_1\end{aligned}$$

where $\|\eta\|_1 = |\eta_1| + |\eta_2|$ and the measure ν is defined by (5.10), is in fact the relaxed functional of E_1 for the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times \mathbf{M}(S_u)$-faible $\times \mathbf{M}(S_u)$-faible. We are also going to prove that the functional defined by :

$$\overline{E} : \mathbf{BV}(\Omega) \to R \tag{5.14}$$

$$\overline{E}(\sigma) = \int_\Omega |\sigma \cdot \nabla u + h_1| dx + \sum_{j=1}^{2} \int_{\Omega^i \cup \Omega^e} \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2 + \int_{S_u} \beta(x, \sigma^i, \sigma^e)$$

where

$$\begin{aligned}\beta(x, \lambda, \theta) = \mathrm{Inf} \Big\{ &|\lambda - s| + |\theta - t| + |s - t| + \left| \frac{s+t}{2} \cdot n(x)(u^+ - u^-) + h_2(x) \right| : \\ &(s,t) \in R^N \times R^N \Big\},\end{aligned} \tag{5.15}$$

is the relaxed functional of E. The expression of $\overline{E}$ will be deduced from $\overline{E_1}$.

Before finding (5.13) and (5.14), we first need to prove some preliminary results. The general idea is that, for technical reasons, we need to work with functions defined on more regular spaces. This is why we introduce the functionals E_2 and $\overline{E}_2$ defined by :

$$E_2 : \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u) \to R \tag{5.16}$$

$$E_2(\sigma, \mu^i, \mu^e) = \begin{cases} E_1(\sigma, \mu^i, \mu^e) \text{ if } \sigma \in \mathbf{W^{1,1}}(\Omega^i \cup \Omega^e) \\ +\infty \text{ otherwise} \end{cases}$$

and

$$\overline{E}_2 : \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u) \to R \tag{5.17}$$

$$\overline{E}_2(\sigma, \mu^i, \mu^e) = \begin{cases} \overline{E}_1(\sigma, \mu^i, \mu^e) \text{ if } (\sigma, \mu^i, \mu^e) \in \mathbf{W^{1,1}}(\Omega^i \cup \Omega^e) \times \mathbf{L^1}(S_u) \times \mathbf{L^1}(S_u) \\ +\infty \text{ otherwise} \end{cases}$$

The justification of considering $E_2, \overline{E}_2$ instead of $E_1, \overline{E}_1$ is given by lemmas 5.2 and 5.3 where we prove that E_j and $\overline{E}_j (j = 1, 2)$ have the same relaxed functional for the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times$ $\mathbf{M}(S_u)$-faible $\times$ $\mathbf{M}(S_u)$-faible. As this is equivalent to say that they have the same dual functional [13], we will use the expressions (5.16) and (5.17) to compute the dual functionals (Lemmas A.1, A.2) and to establish the main relaxation result.
Let us present a version of the Slicing Lemma of De Giorgi that will be useful in the sequel.

THEOREM 5.1. *Let $\phi(\cdot)$ a function verifying hypotheses (4.2) (4.3). Let $u \in BV(\Omega) \cap L^2(\Omega)$. Then, for every open set $A \subset \Omega$ with Lipschitz boundary, we can find a sequence $u^n \in W^{1,1}(\Omega)$ such that :*

$$u^n \xrightarrow[L^2(\Omega)]{} u \tag{5.18}$$

$$u^n = u \quad on \quad \partial A \tag{5.19}$$

$$\lim_{n\to\infty} \int_A \phi(\|\nabla u^n\|)dx = \int_A \phi(Du) \tag{5.20}$$

Notice that this theorem permits to fix the trace at the boundaries.

Proof. The proof of that theorem is a consequence of the Lemma 2.6. proposed in [11] which can be modified to obtain the strong convergence in L^2. □

LEMMA 5.2. *Let E_1 and E_2 defined respectively by (5.11) and (5.16). Then E_1 and E_2 have the same lower semi-continuous relaxed functions for the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times$ $\mathbf{M}(S_u)$-faible $\times$ $\mathbf{M}(S_u)$-faible.*

Proof. The proof contains two steps.
`First step:` Since we have $E_1 \leq E_2$ we deduce that :

$$R(E_1) \leq R(E_2) \tag{5.21}$$

`Second step:` the reverse inequality will be proven using an approximation argument. Let $(\sigma, \mu^i, \mu^e) \in \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)$ such that E_1 be finite. Notice that this forces the mesures μ^i, μ^e to be the traces of σ. We search for a sequence $(\sigma^n, \mu^{i^n}, \mu^{e^n}) \in \mathbf{W^{1,1}}(\Omega^i \cup \Omega^e) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)$ converging to (σ, μ^i, μ^e) for the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times$ $\mathbf{M}(S_u)$-faible $\times$ $\mathbf{M}(S_u)$-faible such that :

$$\lim_{n\to\infty} E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) = E_1(\sigma, \mu^i, \mu^e) \tag{5.22}$$

If we can find such a sequence then the proof is complete since equality (5.22) means that $E_1 \geq R(E_2)$. But, since $R(E_1)$ is the greatest lower semi continuous function less than or equal to E_1, we deduce that :

$$R(E_1) \geq R(E_2) \tag{5.23}$$

Inequalities (5.21) and (5.23) conclude the proof. The difficulty consists in finding such a sequence. The idea is to apply the Theorem 5.1 in Ω^i and Ω^e separately. In Ω^i, we obtain the existence of a sequence (σ^{i^n}), such that :

$$\sigma^{i^n} \xrightarrow[L^2(\Omega^i)]{} \sigma$$

$$\sigma^{i^n}|_{S_u} = \sigma^i|_{S_u}$$

$$\lim_{n\to\infty} \int_{\Omega^i} \phi(\|\nabla \sigma^{i^n}\|)dx = \int_{\Omega^i} \phi(D\sigma) \tag{5.24}$$

We proceed as the same way in Ω^e and we define the sequence (σ^n)-$\mathcal{L}^2$ a.e. on Ω by :

$$\sigma^n(x) = \begin{cases} \sigma^{i^n} & \text{if } x \in \Omega^i \\ \sigma^{e^n} & \text{if } x \in \Omega^e \end{cases}$$

It is easy to check that σ^n belongs to $\mathbf{W^{1,1}}(\Omega^i \cup \Omega^e)$. Using that sequence, we define the sequence of measures μ^{i^n} and μ^{e^n} defined on S_u by :

$$\mu^{i^n} = \sigma^{i^n} ds \tag{5.25}$$

$$\mu^{e^n} = \sigma^{e^n} ds \tag{5.26}$$

Notice that since we have fixed the traces of σ^n on both sides of S_u, the sequences defined by (5.25)-(5.26) are in fact constant. So we have :

$$E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) =$$
$$\underbrace{\int_\Omega |\sigma^n \cdot \nabla u + h_1| dx + + \int_\Omega c(x) \|\sigma^n\|^2 dx}_{\text{Continuous for the } L^2 \text{ strong topology}} + \sum_{j=1}^{2} \int_\Omega \phi(D\sigma_j^n) + \underbrace{\int_{S_u} d|\nu|}_{\text{Constant}}$$

and moreover :

$$\begin{aligned} \int_\Omega \phi(D\sigma_j) &= \int_{\Omega^i} \phi(D\sigma_j) + \int_{\Omega^e} \phi(D\sigma_j) + \int_{S_u} |\sigma_j^i - \sigma_j^e| ds \\ &= \lim_{n\to\infty} \int_{\Omega^i} \phi(\|\nabla \sigma_j^{i^n}\|) dx + \lim_{n\to\infty} \int_{\Omega^e} \phi(\|\nabla \sigma_j^{e^n}\|) dx + \int_{S_u} |\sigma_j^i - \sigma_j^e| ds \\ &= \lim_{n\to\infty} \int_\Omega \phi(\|\nabla \sigma_j^n\|) dx \end{aligned}$$

So condition (5.22) is satisfied and this concludes the proof. □

LEMMA 5.3. *Let $\overline{E}_1$ and $\overline{E}_2$ defined respectively by (5.13) and (5.17). Then $\overline{E}_1$ and $\overline{E}_2$ have the same lower semi continuous relaxed functionals for the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times$ $\mathbf{M}(S_u)$-faible $\times$ $\mathbf{M}(S_u)$-faible.*

Proof. This proof is inspired by the proof of the preceding lemma. The first step is analoguous and the only difficulty is to find, for a given $(\sigma, \mu^i, \mu^e) \in \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)$, a sequence $(\sigma^n, \theta^{i^n}, \theta^{e^n}) \in \mathbf{W^{1,1}}(\Omega^i \cup \Omega^e) \times \mathbf{L^1}(S_u) \times \mathbf{L^1}(S_u)$ such that :

$$\lim_{n\to\infty} \overline{E}_2(\sigma^n, \theta^{i^n}, \theta^{e^n}) = \overline{E}_1(\sigma, \mu^i, \mu^e) \tag{5.27}$$

Construction of the sequence σ^n uses the same arguments as in Lemma 5.2, that is to say the use of Theorem 5.1 on Ω^i and Ω^e. We recall that the traces of σ^n on both sides of S_u are constant. The construction of the sequence approximating μ^i, μ^e is based on a result of Bouchitté-Valadier [14]. We recall that the part depending on the measures μ^i, μ^e in $\overline{E}_1$ is :

$$H(x, \mu^i, \mu^e, \sigma^i, \sigma^e) = \int_{S_u} d|\nu| + \int_{S_u} \|\mu^i - \mu^e\|_1 + \int_{S_u} \|\mu^i - \sigma^i ds\|_1 + \|\mu^e - \sigma^e ds\|_1$$

where the measure ν is defined by (5.10). It is easy to check that the functional H is homogeneous, so that using [14], we can find a sequence $\theta^{i^n}, \theta^{e^n}$ in $L^1(S_u)$ such that :

$$\theta^{i^n} \underset{\mathbf{M}(S_u)}{\longrightarrow} \mu^i$$
$$\theta^{e^n} \underset{\mathbf{M}(S_u)}{\longrightarrow} \mu^e$$
$$\lim_n H(x, \theta^{i^n}, \theta^{e^n}, \sigma^i, \sigma^e) = H(x, \mu^i, \mu^e, \sigma^i, \sigma^e)$$

Consequently, the constructed sequence $(\sigma^n, \theta^{i^n}, \theta^{e^n})$ permits to get (5.27) which concludes the proof. □

5.4. The relaxation results.

PROPOSITION 5.4. *Let E_1 the functional defined by (5.11) with hypotheses (4.23)-(4.24), (4.2)-(4.3), (4.4)-(4.6), (4.7)-(4.8). Then, the relaxed functional of E_1 for the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times$ $\mathbf{M}(S_u)$-faible $\times$ $\mathbf{M}(S_u)$-faible is :*

$$R(E_1) : \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u) \to R \tag{5.28}$$

$$R(E_1)(\sigma, \mu^i, \mu^e) = \int_\Omega |\sigma \cdot \nabla u + h_1| dx + \sum_{j=1}^{2} \int_{\Omega^i \cup \Omega^e} \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2$$
$$+ \int_{S_u} d|\nu| + \int_{S_u} \|\mu^i - \mu^e\|_1 + \int_{S_u} \|\mu^i - \sigma^i ds\|_1 + \|\mu^e - \sigma^e ds\|_1$$
$$\textit{where} \quad \nu = \frac{\mu^i + \mu^e}{2} \cdot n(u^+ - u^-) + h_2 ds$$

We can verify that we have $R(E_1) \leq E_1$ and that they are equal as soon as $\mu^e = \sigma^e ds$ and $\mu^i = \sigma^i ds$.

Proof. To simplify notations, we will note τ the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-fort $\times$ $\mathbf{M}(S_u)$-faible $\times$ $\mathbf{M}(S_u)$-faible and τ^d the topology $\mathbf{L^2}(\Omega^i \cup \Omega^e)$-faible $\times$ $\mathcal{C}^0(S_u)$-fort $\times$ $\mathcal{C}^0(S_u)$-fort. Notice that we will also use the notation M to denote a universal constant appearing in uniform bounds. This value may change from one line to another but we will always write M.

We first remark that using Lemma 5.2, permits us to work on a more regular space, that is to say with E_2. $R(E_2)$ is the relaxed functional of E_2 (or equivalently of E_1, thanks to Lemma 5.2) if and only if, for all $(\sigma, \mu^i, \mu^e) \in \mathbf{W^{1,1}}(\Omega^i \cup \Omega^e) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)$, we have the two conditions :

(i) for all $(\sigma^n, \mu^{i^n}, \mu^{e^n}) \underset{\tau}{\to} (\sigma, \mu^i, \mu^e)$, then

$$\liminf_{n\to\infty} E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) \geq R(E_2)(\sigma, \mu^i, \mu^e) \tag{5.29}$$

(ii) there exists $(\sigma^n, \mu^{i^n}, \mu^{e^n}) \underset{\tau}{\to} (\sigma, \mu^i, \mu^e)$ such that

$$\limsup_{n\to\infty} E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) \leq R(E_2)(\sigma, \mu^i, \mu^e) \tag{5.30}$$

The purpose of the two steps below is to establish that $R(E_2) = \overline{E}_2$

`First step:` This part is devoted to prove that

$$\liminf_{n\to\infty} E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) \geq R(E_2)(\sigma, \mu^i, \mu^e)$$

for all the sequences $(\sigma^n, \mu^{i^n}, \mu^{e^n})$ converging to (σ, μ^i, μ^e).

Let $(\sigma, \mu^i, \mu^e) \in \mathbf{W^{1,1}}(\Omega^i \cup \Omega^e) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)$ and a sequence $(\sigma^n, \mu^{i^n}, \mu^{e^n}) \in \mathbf{BV}(\Omega) \times \mathbf{M}(S_u) \times \mathbf{M}(S_u)$ converging to (σ, μ^i, μ^e) for the $\tau-$topology such that :

$$E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) \leq M$$

where M is a constant. Then, thanks to (4.3), we deduce that :

$$\|\sigma^n\|_{BV(\Omega)} \leq M \tag{5.31}$$

$$\text{On } S_u \text{ we have } \left\{ \begin{array}{l} \mu^{i^n} = \sigma^{i^n} ds \\ \mu^{e^n} = \sigma^{e^n} ds \end{array} \right. \quad \text{so} \quad \left\{ \begin{array}{l} \sigma^{i^n} ds \xrightarrow[\mathbf{M}(S_u)]{} \mu^i \\ \sigma^{e^n} ds \xrightarrow[\mathbf{M}(S_u)]{} \mu^e \end{array} \right. \tag{5.32}$$

Since the sequence (σ^n) is bounded in $BV(\Omega)$, we can deduce that there exists a measure $\tilde{\mu}$ such that :

$$\sigma^n ds \xrightarrow[\mathcal{M}(\partial\Omega)]{} \tilde{\mu} \tag{5.33}$$

Decomposing Ω permits to write :

$$\begin{aligned} E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) = & \int_\Omega |\sigma^n \cdot \nabla u + h_1| dx + \int_\Omega c(x) \|\sigma^n\|^2 dx + \\ & \sum_{j=1}^{2} \left(\int_{\Omega^i \cup \Omega^e} \phi(\|\nabla \sigma_j^n\|) dx \right) + \int_{S_u} \|\sigma^{i^n} - \sigma^{e^n}\|_1 ds \\ & + \int_{S_u} d|\nu^n| \end{aligned}$$

where ν^n is given by (5.10). Thanks to the convergence properties, it is easy to check that :

$$\begin{aligned} \liminf_{n\to\infty} \int_\Omega |\sigma^n \cdot \nabla u + h_1| dx + \int_\Omega c(x) \|\sigma^n\|^2 dx &\geq \int_\Omega |\sigma \cdot \nabla u + h_1| dx + \int_\Omega c(x) \|\sigma\|^2 dx \\ \liminf_{n\to\infty} \int_{S_u} \|\sigma^{i^n} - \sigma^{e^n}\|_1 ds &\geq \int_{S_u} \|\mu^i - \mu^e\|_1 ds \\ \liminf_{n\to\infty} \int_{S_u} d|\nu^n| &\geq \int_{S_u} d|\nu| \end{aligned} \tag{5.34}$$

By classical arguments and for a fixed j, we also have :

$$\liminf_{n\to\infty} \int_{\Omega^i \cup \Omega^e} \phi(\|\nabla \sigma_j^n\|) dx \geq \int_{\Omega^i \cup \Omega^e} \phi(D\sigma_j) dx$$

However, this minoration will not permit to conclude anything because we need to be more precise. Using notations proposed in figure 5.1, and especially the decomposition

$\Omega^i \cup \Omega^e = \Omega^{i,\alpha} \cup \Omega^{e,\alpha} \cup \Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}$ we can write :

$$\int_{\Omega^i \cup \Omega^e} \phi(\|\nabla {\sigma_j}^n\|)dx = \underbrace{\int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} \phi(\|\nabla {\sigma_j}^n\|)dx}_{\mathtt{A}^\alpha} + \underbrace{\int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} \phi(\|\nabla {\sigma_j}^n\|)dx}_{\mathtt{B}^\alpha} \tag{5.35}$$

We study both parts separately.

`Integral` A^α : by classical arguments [25], we can write :

$$\liminf_{n\to\infty} \int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} \phi(\|\nabla \sigma^n_j\|)dx \geq \int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} \phi(D\sigma_j)dx \tag{5.36}$$

`Integral` B^α : thanks to (4.6), we have :

$$\int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} \phi(\|\nabla {\sigma_j}^n\|)dx \geq \int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} \phi^\infty(\|\nabla {\sigma_j}^n\|)dx - k|\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}|$$

But, since ϕ^∞ is convex, we have :

$$\phi^\infty(\|\nabla {\sigma_j}^n\|) = \sup_{q^{Gj} \in R^2} q^{Gj} \cdot \nabla \sigma^n_j - (\phi^\infty)^*(q^{Gj})$$

$$\text{where } (\phi^\infty)^*(x^*) = \begin{cases} 0 \text{ if } x^* \in dom(\phi^*) = \{x^* \in R^2 / |x^*| < 1\} \\ +\infty \text{ otherwise} \end{cases}$$

So, for any function $q^{Gj}(x)$ such that $q^{Gj}(x) \in dom(\phi^*)$ a.e. $x \in \Omega$ with q^{Gj} in K defined by :

$$\begin{aligned} K = \{q \in & \mathbf{L^2}(\Omega) \text{ such that} \\ & \|q^i\|_{L^\infty(\Omega)} \leq 1, \|q^e\|_{L^\infty(\Omega)} \leq 1, \\ & div(q^i) \text{ and } div(q^e) \in L^2(\Omega), \\ & q^i \cdot n|_{S_u} \text{ and } q^e \cdot n|_{S_u} \in \mathcal{C}^0(S_u), \\ & q \cdot n|_{\partial\Omega} = 0\} \end{aligned} \tag{5.37}$$

we have :

$$\int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} \phi^\infty(\|\nabla {\sigma_j}^n\|) \geq \int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} q^{Gj} \cdot \nabla \sigma^n_j. \tag{5.38}$$

The set K has been introduced so that all the integrals that we are going to write below are well defined. All the problem now is to estimate the limit of the right-hand side. To this end, we first integrate by parts the same term, but on Ω. We have :

$$\begin{aligned} \int_\Omega q^{Gj} \cdot \nabla \sigma^n_j dx &= \int_{\Omega^i} q^{Gj} \cdot \nabla \sigma^n_j dx + \int_{\Omega^e} q^{Gj} \cdot \nabla \sigma^n_j dx \\ &= -\int_{\Omega^i \cup \Omega^e} div(q^{Gj}) \sigma^n_j dx + \int_{S_u} ({q^{Gj}}^i {\sigma^i_j}^n - {q^{Gj}}^e {\sigma^e_j}^n) \cdot n ds + \int_{\partial\Omega} q^{Gj} \sigma^n_j \cdot n ds. \end{aligned}$$

Thanks to the strong convergence in $\mathbf{L^2}(\Omega^i \cup \Omega^e)$ of the sequence σ^n and to (5.32) (5.33), we have :

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega q^{Gj} \cdot \nabla \sigma^n_j = \\ &\quad -\int_{\Omega^i \cup \Omega^e} div(q^{Gj}) \sigma_j dx + \int_{S_u} ({q^{Gj}}^i {\mu^i}_j - {q^{Gj}}^e {\mu^e}_j) \cdot n + \int_{\partial\Omega} q^{Gj} \tilde{\mu}_j \cdot n. \end{aligned} \tag{5.39}$$

Moreover, if we consider $q^{Gj} \cdot \nabla \sigma_j^n$ as a measure, for $\varphi \in \mathcal{C}_c^1(\Omega)$, we have by (4.11) :

$$< q^{Gj} \cdot \nabla \sigma_j^n, \varphi >= -\int_\Omega div(q^{Gj}) \sigma_j^n \varphi dx - \int_\Omega q^{Gj} \cdot \nabla \varphi \sigma_j^n dx.$$

When n tends to infinity, we have :

$$\begin{aligned} \lim_{n\to\infty} < q^{Gj} \cdot \nabla \sigma_j^n, \varphi > &= -\int_\Omega div(q^{Gj}) \sigma_j \varphi - \int_\Omega q^{Gj} \cdot \nabla \varphi \sigma_j \\ &=< q^{Gj} \cdot \nabla \sigma_j, \varphi > . \end{aligned}$$

The last result is equivalent to say that the measure $q^{Gj} \cdot \nabla \sigma_j^n$ converges to $q^{Gj} \cdot \nabla \sigma_j$ for the topology $\mathcal{M}(\Omega)$weak$\star$. Since Ω is bounded, we can prove in fact that :

$$\lim_{n\to\infty} \int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} q^{Gj} \cdot \nabla \sigma_j^n dx = \int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} q^{Gj} \cdot \nabla \sigma_j dx \tag{5.40}$$

Consequently, substracting (5.40) to (5.39) permits to write :

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} q^{Gj} \cdot \nabla \sigma_j^n dx = &- \int_{\Omega^i \cup \Omega^e} div(q^{Gj}) \sigma_j dx - \int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} q^{Gj} \cdot \nabla \sigma_j dx \\ &+ \int_{S_u} ({q^{Gj}}^i {\mu^i}_j - {q^{Gj}}^e {\mu^e}_j) \cdot n + \int_{\partial\Omega} q^{Gj} \tilde{\mu}_j \cdot n \end{aligned}$$

and, after integrating by part the term $\int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} q^{Gj} \cdot \nabla \sigma_j dx$, we can rewrite (5.38):

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega^{i,\alpha}_{S_u} \cup \Omega^{e,\alpha}_{S_u}} \phi(\|\nabla \sigma_j^n\|) dx \geq &- \int_{\Omega^i \cup \Omega^e} div(q^{Gj}) \sigma_j dx + \int_{\Omega^{i,\alpha} \cup \Omega^{e,\alpha}} div(q^{Gj}) \sigma_j dx \\ &+ \int_{S_u} ({q^{Gj}}^i {\mu^i}_j - {q^{Gj}}^e {\mu^e}_j) \cdot n \\ &- \int_{S_u^{i,\alpha}} {q^{Gj}}^i \sigma_j^i \cdot n ds - \int_{S_u^{e,\alpha}} {q^{Gj}}^e \sigma_j^e \cdot n ds \\ &+ \int_{\partial\Omega} q^{Gj} \cdot n(\tilde{\mu}_j - \sigma_j ds) \end{aligned} \tag{5.41}$$

Now that we have found a minoration for integrals A^α (5.36) and B^α (5.41), we merge both results and we let α tends to 0. The obtained result is :

$$\begin{aligned} &\liminf_{n\to\infty} E_2(\sigma^n, {\mu^i}^n, {\mu^e}^n) \geq \\ &\int_\Omega |\sigma \cdot \nabla u + h_1| dx + \sum_{j=1}^2 \int_{\Omega^i \cup \Omega^e} \phi(D\sigma_j) + \int_\Omega c(x) \|\sigma\|^2 + \int_{S_u} d|\nu| + \int_{S_u} \|\mu^i - \mu^e\|_1 \\ &+ \sum_{j=1}^2 \int_{S_u} {q^{Gj}}^i \cdot n({\mu^i}_j - \sigma_j^i ds) - \sum_{j=1}^2 \int_{S_u} {q^{Gj}}^e \cdot n({\mu^e}_j - \sigma_j^e ds) \qquad (\blacklozenge) \\ &+ \sum_{j=1}^2 \int_{\partial\Omega} q^{Gj} \cdot n(\tilde{\mu}_j - \sigma_j ds) \qquad (\lozenge) \\ &\text{where} \quad \nu = \frac{\mu^i + \mu^e}{2} \cdot n(u^+ - u^-) + h_2 ds. \end{aligned} \tag{5.42}$$

Since this inequality is true for all $\mathbf{q}^G = (q^{G1}, q^{G2}) \in K \times K$, it is still true when we take the supremum in $\mathbf{q}^G$. This supremum is taken for $q^{G1}, q^{G2} \in K$ defined by (5.37). Next, we introduce the set :

$$C(x) = \text{Closure}\{(q^i(x) \cdot n(x), q^e(x) \cdot n(x)), \quad (q^i, q^e) \in K\}$$

which can be rewritten as :

$$\begin{aligned} C(x) = \text{Closure}\{(z^i, z^e) \in R^2 \text{ such that} \\ & \exists (\varphi^i, \varphi^e) \in [\mathcal{C}^0(S_u)]^2 / \varphi^i(x) = z^i, \varphi^e(x) = z^e, \\ & \exists (q^i, q^e), \|q^i\|_\infty \leq 1, \|q^e\|_\infty \leq 1, div(q^i) \text{ and } div(q^e) \in L^2(\Omega), \\ & q^i(x) \cdot n(x) = \varphi^i(x), q^e(x) \cdot n(x) = \varphi^e(x) \ , \ x \in S_u \\ & q(x) \cdot n = 0 \ \ x \in \partial\Omega.\} \end{aligned}$$

To compute the supremum of (5.42), we only consider in (5.42) the term noted with the symbol ($\blacklozenge$). Notice that the term noted with the symbol ($\lozenge$) will not appear in the minimization thanks to the definition of the set K where we have imposed $q \cdot n|_{\partial\Omega} = 0$. If we note :

$$W = \begin{pmatrix} (\mu^i{}_1 - \sigma_1^i, \mu^e{}_1 - \sigma_1^e) \\ (\mu^i{}_2 - \sigma_2^i, \mu^e{}_2 - \sigma_2^e) \end{pmatrix},$$

we claim that :

$$\begin{aligned} & \sup_{(q^{G1}, q^{G2}) \in K^2} \left\{ \sum_{j=1}^{2} \int_{S_u} q^{Gj^i} \cdot n(\mu^i{}_j - \sigma_j^i ds) - \sum_{j=1}^{2} \int_{S_u} q^{Gj^e} \cdot n(\mu^e{}_j - \sigma_j^e ds) \right\} \\ & = \sup_{Z} \int_{S_u} \sum_{j=1}^{2} Z_j^t \cdot W_j \ \text{ where } \ Z = \begin{pmatrix} (z_1^i, z_1^e) \\ (z_2^i, z_2^e) \end{pmatrix} \in C(x)^2 \qquad (5.43) \\ & = \int_{S_u} \|\mu^i - \sigma^i ds\|_1 + \|\mu^e - \sigma^e ds\|_1. \qquad (5.44) \end{aligned}$$

Equality (5.43) corresponds to the permutation of the supremum. It is based on techniques developed in [15, 17]. We then need to express that supremum giving the expression (5.44) [14, 13]. We refer to [35] for the complete proof.

Finally, taking the supremum in (5.42) with respect to q^{Gj^i},q^{Gj^e}, and using (5.44) permits to have :

$$\liminf_{n \to \infty} E_2(\sigma^n, \mu^{i^n}, \mu^{e^n}) \geq \overline{E}_2(\sigma, \mu^i, \mu^e) \qquad (5.45)$$

where $\overline{E}_2$ has been previously defined in (5.17). The functional $\overline{E}_2$ is then a candidate to be the relaxed functional of E_2. It remains to show the second condition (5.30) (with $R(E_2) = \overline{E}_2$) which is the aim of the second step.

`Second step:` The way to demonstrate (5.30) is based on the following assertion

: showing (5.30) is equivalent to prove that :

$$\begin{aligned}
&\text{for all } \ (f,\varphi^i,\varphi^e) \in \mathbf{L}^\infty(\Omega^i \cup \Omega^e) \times \mathcal{C}^0(S_u) \times \mathcal{C}^0(S_u) \\
&\text{there exists } (f^n,\varphi^{i^n},\varphi^{e^n}) \underset{\tau^d}{\rightarrow} (f,\varphi^i,\varphi^e) \ \text{ such that} \\
&\qquad \liminf_{n\to\infty} E_2^*(f^n,\varphi^{i^n},\varphi^{e^n}) \geq R(E_2)^*(f,\varphi^i,\varphi^e),
\end{aligned} \tag{5.46}$$

where the superscript $*$ denotes the conjugate functionals. This result is due to [8] (see also [13] where this idea has been used).

Naturally, the difficulty is to compute the conjugate functional of E_2 and $\overline{E_2}$. This is done in the Lemmas A.1 and A.2 of the appendix. We have shown that :

$$E_2^*(f,\varphi^i,\varphi^e) = \inf_{q\in\mathcal{A}(f,\varphi^i,\varphi^e)} J(q)$$

where the minimum is computed for $q = (q^\sigma, q^{G1}, q^{G2}, q^u, q^{T1}, q^{T2}, q^{S_u})$ in $(\mathbf{L}^\infty(\Omega^\mathbf{i} \cup \Omega^\mathbf{e}))^3 \times L^\infty(\Omega^i \cup \Omega^e) \times (\mathbf{L}^\infty(\mathbf{S_u}))^2 \times L^\infty(S_u)$ verifying the conditions (A.2)-(A.10) (which defines the set $\mathcal{A}(f,\varphi^i,\varphi^e)$)), and

$$\overline{E_2}^*(f,\varphi^i,\varphi^e) = \inf_{\overline{q}\in\overline{\mathcal{A}}(f,\varphi^i,\varphi^e)} \overline{J}(\overline{q})$$

when the minimum is computed for $\overline{q} = (\overline{q}^\sigma, \overline{q}^{G1}, \overline{q}^{G2}, \overline{q}^u, \overline{q}^{T1}, \overline{q}^{T2}, \overline{q}^{\mu^i}, \overline{q}^{\mu^e}, \overline{q}^{S_u})$ in $(\mathbf{L}^\infty(\Omega^\mathbf{i} \cup \Omega^\mathbf{e}))^3 \times L^\infty(\Omega^i \cup \Omega^e) \times (\mathbf{L}^\infty(\mathbf{S_u}))^4 \times L^\infty(S_u)$ verifying conditions (A.13)-(A.24) (which defines the set $\overline{\mathcal{A}}(f,\varphi^i,\varphi^e)$). For more details about the definitionss of $J, \overline{J}, \mathcal{A}(f,\varphi^i,\varphi^e)$ and $\overline{\mathcal{A}}(f,\varphi^i,\varphi^e)$, we refer to Lemmas A.1 and A.2.

Let $(f^n,\varphi^{i^n},\varphi^{e^n})$ a sequence such that :

$$\liminf_{n\to\infty} E_2^*(f^n,\varphi^{i^n},\varphi^{e^n}) \leq M \tag{5.47}$$

where M is a constant. Then, for each n and using the definition of the conjugate function associated to E_2 (Lemma A.1), there exists a $q^n \in \mathcal{A}(f^n,\varphi^{i^n},\varphi^{e^n})$ so that :

$$E_2^*(f^n,\varphi^{i^n},\varphi^{e^n}) \geq J(q^n) - \frac{1}{n} \tag{5.48}$$

Since $E_2^*(f^n,\varphi^{i^n},\varphi^{e^n})$ is uniformly bounded thanks to (5.47), it is easy to check that we can find an element $q \in \mathcal{A}(f,\varphi^i,\varphi^e)$ such that the sequence q^n converges to q for the weak topology of this space that is to say the topology :

$$(\mathbf{L}^\infty(\Omega^\mathbf{i} \cup \Omega^\mathbf{e}))^3\text{weak} \times L^\infty(\Omega^i \cup \Omega^e)\text{weak} \times (\mathbf{L}^\infty(\mathbf{S_u}))^2\text{weak} \times L^\infty(S_u)\text{weak}$$

As the function J is lower semi continuous for this topology, we have :

$$\liminf_{n\to\infty} J(q^n) \geq J(q) \geq \inf_{q\in\mathcal{A}(f,\varphi^i,\varphi^e)} J(q) \tag{5.49}$$

Now, let us define the application T by :

$$\begin{aligned}
&T : \mathcal{A}(f,\varphi^i,\varphi^e) \to \overline{\mathcal{A}}(f,\varphi^i,\varphi^e) \\
&T(q) = \overline{q}
\end{aligned} \tag{5.50}$$

where $\overline{q}$ is defined by:

$$\begin{aligned}
\overline{q}^{\sigma} &= q^{\sigma} \\
\overline{q}^{Gj} &= q^{Gj} \quad (j=1,2) \\
\overline{q}^{u} &= q^{u} \\
\overline{q}^{S_u} &= q^{S_u} \\
\overline{q}^{Ti} &= \frac{1}{2} q^{S_u} \cdot n(u^+ - u^-) + q^{Ti} \\
\overline{q}^{Te} &= \frac{1}{2} q^{S_u} \cdot n(u^+ - u^-) + q^{Te} \\
\overline{q}^{\mu^i} &= \varphi^i - \frac{1}{2} q^{S_u} \cdot n(u^+ - u^-) \\
\overline{q}^{\mu^e} &= \varphi^e - \frac{1}{2} q^{S_u} \cdot n(u^+ - u^-)
\end{aligned}$$

An easy computation permits to see that if q belongs to $\mathcal{A}(f,\varphi^i,\varphi^e)$, then $T(q)$ belongs to $\overline{\mathcal{A}}(f,\varphi^i,\varphi^e)$. Moreover, we can oberve that :

$$J(q) = \overline{J}(T(q))$$

Consequently, using (5.48), (5.49), the definition of the function T and the Lemma A.2, we have :

$$\begin{aligned}
\liminf_{n\to\infty} E_2^*(f^n,\varphi^{i^n},\varphi^{e^n}) &\geq \inf_{q\in\mathcal{A}(f,\varphi^i,\varphi^e)} J(q) \\
&= \inf_{\overline{q}\in\overline{\mathcal{A}}(f,\varphi^i,\varphi^e)\cap Im(T)} \overline{J}(\overline{q}) \qquad \geq \inf_{\overline{q}\in\overline{\mathcal{A}}(f,\varphi^i,\varphi^e)} \overline{J}(\overline{q}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad = \overline{E}_2^*(f,\varphi^i,\varphi^e) \qquad (5.51)
\end{aligned}$$

which is exactly statement (5.46).

`Conclusion` As we can observe, the functional $\overline{E}_2$ complies with conditions (5.45) and (5.51), that is to say (5.29) (5.30) (or equivalently (5.29) (5.46)). As a conclusion we have :

$$R(E_2) = \overline{E}_2$$

which is the desired statement. □

PROPOSITION 5.5. *The relaxed functional noted $R(E)$ of the functional E defined by (5.3) is given by :*

$$R(E) : \mathbf{BV}(\Omega) \to R \qquad (5.52)$$

$$R(E)(\sigma) = \int_\Omega |\sigma\cdot\nabla u + h_1| dx + \sum_{j=1}^{2} \int_{\Omega^i\cup\Omega^e} \phi(D\sigma_j) + \int_\Omega c(x)\|\sigma\|^2 + \int_{S_u} \beta(x,\sigma^i,\sigma^e)$$

where

$$\beta(x,\lambda,\theta) = \mathrm{Inf}\left\{ |\lambda - s| + |\theta - t| + |s-t| + \left| \frac{s+t}{2}\cdot n(x)(u^+ - u^-) + h_2(x) \right| \ : \ (s,t)\in R^N\times R^N \right\}. \qquad (5.53)$$

Proof. This proposition is a direct consequence of the proposition 5.4 and we will just sketch the proof. Let us define

$$G(\sigma) = \inf_{(\mu^i,\mu^e)\in\mathcal{M}(\Omega)} \overline{E_1}(\sigma,\mu^i,\mu^e) \ .$$

By classical arguments, we prove that the functional G is lower semi continuous, less than E, and also greater that $R(E)$, so in fact :

$$G(\sigma) = R(E)(\sigma).$$

We deduce the final result from a Rockafellar theorem [42, 44] which permits to permute the infimum and the integral. □

6. Existence for the relaxed functional.

PROPOSITION 6.1. *Let $R(E)$ defined by (5.52) and (5.53), where u verifies hypotheses (4.23)-(4.24), $\phi(\cdot)$ satisfies (4.2)-(4.3), (4.4)-(4.6), and $c(x)$ is a function verifying (4.7)-(4.8). Then the problem* $\mathrm{Inf}\{R(E)(\sigma) : \sigma \in \mathbf{BV}(\Omega)\}$ *admits a solution in* $\mathbf{BV}(\Omega)$.

Proof. The functional $R(E)$ is a convex function of measures which is l.s.c. by construction. Moreover, it is coercive so we can uniformly bound minimizing sequences and deduce by classical arguments the existence of a solution. □

The above theorem proves an existence result for the relaxed functional associated to the optical flow problem. The main difficulty came from the product $(\sigma \cdot Du)$ for which we found an explicit integral representation. It will be interesting to study more general functionals involving terms of the form $f((\sigma \cdot Du))$. This question will be considered in a forthcoming paper.

Another challenging problem is the numerical analysis of these abstract results. This induces several difficulties. One of the first is to characterize the solution. No Euler equations can be written but some partial answers have been given using variational [4] or dual [51] formulations. Then, it will be necessary to propose some suitable discretizations to take into account the discontinuities of the solution. These problems will be considered in the future.

Appendix A. The dual functionals E_2^* and $\overline{E}_2^*$. We give in the two lemmas below the detailed expressions of the dual functions associated to E_2 and $\overline{E}_2$.

LEMMA A.1. *Let E_2 given by (5.16). Its dual functional is defined by :*

$$E_2^* : \mathbf{L}^\infty(\Omega^i \cup \Omega^e) \times \mathcal{C}^0(S_u) \times \mathcal{C}^0(S_u) \to R \tag{A.1}$$

$$E_2^*(f,\varphi^i,\varphi^e) = \inf_{q\in\mathcal{A}(f,\varphi^i,\varphi^e)} J(q)$$

where the infimum is taken over $q = (q^\sigma, q^{G1}, q^{G2}, q^u, q^{Ti}, q^{Te}, q^{S_u})$ belonging to $(\mathbf{L}^\infty(\Omega^{\mathbf{i}} \cup \Omega^{\mathbf{e}}))^3 \times L^\infty(\Omega^i \cup \Omega^e) \times (\mathbf{L}^\infty(\mathbf{S_u}))^2 \times L^\infty(S_u)$ and complying the following conditions :

$$|q^u| \le 1 \qquad a.e.\ on\ \Omega \tag{A.2}$$

$$|q^{Gj}| \le 1 \ \ (j=1,2) \qquad a.e.\ on\ \Omega \tag{A.3}$$

$$|q^{S_u}| \leq 1 \qquad a.e.\ on\ \ S_u \tag{A.4}$$
$$|q^{Ti} + \varphi^i| \leq 1 \qquad a.e.\ on\ \ S_u \tag{A.5}$$

$$q^u \nabla u + q^\sigma - div(\mathbf{q^G}) - f = 0 \qquad on\ \ \Omega \tag{A.6}$$
$$\frac{1}{2} q^{S_u} n(u^+ - u^-) + q^{Ti} + \mathbf{q^G}^i n = 0 \qquad on\ \ S_u \tag{A.7}$$
$$\frac{1}{2} q^{S_u} n(u^+ - u^-) + q^{Te} + \mathbf{q^G}^e n = 0 \qquad on\ \ S_u \tag{A.8}$$
$$q^{Ti} + \varphi^i + q^{Te} + \varphi^e = 0 \qquad on\ \ S_u \tag{A.9}$$
$$\mathbf{q^G} n = 0 \qquad on\ \ \partial\Omega \tag{A.10}$$

where $\mathbf{q^G}$ *is the matrix defined by* $\mathbf{q^G} = \begin{pmatrix} q^{G1T} \\ q^{G2T} \end{pmatrix}$, *and where the function* J *is defined by :*

$$J(q) = \int_\Omega q^u h_1 dx + \int_\Omega \frac{1}{4c(x)^2} |q^\sigma|^2 dx + \sum_{j=1}^{2} \int_\Omega \phi^*(|q^{Gj}|) dx + \int_{S_u} q^{S_u} h_2 ds \tag{A.11}$$

LEMMA A.2. *Let* $\overline{E}_2$ *given by (5.17). Its dual functional is defined by :*

$$\overline{E}_2^* : \mathbf{L}^\infty(\Omega^i \cup \Omega^e) \times \mathcal{C}^0(S_u) \times \mathcal{C}^0(S_u) \to R \tag{A.12}$$
$$\overline{E}_2^*(f, \varphi^i, \varphi^e) = \inf_{\overline{q} \in \overline{\mathcal{A}}(f, \varphi^i, \varphi^e)} \overline{J}(\overline{q})$$

where the infimum is taken over $\overline{q} = (\overline{q}^\sigma, \overline{q}^{G1}, \overline{q}^{G2}, \overline{q}^u, \overline{q}^{Ti}, \overline{q}^{Te}, \overline{q}^{\mu^i}, \overline{q}^{\mu^e}, \overline{q}^{S_u})$ *belonging to* $(\mathbf{L}^\infty(\Omega^\mathbf{i} \cup \Omega^\mathbf{e}))^3 \times L^\infty(\Omega^i \cup \Omega^e) \times (\mathbf{L}^\infty(\mathbf{S_u}))^4 \times L^\infty(S_u)$ *and complying the following conditions :*

$$|\overline{q}^u| \leq 1 \qquad a.e.\ on\ \ \Omega \tag{A.13}$$
$$|\overline{q}^{Gj}| \leq 1 \quad (j = 1, 2) \qquad a.e.\ on\ \ \Omega \tag{A.14}$$
$$|\overline{q}^{S_u}| \leq 1 \qquad a.e.\ on\ \ S_u \tag{A.15}$$
$$|\overline{q}^{Ti} + \overline{q}^{\mu^i}| \leq 1 \qquad a.e.\ on\ \ S_u \tag{A.16}$$
$$q^{Ti}\ \ and\ \ q^{Te} \in \mathcal{C}(x) \qquad a.e.\ on\ \ S_u \tag{A.17}$$

$$\overline{q}^u \cdot \nabla u + \overline{q}^\sigma - div(\overline{\mathbf{q}}^\mathbf{G}) - f = 0 \qquad on\ \ \Omega \tag{A.18}$$
$$q^{Ti} + \overline{\mathbf{q}}^{\mathbf{G}^i} : n = 0 \qquad on\ \ S_u \tag{A.19}$$
$$\overline{q}^{Te} + \overline{\mathbf{q}}^{\mathbf{G}^e} : n = 0 \qquad on\ \ S_u \tag{A.20}$$
$$\frac{1}{2} \overline{q}^{S_u} n(u^+ - u^-) + \overline{q}^{\mu^i} - \varphi^i = 0 \qquad on\ \ S_u \tag{A.21}$$
$$\frac{1}{2} \overline{q}^{S_u} n(u^+ - u^-) + \overline{q}^{\mu^e} - \varphi^e = 0 \qquad on\ \ S_u \tag{A.22}$$
$$\overline{q}^{Ti} + \overline{q}^{\mu^i} + \overline{q}^{Te} + \overline{q}^{\mu^e} = 0 \qquad on\ \ S_u \tag{A.23}$$
$$\overline{\mathbf{q}}^\mathbf{G} : n = 0 \qquad on\ \ \partial\Omega \tag{A.24}$$

where $\overline{\mathbf{q}}^{\mathbf{G}}$ *is the matrix defined by* $\overline{\mathbf{q}}^{\mathbf{G}} = \begin{pmatrix} \overline{q}^{G1\,T} \\ \overline{q}^{G2\,T} \end{pmatrix}$, *and where the function* $\overline{J}$ *is defined by :*

$$\overline{J}(\overline{q}) = \int_\Omega \overline{q}^u h_1 dx + \int_\Omega \frac{1}{4c(x)^2} |\overline{q}^\sigma|^2 dx + \sum_{j=1}^{2} \int_\Omega \phi^*(|\overline{q}^{Gj}|) dx + \int_{S_u} \overline{q}^{S_u} h_2 ds \qquad \text{(A.25)}$$

Proof. of Lemmas A.1 and A.2. We refer to [35] for the complete proof which is mainly technical. To get that result, we used classical techniques developed in [21], the Rockafellar's Theorem and suitable choices of dual variables $q, \overline{q}$ to simplify calculus. □

Acknowledgments. We thank M. Bellieud, G. Bouchitté and G. Buttazzo for their usefull suggestions about relaxation results.

REFERENCES

[1] R. Acart and C.R. Vogel. Analysis of bounded variation penalty methods for ill-posed p roblems. *Inverse Problems*, 10:1217–1229, 1994.

[2] L. Ambrosio. A compactness theorem for a new class of functions of bounded variation. *Boll. Unione Mat. Ital.*, VII(4):857–881, 1989.

[3] G. Anzellotti. Pairings between measures and bounded functions and compensated compactness. *Ann. Mat. Pura Appl.*, IV(135):293–318, 1983.

[4] G. Anzellotti. The euler equation for functionals with linear growth. *Trans. Am. Math. Soc.*, 290(2):483–501, August 1985.

[5] G. Aubert, M. Barlaud, L. Blanc-Feraud, and P. Charbonnier. Deterministic edge-preserving regularization in computed imaging. *IEEE Trans. Imag. Process.*, 5(12), February 1997.

[6] G. Aubert, R. Deriche, and P. Kornprobst. Computing optical flow via variational techniques. *SIAM Journal on Applied Mathematics.* To appear.

[7] G. Aubert and L. Vese. A variational method in image recovery. *SIAM J. Numer. Anal.*, 34(5):1948–1979, October 1997.

[8] D. Aze. Convergence of dual variables in problems of transmission through thin layers by epi-convergence methods. *Ric. Mat.*, 35:125–159, 1986.

[9] J.L. Barron, D.J. Fleet, and S.S. Beauchemin. Performance of optical flow techniques. *The International Journal of Computer Vision*, 12(1):43–77, 1994.

[10] A.C. Barroso, G. Bouchitté, G. Buttazzo, and I. Fonseca. Relaxation of Bulk and Interfacial Energies. *Arch. Rat. Mech. Anal.*, 135:107–173, 1996.

[11] G. Bouchitté, I. Fonseca, and L. Mascarenhas. A global method for relaxation. Technical report, Université de Toulon et du Var, May 1997.

[12] G. Bouchitté and G. Dal Maso. Integral representation and relaxation of convex local functionals on $BV(\Omega)$. *Ann. Scuola Norm. Sup.*, 20:483–533, February 1993.

[13] G. Bouchitté and P. Suquet. Homogenization, plasticity and yield design. In *Proc. Int. Cent. Theor. Phys. Workshop*, volume 5 of *Composite media and homogenization theory*, pages 107–133, Trieste, 1991.

[14] G. Bouchitté and M. Valadier. Integral representation of convex functions on a space of measures. *J. Funct. Anal.*, 80(2):398–420, 1988.

[15] G. Bouchitté and M. Valadier. Multi-fonctions s.c.i. et régularisée s.c.i. essentielle. fonctions de mesure dans le cas sous linéaire. In *Proceedings Congrès Franco-Québecois Analyse non linéaire.* Gauthier Villars, Paris, 1989.

[16] G. Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variations.* Longman Scientific & Technical, 1989.

[17] C. Castaing and M. Valadier. *Convex Analysis and Measurable Multifunctions.* Number 580 in Lecture Notes in Mathematics. Springer-Verlag, Berlin . Heidelberg . New York, 1977.

[18] I. Cohen. Nonlinear variational method for optical flow computation. In *Proceedings 8th SCIA*, volume 1, pages 523–530, 1993.

[19] F. Demengel and R. Temam. Convex functions of a measure and applications. *Indiana University Mathematics Journal*, 33:673–709, 1984.

[20] R. Deriche and O. Faugeras. Les EDP en traitement des images et vision par ordinateur.

Technical report, INRIA, November 1995. A more complete version of this Research Report has appeared in the French Revue "Traitement du Signal". Volume 13 - No 6 - Special 1996.
[21] I. Ekeland and R. Temam. *Analyse convexe et problèmes variationnels.* Etudes mathematiques. Dunod; Gauthier-Villars, Paris - Bruxelles - Montreal, 1974.
[22] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* CRC, 1992.
[23] H. Federer. *Geometric Measure Theory.* Classics in Mathematics. Springer-Verlag, Berlin . Heidelberg . New York, 1969.
[24] D.J. Fleet and A.D. Jepson. Computation of component image velocity from local phase information. *The International Journal of Computer Vision*, 5:77–104, 1990.
[25] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation.* Birkhäuser, 1984.
[26] C. Goffman and J. Serrin. Sublinear functions of measures and variational integrals. *Duke Math J.*, 31:159–178, 1964.
[27] F. Heitz and P. Bouthemy. Multimodal estimation of discontinuous optical flow using markov random fields. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 15(12):1217–1232, December 1993.
[28] B.K. Horn. *Robot Vision.* MIT Press, 1986.
[29] B.K. Horn and B.G. Schunk. Determining Optical Flow. *Artificial Intelligence*, 17:185–203, 1981.
[30] Jan J. Koenderink. Optic Flow. *Vision Research*, 26(1):161–180, 1986.
[31] Jan J. Koenderink and A. J. van Doorn. Invariant properties of the motion parallax field due to the movement of rigid bodies relative to an observer. *Optica Acta*, 22:717–723, 1975.
[32] Jan J. Koenderink and A. J. van Doorn. How an ambulant observer can construct a model of the environment from the geometrical structure of the visual inflow. In Hauske G. and Butenandt E., editors, *Kybernetik 1978.* Oldenburg, Muenchen, 1978.
[33] Jan J. Koenderink and Andrea J. van Doorn. Affine structure from motion. *Journal of the Optical Society of America*, A8:377–385, 1991.
[34] J.J. Koenderink and A.J. van Doorn. Dynamic shape. *Biological Cybernetics*, 53:383–396, 1986.
[35] P. Kornprobst. *Contributions à la restauration d'images et à l'analyse de séquences : Approches Variationnelles et Equations aux Dérivées Partielles.* PhD thesis, Université de Nice-Sophia Antipolis, 1998.
[36] G. Dal Maso. *An introduction to* Γ*-convergence.* Progress in Nonlinear Differential Equations and their Applications. Birkhauser, 1993.
[37] H.H. Nagel. Direct estimation of optical flow and of its derivatives. In G.A. Orban and H.-H. Nagel, editors, *Artificial and Biological Vision Systems*, Basic Research Series, pages 191–224. Springer-Verlag, 1992.
[38] H.H. Nagel and W. Enkelmann. An investigation of smoothness constraint for the estimation of displacement vector fiels from images sequences. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 8:565–593, 1986.
[39] M. Orkisz and P. Clarysse. Estimation du flot optique en présence de discontinuités: une revue. *Traitement du Signal*, 13(5):489–513, 1996.
[40] M. Otte and H.H. Nagel. Optical flow estimation: Advances and comparisons. In Jan-Olof Eklundh, editor, *Proceedings of the 3rd European Conference on Computer Vision*, volume 800 of *Lecture Notes in Computer Science*, pages 51–70. Springer-Verlag, 1994.
[41] M. Proesmans, L. Van Gool, E. Pauwels, and A. Oosterlinck. Determination of Optical Flow and its Discontinuities using Non-Linear Diffusion. In *Proceedings of the 3rd ECCV, II*, number 801 in Lecture Notes in Computer Science, pages 295–304. Springer-Verlag, 1994.
[42] R.T. Rockafellar. Integrals which are convex functionals. *Pacific J. Math.*, 24:525–539, 1968.
[43] R.T. Rockafellar. *Convex analysis.* Princeton University Press, Princeton, N. J., 1970.
[44] R.T. Rockafellar. Integral which are convex functionals II. *Pacific J. Math.*, 39:439–469, 1971.
[45] L. Rudin, S. Osher, and E. Fatemi. Nonlinear total variation based noise removal algorithms. *Physica D*, 60:259–268, 1992.
[46] W. Rudin. *Real and complex analysis.* McGraw-Hill, 1966.
[47] R. Temam. Dual Variational Principles in Mechanics and Physics. In A.V. Fiacco and K.O. Kortanek, editors, *Semi-Infinite Programming and Applications*, volume 215 of *Lecture Notes in Economics and Mathematical Systems.* Springer-Verlag, 1981.
[48] A.N. Tikhonov and V.Y. Arsenin. *Solutions of Ill-posed Problems.* Winston and Sons, Washington, D.C., 1977.
[49] A. Verri, F. Girosi, and V. Torre. Differential techniques for optical flow. *Journal of the Optical Society of America A*, 7:912–922, 1990.
[50] A. Verri and T. Poggio. Motion field and optical flow: qualitative properties. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 11(5):490–498, 1989.
[51] L. Vese. *Problèmes variationnels et EDP pour l'analyse d'images et l'évolution de courbes.*

PhD thesis, Université de Nice Sophia-Antipolis, November 1996.
[52] A.I. Vol'pert. The spaces BV and quasilinear equations. *Math. USSR-Sbornik*, 2(2):225–267, 1967.
[53] W.P. Ziemer. *Weakly Differentiable Functions*. Springer-Verlag, 1989.