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Optimal Control of a Plant Disease Model with Harvesting and Clean Seed Usage

Optimal Control of Plant Disease

Bowen, B. · Hamelin, F. · Bernhard, P. · Bokil, V.

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Abstract We consider the question of eradicating disease in plants through the optimal usage of clean seeds. Optimal control theory is used to set up optimization of plant disease S-I model.

1 Introduction

We consider the problem of optimally controlling vectored viral diseases in plant populations, which hamper the production of staple food crops especially in lesser developed countries. Examples are Cassava production, and Cassava Mosaic virus which is spread by whiteflies.

Traditionally control strategies used to combat viral diseases in vegetatively propagated crops include control of vectors by insecticides, roguing or removal of infected plants, breeding plants to be resistant to the virus, using clean seeds and using healthy planting material for vegetative propagation.

Using mathematical modeling and analysis, we consider the question of how to control clean seed usage while optimizing revenue from harvesting of both healthy and infected plants. Using a continuous time $S-I$ model based

B. Bowen
Department of Mathematics, Oregon State University
Tel.: +123-45-678910
E-mail: fauthor@example.com

F. Hamelin
second address

P. Bernhard
Université Côte d'Azur-INRIA, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis CEDEX, France
Tel.: +33-(0)497-155-387
E-mail: Pierre.Bernhard@inria.fr

V. Bokil

on ordinary differential equations, we divide the host plant population into two compartments; susceptible to disease or healthy plant compartment, S, and infected plant compartment I. We assume that the viral disease is transmitted directly, which in the case of a vectored disease, could be interpreted as a model in which the vectors are at a quasi-equilibrium. We assume that growers are not able to detect between healthy and infected seeds and so both are used to propagate the host plant, so that the model incorporates vertical transmission via the usage of infected seeds. Both the healthy and infected plants are harvested though we assume that the infected plants are worth less per unit price than the healthy plants. Using the mathematical theory of optimal control we optimize over the revenue from harvesting by controlling the fraction of clean seeds that are used in planting.

2 $S-I$ Model of Plant Disease with Clean Seeds and Harvesting

PUT the full van den Bosch et al 2006 model with reversion r and roguing ρ and p the probability to detect that a seed is infected to explicitly show that our model is a simplification of vdB's model with $(r=\rho=\omega=0)$ BUT adding vertical transmission ν that they implicitly assumed equal to 1. Try to generalize our approach to $p>0$.

PUT the equivalent I dynamics and criterion equations to make explicit the different with optimal harvesting models in fisheries?

We consider a plant disease model with clean seed usage that is a modification of the model presented in [1]. Our model is an $S-I$ model with harvesting that includes vertical transmission and use of clean seeds. The model equations for the S and I compartments are given as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \sigma\phi + \sigma(1-\phi)\frac{(1-\nu)I+S}{N} - \mu S - \beta IS, \tag{1a}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \sigma(1-\phi)\frac{\nu I}{N} - \mu I + \beta IS, \tag{1b}$$

where $N=I+S$ is the total density of the plant host, with S and I, the density of susceptible, and infected hosts, respectively. We incorporate density dependent direct transmission of disease.

Descriptions of the variables and parameters in the $S-I$ model are given in Table 1 with default values used in our simulations and corresponding ranges of values.

The basic reproduction number for this model is

$$\mathcal{R}_0 = \underbrace{(1-\phi)\nu}_{\text{vertical transmission}} + \underbrace{\frac{\beta}{\mu}}_{\text{horizontal transmission}}, \tag{2}$$

and can be seen as the sum of two modes of transmission of disease.

Variables/ Parameters	Description	Range of Values	Default Value
S	Susceptible Plant Population density	$[0, 1]$	
I	Infected Plant Population density	$[0, 1]$	
N	Total Plant Population density	$[0, 1]$	
ϕ	Fraction of in-vitro seeds used in replanting	$[0, 1]$	
σ	Replanting Rate	$[0, \infty)$	1
μ	Harvest Rate	$[0, \infty)$	1
β	horizontal transmission rate	$[0, \beta_s]$	2
ν	vertical transmission rate	$[0, 1]$	$\frac{2}{3}$ or 1
p_S	Profit from selling uninfected plants	$[0, \infty)$	3
p_I	Profit from selling infected plants	$[0, \infty)$	0
c	Unit Cost of using in-vitro seeds	$[0, \infty)$	1
P_S	End profit of selling remaining uninfected plants	$[0, \infty)$	0 or 2
P_I	End profit of selling remaining infected plants	$[0, \infty)$	0

Table 1 List of variables and parameters in the $S - I$ model (Equation 5) and objective functional (Equation 4).

Our goal in this paper is to look at the optimal control problem with control variable $\phi(t)$, for which the objective functional

$$J(\phi(\cdot)) = P_S S(T) + P_I I(t) + \int_0^T [p_S \mu S(t) + p_I \mu I(t) - c\sigma\phi(t)] \,\mathrm{d}t \qquad (3)$$

is maximized. Thus, our objective if to maximize revenue by controlling the introduction of uninfected in-vitro seeds. The objective functional represents the total profit from the patch of crops. It contains the total profit from the sale of the healthy plants at the final time, selling of both healthy and infected crops over the season, minus the cost of replenishing harvested crops through in-vitro seeds.

In the sequel, we will consider two subcases, 1) $P_S = 0$ (we don't care about the disease prevalence in the end), and 2) $P_S > 0$.

Assuming $\sigma = \mu N$ (there are as many plants sowed as plants harvested), we have from the equation for the total plant density,

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \sigma - \mu N = 0\,,$$

that N is a constant. Assuming $N = 1$ without loss of generality, we have $\mu = \sigma$. Then $I = 1 - S$, and the state equations and the optimal control problem simplify to maximizing

$$J(\phi(\cdot)) = P_S S(T) + P_I(1 - S(T)) \int_0^T [p_S \mu S(t) + p_I \mu (1 - S(t)) - c\mu\phi(t)] \,\mathrm{d}t\,. \qquad (4)$$

subject to the state equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} = (\mu\phi\nu - \beta S + \mu(1 - \nu))\,(1 - S)\,. \qquad (5)$$

The optimal control problem in this form looks similar to the problem of optimal harvesting of a logistically growing population. We can rewrite Equation 4 in the form

$$J(\phi(\cdot)) = (P_S - P_I)S(T) + (P_I + T\mu p_I) + \int_0^T [(p_S - p_I)\mu S(t) - c\mu\phi(t)]\,\mathrm{d}t$$

From this we determine that maximizing this objective functional is the same as maximizing the functional

$$J(\phi(\cdot)) = (P_S - P_I)S(T) + \int_0^T [(p_S - p_I)\mu S(t) - c\mu\phi(t)]\,\mathrm{d}t\,.$$

Therefore, without loss of generality, we assume that both $p_I = 0$ and $P_I = 0$. In the case where either of these quantities is not zero, we can instead consider P_S or p_S to be the difference in the cost of clean plants and the cost of infected plants, in the case of terminal payoff and running payoff respectively.

3 Maximizing the long-term running payoff

In this section, we are interested in maximizing

$$\bar{\ell}(\phi) = p_S\bar{S}(\phi) - c\phi$$

with respect to $\phi \in [0, 1]$, where $\bar{S}(\phi)$ is the equilibrium value asymptotically reached by

$$\frac{\mathrm{d}S}{\mathrm{d}t} = (\mu\phi\nu - \beta S + \mu(1-\nu))(1-S)\,.$$

There are two equilibrium values of S, either $\bar{S} = 1$, or $\bar{S} = \dfrac{\mu}{\beta}(\phi\nu + (1-\nu))$, that are reached depending on whether $R_0 < 1$ or $R_0 > 1$, respectively. We can rewrite the S equation as

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \beta(\bar{S} - S)(1-S). \tag{6}$$

with $\bar{S} = 1 + \dfrac{1 - R_0}{R}$ (EE). Thus, $\bar{S}$ exists and is biologically feasible when $R_0 > 1$. In this case, the equation for S indicates that $\bar{S}$ is locally asymptotically stable, while $S = 1$ is unstable. When $R_0 < 1$, $S = 1$ (DFE) is the only equilibrium that is biologically feasible and is stable.

To analyze the combined effects of horizontal and vertical transmission on the disease as well as the optimal revenue we consider the ratio R/ν, where the quantity

$$R := \frac{\beta}{\mu}. \tag{7}$$

The quantity R/ν is the ratio of the rates of horizontal versus vertical transmission and will be important in the analysis of optimal revenue.

3.1 Dynamics of the epidemic model

In this section, we analyze the $S-I$ model for its equilibria. Two cases are to be distinguished:

(A) $\dfrac{R}{\nu} = \dfrac{\beta}{\mu\nu} > 1$: In this case, S asymptotically reaches the endemic equilibrium (EE), $\bar{S} = \dfrac{\mu}{\beta}(\phi\nu + (1-\nu))$ for all $\phi \in [0,1]$. In particular, for $\phi = 1$, $\bar{S} = \mu/\beta = 1/R_0 = 1/R < 1$, meaning that the prevalence of the disease is $\bar{I} = 1 - 1/R > 0$, i.e. one cannot get rid of the disease.

(B) $\dfrac{R}{\nu} = \dfrac{\beta}{\mu\nu} < 1$: Here S attains the EE, $\bar{S} = \dfrac{\mu}{\beta}(\phi\nu + (1-\nu))$ over the interval $\phi \in \left[0, \dfrac{R}{\nu}\right)$ only. For $\phi \in \left[\dfrac{R}{\nu}, 1\right]$, S approaches the disease free equilibrium (DFE), $\bar{S} = 1$, meaning that the disease goes extinct. That is, $\phi = \dfrac{R}{\nu}$ is sufficient to get rid of the disease.

In either case, in absence of control ($\phi = 0$), then $\bar{S} = \dfrac{1-\nu}{R}$, meaning that the pathogen will invade the entire plant population when there is full vertical transmission, i.e. $\nu = 1$.

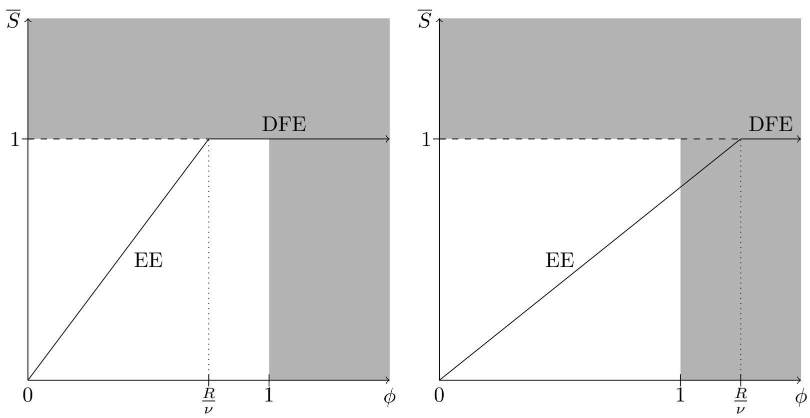

Fig. 1 Dynamics of the $S-I$ disease model. DFE = Disease free equilibrium, EE = endemic equilibrium. The biologically reasonable region is in white. (Left) Case B: Dynamics for the case $R/\nu < 1$. (Right) Case A: Dynamics for the case $R/\nu > 1$.

3.2 Static optimization

In this section, we consider static optimization of the problem. In this case, the relative unit price of healthy plants versus clean seeds also becomes important, and so we define the quantity

$$r := \frac{p_S}{c}. \tag{8}$$

We start with distinguishing the same two cases as in the dynamics of the $S - I$ model. Within each case, we encounter subcases that depend on the relative values of r and $\frac{R}{\nu}$.

(A) $R/\nu = \frac{\beta}{\mu\nu} > 1$. Then, for all $\phi \in [0, 1]$, we have

$$\begin{aligned} \bar{\ell}(\phi) &= p_S \frac{\mu}{\beta}(\phi\nu + (1 - \nu)) - c\phi \\ &= \phi \left(p_S \frac{\mu}{\beta}\nu - c \right) + p_S \frac{\mu(1-\nu)}{\beta} \\ &= \phi c \left(\frac{r\nu}{R} - 1 \right) + \frac{p_s \mu(1-\nu)}{\beta}. \end{aligned} \tag{9}$$

Two sub-cases are then to be distinguished:

(A1) $\frac{R}{\nu} < r$. Then $\phi^\star = 1$.

(A2) $\frac{R}{\nu} > r$. Then $\phi^\star = 0$.

(B) $\frac{R}{\nu} = \frac{\beta}{\mu\nu} < 1$. Then, we have

$$\bar{\ell}(\phi) = \begin{cases} \phi c \left(\frac{r\nu}{R} - 1 \right) + p_S \frac{1-\nu}{R} & \text{for } \phi \in \left[0, \frac{R}{\nu} \right), \\ c(r - \phi) & \text{for } \phi \in \left[\frac{R}{\nu}, 1 \right]. \end{cases}$$

We note that $\bar{\ell}(\phi)$ is continuous over $[0, 1]$. The derivative of $\bar{\ell}$ w.r.t. ϕ then is:

$$\bar{\ell}'(\phi) = \begin{cases} c \left(\frac{r\nu}{R} - 1 \right) & \text{for } \phi \in \left(0, \frac{R}{\nu} \right), \\ -p_3 < 0 & \text{for } \phi \in \left(\frac{R}{\nu}, 1 \right). \end{cases}$$

Note that $\bar{\ell}'(\phi)$ has a discontinuity at $\phi = \frac{R}{\nu}$. Two sub-cases are then to be distinguished:

(B1) If $\frac{R}{\nu} < r$, then $\bar{\ell}'(\phi) > 0$ for $\phi \in \left[0, \frac{R}{\nu} \right]$ and $\bar{\ell}'(\phi) < 0$ for $\phi \in \left(\frac{R}{\nu}, 1 \right]$. Therefore, $\phi^\star = \frac{R}{\nu}$.

(B2) Otherwise (if $\frac{R}{\nu} > r$) then $\bar{\ell}'(\phi) < 0$ for $\phi \in [0, 1]$. Therefore, $\phi^\star = 0$.

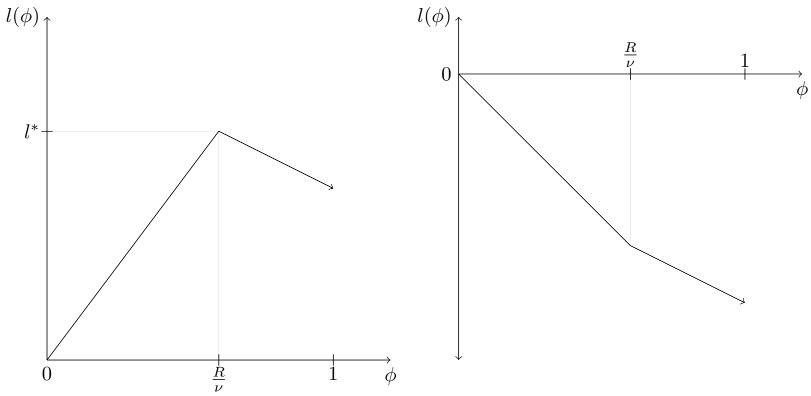

Fig. 2 Long term payoff for different values of r. (Left) Case B1: the maximum occurs at $\phi = \dfrac{R}{\nu}$. (Right) Case B2: the maximum occurs at $\phi = 0$.

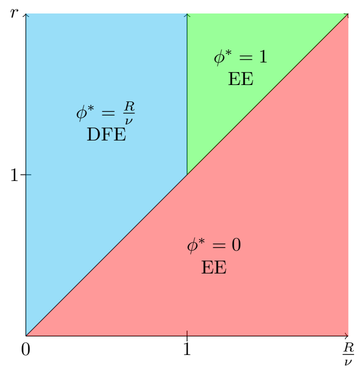

Fig. 3 Results of static optimization: The $\left(\dfrac{R}{\nu}, r\right)$ plane is divided into distinct regions with differing control strategies. The epidemiologic parameter $\frac{R}{\nu}$ represents the ratio of horizontal transmission to vertical transmission, while the economic parameter r represents the ratio of the cost of clean plants to the cost of clean seeds.

3.3 Discussion of Static Optimization Results

The results from static optimization can be summarized in the parameter plane $\left(r, \dfrac{R}{\nu}\right)$ as below.

(S1) $\phi^\star = 0$ if $\dfrac{R}{\nu} > r$.

(S2-1) $\phi^\star = 1$ if $\dfrac{R}{\nu} < r$ and $\dfrac{R}{\nu} > 1$.

(S2-2) $\phi^\star = \dfrac{R}{\nu}$ if $\dfrac{R}{\nu} < r$ and $\dfrac{R}{\nu} < 1$.

Figure 3 displays this summary by dividing the $\left(\frac{R}{\nu}, r\right)$ plane into distinct regions of differing control strategies. Biologically we may interpret the results from static optimization as follows:

(S1) If $\frac{R}{\nu} > r$ then the disease transmission rate (β/μ) is too large relative to the the plant renewal rate or vertical transmission (ν), to make it worthy to use clean seeds (which have a relative cost $1/r$). Hence $\phi^\star = 0$ and the disease is left uncontrolled.

(S2) If $\frac{R}{\nu} < r$ then it is worthy to use clean seeds, and the fraction to be used depends on the relative importance of horizontal versus vertical transmission

  (S2-1) If $R > \nu$, then the optimal strategy is to use as many clean seeds as possible $(\phi^\star = 1)$ to minimize the prevalence of the disease, which remains at an endemic equilibrium. Thus, when horizontal transmission dominates vertical transmission, we use all clean seeds.

  (S2-2) If $R < \nu$, then it is possible to get rid of the disease by using an intermediate proportion of clean seeds (any $\phi \in \left[\frac{R}{\nu}, 1\right]$). Since clean seeds are costly, the optimal proportion of clean seeds is $\phi^\star = \frac{R}{\nu}$, and the disease goes extinct. Thus, the fraction of clean seeds to use is exactly the ratio of the rates of horizontal to vertical transmission.

Prospects

Case (i) is discouraging. One may wonder whether subsidizing clean seeds may help controlling the disease. Let s be the discount on the unit price of clean seeds due to the subsidies, i.e. the unit price of clean seeds is now $c(1-s)$ instead of c. Thus, $r = p_S/c$ is now replaced with $r/(1-s)$, and the condition $\frac{R}{\nu} < r$ becomes

$$\frac{R}{\nu} < \frac{r}{1-s} \,.$$

Therefore, the discount due to subsidies should exceed a critical fraction, i.e.,

$$s > 1 - \frac{r\nu}{R} \,.$$

4 Optimal Control equations

Given the previous system and objective functional, we get the following Hamiltonian:

$$\mathcal{H} = \mu\left[p_S S(t) - c\phi(t)\right] + \lambda(t)\left[(\nu\mu\phi(t) - \beta S(t) + \mu(1-\nu))(1 - S(t))\right] \,. \quad (10)$$

This gives us the following adjoint equation:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial S} = -\mu p_S + \lambda(t)[(1-S(t))\beta + (\nu\mu\phi(t) - \beta S(t) + \mu(1-\nu))] \,. \quad (11)$$

Since the Hamiltonian is linear in ϕ, we define

$$\psi(t) = \frac{\partial \mathcal{H}}{\partial \phi} = \mu\left(-c + \lambda(t)(1-S(t))\nu\right), \quad (12)$$

and let

$$\phi^\star(t) = \begin{cases} 0 & \text{if} \quad \psi(t) \le 0 \\ 1 & \text{if} \quad \psi(t) > 0 \end{cases} .$$

According to Pontryagin's maximum principle, we have

$$\lambda(T) = P_S \,.$$

Defining $S_T = S(T)$,

$$\psi(T) = -\mu c + P_S(1-S_T)\nu\mu \,.$$

Let

$$\hat{S}_T = 1 - \frac{c}{\nu P_S} \,.$$

If $S_T > \hat{S}_T$, then $\psi(T) < 0$ and $\phi^\star(T) = 0$. Otherwise (if $S_T < \hat{S}_T$, which implies $\hat{S}_T > 0$ or equivalently $\nu P_S > c$), $\psi(T) > 0$ and $\phi^\star(T) = 1$.

From now on, we focus on the $\phi^\star(T) = 0$ case to derive a switching curve in the (t, S) plane.

As long as $\phi = 0$ (in backward time), we have, introducing a dot to denote differentiation w.r.t. time,

$$\begin{aligned} \dot{S} &= (1-S)(-\beta S + \mu(1-\nu)) \,, \\ \dot{\lambda} &= \lambda(\beta(1-2S) + \mu(1-\nu)) - \mu p_S \,, \quad \lambda(T) = P_S \,, \end{aligned}$$

and

$$\psi = \mu\left[\nu\lambda(1-S) - c\right] \Longleftrightarrow \nu\lambda(1-S) = \frac{\psi}{\mu} + c \,.$$

This yields

$$\begin{aligned} \dot{\psi} &= \nu\mu\left[\dot{\lambda}(1-S) - \dot{S}\lambda\right] \,, \\ &= \nu\mu(1-S)\left[\lambda(\beta(1-2S) + \mu(1-\nu)) - \mu p_S - \lambda(\mu(1-\nu) - \beta S)\right] \,, \\ &= \nu\mu(1-S)\left[\lambda\beta(1-S) - \mu p_S\right] \,, \\ &= \nu\mu(1-S)\left[\frac{\beta}{\nu}\left(\frac{\psi}{\mu} + c\right) - \mu p_S\right] \,, \\ &= \mu(1-S)\left[\frac{\beta}{\mu}\psi + \beta c - \mu\nu p_S\right] \,, \end{aligned}$$

with $\psi(T) = \mu[\nu P_S(1 - S_T) - c] < 0$ (since we assume $\phi^\star(T) = 0$). One may notice that

$$\dot{\psi}(T) = \nu\mu(1 - S_T)(\beta P_S(1 - S_T) - \mu p_S) < 0$$

if $R/\nu < r$.

Let $t^\star$ be a switching time such that $\psi(t^\star) = 0$. Then

$$\dot{\psi}(t^\star) = \mu(1 - S(t^\star))\left[\beta c - \mu\nu p_S\right] .$$

Consequently, switching from $\phi = 0$ to $\phi = 1$ (in backward time) is possible iff

$$\beta c - \mu\nu p_S < 0 \Longleftrightarrow \frac{R}{\nu} < r .$$

(The adjoint and state variables are necessarily continuous, so the switch function ψ is continuous as well.) As a corollary, the condition $R/\nu > r$ implies that the switch function ψ never crosses zero and so the optimal control is $\phi^\star = 0$ for all $t < T$.

If $R/\nu < r$, then $\dot{\psi} < 0$ (since $\psi(T) < 0$) as long as $\phi = 0$ (in backward time).

Since the Hamiltonian is constant all along the optimal trajectory (see appendix A), we have, for all t such that $\phi^\star = 0$:

$$H(T) = H(t) ,$$
$$\mu p_S S_T + P_S(\mu(1-\nu) - \beta S_T)(1 - S_T) = \mu p_S S + \lambda(\mu(1-\nu) - \beta S)(1 - S) ,$$

which yields

$$\lambda(t) = \frac{\mu p_S(S(t) - S(T)) + P_S(1 - S_T)(\beta S_T - \mu(1-\nu))}{(\beta S(t) - (1-\nu)\mu)(1 - S(t))} .$$

Thus,

$$\psi(t) = \mu\left[\frac{\nu(\mu p_S(S(t) - S_T) + P_S(1 - S_T)(\beta S_T - \mu(1-\nu)))}{\beta S(t) - (1-\nu)\mu} - c\right] .$$

We get

$$S(t^\star) = \frac{c\mu + \nu(P_S\beta S_T(1 - S_T) - \mu((1-\nu)(1 - S_T)P_S + S_T p_S + c))}{\beta c - \nu\mu p_S} . \quad (13)$$

Solving $\dot{S} = (1 - S)(-\beta S + \mu(1-\nu))$ with terminal condition $S(T) = S_T$, we get, for all $t \in [t^\star, T]$:

$$S(t) = \frac{(\beta S_T - \mu(1-\nu))\exp[(\beta - \mu(1-\nu))(T-t)] + \mu(1-\nu)(1 - S_T)}{(\beta S_T - \mu(1-\nu))\exp[(\beta - \mu(1-\nu))(T-t)] + \beta(1 - S_T)} . \quad (14)$$

Equating the expressions of $S(t^\star)$ given by Equations (13) and (14) yields at most 3 possible solutions for S_T as a function of $t^\star$: the first one is $(1-\nu)\mu/\beta$, which corresponds to the endemic equilibrium in absence of control (for $\phi = 0$). Using $S_T = (1-\nu)\mu/\beta$ into Equation (13) yields $S(t^\star) = S_T = (1-\nu)\mu/\beta$.

Since it is an equilibrium line, it cannot be crossed and so does not correspond to a switching curve.

While we can get closed-form expressions of the two remaining solutions, they are not that explicit so they are not presented here at the moment. We conjecture that only one solution is positive and therefore biologically relevant. This is the one we used to draw the switching curves in Figure 4.

4.1 Special case $\nu = 1$ and $P_S = 0$

In the special case $\nu = 1$ and $P_S = 0$, there are only two solutions, zero and

$$S_T = 1 - \frac{\beta c}{\mu p_S} \frac{1}{1 - \exp(\beta(t - T))} .$$

Using Equation (13) finally yields the switching curve $(t^\star, S(t^\star))$ in the plane (t, S) (Figure 4):

$$S(t^\star) = \left(1 - \frac{\beta c}{\mu p_S} \frac{1}{1 - \exp(\beta(t - T))}\right) \frac{\mu p_S}{\mu p_S - \beta c} .$$

4.2 Special case $\nu = 1$ and $P_S > 0$

In the special case $\nu = 1$ and $P_S > 0$, it can be shown that

$$\lim_{t^\star \to T} S(t^\star) = 1 - \frac{c}{P_S} = \hat{S}_T ,$$

meaning that the switching curve crosses the $t = T$ vertical line when $P_S > c$. Moreover, it crosses it exactly at $S = \hat{S}_T$, meaning that trajectories starting (in backward time) with $\phi^\star = 1$ are below the switching curve already, which does not open an empty space in which singular optimal solutions could occur.

5 Discussion

- Bio-economical implications
  - Static optimization yields 3 possible strategies, depending on parameter values: (i) no control, (ii) intermediate control, (iii) full control. Reiterate the biological interpretation from (S1–S2).
  - Subsidizing clean seeds enables switching from the uncontrolled case (i) to the controlled cases (ii) or (iii). We derived a very simple bio-economical threshold for the minimal amount of subsidies to make it economically viable for an individual grower to control the disease.

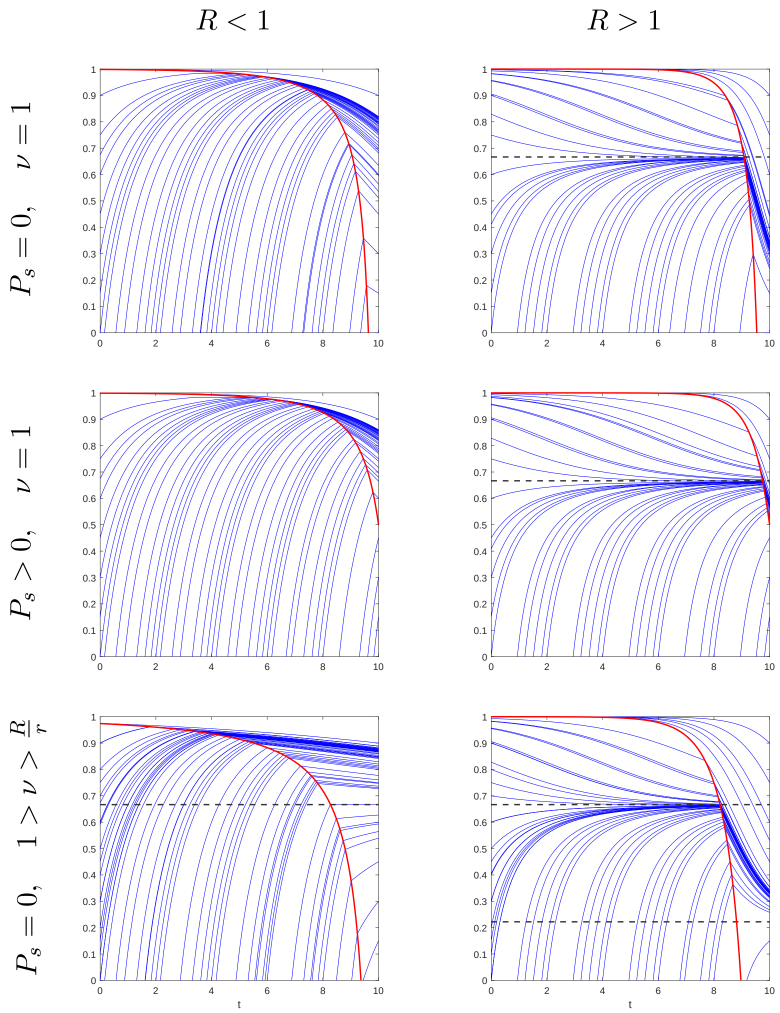

Fig. 4 Dynamics of different combinations of R, P_s, and ν. Blue curves are trajectories under optimal control, Red curve is the switching curve, and dashed lines are equilibrium values for $\phi = 0$ and $\phi = 1$ when they differ from $\bar{S} = 0$ and $\bar{S} = 1$ respectively.

– However, subsidizing clean seeds does not make it possible to switch from full control (iii), where the pathogen persists at endemic equilibrium, to intermediate control (ii), where the pathogen goes extinct. The transition from (iii) to (ii) can only be made possible through other possible control methods decreasing horizontal transmission (e.g. partially resistant plants).
– Decreasing vertical transmission has the opposite effect as it decreases the utility of clean seeds. Conversely, increasing vertical transmission may allow one to get rid of the pathogen (transitions from (i) to (iii)

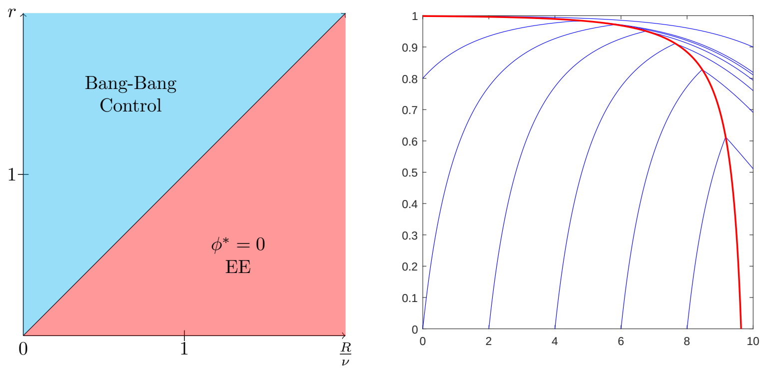

Fig. 5 Results of dynamic optimization: (Right) The $\left(\frac{R}{\nu}, r\right)$ plane is divided into distinct regions with differing control strategies. (Left) Trajectories showing the optimal bang-bang control strategy for specific values of R, r, and ν.

to (ii), or from (i) to (ii), depending on whether the unit price of clean seeds is greater than the unit price of healthy plants) as its makes control economically beneficial. Depending on whether full control enables getting rid of the disease, the epidemiological dynamics converge either to the endemic equilibrium or to the disease-free state. Hence, decreasing vertical transmission can have a counter-productive effect. Breeding for partially resistant plants decreasing the vertical transmission rate of the pathogen should therefore be given caution.

– An alternative way to decrease vertical transmission is to sort out diseased seeds from the local pool of seeds to be replanted, which amounts to considering an additional parameter in the model (the probability not to detect that a seed is infected, p in van den Bosch et al 2006 notations). This opens an interesting avenue for future (if not present) research.
– However, a dynamic optimization approach maximizing an economically relevant finite-time horizon criterion shows that the above distinction between (ii) and (iii) is likely oversimplified as in practice, it may take a long time to reach an equilibrium. Rather, the optimal strategy is either (i) no control or (iv) bang-bang control, i.e. full control followed by no control. Intermediate control is not optimal/rational. If a grower uses clean seeds, she should use only clean seeds.
– Although static and dynamic optimization approaches yield qualitatively contrasting results, the subsidies threshold is the same following both approaches.
– Do not forget to mention that there may be an incentive to control even if the unit cost of clean seeds is greater than the unit benefit of harvested healthy plants. This reflects at population-scale effect (a few clean seeds may apparently protect a greater number of plants).

- Connections to logistic model and differences
- Next paper p, omega
  - Discuss non bang-bang and smoothing on control
- Connections to animal disease control?

6 Conclusions

7 Appendix A

We define an *extremal trajectory* as one where the control ϕ maximizes or minimizes the hamiltonian. We write it $\hat{\phi}$. It is a function of both the state variable(s) (here S) and the adjoint variable(s) (here λ). We want to prove the theorem:

Theorem 1 (First integral of the energy) *If both the dynamics and the running cost are time-invariant, the hamiltonian is constant along an extremal trajectory.*

Proof To avoid any confusion in the notation, we use here Dieudoné's notation for the partial derivatives of the hamiltonian $\mathcal{H}(S, \lambda, \phi)$ as

$$\begin{aligned} \mathrm{D}_1\mathcal{H}(S, \lambda, \phi) &= -\dot{\lambda}(S, \lambda, \phi)\,, \\ \mathrm{D}_2\mathcal{H}(S, \lambda, \phi) &= \dot{S}(S, \lambda, \phi)\,. \end{aligned}$$

Let $\widehat{\mathcal{H}}(S, \lambda) := \mathcal{H}(S, \lambda, \hat{\phi}(S, \lambda))$. It follows from Danskin's theorem (see [2]) that, if the extremalizing $\hat{\phi}$ is unique,

$$\begin{aligned} \mathrm{D}_1\widehat{\mathcal{H}}(S, \lambda) &= \mathrm{D}_1\mathcal{H}(S, \lambda, \hat{\phi}(S, \lambda))\,, \\ \mathrm{D}_2\widehat{\mathcal{H}}(S, \lambda) &= \mathrm{D}_2\mathcal{H}(S, \lambda, \hat{\phi}(S, \lambda)) \end{aligned}$$

Moreover, the only way $\hat{\phi}$ could be non-unique would be on a singular arc, where $\mathrm{D}_3\mathcal{H} = 0$, so that the conclusion of Danskin's theorem would still hold. Hence we have

$$\frac{\mathrm{d}\widehat{\mathcal{H}}}{\mathrm{d}t} = \mathrm{D}_1\widehat{\mathcal{H}}(S, \lambda)\dot{S}(S, \lambda, \hat{\phi}) + \mathrm{D}_2\widehat{\mathcal{H}}(S, \lambda)\dot{\lambda}(S, \lambda, \hat{\phi}) = -\dot{\lambda}\dot{S} + \dot{S}\dot{\lambda} = 0\,.$$

Q.E.D.

8 Appendix B

In order to numerically compute the optimal control for various values for each parameter, we modified a method named Forward-Backward Sweep as described in [3]. This method uses the defined initial conditions of $S(\tau) = S_\tau$ and $\lambda(T) = \lambda_T$ and an initial guess of the control $\phi(t) = \phi_0(t)$ and refines this guess until some convergence criterion has been met. The steps in this method are described in the below steps:

1. Determine an initial guess for the control $\phi_0(t)$.
2. Solve $S(t)$ by moving forward in time on the region $[\tau, T]$ and backward in time on the region $[0, \tau]$ using Runge-Kutta methods (RK4) to approximate the solution to the ODE.
3. Use these values of of $S(t)$ to approximate the solution of the ODE for $\lambda(t)$ by moving backward in time on the region $[0, T]$ and using RK4 with the same timesteps.
4. Calculate an approximation to $\psi(t)$ using $S(t)$ and $\lambda(t)$ and use this to determine an update function $\phi_{fix}(t)$.
5. Update the guess using values of $\phi_{i-1}(T)$ and $\phi_{fix}(t)$, we use $\phi_i(t) = (1-\alpha)\phi_{i-1}(t) + \alpha\phi_{fix}(t)$ for some $\alpha \in (0, 1)$.
6. Continue this process from step 2 until some set of convergence criteria is reached.

We used this method to compute the optimal trajectories in Figure 4. In our case, we found that $\alpha = 0.1$ was sufficient to generate the optimal control trajectories represented, with $\Delta t = 10^{-3}$ the timestep for RK4 and $\phi_0(t) = 0.5$ the initial guess for the control.

9 [Appendix C]

To show that there is no singular region of the optimal control in the cases observed, we turn to proof by contradiction.

Assume ϕ^* is singular. Then, by the definition of ϕ^*, this means that there exists some interval I such that $\psi(t) = 0$ on this interval,

$$\psi(t) = \mu(-c + \lambda(t)(1 - S(t))\nu) = 0.$$

Solving for λ, we get that

$$\lambda(t) = \frac{c}{\nu(1 - S(t))}$$

whenever $S(t) \neq 1$. We can then take the time derivative of this quantity and use the definition of $\frac{dS}{dt}$ to get the relationship

$$\frac{d\lambda}{dt} = \frac{c}{\nu(1 - S(t))}(-\beta S(t) + \mu\phi\nu + \mu(1 - \nu)).$$

We then equate this to the relationship for $\dfrac{d\lambda}{dt}$ derived from the Hamiltonian in Equation 11 and cancel out terms to get that

$$-\mu p_s + \frac{c}{\nu}\beta = 0.$$

However, we can easily use the definitions of R and r to rewrite this as

$$\frac{R}{\nu} = r,$$

which is on the line dividing the regions in Figure 5. Therefore, there is no singular region of the optimal control along the switching curve as we are only considering the cases $\frac{R}{\nu} > r$ and $\frac{R}{\nu} < r$.

10 [Appendix D]

In the case of an infinite horizon, we are interested in maximizing

$$J(\phi(\cdot)) = \mu \int_0^\infty e^{-\delta t}(p_S S(t) - c\phi(t))dt, \tag{15}$$

with respect to $\phi \in [0, 1]$, subject to

$$\frac{dS}{dt} = (1 - S(t))(-\beta S + \mu\phi\nu + \mu(1 - \nu)). \tag{16}$$

To show that the integral in Equation 15 converges, we use the property that both $S(t)$ and $\phi(t)$ are bounded between 0 and 1:

$$\begin{aligned}
\left|\mu \int_0^\infty e^{-\delta t}(p_S S(t) - c\phi(t))dt\right| &\le \mu \int_0^\infty \left|e^{-\delta t}(p_S S(t) - c\phi(t))\right| dt \\
&\le \mu \int_0^\infty e^{-\delta t} \left|(p_S S(t) - c\phi(t))\right| dt \\
&\le \mu \int_0^\infty e^{-\delta t} \max(p_S, c) dt \\
&= \frac{\mu \max(p_S, c)}{\delta}.
\end{aligned}$$

The Hamiltonian $\mathcal{H}$ and the switch function $\psi(t)$ are slightly modified in respect to the finite-horizon problem addressed in Section 4. This results in the adjoint equation

$$\frac{d\lambda}{dt} = -\mu p_S e^{-\delta t} + \lambda(t)[(1 - S(t))\beta + (\nu\mu\phi(t) - \beta S(t) + \mu(1 - \nu))],$$

and switch function

$$\psi(t) = \mu\left(-ce^{-\delta t} + \lambda(t)(1 - S(t))\nu\right).$$

We are interested in a singular control $\phi^* \in (0, 1)$ such that $\psi = 0$ all long the singular part of the optimal trajectory. This yields

$$\lambda(t) = \frac{ce^{-\delta t}}{\nu(1 - S(t))},$$

and differentiating this in respect to time gives us

$$\frac{d\lambda}{dt} = ce^{-\delta t}\frac{-\delta + \mu\phi\nu - \beta S + \mu(1 - \nu)}{\nu(1 - S(t))}.$$

Setting these two equations for the time derivative of λ equal to each other and solving for $S(t)$ results in the following equation for S^*

$$S^* = 1 - \frac{\delta}{\nu\mu\left(r - \frac{R}{\nu}\right)}.$$

A necessary condition for the singular control to exist is $S^* < 1$, which occurs whenever

$$r > \frac{R}{\nu}.$$

This ratio has appeared in previous analysis, and represents the conditions where it is economically beneficial to use clean seeds. From here, we have that $S^* > 0$ if and only if

$$\delta < \nu\mu\left(r - \frac{R}{\nu}\right),$$

meaning the discount rate must be small enough for long term interests to prevail over short term interests.

If $\phi^* \in (0,1)$ is constant, then the dynamics of the system result in the equilibrium point

$$\bar{S} = \frac{1}{R}\left(\phi^*\nu + (1-\nu)\right).$$

Equating $\bar{S}$ and S^* results in the following expression for ϕ^*

$$\phi^* = \frac{1}{\nu}\left(R\left(1 - \frac{\delta}{\nu\mu\left(r - \frac{R}{\nu}\right)}\right) - (1-\nu)\right) = \frac{1}{\nu}\left(RS^* - (1-\nu)\right).$$

From this, we get that $\phi^* > 0$ only if

$$\delta > \frac{1}{R}(1-\nu)\left(\nu\mu\left(r - \frac{R}{\nu}\right)\right),$$

and $\phi^* < 1$ only if

$$\delta > \nu\mu\left(r - \frac{R}{\nu}\right)\left(1 - \frac{\nu}{R}\right).$$

This leads to three conditions on δ for there to be a singular solution. Letting $K = \nu\mu\left(r - \frac{R}{\nu}\right)$, these can be represented as

$$K\left(1 - \frac{\nu}{R}\right) < \delta < K \tag{17}$$

when $R > 1$, and

$$K\frac{1-\nu}{R} < \delta < K \tag{18}$$

when $R < 1$.

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