Geometry of Optimal Paths Around Focal Singular Surfaces in Differential Games

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Abstract

We investigate a special type of singularity in non-smooth solutions of first order partial differential equations, with an emphasis on Isaacs'equation. This type, called *focal manifold*, is characterized by incoming trajectory fields on the two sides and a discontinuous gradient. We provide a complete set of constructive equations under various hypotheses on the singularity, culminating with the case where no a priori hypothesis on its geometry is known, and where the extremal tajectory fields need not be collinear. We show two examples of differential games exhibiting non collinear fields of extremal trajectories on the focal manifold, one with transversal approach and one with tangential approach. (MSC 35F99, 49L99, 91A23.)

1 Introduction

Nonsmooth solutions to HJBI equations (and generally, to nonlinear first-order PDEs) have several interesting singularities representing singular trajectories in differential games. One of such singularities is the so-called focal surface, which is approached by regular characteristics (trajectories) from both sides of the surface. The incoming fields may be either tangential (the so called "regular case", this was the original form discovered [2]) or transverse (the "singular case", predicted in [3] and exhibited in [4]). Such surfaces arise in a number of applied problems and have important theoretical value. They differ from "universal surfaces" of [1] in that the gradient of the solution is discontinuous.

The theory of such surfaces is developed in [3] and [5] for the case of the collinear velocities corresponding to different sides of the surface. In [5], p.77, it was noticed that in generic case these velocities are not collinear and the construction algorithm is given for the case of a focal surface-hyperplane known before hand for reasons of symmetry.

Untill recently, no example with non collinear velocities was known. A first instance was found in [6]. The formulation of the corresponding game problem is given in section 6 below. Section 7 represents a non-linear Hamiltonian satisfying the necessary conditions in [5] for (non-collinear) focal surface.

Finding an explicit construction scheme for such singularities has been standing as an open problem ever since a construction has been known for other singular manifolds of co-dimension 1 [3], that is at least for 26 years. In section 5, we give a solution to this problem, even if the properties of the trajectory fields generated by such Hamiltonians require further investigation.

2 General Description of a Focal Surface

We consider here so-called singular focal surface, which consists of singular trajectories and is approached by regular characteristics (trajectories) from both sides of the surface. The incoming fields may be either tangential (so called "regular case", this was the original form discovered [2]) or transverse (the "singular case, predicted in [3] and exhibited in [4]). Such surfaces arise in a number of applied problems and have important theoretical value. They differ from "universal surfaces" of [1] in the discontinuity of the gradient of the solution.

We will denote a focal $(n-1)$-dimensional surface as Γ and its n-dimensional half-neighborhoods as D_0 and D_1. As stated above, approaching trajectories are either tangent to the surface from both sides or transversal to it. We will distingwish these two cases, though the intermediate case also is possible when tangency takes place in one side and transversality in the other. The theory of the latter case is just a combination of two main cases.

We write the nonlinear first order equation for the whole state space as

$$F(x,p)=0, \quad x\in\Omega\subset\mathbb{R}^n \quad \left(p=\frac{\partial V}{\partial x}\right)$$

and treat it as an abstract mathematical PDE whose continuous solution $V(x)$ is understood in viscosity sense, or as the Bellman-Isaacs equation for the value function $V(x)$ of a differential game. Writing the equation in this form we assume that the time variable may be one of the components of the vector x, so that an equation of the form

$$\frac{\partial V}{\partial t}+H(x,t,p)=0$$

is also accounted for.

It is convenient to use different notations for the game value, its gradient and the Hamiltonian in the two half-neighborhoods D_0 and D_1 of Γ :

$$D_0: \quad V(x), \quad p=\frac{\partial V}{\partial x}, \quad F(x,p)=0$$

$$D_1: \quad W(x), \quad q=\frac{\partial W}{\partial x}, \quad G(x,q)=0$$

Due to continuity of the solution the vector $p-q$ is a normal to Γ. It is known that for the tangency case the following necessary conditions hold at Γ on the D_0 and D_1 sides:

$$D_0-\text{side}: \quad F(x,p)=0, \quad \langle F_p(x,p),p-q\rangle=0,$$

$$D_1-\text{side}: \quad G(x,q)=0, \quad \langle G_q(x,q),p-q\rangle=0$$

For a focal surface with the transversal approach one has another four necessary conditions:

$$D_0-\text{side}: \quad F^+(x,p)=0, \quad F^-(x,p)=0,$$

$$D_1-\text{side}: \quad G^+(x,q)=0, \quad G^-(x,q)=0$$

Here Hamiltonians are supposed to have the form

$$F=\min[F^+,F^-], \quad G=\min[G^+,G^-]$$

with Bellman-Isaacs equation having the superindex plus, i.e. one has

$$F^+(x,p)=0,\ x\in D_0, \quad G^+(x,q)=0,\ x\in D_1$$

The switching condition for the minimizing player has the form $F^+-F^-=0$ in D_0 and $G^+-G^-=0$ in D_1.

3 Singular Surface – Hyperplane

Let the surface Γ be the coordinate hyperplane $x_n = 0$ in the n-dimensional state space. We use for the shortened $(n-1)$-dimensional vectors the following notation:

$$\bar{x} = (x_1, x_2, \ldots, x_{n-1}), \quad \bar{p} = (p_1, p_2, \ldots, p_{n-1}), \quad \bar{q} = (q_1, q_2, \ldots, q_{n-1})$$

Notice that from the continuity condition $V(\bar{x}, 0) = W(\bar{x}, 0)$ on Γ it follows that $\bar{p} = \bar{q}$, and the gradient jump condition only means that $p_n \neq q_n$. The analysis for the cases of tangent and transversal approach are different.

The case of tangent approach The tangency condition takes a simple form:

$$F_{p_n}(\bar{x}, 0, p) = 0$$

We solve this equation for p_n to get the solutions:

$$p_n = P(\bar{x}, \bar{p}), \quad q_n = Q(\bar{x}, \bar{q}) \quad (\bar{p} = \bar{q})$$

Note, that by assumption it should be two solutions corresponding to different sides of Γ. We put the function P into the Hamiltonian to get the shortened Hamiltonian

$$\bar{F}(\bar{x}, \bar{p}) = F(\bar{x}, 0, \bar{p}, P(\bar{x}, \bar{p}))$$

Now equations of motion along Γ are given by regular (classical) characteristics corresponding to $\bar{F}$. The tangency condition simplifies the chain rule, as,

$$\bar{F}_{x_k} = F_{x_k} + F_{p_n} P_{x_k} = F_{x_k}$$

and one gets the following system of regular characteristics:

$$\dot{x}_k = F_{p_k}, \quad \dot{p}_k = -F_{x_k}, \quad k = 1, \ldots, n-1$$

Though the value of p_n can be recovered after integration of the above characteristic system as $p_n = P(\bar{x}, \bar{q})$, it is convenient to have a differential equation for p_n as well. We take the full time derivative of F_{p_n} along that system to get:

$$0 = \frac{d}{dt} F_{p_n} = \sum_{k=1}^{n} F_{p_n x_k} \dot{x}_k + \sum_{k=1}^{n} F_{p_n p_k} \dot{p}_k =$$

$$\sum_{k=1}^{n} F_{p_n x_k} F_{p_k} - \sum_{k=1}^{n-1} F_{p_n p_k} F_{x_k} + \dot{p}_n F_{p_n p_n} =$$

$$\sum_{k=1}^{n} F_{p_n x_k} F_{p_k} - \sum_{k=1}^{n} F_{p_n p_k} F_{x_k} + F_{p_n p_n} F_{x_n} + \dot{p}_n F_{p_n p_n} =$$

$$\{F_{p_n} F\} + F_{p_n p_n} F_{x_n} + \dot{p}_n F_{p_n p_n} = 0$$

where $\{F_{p_n} F\} := \sum_{k=1}^{n} F_{p_n x_k} F_{p_k} - \sum_{k=1}^{n} F_{p_n p_k} F_{x_k}$ is the *Poisson bracket* of F_{p_n} and F.

Solving this equation for $\dot{p}_n$ and using tangency condition gives the following differential equations for x_n and p_n:

$$\dot{x}_n = F_{p_n} = 0, \quad \dot{p}_n = -F_{x_n} - \frac{\{F_{p_n} F\}}{F_{p_n p_n}}$$

Equations for (W, q)-side have the same form with the corresponding interchange of p, q, F, G and V, W.

The case of transversal approach We solve for q_n the equation $G^-(\bar{x}, 0, q) = 0$ and find the function $q_n = Q(\bar{x}, \bar{q})$. We put this function into the Hamiltonian G^+ to get

$$\bar{G}^+(\bar{x}, \bar{q}) = G^+(\bar{x}, 0, \bar{q}, Q(\bar{x}, \bar{q}))$$

Now the singular trajectories in Γ are regular characteristics in terms of the Hamiltonian $\bar{G}^+$. Instead of using the chain rule in differentiating the function $\bar{G}^+$ it is simpler to use the Lagrange multipliers technique through the augmented Hamiltonian

$$H = \lambda^+ G^+ + \lambda^- G^-, \quad \lambda^+ + \lambda^- = 1$$

We use here two multipliers instead of one with a normalization condition. Now one has:

$$\dot{x}_k = \bar{G}^+_{q_k} = \lambda^+ G^+_{q_k} + \lambda^- G^-_{q_k},$$

$$\dot{q}_k = -\bar{G}^+_{x_k} = -\lambda^+ G^+_{x_k} - \lambda^- G^-_{x_k}, \quad k = 1, \ldots, n-1$$

To find the expressions for λ^+ and λ^- let us derive a differential equation for q_n. We take the full time derivative of $G^+(x, q)$ along that system to get:

$$0 = \frac{d}{dt} G^+ = \sum_{k=1}^{n} G^+_{x_k} \dot{x}_k + \sum_{k=1}^{n} G^+_{q_k} \dot{q}_k =$$

$$\sum_{k=1}^{n} G^+_{x_k} (\lambda^+ G^+_{q_k} + \lambda^- G^-_{q_k}) - \sum_{k=1}^{n-1} G^+_{q_k} (\lambda^+ G^+_{x_k} + \lambda^- G^-_{x_k}) + \dot{q}_n G^+_{q_n} =$$

$$\lambda^+ G^+_{x_n} G^+_{q_n} + \lambda^- \left[\sum_{k=1}^{n} G^+_{x_k} G^-_{q_k} - \sum_{k=1}^{n} G^+_{q_k} G^-_{x_k} + G^+_{q_n} G^-_{x_n} \right] + \dot{q}_n G^+_{q_n} =$$

$$\lambda^+ G^+_{x_n} G^+_{q_n} + \lambda^- \{G^+ G^-\} + \lambda^- G^+_{q_n} G^-_{x_n} + \dot{q}_n G^+_{q_n} = 0$$

Here $\{G^+ G^-\}$ is the Poisson bracket of G^+ and G^-.

Solving this equation for $\dot{q}_n$ gives the differential equation:

$$\dot{q}_n = -\lambda^+ G^+_{x_n} - \lambda^- G^-_{x_n} - \frac{\lambda^-}{G^+_{q_n}} \{G^+ G^-\}$$

Similarly, from the condition $d/dt(G^-) = 0$ one can derive:

$$\dot{q}_n = -\lambda^+ G^+_{x_n} - \lambda^- G^-_{x_n} - \frac{\lambda^+}{G^-_{q_n}} \{G^- G^+\}$$

Equating these two expressions one gets the condition for the Lagrange multipliers:

$$\frac{\lambda^-}{G^+_{q_n}} = \frac{\lambda^+}{G^-_{q_n}}$$

which together with the normalization condition $\lambda^+ + \lambda^- = 1$ gives:

$$\lambda^+ = \frac{G^-_{q_n}}{G^-_{q_n} - G^+_{q_n}}, \quad \lambda^- = \frac{G^+_{q_n}}{G^+_{q_n} - G^-_{q_n}}$$

With such values of λ one has two differential equations for x_n and q_n:

$$\dot{x}_n = \lambda^+ G^+_{q_n} + \lambda^- G^-_{q_n} = 0$$

$$\dot{q}_n = -\lambda^+ G^+_{x_n} - \lambda^- G^-_{x_n} - \frac{\{G^+ G^-\}}{G^+_{q_n} - G^-_{q_n}}$$

Similar equations are true for (V, p)-side.

The system of singular characteristics for both tangent and transversal approaches coincide with the ones in [5] obtained by the method of singular characteristics. The above derivation does not use that method and is based on classical characteristics and Poisson brackets.

4 General Surface, Collinear Fields

In this case constructions are essentially based upon the method of singular characteristics for the equivocal surface, i.e. a (hyper-)surface which is approached by the characteristics on one side and left on the other side.

Theory of Singular Equivocal Surface Following the methodology in [5] to write the equations of singular motion one has to specify the three (in case of hyperplane) necessary optimality conditions of the form:

$$F_0(x, V, p) = 0, \quad F_1(x, V, p) = 0, \quad F_{-1}(x, V, p) = 0$$

fulfilled on the surface Γ. These conditions, generally, involve the value function $V(x)$ as well. In terms of these three functions so-called singular Hamiltonian H is defined as:

$$\nu H = \{F_0 F_1\} F_{-1} + \{F_1 F_{-1}\} F_0 + \{F_{-1} F_0\} F_1$$

Here ν is a homogeinity multiplier normalizing independent (time) variable and the Jacobi brackets

$$\{FG\} = \langle F_x + pF_V, G_p\rangle - \langle G_x + pG_V, F_p\rangle$$

are used rather than Poisson ones due to presence of V. Now singular characteristics are regular ones in terms of H which have the form:

$$\dot{x} = H_p, \quad \dot{p} = -H_x - pH_V, \quad \dot{V} = \langle p, H_p\rangle$$

This system should be considered on the manifold $F_i = 0$, $|i| \leq 1$. The last equation for V, generally, is separated from the first two ones.

The value function to that side of Γ where the trajectories leave the surface (in direct time) is considered to be known prior to construction of the surface. We denote it and its gradient by $W(x), q(x)$, using the notation V, p for the other (approaching) side. The functions F_i have different character for the case of smooth or nonsmooth Hamiltonian in the original differential game problem. A smooth Hamioltonian usually arises when optimal controls of both players $u(x, p)$ and $v(x, p)$ smoothly depend on x, p. In that case approaching to Γ (in ditect time) trajectories must be tangent to the surface and the set of the necessary conditions have the form:

$$F_0 = F(x, p) = 0, \quad F_1(x, V) = V - W(x) = 0,$$

$$F_{-1}(x, p) = \{F_1 F\} = \langle F_p, p - q(x)\rangle = 0$$

Then the above system of singular characteristics takes the form:

$$\dot{x} = F_p, \quad \dot{p} = -F_x - \frac{\{\{FF_1\}F\}}{\{\{F_1 F\}F_1\}}(p - q) \quad (\nu = \{\{F_1 F\} F_1\})$$

Note, that second partials arise here in double Jacobi brackets, W_{xx} in numerator, and F_{pp} in denominator, while in regular characteristics only first partials F_x, F_p are present.

We consider a nonsmooth Hamiltonian of the form

$$F(x,p) = \min[F^+(x,p), F^-(x,p)]$$

with smooth branches F^+, F^-. Such a Hamiltonian arises, for instance, in case of bang-bang control of the minimizing player, the switching condition having the form $F^+ - F^- = 0$. In this case the three necessary conditions are the following ones:

$$F_0 = F^+(x,p) = 0, \quad F_1(x,V) = V - W(x) = 0, \quad F_{-1}(x,p) = F^-(x,p) = 0$$

and the system of singular characteristics takes the form:

$$\dot{x} = \lambda^+ F_p^+ + \lambda^- F_p^-, \quad \dot{p} = -\lambda^+ F_x^+ - \lambda^- F_x^- - \frac{\{F^- F^+\}}{\nu}(p-q)$$

$$\lambda^+ = \{F_1 F^-\}/\nu, \quad \nu = \{F_1 F^-\} + \{F^+ F_1\}, \quad \lambda^+ + \lambda^- = 1$$

Focal surface with tangent approach We give here the equation of singular motion along the surface for the case of tangent approach under the collinearity assumption

$$F_p(x,p) = cG_q(x,q), \quad c = c(x,p,q)$$

following the considerations in [5]. Here c is a proportionality coefficient. Depending upon the physical meaning of the problem the independent variable of differentiation parametrizing the trajectory (usually time) may be different for each side of Γ. If we denote these variables for different sides as t and v then one has

$$\frac{dv}{dt} = c(x,p,q)$$

If, say, time is the independent variable for both sides then $c = 1$. Generally, one can try to use two copies of the equations for equivocal surface for each side of the focal one. There are two obstacles for that: 1) trajectories in each side may have different (non-collinear) directions; 2) except for the variables (x,p,q) one has second partials as well. The first obstacle is removed by the collinearity assumption. Using the tangency condition one can express the term with second partials through the variables (x,p,q), see [5], p. 82.

As a result one can get the following system of of the order $3n$ for singular focal characteristics in terms of x, p, q:

$$\dot{x} = F_p(x,p) = cG_q(x,q)$$

$$\dot{p} = -F_x - \frac{\{\{FF_1\}F\}^*}{\{\{F_1F\}F_1\}}(p-q) \equiv K(x,p,q)$$

$$\dot{q} = -c(G_x + \frac{\{\{GG_1\}G\}^*}{\{\{G_1G\}G_1\}}(q-p)) = cK(x,q,p)$$

$$F_1 = V - W(x), \quad G_1 = W - V(x)$$

Note the interchange of p and q in K here. In this relations one has to use the Jacobi brackets

$$F = F(x,V,p), \quad H = H(x,V,p),$$

$$\{FH\} = \langle F_x + pF_V, H_p\rangle - \langle H_x + pH_V, F_p\rangle\,,$$

rather than the Poisson ones because of the presense of the variable V (or W) in function F_1 (or G_1). Here the expression $\{\{GG_1\}G\}^*$ stands for the Jacobi bracket $\{\{GG_1\}G\}$ in which the term $W_{xx}F_p, F_p$ is substituted by the following one: $-c^2\langle G_x, G_q\rangle$.

Focal surface with transversal approach Denote by μ^+, μ^- corresponding multipliers in the neighborhood D_1 and write the collinearity assumption (with $c = 1$):

$$\lambda^+ F_p^+ + \lambda^- F_p^- = \mu^+ G_q^+ + \mu^- G_q^-$$

In this case no complexification takes place and the system of two equations for equivocal surface gives the following $3n$ equations for the focal surface under consideration:

$$\dot{x} = \lambda^+ F_p^+ + \lambda^- F_p^- = \mu^+ G_q^+ + \mu^- G_q^-$$

$$\dot{p} = -\lambda^+ F_x^+ - \lambda^- F_x^- - \frac{\{F^-F^+\}}{\{F_1F^-\} + \{F^+F_1\}}(p-q) = P^*(x,p,q)$$

$$\dot{q} = -\mu^+ G_x^+ - \mu^- G_x^- - \frac{\{G^-G^+\}}{\{G_1G^-\} + \{G^+G_1\}}(q-p) = Q^*(x,q,p)$$

$$\lambda^+ = \{F_1F^-\}/(\{F_1F^-\} + \{F^+F_1\}), \quad \lambda^+ + \lambda^- = 1,$$

$$\mu^+ = \{G_1G^-\}/(\{G_1G^-\} + \{G^+G_1\}), \quad \mu^+ + \mu^- = 1$$

5 A System of two PDEs for a general Focal Manifold

Consider a focal surface Γ with transversal approach of the trajectories from both sides in a generic position, i.e. it is not a known hyperplane and collinearity assumption is not necessarily fulfilled. The viscosity solution (game value) on the different sides of Γ is denoted as $V(x)$ and $W(x)$, correspondingly, with the gradients p and q.

On Γ one has the continuity condition

$$V(x) = W(x)$$

Suppose we can solve this equation for x_n to find the function $x_n = X(\bar{x})$. Generally, the jump condition on Γ, $p \neq q$, ensures that this could be done for some coordinate x_k. By definition of the function X one has the identities on Γ:

$$Y(\bar{x}) \equiv V(\bar{x}, X(\bar{x})) \equiv W(\bar{x}, X(\bar{x})), \quad \bar{x} \in R^{n-1}$$

the first one being the definition of a new scalar function of $\bar{x}$ – the restriction of the game value to Γ.

We denote the $(n-1)$-dimensional gradients of the two scalar functions X and Y by r and s respectively. Differentiating the above identities with respect to each component x_k, $k = 1, \ldots, n-1$, and involving the two equalities for each side of Γ one can consider the following system of $2n+2$ equations:

$$s_k = p_k + p_n r_k = q_k + q_n r_k, \quad k = 1, \ldots, n-1,$$

$$F^+(x,p) = 0, \quad F^-(x,p) = 0, \qquad G^+(x,q) = 0, \quad G^-(x,q) = 0,$$

One can eliminate the $2n$ variables p and q from this $2n+2$ equations to get 2 equalities in $(\bar{x}, X, r, s)$, i.e. two coupled first order PDEs with respect to two functions X and Y. We first solve the equations for the gradients to get

$$p_k = s_k - p_n r_k, \quad q_k = s_k - q_n r_k, \quad k = 1, \ldots, n-1$$

and put this into the equations:

$$F^+(\bar{x}, X, s_1 - p_n r_1, s_2 - p_n r_2, \ldots, s_{n-1} - p_n r_{n-1}, p_n) = 0$$

$$G^+(\bar{x}, X, s_1 - q_n r_1, s_2 - q_n r_2, \ldots, s_{n-1} - q_n r_{n-1}, q_n) = 0$$

Now we observe that these equations are solvable with respect to p_n and q_n. Indeed, one has for the partial derivative of the left hand side of the first equation:

$$\frac{\partial}{\partial p_n} F^+ = \left\langle F_p^+, N \right\rangle, \quad N = (-r_1, -r_2, \ldots, -r_{n-1}, 1) \in \mathbb{R}^n$$

where N is the gradient of the function $x_n - X(\bar{x})$ in $\mathbb{R}^n$, i.e. a normal to Γ, and thus $N = \nu(p - q)$ with some $\nu \neq 0$. This scalar product is nonzero in case of the transversal approach. Thus, through the implicit function theorem these equations are solvable and one can find solutions in the form:

$$p_n = P(\bar{x}, X, r, s), \quad q_n = Q(\bar{x}, X, r, s)$$

We put this expressions into the functions F^- and G^- and obtain two first order PDEs with respect to two unknown functions X and Y:

$$R(\bar{x}, X, r, s) = 0, \quad S(\bar{x}, X, r, s) = 0$$

where

$$R(\bar{x}, X, r, s) \equiv F^-(\bar{x}, X, s_1 - r_1 P, s_2 - r_2 P, \dots, s_{n-1} - r_{n-1} P, P)$$

$$S(\bar{x}, X, r, s) \equiv G^-(\bar{x}, X, s_1 - r_1 P, s_2 - r_2 P, \dots, s_{n-1} - r_{n-1} P, P)$$

These two PDEs involve explicitely only one of the unknown functions X and are interconnected through the partials r, s of the functions $X(\bar{x}), Y(\bar{x})$.

Remark 1. In case of tangent approach to Γ to get the implicit function theorem condition fulfilled one needs to find the functions

$$p_n = P(\bar{x}, X, r, s), \quad q_n = Q(\bar{x}, X, r, s)$$

from the equations (tangency conditions)

$$\langle F_p, p - q \rangle = 0, \quad \langle G_q, p - q \rangle = 0$$

and put them into $F(x, p) = 0$ and $G(x, q) = 0$ (Bellman-Isaacs equation). Indeed, putting the expressions for p_k and q_k into $\langle F_p, p - q \rangle$ and differentiating with respect to p_n one gets:

$$\frac{\partial}{\partial p_n} \langle F_p, p - q \rangle = \langle F_p, N \rangle + \langle F_{pp}(p - q), N \rangle =$$

$$= \nu \langle F_p, p - q \rangle + \nu \langle F_{pp}(p - q), p - q \rangle = -\nu \{\{F_1 F\} F_1\} \neq 0$$

The latter Jacobi bracket is a denominator in equations of singular motion and in generic case must be non-zero. Note, that the linear form in $p - q$ vanishes here due to tangency condition and the bracket equals to the remaining quadratic form.

Remark 2. For $n = 3$ the solution of this coupled system of first order PDEs can be, generally, reduced to the solution of one second order PDE in two independent variables. To do this one has to differentiate the above PDEs $R = 0$ and $S = 0$

with respect to x_i, $i = 1, \dots, n-1$, to get alltogether $2m+2$ equations (with $m = n-1$). The total number of different second partials $W_{x_i x_j}$ is $m(m+1)/2$ which together with m first partials gives $m(m+1)/2+m$ variables. If the number of variables $W_{x_i x_j}$,W_{x_i} is less by one than the number of equations:

$$\frac{m(m+1)}{2} + m = 2m + 2 - 1$$

i.e. if $m = 2$, or $n = m + 1 = 3$, then one can, generally, solve the all but one equations for $W_{x_i x_j}$,W_{x_i} and put the result into the remaining one equation. This will give a second order PDE.

Remark 3. A system of two first order PDEs one can get in the case of the surface-hyperplane and in the case with collinearity condition, but in these cases one has certain simplification. In case of hyperplane the function $X(\bar{x})$ is identically zero, $x_n = 0$, so that one has to find only the function $Y(\bar{x})$. This function satisfies the equation with shortened Hamiltonian used above, like $\bar{F}(\bar{x}, \bar{p}) = 0$, $Y(\bar{x}) = V(\bar{x}, 0), \bar{p} = \bar{q} = s$.

Collinearity condition actually specifies the situation when the solution of a system of two PDEs can be reduced to a characteristic ODE system like in case of one PDE. Generally, a system of several PDEs requires more complicated technique for its solution, see for instance [10].

6 An Impulsive Differential Game Arising in Finance

6.1 Previous work

In [6] an impulse control differential game is investigated. The geometrical form of the Isaacs Breakwell theory, using the concept of semi-permeability, is shown to apply essentially unchanged, and a 2D focal singular surface is constructed. (It can also be seen as the solution of an equivalent non-impulsive differential game, whose Isaacs equation is a differential form of the quasi-variational inequality associated with the impulse control game.)

The focal surface is approached by regular characteristics (trajectories) from both sides in a non-tangential non-collinear manner. Such a geometrical picture was conjectured in [5]. We have only partly shown that the optimality conditions in the form [5] are fulfilled for the problem in [6]. This allowed us to suggest a closed form ODE system for the construction of that surface (and, indeed all singular surfaces of that game), and, in [11], to use the technique of viscosity solutions to get a complete proof of optimality.

More specifically, the problem is as follows. The state is 2-dimensional plus time. Let it be (x, y). The controls are v for the maximizer, and u for the minimizer, this appearing either as a "continuous component" $u(t)$ in the dynamics, or as impulses or jumps of amplitude u_k at time instants t_k, the choice of which is part of the minimizer's control.

The definition of the game involves parameters $v^- < C^- < 0 < C^+ < v^+$, and in such expressions as $C^\varepsilon u$, ε stands for $\mathrm{sign}(u)$. The game is defined by the dynamics

$$\begin{aligned} \dot{x} &= (1+v)x\,, \qquad v^- \le v \le v^+\,, \\ \dot{y} &= (1+v)y + u\,, \quad y(t_k^+) = y(t_k^-) + u_k\,, \end{aligned}$$

and by the cost function involving the function $M(s) = \max\{0, s-K\}$ for some positive parameter K :

$$J = M(x(T)) + \int_0^T (-v(t)y(t) + C^\varepsilon u(t))\,\mathrm{d}t + \sum_k C^{\varepsilon_k} u_k\,.$$

This game is first transformed into a non-impulsive game by introducing a fictitious "time" θ, and making t a state variable, with derivative 1 when there is no jump and 0 when there is a jump (at the will of the minimizer) during which, $\dot{x} = 0$ also, while $\dot{y} = \pm 1$ (again at the will of the minimizer), thus introducing a jump in y when watched in t time. The same trick applies to the cost function.

The Isaacs equation becomes a "differential quasi-variational inequality"

$$\min\left\{ \frac{\partial W}{\partial t} + \max_{v\in[v^-,v^+]} v\left[\frac{\partial W}{\partial x}x + \left(\frac{\partial W}{\partial y} - 1\right)y\right], \right. \left. ,\frac{\partial W}{\partial y} + C^+\,,\ -\frac{\partial W}{\partial y} - C^- \right\} = 0\,.$$

$$W(T,x,y) = M(x)\,.$$

It has been shown that for $t \le T - (1/v^+)\ln(1+C^+)$, this game has a focal surface with non collinear characteristic fields. This focal surface can be computed through a system of two linear PDE's. As a matter of fact, we showed [11] that the solution of the above Isaacs DQVI is given by

$$W(t,x,y) = Y(t,x) + q^\varepsilon(t)(X(t,x) - y)$$

where $\varepsilon = \mathrm{sign}(X-y)$, and with $m := T-t$, $q^-(t) = \max\{\mathrm{e}^{v^- m} - 1, C^-\}$, $q^+(t) = \min\{\mathrm{e}^{v^+ m} - 1, C^+\}$, and X and Y satisfy a pair of coupled linear PDE's the investigation of which is still in progress:

$$\begin{pmatrix} X_t \\ Y_t \end{pmatrix} + \mathcal{T}\left(\begin{pmatrix} X_x \\ Y_x \end{pmatrix} x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} y\right) = 0\,, \tag{1}$$

$$\begin{pmatrix} X(T,x) \\ Y(T,x) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ if } x < K\,, \quad \begin{pmatrix} x \\ x-K \end{pmatrix} \text{ if } x > K\,.$$

where

$$\mathcal{T} = \frac{1}{q^+ - q^-} \begin{pmatrix} v^+q^+ - v^-q^- & v^+ - v^- \\ (v^- - v^+)q^-q^+ & v^-q^+ - v^+q^- \end{pmatrix}.$$

The whole manifold $y = X(t,x)$ is therefore a discontinuity of the gradient of W, and is a focal manifold in the region where both q^- and q^+ are constant. (For larger $t's$, this is first an equivocal manifold and finally a dispersal manifold.)

6.2 Applying the theory of the previous section

Using the notations of the previous section, where (t,x,y) plays the role of (x_1,x_2,x_3) take

$$\begin{aligned} F^+ &= V_t + v^+[V_x x + (V_y - 1)y], \\ F^- &= V_y + C^+\,, \\ G^+ &= W_t + v^-[W_x x + (W_y - 1)y], \\ G^- &= W_y + C^-\,. \end{aligned}$$

In $F^+ = 0$ and $G^+ = 0$, replace (V_t, V_x) and (W_t, W_x) respectively using

$$\begin{aligned} V_t &= Y_t - V_y X_t\,, & V_x &= Y_x - V_y X_x\,, \\ W_t &= Y_t - W_y X_t\,, & W_x &= Y_x - W_y X_x\,. \end{aligned}$$

It comes

$$\begin{aligned} Y_t - V_y X_t + v^+[(Y_x - V_y X_x)x + (V_y - 1)X] &= 0\,, \\ Y_t - W_y X_t + v^-[(Y_x - W_y X_x)x + (W_y - 1)X] &= 0\,. \end{aligned}$$

Finally, eliminate V_y and W_y between these and $F^- = 0$ and $G^- = 0$, it comes

$$\begin{aligned} Y_t + C^+X_t + v^+[(Y_x + C^+X_x)x - (C^+ + 1)X] &= 0\,, \\ Y_t + C^-X_t + v^-[(Y_x + C^-X_x)x - (C^- + 1)X] &= 0\,. \end{aligned} \tag{2}$$

This is our pair of first order PDE's, which it is a simple matter to rearrange in (1) by a left multiplication by the matrix

$$\frac{1}{C^+ - C^-} \begin{pmatrix} 1 & -1 \\ -C^- & C^+ \end{pmatrix}.$$

It should be noticed that this calculation is much simpler than the original derivation in [6].

6.3 The equivalent scalar second order PDE

As shown above (Section 5 Remark 2), in the case where, as here, the state space dimension is 3, one can replace this system of two first order PDE's by a unique, scalar second order PDE. Differentiating the two PDE's with respect to the two independant variables (here t and x) yields 4 more relations, i.e. 6 altogether. Because Y itself does not appear in the original equations (but only its derivatives), it does not enter either in this system. The dictum is therefore to eliminate its five first and second partials between these 6 equations.

Here, everything is linear, and that program involves only elementary linear algebra. Starting either from (2) or from (1), it turns out that the last four equations, obtained by differentiating the first order system with respect to t and to x, suffice to elminate all terms in Y. They write:

$$\begin{pmatrix} -v^+(C^+ + 1) & 0 & C^+ & v^+C^+ & 0 \\ -v^-(C^- + 1) & 0 & C^- & v^-C^- & 0 \\ 0 & -v^+(C^+ + 1) & 0 & C^+ & v^+C^+ \\ 0 & -v^-(C^- + 1) & 0 & C^- & v^-C^- \end{pmatrix} \begin{pmatrix} X_t \\ X_x x \\ X_{tt} \\ X_{tx} x \\ X_{xx} x^2 \end{pmatrix} + \begin{pmatrix} 1 & v^+ & 0 \\ 1 & v^- & 0 \\ 0 & 1 & v^+ \\ 0 & 1 & v^- \end{pmatrix} \begin{pmatrix} Y_{tt} \\ Y_{tx} x \\ Y_{xx} x^2 \end{pmatrix} = 0.$$

Left multiply by the vector $(1 \ -1 \quad v^- \ -v^+)$ and divide by $(C^+ - C^-)$ to get

$$-\frac{v^+(1+C^+) - v^-(1+C^-)}{C^+ - C^-} X_t - v^+v^- X_x x + X_{tt} + (v^+ + v^-) X_{tx} x + v^+v^- X_{xx} x^2 = 0\,.$$

And in the complete system of 6 equations, the matrix of coefficients of the terms involving the partials of Y is of rank 5, so that this is the only PDE in X only that can be derived this way.

Notice also that writing everything in terms of $\ln(x)$ instead of x, all these linear partial differential equations have constant coefficients, making an investigation via Laplace transforms in both time and space possible.

Finally, we point out that the determinant of the higher order terms is just $-(v^+ - v^-)^2 x^2/4 < 0$, so that the PDE is hyperbolic, and that it can be written

$$\left(\frac{\partial}{\partial t} + v^+ x \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} + v^- x \frac{\partial}{\partial x}\right) X - \frac{v^+(1+C^+) - v^-(1+C^-)}{(1+C^+) - (1+C^-)} X_t - v^+v^- x X_x = 0\,,$$

which displays the two different (real) characteristic fields as the extreme trajectory fields $\dot{x} = v^+ x$ and $\dot{x} = v^- x$. This means that the optimal motion along the singular surface is nonunique, as shown in [6].

7 A 3D Differential Game with Smooth Hamiltonian

We consider a fixed time interval differential game with a terminal cost function. Let the phase space be three-dimensional with phase vector $x = (x_1, x_2, x_3)$, where the last component stands for the time variable, $t = x_3$. The game dynamics is given by the equations and control constraints:

$$\dot{x}_1 = k(bu_1 - u_2) + v_1 - v_2, \quad u_1^2 + u_2^2 \leq 1$$

$$\dot{x}_2 = ku_2 + v_2, \quad v_1^2 + v_2^2 \leq 1$$

where b is a positive constant and $k = k(t)$ a positive time function, which can be taken as $k = t$ for $t \geq 0$.

The extended minimaximized Hamiltonian (including the time partial of the value function $p_3 = \partial V/\partial x_3$) depends on the costate vector $p = (p_1, p_2, p_3)$ and the last component x_3 of the state vector x:

$$H(x, p) = p_3 + \sqrt{p_1^2 + (p_2 - p_1)^2} - k\sqrt{b^2 p_1^2 + (p_2 - p_1)^2}$$

The HJBI equation with terminal conditions has the form:

$$H(x, \partial V/\partial x) = 0, \quad V(x_1, x_2, T) = M(x_1, x_2)$$

where T is the termination time and M is the terminal cost function.

We suppose that the function M is such that a part of the surface $x_2 = 0$ is a focal surface. Since the above Hamiltonian $H(x, p)$ is a smooth function the surface is with the tangent approach.

The (simplified) tangency condition has the form:

$$\frac{\partial H}{\partial p_2} = \frac{p_2 - p_1}{\sqrt{p_1^2 + (p_2 - p_1)^2}} - k\frac{p_2 - p_1}{\sqrt{b^2 p_1^2 + (p_2 - p_1)^2}} = 0$$

One can show that this equation has the following three real roots in p_2 for $1 < k < b$:

$$p_2 = p_1, \quad p_2 = p_1 + h, \quad p_2 = p_1 - h \quad \left(h = |p_1|\sqrt{(b^2 - k^2)/(k^2 - 1)}\right)$$

The latter two values may produce two following non-collinear fields lying on different sides of the hyperplane $x_2 = 0$:

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1 - h}{\sqrt{p_1^2 + h^2}} - k\frac{b^2 p_1 - h}{\sqrt{b^2 p_1^2 + h^2}}, \quad \dot{x}_3 = \frac{\partial H}{\partial p_3} = 1$$

on one side and

$$\dot{x}_1 = \frac{\partial H}{\partial p_1} = \frac{p_1 + h}{\sqrt{p_1^2 + h^2}} - k\frac{b^2 p_1 + h}{\sqrt{b^2 p_1^2 + h^2}}, \quad \dot{x}_3 = \frac{\partial H}{\partial p_3} = 1$$

on the other side. To complete the constructions one needs to formulate appropriate boundary conditions. This is the matter for further investigations.

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