Robust control approach to digital option pricing: synthesis approach

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Abstract

In the framework of an interval market model already used in [6,7,5], we tackle the case of a digital option with its discontinuous terminal payment. In this paper we develop the synthesis approach via the investigation of the trajectory field à la Isaacs-Breakwell.

Key words. Differential games, Isaacs' equation, option pricing

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1 Introduction

In [3,4,6], we introduced a robust control approach to option pricing using a non stochastic model of the market uncertainties and a minimax approach. Several authors used simultaneously and independently similar approaches. The same market model has previously been used in [12][1], who coined the phrase "interval model", and also in [9], where the author stresses that due to the incompleteness of the market, the hedge is a super-replication,

[1] available as a preprint as early as 2000

and the premium computed is a "seller's price". While these references use tools quite different from ours, a very similar theory (actually strictly more general, and consequently less detailed) has been developed in [11,1] and related papers. See also a further development of similar ideas in [13]. A rather exhaustive survey of our own theory can be found in [5]. A discussion of the strengths and weaknesses of this approach, as compared to the classical Black and Scholes theory, can be found in [6].

We tackle here the case of a digital call. This is a contract whereby the writer (also called here the *trader*) pledges to pay the buyer a fixed amount D at a prescribed final time (or *exercise time*) T if the market price of a specified underlying stock at this time is not less than a given price K (also called the *strike*). This defines a *payoff* of the option which is a discontinuous function of the final price of the underlying stock. This is to be contrasted with classical —or "vanilla"— options where the payoff is continuous and even convex.

In this paper, we develop the analysis in terms of the field of extremal trajectories, reconstructing the Value function via a classical synthesis approach à la Isaacs-Breakwell. This allows us to completely solve the problem and to exhibit a representation theorem similar to that of [8,5], with the same linear vector PDE, but with a different coefficient q^+, and different sets of initial conditions and gradient discontinuities. [2]

2 The model

We quickly recall the overall framework. The reader is referred to our previous works, say [5], for a more detailed description.

Parameters The parameters of our problem are the exercise time T, the strike K, the amount [3] D, the transaction cost rates C^+ and C^-, depending on whether it is a buy or a sale, and the similar c^+ and c^- that apply to the closure costs, with $C^- \le c^- \le 0 \le c^+ \le C^+$, and $C^-C^+ \neq 0$. We let $\xi > 0$ denote the amount of a buy of underlying stock, and $\xi < 0$ denote a sale of an amount $-\xi$. The transaction costs are $C^+\xi$ or $C^-\xi$ respectively, both positive. Moreover, we shall (realistically) assume that $1 + C^- > 0$.

We shall use the notation $C^\varepsilon\xi$ to mean $C^+\xi$ if $\xi > 0$ and $C^-\xi$ if $\xi < 0$. A similar convention will apply to such notations as $q^\varepsilon(\check{v} - v)$ where q^ε will mean q^+ or q^- (to be defined) depending on the sign ε of $(\check{v} - v)$.

Furthermore, two constant bounds $\tau^- < 0$ and $\tau^+ > 0$ on the relative stock price rate of change are known. See the market model below.

[2] A much more detailed and complete analysis is to appear in [14].

[3] Without loss of generality, this could have been taken as 1. Keeping it as D helps one keep track of the physical dimensions in the calculations. The meaningful dimensionless quantity is the ratio D/K.

We assume that via the classical change of variables to end-time values, the riskless interest rate has been factored out.

State variables In these end-time values, the state variables are
- the **u**nderlying stock price u,
- the **v**alue of the portfolio's underlying stock content v,
- the total **w**orth of the portfolio w.

The underlying stock price u is by essence positive. Moreover, since we consider a call (an increasing payment function), there is no point in considering negative v's either, so that we shall always assume that u and v are both non negative.

Market model We use the *interval model*. In that model, it is assumed that the stock price is an absolutely continuous function, and furthermore that two numbers, τ^+ and τ^-, with $\tau^- < 0 < \tau^+$, are known, such that

$$\forall t_1, t_2 \,, \quad \mathrm{e}^{\tau^-(t_2-t_1)} \le \frac{u(t_2)}{u(t_1)} \le \mathrm{e}^{\tau^+(t_2-t_1)} \,. \tag{1}$$

We call Ω the set of such admissible stock price trajectories. Alternatively, we shall let $\tau = \dot{u}/u$ and let $\tau(t)$ be a measurable function with for all t, $\tau^- \le \tau(t) \le \tau^+$. We shall call Ψ the set of all such admissible rate functions.

Trader controls and strategy The control of the trader is through buying or selling underlying stocks. He may either do so in a *continuous trading* mode, a classical fictitious mode whereby one buys stocks at a continuous rate $\xi(t)$, (a sale if $\xi < 0$,) or in an impulsive mode or *discrete trading*. In that later mode, the trader buys (or sells) lump sums at finitely many freely chosen time instants. We shall call these times t_k, $k = 1, 2, \ldots$ and the amount traded at these instants ξ_k, of signs ε_k. Hence, the function $\xi(\cdot)$ will be considered as a sum of a measurable *continuous component* and finitely many weighted translated Dirac impulses. We shall call Ξ the set of such admissible trader's controls.

The trader acts knowing the market situation, (fictitiously) with no time delay. The mathematical metaphor of that hypothesis is as follows: a strategy is a non-anticipative function $\phi : \Omega \to \Xi$. (The initial portfolio content $v(0) = v_0$ will usually be considered as zero. Yet, to be more general, we must let ϕ also depend on it.) In practice, we shall implement it as a state feedback $\xi(t) = \varphi(t, u(t), v(t))$. We do not attempt to describe all admissible state feedbacks, being content to check that the one we exhibit actually yields an admissible non-anticipative strategy. We let Φ be the set of admissible non-anticipative strategies, and use the notation $\varphi \in \Phi$ to mean that a feedback strategy φ generates an admissible strategy ϕ in Φ.

Dynamics The dynamics are

$$\dot{u} = \tau u\,, \tag{2}$$

$$\dot{v} = \tau v + \xi\,, \tag{3}$$

$$\dot{w} = \tau v - C^{\varepsilon}\xi\,, \tag{4}$$

and in case the trader decides to make a block buy or sale of stocks of magnitude ξ_k at time t_k,

$$v(t_k^+) = v(t_k) + \xi_k\,, \tag{5}$$

$$w(t_k^+) = w(t_k) - C^{\varepsilon}\xi_k\,. \tag{6}$$

Payoff At the terminal time T, the trader sells any remaining underlying stock $v(T)$ in its portfolio, at a closure cost $-c^-v(T)$, and pays its due D to the buyer if $u(T) \geq K$. Thus, its incurred cost is $N(u(T), v(T))$ with

$$N(u,v) = \begin{cases} -c^-v & \text{if } u < K\,, \\ D - c^-v & \text{if } u \geq K\,. \end{cases} \tag{7}$$

The total expense born by the trader due to possible losses (or gains) due to stock price variations, transaction costs, and the terminal costs, if the contract has been written at time t_0 with $u(t_0) = u_0$ and $v(t_0) = v_0$ is therefore

$$J(t_0, u_0, v_0; \xi(\cdot), \tau(\cdot)) = N(u(T), v(T)) + \int_{t_0}^{T} (C^{\varepsilon}\xi(t) - \tau(t)v(t)) + \sum_k C^{\varepsilon_k}\xi_k\,. \tag{8}$$

Let

$$W(t_0, u_0, v_0) := \inf_{\phi\in\Phi} \sup_{\tau\in\Psi} J(t_0, u_0, v_0; \phi(u(\cdot)), \tau(\cdot)) \tag{9}$$

be the Value function of that differential game problem. The premium that the writer should charge for this contract, written at time 0, is $P(u(0)) = W(0, u(0), 0)$.

The limit case $u(T) = K$. We chose to let the terminal payment be u.s.c., deciding that the amount D is owed if $u(T) = K$. This is no serious restriction in terms of modelization. But it simplifies the analysis in that it lets the supremum in (9) be a maximum.

For a given pair (t, u), we shall call $\Psi_K(t, u)$ the subset of Ψ of controls $\tau(\cdot) : [t, T] \to [\tau^-, \tau^+]$ that drive the price from $u(t) = u$ to $u(T) = K$, (i.e $\exp \int_t^T \tau(s)\,\mathrm{d}s = K/u$). Finally, we call $\Lambda \subset [0, T] \times \mathbb{R}_+$ the set of (t, u)'s for which $\Psi_K(t, u) \neq \emptyset$. This is equivalent to $u \in [u_\ell(t), u_r(t)]$, with

$$u_\ell(t) = K\mathrm{e}^{-\tau^+(T-t)}\,, \qquad u_r(t) = K\mathrm{e}^{-\tau^-(T-t)}\,. \tag{10}$$

Further notations We give here for ease of reference some (strange) notations that we shall use all along hereafter. We let

$$q^-(t) = \max\{(1+c^-)\mathrm{e}^{\tau^-(T-t)} - 1\,, C^-\}\,, \tag{11}$$

$$q^+(u) = \min\Big\{\max\{(1+c^-)\frac{K}{u} - 1\,, C^-\}\,, C^+\Big\}\,. \tag{12}$$

These two numbers will appear as the opposite of the partial derivative $\partial W/\partial v$, or loss coefficients associated to the fact of having too much, respectively not enough, of the underlying stock in the portfolio as compared to an "ideal" content $\check{v}(t,u)$.

We let also

$$t_- = T - \tfrac{1}{\tau^-}\ln\left(\tfrac{1+C^-}{1+c^-}\right)\,,\ \ u_- = K\tfrac{1+c^-}{1+C^-}\,, \tag{13}$$

$$t_+ = T - \tfrac{1}{\tau^+}\ln\left(\tfrac{1+C^+}{1+c^-}\right)\,,\ \ u_+ = K\tfrac{1+c^-}{1+C^+}\,. \tag{14}$$

t_-, u_-, and u_+ are the switch values in the definitions (11,12) of q^- and q^+:

$$q^-(t) = \begin{cases} C^- & \text{if } t \le t_-\,, \\ (1+c^-)\mathrm{e}^{\tau^-(T-t)} - 1\,, & \text{if } t \ge t_-\,, \end{cases}$$

and

$$q^+(u) = \begin{cases} C^+ & \text{if } u \le u_+\,, \\ (1+c^-)\frac{K}{u} - 1 & \text{if } u_+ \le u \le u_-\,, \\ C^- & \text{if } u \ge u_-\,, \end{cases}$$

and t_+ is related to u_+ as shown in Figure 1 and Figure 2.

We shall also let

$$Q^\varepsilon = (q^\varepsilon \quad 1)\,,\ \varepsilon = \pm\,, \quad Q = \begin{pmatrix} Q^+ \\ Q^- \end{pmatrix}\,, \tag{15}$$

and, wherever $q^- \neq q^+$ (and hence Q invertible),

$$\mathcal{T} = \frac{1}{q^+ - q^-}\begin{pmatrix} \tau^+q^+ - \tau^-q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-)q^+q^- & \tau^-q^+ - \tau^+q^- \end{pmatrix} = Q^{-1}\begin{pmatrix} \tau^+ & 0 \\ 0 & \tau^- \end{pmatrix} Q\,. \tag{16}$$

This matrix appears in Theorem 3.1 below, and, as a consequence, in the representation theorem 4.1. The second form in the equation above makes it, if not intuitive, at least logically connected to the context.

Finally, we set

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\,, \quad \mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\,. \tag{17}$$

$$\mathcal{V} = \begin{pmatrix} v \\ w \end{pmatrix}\,, \quad \check{\mathcal{V}} = \begin{pmatrix} \check{v} \\ \check{w} \end{pmatrix}\,. \tag{18}$$

The notations $\check{v}$ and $\check{w}$ will appear later.

3 Analysis

3.1 Isaacs equation

One of the issues in investigating that game is to decide how to cope with the impulse controls. One avenue is to consider the three-dimensional game of degree with state space (t, u, v) and payoff (8), and to use the differential quasi-variational inequality (DQVI) associated with that game, as done in [7]. The other avenue, that we follow here, and also used in [7], makes use of the 4-dimensional representation in (t, u, v, w), and finds the graph of W as the boundary of the *capturable states*. By this, we mean states that can be driven by the trader to the target set $\{w \geq N(u, v)\}$ at the terminal time T. In that framework, we use the tools of semipermeability to construct that boundary, and jump trajectories are just trajectories orthogonal to the (t, u) plane.

We shall use normals to the boundary of the capturable states of the general form

$$\nu = \begin{pmatrix} n \\ p \\ q \\ r \end{pmatrix}$$

with usually $r = 1$. i.e. an *inward* normal. The extremalizing controls will be given by Isaacs main equation. It can be written in terms of the hamiltonian

$$H(t, u, v, w, n, p, q, \tau, \xi) = n + \tau[pu + (1 + q)v] + (q - C^\varepsilon)\xi$$

as

$$\sup_{\xi} \min_{\tau \in [\tau^-, \tau^+]} H(t, u, v, w, n, p, q, \tau, \xi) = 0\,. \tag{19}$$

The minimum in τ is always reached at $\tau = \tau^\varepsilon$, with $\varepsilon = \mathrm{sign}(\sigma)$ decided by the switch function $\sigma = -pu - (1 + q)v$. Singular trajectories will involve $\sigma = 0$.

The supremum in ξ is reached at $\xi = 0$ if $C^- \leq q \leq C^+$. Otherwise, the supremum is $\pm\infty$, according to the sign of q. (This corresponds to a jump, of the same sign as q.)

It follows directly from the dynamics that jump trajectories lie in a (v, w) plane, and have a slope C^ε, ε the sign of the jump. Consequently, any hypersurface made up of such trajectories has a normal of the form $\nu^t = (n\ \ p\ \ C^\varepsilon\ \ 1)$. This yields a singular hamiltonian in ξ.

The adjoint equations read

$$\begin{aligned} \dot{n} &= 0\,, \\ \dot{p} &= -\tau p\,, \\ \dot{q} &= -\tau(q + 1)\,, \\ \dot{r} &= 0\,. \end{aligned}$$

Hence we may set $r = 1$ if we manage to choose it such at terminal time.

3.2 Primary field

3.2.1 The sheets (τ_ℓ^-) and (τ_r^-)

Trajectories ending in the region $u < K$, $w + c^- v = 0$ must be with $p(T) = 0$, $q(T) = c^-$, yielding $\sigma(T) = -(1 + c^-)v(T) < 0$. Therefore, the final τ should be τ^- and $\xi = 0$. The switch function is constant along such a trajectory. But these trajectories cannot be integrated beyond t_- (see (13)). As a matter of fact, the adjoint equations yield $q(t) = q^-(t)$ (see (11)), so that before t_-, we would have $q < C^-$, and the control $\xi = 0$ would no longer maximize the hamiltonian.

This 2-D set of trajectories (parametrized by $u(T), v(T)$) creates a 3-D manifold, a semipermeable hypersurface, that we call the *sheet* (τ_ℓ^-). This manifold obeys the equation

$$(\tau_\ell^-): \quad q^- v + w = 0\,,$$

and its normal ν_{τ^-} is

$$\nu_{\tau^-}^t = (-\tau^-(q^- + 1)v \quad 0 \quad q^- \quad 1) \tag{20}$$

or, using the notations (15)(18) as

$$Q^- \mathcal{V} = 0\,, \qquad \text{and} \qquad \nu_{\tau^-}^t = (-\tau^- Q^- \mathbf{1} v \quad 0 \quad q^- \quad 1)\,.$$

A similar construction holds for trajectories ending in the region $u > K$, resulting in a semipermeable surface (τ_r^-): $Q^- \mathcal{V} = D$, with the same normal (20).

3.2.2 The singular sheet (K)

At $u = K$, the final payment N is non differentiable. The semipermeable normal can be any element of the super-differential, hence of the form $(n \;\; p \;\; c^- \;\; 1)$, $p \le 0$. The final switch function is thus of the form $\sigma(T) = -[pK + (1 + c^-)v]$. Again, it is constant along a trajectory. If $\sigma < 0$, we get the trajectory $u_r(t)$ (see(10)) this is the "right" boundary of the sheet (τ_ℓ^-). If $\sigma > 0$, we get $u_\ell(t)$. This will be seen to be the "left" boundary of the sheet (K) in the region $t \ge t_+$.

If $p(T)$ is chosen to make $\sigma(T) = 0$, then sigma will remain zero along any trajectory (as long as $\xi = 0$), and hence any $\tau(t) \in [\tau^-, \tau^+]$ is permissible. Let $\theta := \int_t^T \tau(s)\mathrm{d}s$. We generate that way a new 3-D manifold parametrized by $(v(T), \theta, t)$.

It is a valid semipermeable hypersurface as long as q remains between C^- and C^+. One easily sees that $q = (1+c^-)K/u - 1$, so that the condition

$q \le C^+$ translates into $u \ge u_+$ (see (14)), —this is the left boundary of that sheet for $t \le t_+$—, and $q \ge C^-$ translates into $u \le u_-$, —this is the right boundary of that sheet for $t \le t_-$. For $t \ge t_+$, the left boundary is the trajectory $u_\ell(t)$. For $t \ge t_-$, the right boundary is on $u = u_r(t)$. On this whole sheet, $q = q^+(u)$ (see(12)).

We call this semipermeable hypersurface the *sheet* (K). It is also characterized by $Q^+\mathcal{V} = D$, and its normal is $\nu_K^t = (0 \quad -Q^+\mathbb{1}v/u \quad q^+ \quad 1)$.

3.2.3 Projection in the (t, u) space

It is useful to look at the projection of the different sheets in the (t, u) space. The domain of validity of each sheets is :

$$\begin{array}{rcl} (\tau_\ell^-) & : & \{t \ge t_-\} \cap \{u < u_r(t)\}\,, \\ (\tau_r^-) & : & \{t \ge t_-\} \cap \{u \ge u_r(t)\}\,, \\ (K) & : & \max\{u_+, u_l(t)\} \le u \le \min\{u_-, u_r(t)\}\,. \end{array}$$

This shows that (K) and (τ_l^-) coexist in the domain

$$\{t \ge t_-\} \cap \{\max\{u_+, u_l(t)\} \le u \le u_r(t)\}\ ,$$

and that (K) and (τ_r^-) exist in disjoint (t, u) domains, except along their common trajectory $u = u_r(t)$ for $t \ge t_-$ where they join smoothly since they share $q = (1 + c^-)K/u_r - 1 = q^-$.

3.2.4 The dispersal manifold $\mathcal{D} = (\tau_\ell^-) \cap (K)$.

In the region $t \ge t_-$, $\max\{u_+, u_\ell(t)\} \le u < u_r(t)$, capturable states are characterized by the two conditions $w \ge -q^-v$ and $w \ge D - q^+v$. The boundary of capturable states (the graph of W) is made of the two sheets (τ_ℓ^-) and (K), between the end time and their intersection $\mathcal{D}$ characterized by $v = \check{v}(t, u)$, $w = \check{w}(t, u)$ with

$$\check{v}(t,u) = \frac{D}{q^+(u) - q^-(t)}\,, \qquad \check{w}(t,u) = -q^-(t)\check{v}(t,u)\,. \tag{21}$$

This is a dispersal manifold $\mathcal{D}$, somewhat degenerate in that, on the one hand, one of the outgoing fields, the sheet (K), is traversed by trajectories generated by any control $\tau(\cdot)$, and, on the other hand, the trajectories of the other "outgoing" field, the sheet (τ_ℓ^-), actually traverse the dispersal manifod $\mathcal{D}$ itself.

This manifold, thus, is born by the sheet (K), and traversed by trajectories generated by $\tau = \tau^-$. We shall show hereafter the following fact.

Theorem 3.1. *If a manifold* $\mathcal{V} = \check{\mathcal{V}}(t, u)$ *is either*

(1) traversed by trajectories τ^- and lying on the sheet (K),
(2) traversed by trajectories τ^+ and lying on a sheet (τ^-),
(3) traversed by trajectories τ^+ and trajectories τ^-,

it satisfies the partial differential equation

$$\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}}) = 0\,. \tag{22}$$

In the present case, as for all other closed formulas obtained thereafter, the PDE (22) can be checked by direct differentiation.

This PDE was first introduced in [7] for the Focal manifold. In [8], we showed that it is satisfied by all the singular surfaces of that game, this coming as a surprise. In [6], we showed that it is a necessary consequence of (23) if this formula is to be that of a viscosity solution of the Isaacs DIQV. The present paper will prove the theorem above via the investigation of the field of optimal trajectories, thus explaining it, perhaps, in a more natural way.

The Value function in that region is therefore

$$W(t,u,v) = \check{w}(t,u) + q^{\varepsilon}(t,u)(\check{v}(t,u) - v)\,, \tag{23}$$

where we recall that in such an expression, $\varepsilon = \mathrm{sign}(\check{v}(t,u) - v)$.

3.2.5 Trivial regions

Outside of Λ. If $u(t) < u_\ell(t)$ or $u(t) \geq u_r(t)$, the terminal $u(T)$ is less than K or respectively, larger or equal to K irrespectively of what the players (market and trader) do. Hence, the final payment to the buyer is certain, the option is actually without any merit. The only hedge is $v = 0$, and $w = 0$ if $u(t) < u_\ell(t)$, or $w = D$ if $u(t) \geq u_r(t)$. If at a given time t, $v(t)$ happens to be positive, then the trader should sell it, either immediately at a cost $-C^-v$ if $t \leq t_-$, or at terminal time at a cost $-q^-v$ at worst if $t \geq t_-$. Hence it always incurs a cost $-q^-v$ (see (11)), and thus the Value function is

$$W(t,u,v) = -q^-(t)v\,, \quad \text{or} \quad W(t,u,v) = D - q^-(t)v\,,$$

depending on whether u lies to the left of Λ or to its right. For $t \geq t_-$, these regions are each covered by a single sheet (τ^-), which is the graph of the function W. Formally, this can be written as (23) with $\check{v} = 0$ and $\check{w} = 0$ or D. One may notice that this $\check{\mathcal{V}}$ still trivially satisfies (22), or at least $\check{\mathcal{V}}_t = 0 = \check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}}$.

Region $t \leq t_-$, $u \geq u_-$ Again, that region needs no big theory. There the loss resulting from keeping a positive v if $u(T) = K$ is larger than C^-v. Hence the only sensible strategy is to sell any v at once. The value is $W(t,u,v) = D - C^-v$, as already seen for larger u's. The representation (23) can be preserved as in the previous paragraph.

3.3 Equivocal manifolds

To further analyze that game, we need to distinguish whether $t_- < t_+$ or $t_- > t_+$. We choose here to show the detailed analysis for the case $t_- < t_+$ because it displays a richer set of singularities, although it is the less likely in a real life application. We shall only sketch the (more realistic) case $t_- > t_+$, stressing the main difference. (See the figures at the end.)

3.3.1 The equivocal manifold $\mathcal{E}^+ = (\tau_\ell^-) \cap (\uparrow)$

In the region $t \in [t_-, t_+]$, $u \le u_+$, the sheet (τ_ℓ^-) still exists, but not the sheet (K). We have reached (backward) $q = C^+$, therefore we may expect a positive jump manifold $(\uparrow)$. Such a 3-D manifold must join onto the sheet (τ_ℓ^-) along a 2-D junction manifold that we call $\mathcal{E}^+$.

We again call $\check{v}(t,u), \check{w}(t,u)$ the values of v and w on $\mathcal{E}^+$. This way, the boundary of the capturable states, and hence the graph of W, will still be described by (23), but now $q^+ = C^+$.

Staying on (τ^-) for a $\tau \ne \tau^-$ requires that $\mathrm{d}(Q^-\mathcal{V})/\mathrm{d}t = 0$, hence that (assuming $\xi \ge 0$)

$$-\tau^-(1+q^-)v + q^-(\tau v + \xi) + \tau v - C^+\xi = 0\,,$$

that yields

$$\xi = (\tau - \tau^-)\frac{1+q^-}{C^+ - q^-}\check{v} \tag{24}$$

which is actually non-negative. We conjecture (and will check later) that the junction will actually be with $\tau = \tau^+$, and let $\xi = \xi^+$ be the corresponding control.

The requirement that $\mathcal{E}^+$ be traversed by trajectories τ^+ implies that the dynamics are satisfied with τ^+, hence that

$$\check{v}_t + \check{v}_u\tau^+u = \tau^+\check{v} + \xi^+\,,$$
$$\check{w}_t + \check{w}_u\tau^+u = \tau^+\check{v} - C^+\xi^+\,.$$

Multiplying the first equation by C^+ and summing, we get

$$C^+\check{v}_t + \check{w}_t + \tau^+[(C^+\check{v}_u + \check{w}_u)u - (C^+ + 1)\check{v}] = 0\,.$$

We now notice that it follows from (16), that for $\varepsilon = \pm$, it holds that $Q^\varepsilon\mathcal{T} = \tau^\varepsilon Q^\varepsilon$. Thus this equation can also be written

$$Q^+[\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = 0\,. \tag{25}$$

Now, $\mathcal{E}^+$ admits the two tangent vectors

$$\begin{pmatrix} 1 \\ 0 \\ \check{v}_t \\ \check{w}_t \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ \check{v}_u \\ \check{w}_u \end{pmatrix}.$$

They must be orthogonal to ν_{τ^-} as given by (20). This yields two equations:

$$-\tau^-(q^-+1)\check{v} + q^-\check{v}_t + \check{w}_t = Q^-\check{\mathcal{V}}_t - \tau^- Q^-\mathcal{S}\check{\mathcal{V}} = 0\,, \qquad (26)$$
$$q^-\check{v}_u + \check{w}_u = Q^-\check{\mathcal{V}}_u = 0\,.$$

Multiplying the second one by $\tau^- u$ and summing and using again $\tau^- Q^- = Q^-\mathcal{T}$, we get

$$Q^-[\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = 0\,. \qquad (27)$$

The two equations (25) and (27) together can be written

$$Q[\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = 0\,,$$

and as Q is invertible, we get (22) for $\mathcal{E}^+$. *This proves assertion 2 in Theorem 3.1.*

We expect this manifold to join continuously with $\mathcal{D}$ on the boundary $u = u_+$, and as this is not a characteristic curve of that equation, it specifies uniquely its solution. But we shall also find a more explicit construction via the trajectories.

We still have to check that the junction "does not leak", or that for a $\tau < \tau^+$, the trajectory remaining on (τ^-) drifts towards the capturable states, or equivalently that $\tau = \tau^+$ minimizes the relevant hamiltonian.

For an intermediate value of ξ to be optimal, we need that at the junction, $q = C^+$ on the incoming trajectory. (We have seen that anyhow, the normal to a positive jump manifold has to have $q = C^+$.) Hence, this will be an equivocal junction, in the parlance of Isaacs. Let $\nu^+ = (n \quad p \quad C^+ \quad 1)^t$ be the normal to the (positive) jump manifold on $\mathcal{E}^+$. Isaacs main equation therefore reads

$$\max_{\xi}\min_{\tau}\{n + \tau[pu + (1+C^+)\check{v}]\} = 0\,,$$

and $\tau = \tau^+$ is indeed the minimizing τ if $[pu + (1+C^+)\check{v}]$ as computed along our trajectories τ^+ is non-positive, or equivalently, $n \geq 0$, since the hamiltonian remains zero by construction.

We shall derive ν^+ from the theory of the generalized adjoint equations developed in [2]. Let us first investigate the conditions at the boundary $u = u_+$. There $\check{v}$ and $\check{w}$ are given by (21). Let s be the time at which a trajectory reaches that boundary. The boundary of $\mathcal{D}$ and its tangent are obtained as follows:

$$\mathcal{D} \cap \{u = u_+\}: \begin{pmatrix} s \\ u_+ \\ \frac{D}{C^+ - q^-(s)} \\ -q^-\frac{D}{C^+ - q^-(s)} \end{pmatrix}, \quad \text{tangent:} \begin{pmatrix} 1 \\ 0 \\ -\frac{D\tau^-(1+q^-)}{(C^+-q^-)^2} \\ C^+\frac{D\tau^-(1+q^-)}{(C^+-q^-)^2} \end{pmatrix} \qquad (28)$$

Since that manifold is to be contained in $\mathcal{E}$, its tangent must be orthogonal to ν^+. This simply yields $n = 0$, which is therefore a terminal condition for the generalized adjoint equation. This equation reads, writing x for (t, u, v, w)

$$\dot{\nu}^+ = -\frac{\partial H}{\partial x} + \alpha(t)(\nu^+ - \nu_{\tau^-})$$

where $\alpha(t)$ must be chosen so as to maintain the singularity while integrating backward, i.e. here, $q = C^+$, hence $\dot{q} = 0$. This gives

$$\begin{aligned}
\dot{n} &= \alpha(n + \tau^-(1 + q^-)\check{v})\,, \\
\dot{p} &= (-\tau^+ + \alpha)p\,, \\
\dot{q} &= -\tau^+(1 + C^+) + \alpha(C^+ - q^-)\,, \\
\dot{r} &= 0\,.
\end{aligned}$$

The requirement that $\dot{q} = 0$ gives $\alpha = \tau^+(1 + C^+)/(C^+ - q^-) > 0$. And the differential equation for n gives $n \geq 0$, since for $n = 0$, $\dot{n} < 0$ and we integrate *backward* from $n = 0$. This provides the sign information we needed.

A final remark, useful in checking that this is a viscosity solution (to appear in a forthcoming paper) is that equation (26) shows that $Q^-\check{\mathcal{V}}_t < 0$, and that, writing that the two tangents to $\mathcal{E}^+$ are orthogonal to ν^+, we get that $Q^+\check{\mathcal{V}}_t = -n \leq 0$.

Placing (24) in the dynamics, and using the initial conditions (28), the equations for the trajectories can be integrated in closed form with $\tau = \tau^+$, leading to the closed form formulas:

$$\begin{aligned}
\tilde{u}(t,s) &= u_+ e^{\tau^+(t-s)} \quad \text{hence} \quad s = t - \tfrac{1}{\tau^+}\ln\left(\tfrac{u}{u_+}\right) \\
\check{v}(t,\tilde{u}(t,s)) = \tilde{v}(t,s) &= \tfrac{D}{u_+(C^+ - q^-(s))}\left(\tfrac{C^+ - q^-(s)}{C^+ - q^-(t)}\right)^{\frac{\tau^+ - \tau^-}{-\tau^-}} \tilde{u}(t,s) \\
\check{w}(t,\tilde{u}(t,s)) = \tilde{w}(t,s) &= -q^-(t)\tilde{v}(t,s)
\end{aligned}$$

3.3.2 The equivocal manifold $\mathcal{E}^- = (\downarrow) \cap (K)$

We now investigate the region $t \leq t_-$, $u \in [u_+, u_-]$. The situation is somewhat symmetrical to that of the preceding paragraph, as in that region the sheet (K) exists while the sheet (τ_ℓ^-) does not. We therefore expect that a negative jump manifold $(\downarrow)$ joins onto the sheet (K). We conduct a similar analysis of the junction $\mathcal{E}^-$, still calling $\check{v}(t,u)$, $\check{w}(t,u)$ the equations of that 2-D manifold, and formula (23) will still hold, but now with $q^- = C^-$, and $q^+ = (1 + c^-)K/u - 1$.

Trajectories staying on (K) must satisfy $\mathrm{d}(Q^+\mathcal{V})/\mathrm{d}t = 0$. Hence (assuming $\xi \leq 0$) by differentiation $(q^+ - C^-)\xi = 0$, thus $\xi = 0$.

We conjecture (and shall check later on) that the junction is with $\tau = \tau^-$. The fact that $\mathcal{E}^-$ be traversed by trajectories τ^- now yields

$$\check{v}_t + \check{v}_u \tau^- u = \tau^- v\,,$$
$$\check{w}_t + \check{w}_u \tau^- u = \tau^- v\,.$$

Proceeding as in the previous case, we infer that $Q^-[\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = 0$.

Writing that the two natural tangent vectors to $\mathcal{E}^-$ are orthogonal to the normal $\nu_K^t = (0 \quad -(1+q^+)v/u \quad q^+ \quad 1)$ yields the two equations $Q^+\check{\mathcal{V}}_t = 0$ and $Q^+(\check{\mathcal{V}}_u - \mathbf{1}v/u) = 0$. Multiplying the second one by $\tau^+ u$ and adding, we get $Q^+[\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = 0$. This together with the previous similar equation again yields (22). *This proves assertion 1 of Theorem 3.1.*

We expect this manifold to join continuously with $\mathcal{D}$ at $t = t_-$. This initial condition can be seen to uniquely specify $\mathcal{E}^-$ in the region $u \geq u_+ \mathrm{e}^{-\tau^-(t_- - t)}$. We call $\mathcal{E}_1^-$ the manifold thus generated. Indeed, integrating with $\tau = \tau^-$ and $\xi = 0$ yields the same formulas for $\check{v}$, $\check{w}$ as in $\mathcal{D}$, but with q^- replaced by $\tilde{q}^- := [(1+c^-)\exp(\tau^-(T-t)) - 1]$. This formula *is* indeed that of q^- in the region $t \geq t_-$, while here, $q^- = C^-$.

A direct calculation then shows that $Q^-\check{\mathcal{V}}_t < 0$. Writing that the tangents to $\mathcal{E}^-$ are orthogonal to the normal ν^- to the jump manifold again yields $n + Q^-\check{\mathcal{V}}_t = 0$, showing that our construction does give $n > 0$ which shows that $\tau = \tau^-$ indeed minimizes the hamiltonian. (The corner does not leak.)

In the region $u \leq u_+ \exp(-\tau^-(t_- - t))$, the trajectories of $\mathcal{E}^-$, if they are still generated by $\tau = \tau^-$ as we shall show, end up on the line $u = u_+$. Therefore, the terminal conditions to integrate them retrogressively will be provided by the analysis of the region $u \leq u_+$ of the next subsection. For that reason, we do not have a closed form for the trajectories of that part, $\mathcal{E}_3^-$, of the equivocal manifold. We check that indeed $\tau = \tau^-$ will be minimizing the hamiltonian.

Use again the theory of generalized adjoint equations. It yields here $\dot{n} = \alpha n$ so that n cannot change sign along a trajectory. And, as it has to be positive at the boundary, it will stay so, proving that τ^- is indeed minimizing in the hamiltonian. (It can easily be seen that $\alpha = -\tau^-(1 + C^-)/(q^+ - C^-) > 0$.)

Again we remark that on the whole equivocal junction, we have $Q^\varepsilon \check{\mathcal{V}}_t \leq 0$, $\varepsilon = \pm$, a property needed in the viscosity solution analysis.

3.4 Focal manifold $\mathcal{F} = (\downarrow) \cap (\uparrow)$

We now investigate the only region left: $t \leq t_-$, $u \leq u_+$. There, neither sheet (τ^-) nor (K) exist to construct an equivocal junction on. We will have two jump manifolds, one of each sign, joining on a 2-dimensional focal manifold $\mathcal{F}$. The theory of a similar manifold in the case of a vanilla option was introduced in [7]. A more general theory of higher dimensional focal manifolds was developed in [10]. Here, we may just notice that this manifold has to be traversed by both τ^- and τ^+ trajectories. According to the analysis provided for $\mathcal{E}^+$ and $\mathcal{E}^-$ above, this shows that necessarily $Q^\varepsilon[\check{\mathcal{V}}_t + \mathcal{T}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = 0$, for both $\varepsilon = \pm$, hence resulting in (22). *This proves assertion 3 of Theorem 3.1.*

We need to provide (22) with boundary values to uniquely specify $\mathcal{F}$. We notice that $\mathcal{E}_3^-$ satisfies the same set of coupled PDE's. We may therefore consider that we have a single set of PDE's to solve in the domain $t \le t_-$, $\in [u_\ell(t), u_+ \exp(-\tau^-(t_- - t))]$. Notice that the coefficients of this linear vector PDE are continuous. We showed in [10] that the trajectories τ^- and τ^+ are its characteristic curves (or, more classically, the characteristic curves of an equivalent scalar second order PDE). This is a Goursat problem. The solution will be specified if we give consistent boundary conditions on two such curves. Concerning the right hand boundary, we have found $\mathcal{E}_1^-$ (in closed form) on its right, and by continuity this provides our boundary condition. It remains to find boundary conditions on the trajectory $u_\ell(t)$, which is our left hand boundary.

On u_ℓ, if at any time, $\tau < \tau^+$, the state drifts outside of Λ, and the optimal strategy is to sell v at once at a cost $-C^- v$, and do nothing ($\xi = 0$) thereafter. For this strategy to drive the state to the admissible end states (provide a hedge), it is necessary that $Q^-\mathcal{V} \ge 0$. This must be maintained along the trajectory u_ℓ (which has $\tau = \tau^+$), and still allow the state to reach $\mathcal{E}^+$ at $t = t_-$. The limit trajectory thus satisfies $Q^-\mathcal{V} = 0$. Differentiating with respect to time, this yields $\xi = (\tau^+ - \tau^-)\check{v}(1 + C^-)/(C^+ - C^-)$, and we can integrate that limit trajectory backward from the boundary of $\mathcal{E}^+$. (This is the same rule as in $\mathcal{E}^+$, thus everything is smooth.)

There remains to check the signs of the time components of the normals to the jump manifolds, as we know that their being non-negative insures that the controls τ used in the construction of the manifold do minimize the relevant hamiltonian.

Let $\nu^\varepsilon = (n^\varepsilon\ p^\varepsilon\ C^\varepsilon\ 1)^t$ be the normal to the jump manifold where ε is the sign of the jump. We shall denote $\bar{\varepsilon}$ the opposite sign to ε. The components in time and v of the generalized adjoint equations applied to $\mathcal{F}$ give

$$\dot{n}^\varepsilon = \alpha^\varepsilon(n^\varepsilon - n^{\bar{\varepsilon}}) \,,$$
$$0 = -\tau^\varepsilon(1 + C^\varepsilon) + \alpha^\varepsilon(C^\varepsilon - C^{\bar{\varepsilon}}) \,,$$

thus

$$\alpha^\varepsilon = \varepsilon\tau^\varepsilon \frac{1 + C^\varepsilon}{C^+ - C^-} > 0 \,.$$

The differential equations for n^+ and n^- are to be integrated backward (not on the same trajectories, though), so that by a standard inward field argument, if $n^{\bar{\varepsilon}}$ is positive, so remains n^ε.

We now check the initial conditions for these backward integrations. We have stressed that $\check{\mathcal{V}}_t$ and $\check{\mathcal{V}}_u$ will be continuous over the whole region of integration of our PDE, and noticeably at $u = u_+$. On $\mathcal{E}^-$, we have seen that necessarily $Q^+\check{\mathcal{V}}_t = 0$. The normal ν^+ has to be orthogonal the tangent vector to $\mathcal{F}$ (and $\mathcal{E}^-$), which yields $n^+ + Q^+\check{\mathcal{V}}_t = 0$, hence $n^+ = 0$ on $u = u_+$. Integrating backwards will indeed provide $n > 0$ in $\mathcal{F}$ as long as $n^- > 0$.

On the left boundary, we have both $H = 0$ on the incoming negative jump manifold, and that its normal ν^- is orthogonal to the trajectory we specified. This yields respectively

$$n^- + \tau^-[p^- u_\ell + (1 + C^-)\check{v}] = 0\,,$$
$$n^- + \tau^+ p^- u_\ell + \tau^-(1 + C^-)\check{v} = 0\,,$$

so that we get that $p^- = 0$ and $n^- = -\tau^-(1 + C^-)\check{v} > 0$.

On the "top" boundary, at $t = t_-$, $u \in [u_\ell(t_-), u_+]$, the jump manifolds join continuously on the jump manifold towards $\mathcal{E}^+$ for the positive jump, to (τ^-) for the negative one. All components of the normal other than the n time-component are therefore continuous, and the requirement that the hamiltonian be zero provides the continuity of the first component.

Thus both n^- and n^+ are positive everywhere.

It is a simple matter to recover the controls ξ^ε on $\mathcal{F}$ according to the trajectory τ^ε considered:

$$\xi^\varepsilon = \frac{\bar{\varepsilon}\,1}{C^+ - C^-}[Q^{\bar{\varepsilon}}\check{\mathcal{V}}_t + \tau^\varepsilon Q^{\bar{\varepsilon}}(\check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}})] = \frac{\bar{\varepsilon}\,1}{C^+ - C^-}[\frac{\tau^{\bar{\varepsilon}} - \tau^\varepsilon}{\tau^{\bar{\varepsilon}}} Q^{\bar{\varepsilon}}\check{\mathcal{V}}_t].$$

The square bracket is nonpositive because, as we have seen, $Q^{\bar{\varepsilon}}\check{\mathcal{V}}_t \le 0$, so that ξ^ε indeed has the sign of ε as it should. There does not seem to be closed form formulas for that manifold.

3.5 Case $t_+ < t_-$

We only sketch some features of that case.

The dispersal manifold is similar to the previous one, holding for $t \ge t_-$, $u \in [u_\ell, \min\{u_-, u_r(t)\}]$. For $t \le t_-$, a negative jump manifold joins on the sheet (K) with an equivocal junction, involving $\tau = \tau^-$ and the "singular" control $\xi = 0$. In the region "above" the trajectory τ^- through $(t_-, u_\ell(t_-))$, i.e. $u \ge u_\ell(t_-)\exp[-\tau^-(t - t_-)]$, it is completely similar to the junction $\mathcal{E}_1^-$ of the previous case, its being given by the same formulas as $\mathcal{D}$ but with $\tilde{q}^-$ instead of q^-.

The region accounted for by the equivocal manifold $\mathcal{E}^+$ is empty. But a new case arises "below" the separating trajectory τ^-, as junction trajectories, still built with $\tau = \tau^-$, $\xi = 0$, reach the left boundary of Λ, i.e.

$u = u_\ell(t)$ before time t_-. We need therefore a boundary condition for $\mathcal{E}^-$ in that region, generating a part $\mathcal{E}_2^-$ of the equivocal manifold that comes between $\mathcal{E}_1^-$ and $\mathcal{E}_3^-$.

On the boundary u_ℓ, two conditions must hold. On the one hand, we must insure that $w(T) \geq 0$ even if $\tau = \tau^-$ up to time T. Since we are at $t < t_-$, the trader should sell its stocks as soon as the state drifts off Λ, hence making a negative jump in v at a cost $-C^- v$. We must therefore have $w + C^- v \geq 0$. On the other hand, we want to be on the sheet (K). The limiting states are thus in $\{Q^-\mathcal{V} = 0\} \cap \{Q^+\mathcal{V} = D\}$, i.e. given by formulas similar to that of $\mathcal{D}$, but this time with $q^- = C^-$: $u = u_\ell(s)$, $s \in [t_-, t_+]$,

$$\check{v}(s,u) = \frac{D}{q^+(u) - C^-}\,, \quad \check{w}(s,u) = -C^- \check{v}(s,u)\,.$$

This is indeed a τ dispersal manifold. For $\tau = \tau^+$, the trader responds with $\xi = 0$, the state leaves the above manifold on the sheet (K), and we check that indeed it remains above the sheet $Q^-\mathcal{V} = 0$ since $\mathrm{d}(Q^-\mathcal{V})/\mathrm{d}t = (1 + C^-)\tau^+ v > 0$.

From this 1-D manifold of "terminal" conditions, we integrate backwards the 2-D equivocal junction with $\tau = \tau^-$, $\xi = 0$. This whole construction can be explicitly performed with closed form formulas, allowing one to check the no-leakage condition (that τ^- actually is the minimizing control in the hamiltonian):

$$\begin{aligned}
\tilde{u}(t,s) &= u_l(s) e^{\tau^-(t-s)} \Longrightarrow s = \tfrac{\tau^+ T - \tau^- t + \ln(\frac{u}{K})}{\tau^+ - \tau^-} \\
\check{v}(t, \tilde{u}(t,s)) = \tilde{v}(t,s) &= \tfrac{D}{q^+(\tilde{u}(t,s)) - \tilde{q^-}(t,s)} \\
\check{w}(t, \tilde{u}(t,s)) = \tilde{w}(t,s) &= -\tilde{q^-}(t,s)\tilde{v}(t,s)
\end{aligned}$$

with $\tilde{q^-}(t,s) = (1 + C^-)e^{\tau^-(s-t)} - 1 < C^-$ since $\tau^-(s-t) < 0$.

The other regions bear a close resemblance to the previous case.

4 Conclusion

We have a complete description of the field of optimal trajectories, via a trajectory-wise description of the singular surfaces. The required sign checks are provided by the generalized adjoint equations. It proves the following representation theorem :

Theorem 4.1. *The value function is everywhere given by (23), or equivalently*

$$W(t,u,v) = Q^\varepsilon \check{\mathcal{V}} - q^\varepsilon v\,, \quad \varepsilon = \mathrm{sign}(\check{v}(t,u) - v)\,,$$

where $\check{\mathcal{V}}$ satisfies the pair of coupled linear PDEs (22), —or $\check{\mathcal{V}}_t = 0 = \check{\mathcal{V}}_u u - \mathcal{S}\check{\mathcal{V}}$ in the region where $\mathcal{T}$ is not defined because $q^+ = q^-$— with appropriate boundary values as discussed above.

Concerning $\mathcal{F}$ and $\mathcal{E}_3^-$, we know of no other way to actually compute them than integrating these PDEs.

In a forthcoming paper, we shall show through a detailed analysis of the discontinuities of the Value function and of its gradient that it actually is a viscosity solution of the corresponding Isaacs quasi-variational inequality.

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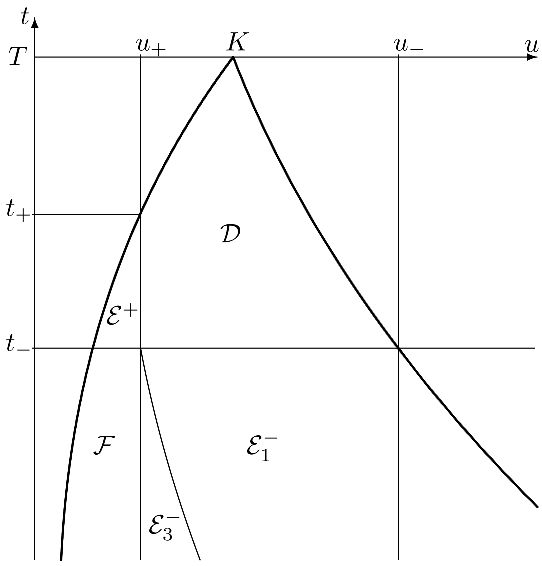

Figure 1: The various regions, $t_- < t_+$

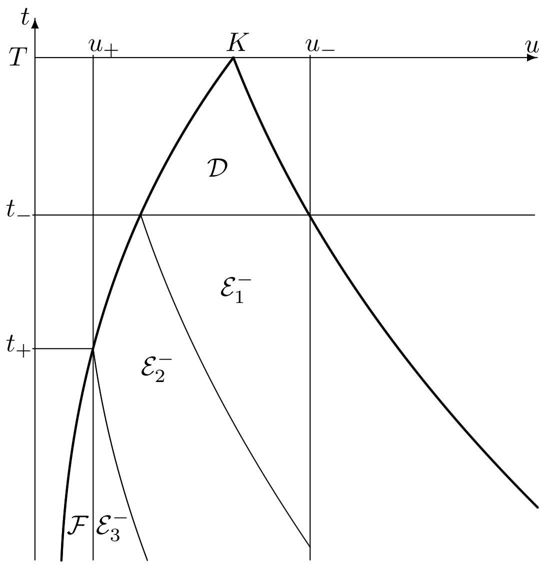

Figure 2: The various regions, $t_+ < t_-$