Evolutionarily Robust Strategies: Two Nontrivial Examples and a Theorem

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Abstract

We revisit the relationship between evolutionarily stable strategy (ESS) and related topics such as evolutionarily robust strategy (ERS) on the one hand, and stability of the replicator dynamics on the other hand, when the phenotypic set is continuous. The state of the population considered is a measure over the phenotypic set. Thus topological considerations come into play that make the situation much more difficult than for a finite phenotypic set. As a consequence, the issue of the asymptotic stability of an ERS is not settled at this time. We give one partial new result in that direction.

It has also been noticed in the literature that there is a dearth of concrete examples of mixed ESS or ERS in the literature. Actually none seems to be known if the "kernel" of the game is continuous. We provide two such examples, one a convex combination of two Dirac measures and one family with the Lebesgue measure as an ERS.

Keywords Evolutionary games, ESS, Replicator dynamics, darwinian dynamics, stability.

1 Introduction

The concept of evolutionarily stable strategy (ESS) as the core concept of Evolutionary Game Theory was introduced by Maynard-Smith and Price in [6]. The very intent of the definition is to imply that in a population in an ESS state, a small

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sub-population in a different state will die out for lack of fitness in Darwinian dynamics.

Taylor and Jonker [11] have given a mathematical foundation to that intuition via their *Replicator Dynamics*, the dynamics of phenotyes distribution if the fitness measures reproductive success as a relative rate of population growth. In the case of a finite phenotypic set, an ESS is an asymptotically stable state of the associated replicator dynamics. But analogous results for the case of a continuous phenotypic set are not proved, in general, and this remains an open conjecture.

As a matter of fact, the state of a population is a probability measure over the phenotypic set. Thus, if this set is finite, the state space is isomorphic to $\mathbb{R}^n$, while if the phenotypic set is infinite, say continuous, one ends up with a true measure dynamics. Hence there are as many concepts of asymptotic stability as there are topologies on the set of measures. We seek results in the weak topology, which seems to be the most significant one as far as the modelization is concerned.

The result is known for an evolutionarily robust strategy (ERS, a strengthening of ESS, strictly stronger in the infinite case) if it is a Dirac measure or if the underlying game is "doubly symmetric". We prove the result for the case of a global ERS strategy.

A further difficulty in investigating this topic is that there does not seem to be in the literature an example of an ERS which is not a Dirac measure. (This point itself has been raised in the literature.) We provide two very different (families of) such examples, and the analysis that substantiates the claim that they are indeed ERS.

The rest of this paper is organized as follows. Section 2 is devoted to present the definitions and some properties of ESS, ERS and related topics. In Section 3, we give a brief description of strong and weak topologies on Δ, the set of all strategies (probability measures on the phenotypic set) which will be useful in subsequent discussions. Section 4 begins with the description of replicator dynamics and its stability. At the end of this section, namely in Subsection 4.6, we prove our first result —the weak asymptotic stability of global ERS. Then, in Section 5, we present two new (families of) examples of ERS that are not Dirac measures.

2 ESS and related concepts

2.1 Definition of ESS

We investigate an evolutionary game in a classical linear framework. The *pure strategy set* of the underlying evolutionary game is a compact subset K of the Euclidean space $\mathbb{R}^d$. We denote a pure strategy using letters $x, y, z, \ldots$. A *mixed strategy* is therefore a probability measure on the Borel σ-algebra $\mathcal{B} = \mathcal{B}(K)$ of

K. The mixed strategy simplex (that is, the set of all probability measures on K) is denoted by Δ. We denote a particular mixed strategy using capital letters $P, Q, R, \ldots$. We are given a *fitness map*, a continuous function $u : K \times K \to \mathbb{R}$. Here $u(x, y)$ represents the fitness gained by the animal adopting x when compete against another animal adopting y. It will be naturally extended to $K \times \Delta$ and to $\Delta \times \Delta$ as

$$u(x, R) = \int_K u(x, y) R(\mathrm{d}y) \,, \qquad u(Q, R) = \iint_{K\times K} u(x, y) Q(\mathrm{d}x) R(\mathrm{d}y) \,.$$

Remark 1 *We shall think of a mixed strategy as a distribution of behaviours among a polymorphic population of animals, each using a fixed pure strategy. It could as well be the probability distribution of the strategies used by every animals of a monomorphic population where all individuals are random players.*

We are now ready for the definition of ESS.

Definition 1 *$P \in \Delta$ is called an evolutionarily stable strategy (ESS), if for any mutant strategy $R \neq P$, there is a $\varepsilon(R) \in (0, 1]$ such that*

$$u(P, \varepsilon R + (1 - \varepsilon)P) > u(R, \varepsilon R + (1 - \varepsilon)P) \qquad \textit{for all } 0 < \varepsilon < \varepsilon(R). \quad (1)$$

Note that (1) can also be stated as

$$\varepsilon[u(P, R) - u(R, R)] + (1-\varepsilon)[u(P, P) - u(R, P)] > 0, \qquad \text{for all } 0 < \varepsilon < \varepsilon(R). \quad (2)$$

The definition (1) is essentially by Taylor & Jonker [11], and, in view of (2), is equivalent to the original definition by Maynard Smith [7]:

$$\forall Q \in \Delta, \qquad u(Q, P) \le u(P, P) \,, \quad (3)$$

$$\forall R \neq P, \qquad [u(R, P) = u(P, P)] \Rightarrow [u(P, R) > u(R, R)] \,. \quad (4)$$

It is convenient to introduce the *best response* map $\mathbb{BR}(P)$ defined as $\mathbb{BR}(P) = \{R \mid u(R, P) = \max_Q u(Q, P)\}$. Then, $P \in \Delta$ is an ESS iff

$$P \in \mathbb{BR}(P) \text{ and } \forall R \in \mathbb{BR}(P) \setminus \{P\},\ u(R, R) < u(P, R). \quad (5)$$

It shall be convenient to introduce the notation, for any P and Q in Δ,

$$\begin{aligned} \sigma(x, P) &:= u(x, P) - u(P, P) \,, \\ \sigma(Q, P) &:= \textstyle\int_K \sigma(x, P) Q(\mathrm{d}x) = u(Q, P) - u(P, P) \,, \end{aligned} \quad (6)$$

so that condition (3) above writes $\sigma(Q, P) \le 0$, $\forall Q \in \Delta$.

Any P satisfying that condition, i.e. $P \in \mathbb{BR}(P)$, is said to be a *symmetric Nash equilibrium*, and the set of symmetric Nash strategies is denoted by Δ^{NE}.

Remark 2 *We note that $P \in \Delta^{NE}$ iff (P,P) is a Nash equilibrium of the two-person game with pure strategy set K, and payoff functions $u_1(x,y) = u(x,y)$, $u_2(x,y) = u(y,x)$.*

P is a *strict symmetric Nash equilibrium* if $\mathbb{B}\mathbb{R}(P) = \{P\}$. The set of such strategies is denoted by Δ^{SNE}. The set of ESS strategies is denoted by Δ^{ESS}. From (5), it now follows that

$$\Delta^{SNE} \subset \Delta^{ESS} \subset \Delta^{NE}. \tag{7}$$

We can also show, as in the finite case, that there always exists a symmetric Nash equilibrium; that is, $\Delta^{NE} \neq \emptyset$. But Δ^{ESS} can be empty, as the next example illustrates.

Example $K = [-1,1]$, $u(x,y) = x^2 y$. For any $x \in K$ and $R \in \Delta$,

$$u(x,R) = x^2[R],$$

where $[R] := \int_K y\, R(dy)$ is the average value of R. Now

$$\begin{aligned}
\Delta^{NE} &= \Big\{R \in \Delta \,:\, R \in \mathbb{B}\mathbb{R}(R)\Big\} \\
&= \Big\{R \in \Delta \,:\, u(x,R) \le u(R,R) \quad \text{for all } -1 \le x \le 1\Big\} \\
&= \Big\{R \in \Delta \,:\, x^2[R] \le [R]\int_{-1}^{1} z^2\, R(dz), \quad \text{for all } -1 \le x \le 1\Big\} \\
&= \Big\{R \in \Delta \,:\, x^2[R] \le [R]\int_{-1}^{1} z^2\, R(dz), \quad \text{for all } -1 \le x \le 1\Big\}
\end{aligned}$$

This suggests that we can write Δ^{NE} as the union of two disjoint sets Δ_1^{NE} and Δ_2^{NE}; where

$$\Delta_1^{NE} = \Big\{R \in \Delta \,:\, [R] = 0\Big\},$$

and

$$\begin{aligned}
\Delta_2^{NE} &= \Big\{R \in \Delta \,:\, [R] > 0 \text{ and } \int_{-1}^{1} z^2\, R(dz) = 1\Big\} \\
&= \Big\{R \in \Delta \,:\, R = \alpha\delta_{-1} + (1-\alpha)\delta_1, \quad 0 \le \alpha < \frac{1}{2}\Big\}.
\end{aligned}$$

Here δ_x denote the Dirac measure at x.
To prove $\Delta^{ESS} = \emptyset$, in view of (7), it suffices to show that no strategy $P \in \Delta^{NE}$ is

an ESS. We have only two cases to consider here. In both cases (that is when $P \in \Delta_1^{NE}$ or $P \in \Delta_2^{NE}$) $\delta_1 \in \mathbb{R}(P)$. But $u(\delta_1, \delta_1) = 1 \geq \int_K x^2 \, P(dx) = u(P, \delta_1)$. Therefore $P \notin \Delta^{ESS}$. Hence no symmetric Nash equilibrium $P \in \Delta^{NE}$ is an ESS, and $\Delta^{ESS} = \emptyset$. ∎

Nevertheless, there are many games with $\Delta^{ESS} \neq \emptyset$. We give here two almost trivial examples.

i. $K = [a, b]$, $u(x, y) = f(x) + g(y)$, where f is a continuous function on $[a, b]$ with a strict maximum at x_0, and g is any continuous function.

   Now, for any $x \in K$,

$$u(x, \delta_{x_0}) - u(\delta_{x_0}, \delta_{x_0}) = f(x) - f(x_0).$$

   Therefore δ_{x_0} is a strict Nash equilibrium, and hence an ESS. ∎

ii. $K = [0, 1]$, $u(x, y) = -xy$. Then $\mathbb{R}(\delta_0) = \Delta$ and for any $R \neq \delta_0$, and

$$u(\delta_0, R) - u(R, R) = [R]^2 > 0.$$

   Therefore δ_0 is an ESS, but not a strict Nash equilibrium. ∎

2.2 ESS and uninvadability

Definition 2 *In definition (1), let* $\varepsilon_P(R) := \max\{\varepsilon(R)\}$. *It is called the* invasion barrier *of P against R*

We define a strategy to be *uninvadable* if it has a uniform invasion barrier, i.e.

Definition 3 *(Vickers & Cannings [12]) A strategy P is* uninvadable *if* $\inf_{R \neq P} \varepsilon_P(R) > 0$.

Notation: To simplify the presentation, we use the notation

$$h_{R,P}(\varepsilon) := u(R, \varepsilon R + (1-\varepsilon)P) - u(P, \varepsilon R + (1-\varepsilon)P).$$

An uninvadable strategy is clearly an ESS. However, unlike in the finite case, the converse is not true, as the following example shows.

Example K any compact interval containing 0, $0 < a < b$, $u(x,y) = bxy - ax^4$. Since $u(x,0) = -ax^4$, δ_0 is a strict Nash equilibrium, and hence an ESS. Now, for any nonzero x in K,

$$\begin{aligned} h_{\delta_x,\delta_0}(\varepsilon) &= \varepsilon u(x,x) + (1-\varepsilon)u(x,0) \\ &= \varepsilon(bx^2 - ax^4) + (1-\varepsilon)(-ax^4) \\ &= (\varepsilon b - ax^2)x^2. \end{aligned}$$

This implies that $\varepsilon_{\delta_0}(\delta_x) = \min(1, \frac{ax^2}{b})$. Clearly this invasion barrier tends to zero with x, and so δ_0 is not uninvadable. ∎

2.3 Strong uninvadability and evolutionary robustness

For a fixed P, let $R_\varepsilon := \varepsilon R + (1-\varepsilon)P$. We first observe that $P \in \Delta$ is uninvadable iff there exists $\varepsilon_0 > 0$ such that

$$\forall R \neq P,\ \forall \varepsilon \in (0, \varepsilon_0), \quad h_{R,P}(\varepsilon) = u(R, R_\varepsilon) - u(P, R_\varepsilon) < 0\,.$$

Notice that $R - P = (1/\varepsilon)(R_\varepsilon - P)$, so that

$$h_{R,P}(\varepsilon) = u(R - P, R_\varepsilon) = \frac{1}{\varepsilon}[u(R_\varepsilon, R_\varepsilon) - u(P, R_\varepsilon)].$$

Therefore, P is uninvadable iff there exists $\varepsilon_0 > 0$ such that

$$\forall R \neq P,\ \forall \varepsilon \in (0, \varepsilon_0), \quad \sigma(P, R_\varepsilon) := u(P, R_\varepsilon) - u(R_\varepsilon, R_\varepsilon) > 0\,. \tag{8}$$

This condition says that for Q ($Q \neq P$) close to P in some sense,

$$\sigma(P, Q) > 0.$$

In order to make the concept of 'nearness' precise, we need to equip $\Delta = \Delta(K)$ with a topology (say τ). With respect to this topology τ, we can define evolutionary stability as follows.

Definition 4 *A strategy $P \in \Delta$ is called*

- locally superior *(w.r.t. τ) if there exists a τ-neighborhood G of P such that*

$$\sigma(P, Q) > 0 \quad \textit{for all } Q \in G \setminus \{P\}\,,$$

- strongly uninvadable *if it is locally superior w.r.t. the strong (variational) topology (Bomze [2]),*

- evolutionary robust *if it is locally superior w.r.t. the weak topology. (Oechssler and Riedel [9]),*
- globally evolutionarily robust *if* $\sigma(P, R) > 0$ *for all* $R \in \Delta \setminus \{P\}$.

We state the following easy theorem :

Theorem 1 *Let* $P \in \Delta$. *Then*
P *evolutionary robust* $\Longrightarrow$ P *strongly uninvadable* $\Longrightarrow$ P *is an ESS.*

Proof This is, respectively, because of the fact that a weak neighborhood is a strong neighborhood, and $\varepsilon R + (1-\varepsilon)P \to P$ strongly as $\varepsilon \to 0$. [1] ∎

But the reverse implications are not true, in general. Nevertheless these reverse implications do hold true in the finite case; that is, when K is a finite set. In this case, strong and weak topologies coincide. Furthermore, in this case, a neighborhood of P consists only of Q of the form $Q = \varepsilon R + (1-\varepsilon)P$. This is not true in the infinite case.

3 Strong and weak topologies on Δ

For the sake of completeness, we provide more details regarding strong and weak topologies on Δ. We view Δ as a subset of the linear space $\mathcal{M}$ of all finite signed measures on K.

If $\mathcal{M}$ is equipped with the *variational norm*

$$\|\mu\|_{var} = \sup_{|f|\leq 1,\, f \text{ measurable}} \left| \int_K f(x)\, \mu(dx) \right|,$$

then $\mathcal{M}$ is a Banach space. The topology generated by this norm on $\mathcal{M}$, is referred to as the *strong topology*.

We denote $\int_K f(x)\mu(dx)$ by $\langle f, \mu \rangle$. $\mu_n \to \mu$ in the strong topology iff $\langle f, \mu_n \rangle \to \langle f, \mu \rangle$ uniformly for all continuous f with $|f| \leq 1$.

For $Q, R \in \Delta$, it can be shown (see Lemma 1, p.360, Shiryaev [10]) that

$$\|Q - R\|_{var} = 2 \sup_{B \in \mathcal{B}(K)} |Q(B) - R(B)|.$$

For this reason, $R_n \to R$ in the strong topology iff $R_n(B) \to R(B)$ uniformly for $B \in \mathcal{B}(K)$. For instance, if $x \neq y$, $\|\delta_x - \delta_y\| = 2$. This implies that Δ, equipped with this topology, is not compact.

[1] This will be clearer when we define explicitly the strong and weak topologies, in the next section

On the other hand, the Banach space $(\mathcal{M}, \|\cdot\|_{var})$ is the dual space of $C^0(K)$.[2] The weak topology on probabilities is its $weak^*$ topology. Therefore, equipped with that weak topology, it is compact (and metrizable).

If R_n, R are probability measures, then $R_n \to R$ in the weak topology iff $\langle f, R_n\rangle \to \langle f, R\rangle$ for every measurable $f \in C(K)$ with $|f| \leq 1$. It can be shown that $R_n \to R$ in the weak topology iff $R_n(B) \to R(B)$ for all $B \in \mathcal{B}(K)$ with $R(\partial B) = 0$.[3]

There are various (equivalent, of course) metrics which generate the weak topology on Δ. One is the *Prohorov metric*:

$$\rho(Q,R) = \\ \inf\{\varepsilon > 0 \mid Q(C) \leq R(C^\varepsilon) + \varepsilon,\ R(C) \leq Q(C^\varepsilon) + \varepsilon, \text{for all closed } C \subset K\},$$

where $C^\varepsilon = \{x \in K \mid \inf_{y\in K} |y - x| < \varepsilon\}$.

Another metric generating the weak topology is

$$d(Q,R) = \sup\{|\langle f, Q - R\rangle| \mid f \text{ Lipschitz continuous}, \|f\|_\infty + L(f) \leq 1\},$$

where $\|f\|_\infty = \sup_{x\in K} |f(x)|$ and $L(f) = \sup_{x\neq y} \frac{|f(x)-(f(y)|}{|x-y|}$.

Yet another one is defined using the set $M(Q,R)$ of measures over $K \times K$ whose marginals are Q and R respectively, as

$$d^2(Q,R) = \inf_{\mu\in M(Q,R)} \int_{K\times K} \|x-y\|^2\, \mu(dx,dy)\,.$$

For more details on strong and weak convergence of probability measures, we refer to Shiryaev [10].

4 Replicator dynamics

4.1 Definition and existence

Let $Q(t) \in \Delta$ be the population state at time t. For a given state Q, the difference between the average fitness of a subpopulation B (a Borel subset of K) and the population average fitness is

$$\frac{1}{Q(B)} \int_B u(y,Q)\, Q(dy) - u(Q,Q) = \frac{1}{Q(B)} \int_B \sigma(y,Q)\, Q(dy).$$

[2] As usual, $C^0(K)$ denotes the space of continuous real valued functions on K, equipped with uniform norm

[3] ∂B is the boundary of B

The replicator equation, as introduced in [11], now reads

$$\dot{Q}(t)(B) = \int_B \sigma(y, Q(t))\, Q(t)(dy) \quad B \in \mathcal{B}(K).$$

In order to make the replicator equation look simpler, we introduce the notation:

$$F(Q)(B) := \int_B \sigma(y, Q)\, Q(dy) \quad B \in \mathcal{B}(K).$$

That is $F(Q)$ is the measure which is absolutely continuous w.r.t. Q (denoted $F(Q) \ll Q$) with the Radon-Nikodym derivative $\frac{dF(Q)}{dQ} = \sigma(\cdot, Q)$.

The replicator dynamics can now be written as

$$\dot{Q}(t) = F(Q(t)). \tag{9}$$

Now, for any measurable f with $|f| \leq 1$, and for $Q, R \in \Delta$,

$$\begin{aligned}
|\langle f, F(Q) - F(R)\rangle| &= |\langle f\sigma(\cdot, Q), Q\rangle - \langle f\sigma(\cdot, R), R\rangle| \\
&\leq |\langle f\sigma(\cdot, Q), Q - R\rangle| + |\langle f[\sigma(\cdot, Q) - \sigma(\cdot, R)], R\rangle| \\
&\leq \|\sigma\|_\infty \|Q - R\|_{var} + \|u\|_\infty \|Q - R\|_{var} + |\langle f[u(Q, Q) - u(R, R)], R\rangle| \\
&\leq 3\|u\|_\infty \|Q - R\|_{var} + |u(Q, Q) - u(R, R)| \\
&= 3\|u\|_\infty \|Q - R\|_{var} + |u(Q, Q - R)| + |u(Q - R, R)| \\
&\leq 5\|u\|_\infty \|Q - R\|_{var}.
\end{aligned}$$

This yields

$$\|F(Q) - F(R)\|_{var} \leq 5\|u\|_\infty \|Q - R\|_{var}.$$

In a similar fashion, one can show that the map F is Lipschitz Continuous (w.r.t. the strong topology) on any bounded ball of the space $\mathcal{M}$ of all finite signed measures (Lipschitz constant depends on the radius of the ball). Therefore F is locally Lipschitz continuous on the Banach space $(\mathcal{M}, \|\cdot\|_{var})$, and hence the replicator dynamics admits a unique solution, for each initial distribution $Q(0)$.

Furthermore,

$$\frac{d}{dt}\int_K Q(t)(dy) = \int_K \dot{Q}(t)(dy) = \int_K F(Q(t))(dy) = \sigma(Q(t), Q(t)) = 0.$$

As a result, the simplex Δ of all probability measures is invariant under the replicator dynamics. Thus, for any initial condition $Q \in \Delta$, the replicator dynamics admits a unique solution $Q(\cdot)$ defined (and stays in Δ) for all times t with $Q(0) = Q$.

4.2 Stationary states of the replicator dynamics

Let Δ^0 denote the set of all stationary points of the replicator dynamics in Δ. Now $Q \in \Delta^0$ iff $F(Q)(B) = 0$ for all B iff $\sigma(\cdot, Q) = 0$ a.e.(Q). That is, $\sigma(y, Q) = 0$ except on a Borel set $N \subset K$ with $Q(N) = 0$. This implies that,

$$\Delta^0 = \{Q \in \Delta \,:\, \sigma(\cdot, Q) = 0 \text{ a.e.}(Q)\}.$$

For every $x \in K$,

$$\frac{\mathrm{d}F(\delta_x)}{\mathrm{d}\delta_x} = \sigma(\cdot, x) = 0 \quad \text{a.e. } (\delta_x),$$

and so, as in the finite case, Δ^0 contains all pure strategies. It now follows that:

Theorem 2

i. $\{\delta_x \,:\, x \in K\} \cup \Delta^{NE} \subset \Delta^0$,

ii. $\Delta^{NE} \cap \mathrm{int}(\Delta) = \Delta^0 \cap \mathrm{int}(\Delta)$.

4.3 Lyapunov stable states of replicator dynamics

A state Q is Lyapunov stable, if any trajectory starting "nearby" Q stays "nearby" Q. Hence stability crucially depends on the topology we use on Δ.

We now prove that any Lyapunov stable state (w.r.t. weak topology) is a Nash equilibrium.

Theorem 3 *If $Q \in \Delta$ is Lyapunov stable under (9) w.r.t the weak topology, then $Q \in \Delta^{NE}$.*

Proof If $Q \notin \Delta^{NE}$, then there exists a pure strategy $x \in K$ such that

$$\sigma(x, Q) = u(x, Q) - u(Q, Q) > 0.$$

Since in the weak topology, $r \mapsto \sigma(x, R)$ is continuous, this implies that there exists a weak neighborhood G of Q and $\delta > 0$ such that

$$\sigma(x, R) \geq \delta \quad \text{for all } R \in G. \tag{10}$$

For $\varepsilon > 0$ small enough, the strategies $Q_\varepsilon(0) := \varepsilon\delta_x + (1 - \varepsilon)Q \in G$. Now,

$$Q_\varepsilon(t)(x) = Q_\varepsilon(0)(x)\mathrm{e}^{\int_0^t \sigma(x,Q(s))\, ds} = \varepsilon \mathrm{e}^{\int_0^t \sigma(x,Q_\varepsilon(s))\, ds}$$

is the trajectory emanating from $Q_\varepsilon(0)$. This, in view of (10), contradicts the Lyapunov stability of the state Q. ∎

4.4 Limit states of replicator dynamics

Theorem 4 *If there is an interior trajectory $Q(t)$ with $\lim_{t\to\infty} Q(t) = P \in \mathrm{int}(\Delta)$ (in the weak topology), then $P \in \Delta^{NE}$.*

Proof If $Q \notin \Delta^{NE}$ then, as in the proof of Theorem 3, there exist a pure strategy $x \in K$, a weak neighborhood G of P and $\delta > 0$ such that

$$\sigma(x, R) \geq \delta \quad \text{for all } R \in G.$$

This implies that, by continuity, that there exists an open set I containing x, with $P(I) \neq 0$ (because P is an interior point), such that

$$F(R)(I) = \int_I \sigma(y, R)\, R(dy) \geq \frac{\delta}{2} P(I) \quad \text{for all } R \in G,$$

Hence, since for some t_0 and $t \geq t_0$ $Q(t) \in G$,

$$Q(t)(I) = Q(0)(I) + \int_0^t F(Q(s))\, \mathrm{d}s \geq Q(t_0)(I) + \frac{\delta}{2}(t - t_0)P(I)$$

and $Q(t)(I)$ diverges. This contradiction proves the theorem. ∎

4.5 Asymptotically stable states of replicator dynamics and locally superior strategies

One would like to prove, as in the finite case, that an evolutionary robust strategy (that is, locally superior w.r.t. the weak topology) P is Lyapunov stable and attracting in the weak topology (henceforth called *weakly asymptotically stable*). The main difficulty here is that the analogue of the Lyapunov function used in the finite case, is not continuous in the weak topology. More precisely, the function[4]

$$V_P(Q) := \begin{cases} \int_K \ln(\frac{\mathrm{d}P}{\mathrm{d}Q})\mathrm{d}P & \text{if } P \ll Q \\ +\infty & \text{otherwise} \end{cases} \tag{11}$$

is not well behaved w.r.t. weak topology Nevertheless, one can establish that this map is well behaved along the trajectories of the replicator dynamics (9). We use the next lemma to establish this.

Lemma 1 *Let $Q(\cdot)$ solve the replicator equation (9) with initial state $Q(0)$. Then, for every $t > 0$, $Q(t) \approx Q(0)$ [5] and*

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}Q(0)} = \mathrm{e}^{\int_0^t \sigma(\cdot, Q(s))\, \mathrm{d}s}.$$

[4] $V_P(Q) = \int_K log(\frac{dP}{dQ}\, dP)$ is the Kullback-Leibler distance between P and Q

[5] $Q(t) \approx Q(0)$ means that $Q(t) \ll Q(0)$ and $Q(0) \ll Q(t)$

Proof See Bomze [3]. ∎

One also has the following result :

Theorem 5

i. $V_P(Q) \geq 0$ *with equality iff* $Q = P$.

ii. *If* $V_P(Q(0)) < \infty$*, then, for all* $0 < t < \infty$*,* $V_P(Q(t)) < \infty$ *and*

$$\frac{d}{dt} V_P(Q(t)) = -\sigma(P, Q(t)).$$

Proof Clearly $V_P(P) = 0$. Let S be the support of P. For $Q \neq P$,

$$\begin{aligned} V_P(Q) &= -\textstyle\int_S \ln(\frac{\mathrm{d}Q}{\mathrm{d}P})\,\mathrm{d}P \\ &> -\textstyle\int_S (\frac{\mathrm{d}Q}{\mathrm{d}P} - 1) \\ &= 1 - Q(S) \\ &\geq 0. \end{aligned}$$

To prove (ii), let $V_P(Q(0)) < \infty$. For $0 < t < \infty$, by the above lemma,

$$\begin{aligned} \ln(\tfrac{\mathrm{d}P}{\mathrm{d}Q(t)}) &= \ln \tfrac{\mathrm{d}P}{\mathrm{d}Q(0)} + \ln \tfrac{\mathrm{d}Q(0)}{\mathrm{d}Q(t)} \\ &= \ln \tfrac{\mathrm{d}P}{\mathrm{d}Q(0)} - \textstyle\int_0^t \sigma(\cdot, Q(s))\,\mathrm{d}s. \end{aligned}$$

This equality of functions holds for $y \in K$ a.e (Q(0)). This implies that

$$V_P(Q(t)) = V_P(Q(0)) - \int_0^t \sigma(P, Q(s))\,\mathrm{d}s.$$

This gives the required result. ∎

Because of the lack of continuity of this Lyapunov function, the stability result for the continuous case has not been settled completely. Nevertheless, we can prove the stability in some special cases, as the next two theorems show.

Theorem 6 (Oechssler & Riedel [9]) *If* $u(x,y) = u(y,x)$*,* $\forall x, y \in K$ *(that is, if the game is doubly symmetric), then an evolutionarily robust strategy is weakly asymptotically stable.*

Proof Let $P \in \Delta$ be an evolutionarily robust strategy, and G a weak neighborhood of P such that

$$u(P,Q) - u(Q,Q) > 0 \quad \forall Q \in G \setminus \{P\}.$$

Consider the function

$$W(Q) = u(P,P) - u(Q,Q) = u(P,P) - u(Q,P) + u(P,Q) - u(Q,Q).$$

Therefore, $W(Q) \geq 0$ for all $Q \in G$, with equality iff $Q = P$. In addition, W is weakly continuous. Since the weak topology is compact, it is enough to prove that, for every trajectory $Q(\cdot)$ in G, the derivative of $t \mapsto W(Q(t))$ is negative.

$$\begin{aligned} -\tfrac{\mathrm{d}}{\mathrm{d}t}W(Q(t)) &= \tfrac{\mathrm{d}}{\mathrm{d}t}u(Q(t),Q(t)) \\ &= 2u(F(Q(t)),Q(t)) \\ &= 2\int_K u(y,Q(t))\sigma(y,Q(t))\,Q(t)(\mathrm{d}y) \\ &= 2\int_K \sigma^2(y,Q(t))\,Q(t)(dy) \end{aligned}$$

This always positive, as long as the trajectory stays inside G and $Q(0) \neq P$. ▮

Theorem 7 *Let $P = \delta_x$ be an ERS. Then P is a stable state (in the weak topology) of the replicator dynamics. Furthermore, $Q(t) \to P$ (in the weak topology) as $t \to \infty$, whenever $Q(0)$ is weakly near P and $Q(0)(\{x\}) > 0$.*

Proof We have

$$\dot{Q}(t)(x) = \sigma(x, Q(t))Q(t)(x),$$

and hence

$$Q(t)(x) = Q(0)(x)\mathrm{e}^{\int_0^t \sigma(x,Q(s))\,ds}.$$

Now let $P = \delta_x << Q(0)$, since $Q(0)(x) > 0$. If $Q(0)$ is close to δ_x, then $\sigma(x, Q(0)) > 0$ and hence $t \mapsto Q(t)(x)$ is initially (strictly) increasing. As a result, $Q(t)$ becomes (weakly) closer to δ_x than $Q(0)$, and therefore $\sigma(x, Q(t)) > 0$ for all t.

Now, since $Q(t)(x) \leq 1$, $\forall t$, we must have

$$\sigma(x, Q(t)) \to 0 \quad \text{as } t \uparrow \infty.$$

Since Δ is compact in the weak topology, any trajectory $t \mapsto Q(t)$ must have limit points. Any such limit point R (this has to be near δ_x, by above arguments) satisfies $\sigma(x, R) = 0$. This can happen only if $R = \delta_x$. This implies that if we start with a $Q(0)$ weakly close to δ_x, then $Q(t)$ converges weakly to δ_x as $t \to \infty$. ▮

4.6 Stability of global evolutionarily robust strategies

We establish now a new result concerning global ERS.

Theorem 8 *Let* $P \in \Delta$, $\Omega_P = \{R \in \Delta : \sigma(P,R) = 0\}$, *and* $\Sigma_P = \{R \in \Delta : \sigma(P,R) \geq 0\}$. *Let* $Q(\cdot)$ *be a trajectory of the replicator dynamics (9) with* $V_P(Q(0)) < \infty$. *If, in addition,* $Q(t) \in \Sigma_P$ *for all* t *large enough, then,* Ω_P *contains all the weak* ω*-limit points of the trajectory* $Q(t)$.

Proof Let $t_0 \geq 0$ be such that

$$Q(t) \in \Sigma_P \quad \forall t \geq t_0.$$

From the proof of Theorem 5, it follows that, for $t \geq t_0$

$$V_P(Q(t)) = V_P(Q(t_0)) - \int_{t_0}^{t} \sigma(P, Q(s))\, \mathrm{d}s.$$

In particular

$$\sup_{t \geq t_0} \left[\int_{t_0}^{t} \sigma(P, Q(s))\, \mathrm{d}s \right] \leq V_P(Q(t_0)) < \infty.$$

If, after time t_0, the trajectory $Q(\cdot)$ lies in Σ_P, then the map

$$[t_0, \infty) \ni t \mapsto \int_{t_0}^{t} \sigma(P, Q(s))\, \mathrm{d}s$$

is nondecreasing, and hence has to converge to a finite number, as $t \to \infty$. Since

$$s \mapsto \sigma(P, Q(s))$$

is Lipschitz continuous, it follows that

$$\lim_{s \to \infty} \sigma(P, Q(s)) = 0.$$

Now

$$R \mapsto \sigma(P, R)$$

is weakly continuous, and hence, for any weak limit point Q of the trajectory $Q(\cdot)$,

$$\sigma(P, Q) = 0.$$

∎

The next result is an immediate consequence of Theorem 8, since then $\Omega_P = \{P\}$ and $\Sigma_P = \Delta$:

Theorem 9 *Let* $P \in \Delta$ *be a global evolutionarily robust strategy. Let* $Q(\cdot)$ *be a trajectory of the replicator dynamics (9) with* $V_P(Q(0)) < \infty$. *Then, as* $t \to \infty$, *the trajectory* $Q(t)$ *converges weakly to* P.

5 Examples

In any area of mathematical research, the importance of ample analytical examples cannot be overemphasized. To the best of our knowledge, there is no nontrivial (that is, non-Dirac) examples of evolutionary robust strategies, in the literature, at least with a continuous fitness function u. We provide here, two such examples: one is a convex combination of Diracs, and the other one has density w.r.t. Lebesgue measure.

5.1 Example 1

We take $K = [0, 1]$. Let $\lambda \in (0, \infty)$, and $u(x, y) = \max\{x - y, \lambda(y - x)\}$. For convenience, let $\alpha = \frac{\lambda}{1+\lambda}$, $\beta = \frac{1}{1+\lambda}$.[6] Let δ_0 and δ_1 denote the Dirac measures concentrated at the points $\{0\}$ and $\{1\}$ respectively.

We claim the following :

Theorem 10 *The strategy $P = \alpha\delta_0 + \beta\delta_1$ is globally superior.*

Proof The rest of this subsection is devoted to the proof of that theorem.

For any $0 \le x \le 1$,

$$u(x, 0) = x, \quad \text{and} \quad u(x, 1) = \lambda(1 - x).$$

Therefore, for $P = \alpha\delta_0 + \beta\delta_1$ and $0 \le x \le 1$,

$$\begin{aligned} u(x, P) &= \alpha u(x, 0) + \beta u(x, 1) \\ &= \alpha x + \beta\lambda(1 - x) \\ &= \alpha. \end{aligned}$$

This implies that $P \in \Delta^{NE}$ and $BR(P) = \Delta$; that is, all strategies gain the same fitness α against P. Our aim is to show that

$$\sigma(P, R) = u(P, R) - u(R, R) > 0 \qquad \forall R \in \Delta \setminus \{P\}. \tag{12}$$

Since for each $0 \le y \le 1$, the map $x \mapsto u(x, y)$ is convex, it follows that, for every $R \in \Delta$, the map $x \mapsto u(x, R)$ is also convex. As a result, the function

$$[0, 1] \ni x \mapsto u(x, R)$$

[6] Notice that it would be equivalent to start with two positive numbers α and $\beta = 1 - \alpha$ and to set $u(x, y) = \max\{\alpha(y - x), \beta(x - y)\}$

attains its maximum at either $x = 0$ or $x = 1$.

We denote $R(\{x\})$ by $R(x)$. We also denote the "average value" of R as

$$[R] := \int_{[0,1]} x\, R(\mathrm{d}x).$$

For $0 \leq x \leq 1$,

$$u(0, x) = \lambda x, \quad \text{and} \quad u(1, x) = 1 - x.$$

This gives

$$u(0, R) = \lambda[R], \quad \text{and} \quad u(1, R) = 1 - [R].$$

For this reason, $u(0, R) = u(1, R)$ (resp. $u(0, R) < u(1, R)$, $u(0, R) > u(1, R)$) iff $[R] = \beta$ (resp. $[R] < \beta$, $[R] > \beta$).

It is also true that

$$[R] \leq 1 - R(0), \quad \forall R \in \Delta,$$

with equality iff $R(0) + R(1) = 1$.

We prove (12) by considering several cases.

Case 1: $R(0) > \alpha$.
In this case, $\alpha < 1 - [R]$, and so $u(0, R) < u(1, R)$. Because of this, $u(\cdot, R)$ attains maximum at $x = 1$. Now

$$\begin{aligned} u(R, R) &= R(0)u(0, R) + \textstyle\int_{(0,1]} u(x, R)\, R(\mathrm{d}x) \\ &\leq R(0)u(0, R) + (1 - R(0))u(1, R) \\ &< \alpha u(0, R) + \beta u(1, R) \\ &= u(P, R). \end{aligned}$$

Case 2: $R(0) = \alpha$.
For $R \neq P$, we must have $R((0, 1)) > 0$, and hence $R(1) < \beta$. Furthermore $[R] < \beta$. Therefore, $u(0, R) < u(1, R)$. The convexity of $u(\cdot, R)$ now yields, for $0 < x < 1$,

$$\begin{aligned} u(x, R) &\leq (1 - x)u(0, R) + xu(1, R) \\ &< u(1, R). \end{aligned}$$

This implies that

$$\begin{aligned} u(R, R) &= R(0)u(0, R) + \textstyle\int_{(0,1]} u(x, R)\, R(\mathrm{d}x) \\ &< R(0)u(0, R) + (1 - R(0))u(1, R) \\ &= \alpha u(0, R) + \beta u(1, R) \\ &= u(P, R). \end{aligned}$$

Case 3: $R(0) < \alpha$ and $R(1) > \beta$.
In this case, $[R] > \beta$. Hence $u(0,R) > u(1,R)$, and so $u(\cdot,R)$ attains maximum at $x = 0$. Now

$$\begin{aligned} u(R,R) &= R(1)u(1,R) + \textstyle\int_{[0,1)} u(x,R)\,R(\mathrm{d}x) \\ &\le R(1)u(1,R) + (1-R(1))u(0,R) \\ &= u(1,R) + (1-R(1))[u(0,R) - u(1,R)] \\ &< u(1,R) + \alpha[u(0,R) - u(1,R)] \\ &= \alpha u(0,R) + \beta u(1,R) \\ &= u(P,R). \end{aligned}$$

Case 4: $R(0) < \alpha$ and $R(1) = \beta$.
In this case, $R((0,1)) > 0$ and $[R] > \beta$. This yields $u(0,R) > u(1,R)$ and $x = 0$ is the maximizer of $u(\cdot,R)$. Now, as in Case 2, $u(x,R) < u(0,R)$ for $0 < x < 1$, and

$$\begin{aligned} u(R,R) &= R(1)u(1,R) + \textstyle\int_{[0,1)} u(x,R)\,R(\mathrm{d}x) \\ &< R(1)u(1,R) + (1-R(1))u(0,R) \\ &= \alpha u(0,R) + \beta u(1,R) \\ &= u(P,R). \end{aligned}$$

Case 5: $R(0) < \alpha$ and $R(1) < \beta$.
Define the numbers $x_0 = x_0(R)$, $x_1 = x_1(R)$ as

$$\begin{aligned} x_0 &:= \inf\{x \in [0,1] \,:\, R([0,x]) \ge \alpha\}, \\ x_1 &:= \sup\{x \in [0,1] \,:\, R([x,1]) \ge \beta\}. \end{aligned}$$

Clearly $x_0 > 0$ and $x_1 < 1$. Furthermore, $R([0,x_0]) \ge \alpha$ and $R([x_1,1]) \ge \beta$. However, we have more. More precisely $x_0 \le x_1$. To see this, let $x_0 > x_1$. Now for $x_0 > x > x_1$,

$$R([0,x]) < \alpha \quad \text{and } R([0,x)) > \alpha.$$

This is a contradiction. Thus $0 < x_0 \le x_1 < 1$.

We first consider the case $x_0 < x_1$. In this case, we must have

$$R([0,x_1]) = \alpha \quad \text{and} \quad R([x_1,1]) = \beta,$$

and hence, $R((x_0,x_1)) = 0$.

We now define

$$a := \int_{[0,x_0]} x\, R(\mathrm{d}x) \quad \text{and} \quad b := \int_{[x_1,1]} (1-x)\, R(\mathrm{d}x).$$

Note that a and b cannot vanish at the same time ($a = b = 0$ would imply that $1 = R(0) + R(1) < \alpha + \beta = 1$, a contradiction). Moreover, for $x \in (x_0, x_1)$, we have

$$\begin{aligned} u(x,R) &= \textstyle\int_{[0,x_0]} (x-y)\, R(\mathrm{d}y) + \int_{[x_1,1]} \lambda(y-x)\, R(\mathrm{d}y) \\ &= \textstyle\int_{[0,x_0]} (x-y)\, R(\mathrm{d}y) + \lambda \int_{[x_1,1]} [(1-x)-(1-y)]\, R(\mathrm{d}y) \\ &= xR([0,x_0]) + \lambda(1-x)R([x_1,1]) - a - \lambda b \\ &= \alpha x + \lambda\beta(1-x) - a - \lambda b \\ &= \alpha - a - \lambda b. \end{aligned}$$

In particular,

$$u(x_0,R) = u(x_1,R) = \alpha - a - \lambda b.$$

Similar calculations also yield

$$u(0,R) = \alpha + \lambda a - \lambda b, \quad u(1,R) = \alpha - a + b.$$

We now use the convexity of $u(\cdot,R)$ to get

$$u(x,R) \le \begin{cases} u(0,R) - \frac{x}{x_0}[u(0,R) - u(x_0,R)] & ; \quad 0 \le x \le x_0 \\ u(1,R) - \frac{1-x}{1-x_1}[u(1,R) - u(x_1,R)] & ; \quad x_1 \le x \le 1. \end{cases}$$

Integrating $u(\cdot,R)$ w.r.t. the probability measure R, using the above calculations, yield

$$\begin{aligned} u(R,R) &\le \alpha u(0,R) + \beta u(1,R) - \tfrac{a}{x_0}[u(0,R) - u(x_0,R)] - \\ &\qquad \tfrac{b}{1-x_1}[u(1,R) - u(x_1,R)] \\ &= u(P,R) - \tfrac{a}{x_0}(1+\lambda)a - \tfrac{b}{1-x_1}(1+\lambda)b \\ &= u(P,R) - (1+\lambda)[\tfrac{a^2}{x_0} + \tfrac{b^2}{1-x_1}] \\ &< u(P,R). \end{aligned}$$

Finally, when $x_0 = x_1$, we let $r_0 := R(x_0)$, and let $\gamma \in [0,1]$ such that

$$R([0,x_0)) = \alpha - \gamma r_0, \quad \text{and} \quad R((x_1,1]) = \beta - (1-\gamma) r_0.$$

Define now,

$$a = \int_{[0,x_0)} x\, R(dx) + x_0 \gamma r_0, \quad \text{and} \quad b = \int_{(x_0,1]} (1-x)\, R(dx) + (1-x_0)(1-\gamma) r_0.$$

Now we can proceed as above to complete the proof. ∎

5.2 Example 2

Let again $K = [0, 1]$ and P, the Lebesgue measure on $[0, 1]$. For $n = 1, 2, \ldots$, let

$$f_n(x) := \cos(n\pi x).$$

Let $a_n > 0$ for all n, and $\sum_{n=1}^{\infty} a_n < \infty$, g be any continuous function with $\langle g, P\rangle = 0$. Define

$$u(x, y) := g(y) - \sum_{n=1}^{\infty} a_n f_n(x) f_n(y).$$

We claim the following

Theorem 11 *The lebesgue measure P is a globally superior strategy.*

Proof The rest of this subsection is devoted to the proof of that theorem.

Clearly $\langle f_n, P\rangle = 0$ for all n, and so $u(P, R) = \langle g, R\rangle$ for every $R \in \Delta$. Now

$$u(R, R) = \langle g, R\rangle - \sum_{n=1}^{\infty} a_n |\langle f_n, R\rangle|^2.$$

This gives

$$u(P, R) - u(R, R) = \sum_{n=1}^{\infty} a_n |\langle f_n, R\rangle|^2.$$

This is obviously nonnegative, and vanishes iff

$$\langle f_n, R\rangle = 0 = \langle f_n, P\rangle \quad \forall n = 1, 2, \ldots.$$

This can happen iff

$$\langle f, R\rangle = \langle f, P\rangle$$

for all continuous functions f on $[0, 1]$ (because any continuous function can be approximated pointwise by a finite linear combination of f_n's). Therefore $R = P$.
∎

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