Uncoupling Isaacs equations in two-player nonzero-sum differential games. Parental conflict over care as an example.

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Abstract

We consider a two-player nonzero-sum differential game in mixed strategies between two players who have, at each instant of time, two pure strategies to mix between. The search for a Nash equilibrium in state feedback leads to a pair of uncoupled first order PDE's. We give an example in Behavioural Ecology.

Key words: Differential Games, Mixed strategies, Nash equilibrium, Behavioral Ecology.

1 Introduction

We deal here with two player differential games with scalar controls, dynamics and payoff affine in each of the controls (and not necessarily jointly affine). A knowledgeable situation where this arises, which will provide our example, is a game situation where each player has only two pure strategies and mixes among them. Thus this is also a result about mixed strategy Nash equilibria in differential games.

The existing literature on mixed strategies in differential games is rather scarce, and even more so concerning nonzero-sum games. Most of it concerns open-loop strategies and existence theorems (see **??**). **?** do consider state feedback mixed strategies, in the context of zero sum two person differential games, and extend their "alternative" theorem, which is the equivalent of the existence of the Value.

As a matter of fact, mixed strategies in dynamical problems are closely related to the concept of relaxed controls, which amount to convexifying the holograph domain (or "vectorgram" in Isaacs' parlance). The reader is referred to **?** and the references therein.

Here, we use the Isaacs equation approach. It is known that it leads to a pair of coupled first order partial differential equations, for which no simple characteristics theory is available in general nor any existence result. Yet, we exploit the linearity in the control variables to uncouple these two PDE's in a fashion very much reminiscent of the equalization theorem of static games.

We provide as an example a simplified model of a problem in Behavioral Ecology.

2 The game

We consider the following two-player nonzero-sum differential game (**?**):

$$\dot{x} = f(x,u,v)\,, \quad x(0) = \xi\,, \tag{1}$$

$$T = \inf\{t : x(t) \in \mathcal{C}\} \quad \text{and} \quad \forall i \in \{1,2\}\,,$$

$$J_i(\xi, u(\cdot), v(\cdot)) = \int_0^T \ell_i(x(t),u(t),v(t))\mathrm{d}t + K_i(x(T))\,.$$

Here, the controls u and v are scalars belonging to compact intervals U and V respectively, the control functions $u(\cdot) \in \mathcal{U}$ (resp. $v(\cdot) \in \mathcal{V}$) are measurable functions from $\mathbb{R}$ to U (resp. V). The functions f and ℓ_i satisfy standard regularity and growth properties that guarantee existence and uniqueness of the generated trajectories and of the payoffs.

We are interested in a Nash equilibrium, in feedback strategies. To handle state feedback strategies in a dynamic game context, we extend the set up of **??** to non

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zero sum games. We call $\phi(x)$ and $\psi(x)$ feedback strategies for player one and two respectively, i.e. maps from $\mathbb{R}^n$ to U and V, and provided that (1) has a unique solution, we use such notations as $J_1(\xi, \phi, v(\cdot))$ to mean the payoff obtained by player one when it uses the feedback $u(t) = \phi(x(t))$ while player two uses $v(t)$.

Therefore, in keeping with the set up of **?** we shall exhibit a pair of feedback maps $(\phi^\star, \psi^\star)$ such that, on the one hand, the differential equation (1) with $u = \phi^\star(x)$ and $v = \psi^\star(x)$ has a unique solution (we say that $J_i(\xi, \phi^\star, \psi^\star)$ is well defined) for all ξ, as well as $J_1(\xi, u(\cdot), \psi^\star)$ for all $u(\cdot) \in \mathcal{U}$ and $J_2(\xi, \phi^\star, v(\cdot))$ for all $v(\cdot) \in \mathcal{V}$. On the other hand, they satisfy the following Nash inequalities: for all ξ,

$$\begin{cases} \forall u(\cdot) \in \mathcal{U}\,,\ J_1(\xi, u(\cdot), \psi^\star) \le J_1(\xi, \phi^\star, \psi^\star) =: V_1(\xi)\,, \\ \forall v(\cdot) \in \mathcal{V}\,,\ J_2(\xi, \phi^\star, v(\cdot)) \ \le J_2(\xi, \phi^\star, \psi^\star) =: V_2(\xi)\,. \end{cases}$$

We call V_i the *value function* of player i.

Each player faces one of the following stationary Hamilton-Jacobi-Bellman equations:

$$\begin{cases} \forall x \in \mathcal{C}\,,\ V_1(x) = K_1(x) \quad \text{and} \quad \forall x \notin \mathcal{C}\,, \\ 0 = \max_u\{\langle \nabla V_1(x), f(x, u, \psi^\star(x))\rangle + \ell_1(x, u, \psi^\star(x))\} \end{cases},$$

$$\begin{cases} \forall x \in \mathcal{C}\,,\ V_2(x) = K_2(x) \quad \text{and} \quad \forall x \notin \mathcal{C}\,, \\ 0 = \max_v\{\langle \nabla V_2(x), f(x, \phi^\star(x), v)\rangle + \ell_2(x, \phi^\star(x), v)\} \end{cases}.$$

For $i = 1, 2$, let $\lambda_i = \nabla V_i$, $H_i = \langle \lambda_i, f\rangle + \ell_i$, and let $\sigma_1 = \partial H_1/\partial u$ and $\sigma_2 = \partial H_2/\partial v$.

Assume that f and the ℓ_i's are affine with respect to both u and v, i.e. $f = A + Bu + Cv + Duv$ and $\ell_i = p_i + q_i u + r_i v + s_i uv$. Write the hamiltonians as $H_1(x, \lambda_i, u, v) = \rho_1(x, \lambda_1, v) + \sigma_1(x, \lambda_1, v)u$ and symmetrically for H_2. Thus v arises linearly in ρ_1 and σ_1, as well as u in ρ_2 and σ_2.

We shall make use of the following conditions, for $i = 1, 2$, each of which may or may not be satisfied in any particular problem:

$\mathsf{C1.i}$: $\langle \lambda_i, A\rangle + p_i \ne 0$	$\mathsf{C2.i}$: $\langle \lambda_i, B\rangle + q_i \ne 0$
$\mathsf{C3.i}$: $\langle \lambda_i, C\rangle + r_i \ne 0$	$\mathsf{C4.i}$: $\langle \lambda_i, D\rangle + s_i \ne 0$

2.1 A bi-singular solution

We are looking for a solution where both controls are singular, or, in the interpretation of mixed strategies, where both optimal strategies are mixed.

Therefore, the equalities $\rho_1 = \sigma_1 = \rho_2 = \sigma_2 = 0$ should hold. This translates into

$$\rho_1 = \langle \lambda_1, A\rangle + p_1 + (\langle \lambda_1, C\rangle + r_1)v = 0\,,$$
$$\sigma_1 = \langle \lambda_1, B\rangle + q_1 + (\langle \lambda_1, D\rangle + s_1)v = 0\,,$$

and symmetrically for ρ_2 and σ_2 mutatis mutandis. These equations can have a solution only if the following two determinants, for $i = 1, 2$, are null :

$$\zeta_i(x, \lambda_i) = (\langle \lambda_i, A\rangle + p_i)(\langle \lambda_i, D\rangle + s_i) \\ \qquad - (\langle \lambda_i, B\rangle + q_i)(\langle \lambda_i, C\rangle + r_i) = 0\,.$$

Assume that both conditions $\mathsf{C4.i}$, $i = 1, 2$, hold. The players may choose $\phi^\star(x) = \hat\phi(x, \lambda_2(x))$ and $\psi^\star(x) = \hat\psi(x, \lambda_1(x))$, according to

$$\hat\psi(x, \lambda_1) = -\frac{\langle \lambda_1, B\rangle + q_1}{\langle \lambda_1, D\rangle + s_1}\,, \quad \hat\phi(x, \lambda_2) = -\frac{\langle \lambda_2, C\rangle + r_2}{\langle \lambda_2, D\rangle + s_2}\,.$$

Assume to the contrary that, say, $\mathsf{C4.1}$ is violated. Then the equation $\zeta_i = 0$ implies that either $\mathsf{C2.i}$ or $\mathsf{C3.i}$ is also. If it is $\mathsf{C3.i}$, there is no singular solution to our problem. Otherwise, we may choose

$$\hat\psi(x, \lambda_1) = -\frac{\langle \lambda_1, A\rangle + p_1}{\langle \lambda_1, C\rangle + r_1}\,, \quad \hat\phi(x, \lambda_2) = -\frac{\langle \lambda_2, A\rangle + p_2}{\langle \lambda_2, B\rangle + q_2}\,.$$

which, in general, coincides with the previous form.

Now, the Isaacs equations degenerate into two uncoupled partial differential equations, for $i = 1, 2$:

$$\begin{cases} \forall x \in \mathcal{C}\,, \quad V_i(x) = K_i(x)\,, \\ \forall x \notin \mathcal{C}\,, \quad \zeta_i(x, \nabla V_i(x)) = 0\,. \end{cases}$$

Notice that if x is scalar then each candidate adjoint is solution of an (at most) quadratic equation. Otherwise, one may attempt to solve these PDE's via their characteristic curves $x_1(s)$ and $x_2(s)$ [1] according to the equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}s} = \nabla_{\lambda_i}\zeta_i \qquad \frac{\mathrm{d}\lambda_i}{\mathrm{d}s} = -\nabla_x\zeta_i\,.$$

Notice that the characteristic curves are generally not game trajectories.

Once the candidate adjoints are obtained, one has to verify whether conditions $\mathsf{C3.i}$ or $\mathsf{C4.i}$ hold, so that one can calculate a candidate equilibrium feedback strategy.

[1] the subscript of x denotes a field of characteristics, not a coordinate.

If so, it remains to check wether these are admissible: lying in U resp. V, and regular enough to get the existence properties sought for (1). If yes, we have exhibited a Nash equilibrium in state feedback.

3 Enforcing bilinearity via mixed strategies

Assume that the players' controls are $\mathsf{u},\mathsf{v} \in \{0,1\}$. Let $f_{\mathsf{uv}}(x) = f(x,\mathsf{u},\mathsf{v})$ and the dynamics can be written

$$\dot{x} = (1-\mathsf{u})(1-\mathsf{v})f_{00}(x) + (1-\mathsf{u})\mathsf{v}f_{01}(x) + \mathsf{u}(1-\mathsf{v})f_{10}(x) + \mathsf{uv}f_{11}(x)\,.$$

In a similar fashion, let $\ell^i_{\mathsf{uv}}(x) = \ell^i(x,\mathsf{u},\mathsf{v})$; ℓ can thus also be written

$$\ell^i(x,\mathsf{u},\mathsf{v}) = (1-\mathsf{v})(1-\mathsf{u})\ell^i_{00}(x) + (1-\mathsf{u})\mathsf{v}\ell^i_{01} + \mathsf{u}(1-\mathsf{v})\ell^i_{10} + \mathsf{uv}\ell^i_{11}(x).$$

We follow **?** and, to some extent, **?**, in identifying mixed strategies with relaxed controls in the tradition of **?**. In a dynamic setting, this can be seen as the limit of a very high frequency chatter between the two possible controls, provided, though, that it be specified that the two chatters cannot be synchronized. Then, we let u and $v \in [0,1]$ be the mathematical expectations of u and v respectively. (Linearity allows us to take the mathematical expectations on the controls rather than on f and the ℓ_i's.)

Connecting these notations to those of the previous section lets $A = f_{00}$, $B = f_{10} - f_{00}$, $C = f_{01} - f_{00}$ and $D = f_{11} - f_{10} - f_{01} + f_{00}$. Similarly, we have $p_i = \ell^i_{00}$, $q_i = \ell^i_{10} - \ell^i_{00}$, $r_i = \ell^i_{01} - \ell^i_{00}$ and $s_i = \ell^i_{11} - \ell^i_{10} - \ell^i_{01} + \ell^i_{00}$.

4 Parental conflict over care as an example

"*Nothing in Biology makes sense except in the light of evolution*" [2] . In this respect, Behavioral Ecology interprets animal behavior from an evolutionary point of view, i.e., in terms of being a stable outcome of the Natural Selection process. As this is genetic information which is conserved through Selection, animal behavior is expected to be shaped in such a way that it is particularly efficient at transmitting individual's genetic inheritance throughout generations (**?**).

"*Conflict between parents over care of young arises when the young benefit from the effort of both parents, but each parent suffers a reduction in future reproductive success as a consequence of its own effort*" (**?**). Such a situation is addressed as a nonzero sum differential game by **?**. The authors are interested in a symmetric Nash Equilibrium in a symmetric game (nothing, but the name, distinguishes the male from the female). Also, they do not allow the offspring to suffer from having no care from its parents: in this case, its welfare remains invariant. The time horizon corresponds to the length of the breeding season. Interestingly, the solution of the game consists in providing an increasing effort all along the season.

Although considering a fixed time horizon is likely to be biologically more relevant, our ambition in this paper is restricted to providing a simple example of how *stationary* [3] Isaacs equations can be uncoupled. We shall thus consider that the game ends when the offspring development has reached a given state. We, being profane in ornithology, have in mind a young bird for who two outcomes are possible: either its has been sufficiently cared by its parents to be able to fly, or it is doomed.

We assume that both parents initially invested in the production of an offspring. Its initial welfare is $x(0) = \xi > 0$. If no one cares the young, its welfare x decreases at a rate $\delta(x) > 0$ down to zero. Caring the young requires an effort whose cost is $\epsilon(x) > 0$ by time unit, for both parents. Nevertheless, we allow for a difference between them in terms of their individual caring ability $\alpha_i(x)$. Assume therefore that $\forall x\,,\ \alpha_1(x) > \alpha_2(x) > 0$. If both parents care, the growth rate of the young is $\forall x\,,\ \gamma(x) > \alpha_1(x)$. The functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot)$ and $\epsilon(\cdot)$ are all supposed to be of class C^1. The problem terminates when either the young dies, i.e. $x = 0$, or it is able to fly on its own wings, i.e. when x reaches a prescribed level Ξ, of course larger than ξ. Moreover, we assume that both parents have better see the game terminate, so that we give them both an unfavorable payoff $\mathcal{N}$, e.g. 0, if it does not.

We thus have $\mathcal{C} = \{0,\Xi\}$, $x \in (0,\Xi)$, $(u,v) \in [0,1]$,

$$\forall i \in \{1,2\}\,,\ K_i(x) = x(T) \quad \text{and}$$

$$\left|\begin{array}{l} f_{00}(x) = -\delta(x), \\ f_{01}(x) = \alpha_2(x), \\ f_{10}(x) = \alpha_1(x), \\ f_{11}(x) = \gamma(x), \end{array}\right. \quad \left|\begin{array}{l} \ell^1_{00}(x) = 0, \\ \ell^1_{01}(x) = 0, \\ \ell^1_{10}(x) = -\epsilon(x), \\ \ell^1_{11}(x) = -\epsilon(x), \end{array}\right. \quad \left|\begin{array}{l} \ell^2_{00}(x) = 0, \\ \ell^2_{01}(x) = -\epsilon(x), \\ \ell^2_{10}(x) = 0, \\ \ell^2_{11}(x) = -\epsilon(x). \end{array}\right.$$

[2] Theodosius Dobzhansky, geneticist, 1900-1975.

[3] Nevertheless, it is well-known that any game can me remade up as stationary, via an augmented state.

Thus [4]

$$
\left|\begin{array}{l} A=-\delta \\ B=\alpha_1+\delta, \\ C=\alpha_2+\delta, \\ D=\gamma-\alpha_1-\alpha_2-\delta, \end{array}\right| \begin{array}{l} p_1=0, \\ q_1=-\epsilon, \\ r_1=0, \\ s_1=0, \end{array} \left|\begin{array}{l} p_2=0, \\ q_2=0, \\ r_2=-\epsilon, \\ s_2=0. \end{array}\right.
$$

As $\forall i \in \{1,2\}$, $p_i s_i - q_i r_i = 0$, $\zeta_1 = 0$ and $\zeta_2 = 0$ respectively yield

$$
\lambda_1 = \frac{(\alpha_2+\delta)\epsilon}{\gamma\delta+\alpha_1\alpha_2} \quad \text{and} \quad \lambda_2 = \frac{(\alpha_1+\delta)\epsilon}{\gamma\delta+\alpha_1\alpha_2}.
$$

Hence

$$
\psi^\star(x) = \frac{\delta(x)}{\alpha_2(x)+\delta(x)} \quad \text{and} \quad \phi^\star(x) = \frac{\delta(x)}{\alpha_1(x)+\delta(x)}.
$$

Both controls are admissible. Noticeably, they do not depend on ϵ. However, one would be expected to make no effort under a sufficiently large ϵ. Let us investigate under what circumstances this could be optimum, say for player 1.

Notice that $f(x,u,\psi^\star) \geq 0$, with equality only for $u = 0$. Therefore, the trajectory driven by $(u(\cdot),\psi^\star)$ cannot reach $x = 0$. Assume that $u(\cdot)$ is such that the game ends at $x(T) = \Xi$. Then

$$
J_1(\xi, u(\cdot), \psi^\star) = x(T) - \int_0^T \epsilon(x(t))u(t)\mathrm{d}t\,.
$$

The dynamics with $v = \psi^\star(x)$ can be written as

$$
\mathrm{d}x = \left[(\alpha_1+\delta) + \frac{\delta}{\alpha_2+\delta}(\gamma-\alpha_1-\alpha_2-\delta)\right] u\,\mathrm{d}t\,,
$$

so that we also get

$$
J_1(\xi, u(\cdot), \psi^\star) = \Xi - \int_\xi^\Xi \frac{(\alpha_2+\delta)\epsilon}{\gamma\delta+\alpha_1\alpha_2}\mathrm{d}x\,.
$$

Therefore $\psi^\star$ is equalizing for player 1 against all controls that make the game terminate. Thus it is clear that playing $(\phi^\star,\psi^\star)$ is better than caring alone. Also, it is better for player 1 to desert than expend resources to breed the young if

$$
\mathcal{N} \geq \Xi - \int_\xi^\Xi \frac{(\alpha_2+\delta)\epsilon}{\gamma\delta+\alpha_1\alpha_2}\mathrm{d}x\,.
$$

[4] we shall sometimes omit to write the dependance in x due to page width constraint or readability.

Hence, an increase in ϵ increases the likelihood that a parent will defect. We get symmetrically, mutatis mutandis, the deserting rule of player 2. If $\alpha_2 < \alpha_1$, it follows that player 2 has a greater incentive to desert than player 1. If no one deserts, the Nash equilibrium consists, for each parent, in investing the effort that, if provided alone, stabilizes the young's welfare. Hence the trajectory goes to Ξ. Notice that it is the most efficient parent that cares the less. Thus arises the following question: could the best parent prevent the desertion of its partner by making a greater offer? The answer is negative, as its offer is precisely the greatest one if expecting its partner to care. Lastly, notice that although $\phi^\star$ and $\psi^\star$ do not depend on α_2 and α_1 respectively, the decision wether to care or desert does depend on both parameters. In which fashion ? The article (?) explores the robustness of the present solution to a time horizon, such as the length of the breeding season, and adresses the existence of other equilibria. In this forthcoming paper, no Greek letter is escorted by x ; it turns out that, in the region where the here discussed Nash Equilibrium still holds, the way the partner's efficiency acts on the propensity to desert depends on the sign of $\sigma := \gamma-\alpha_1-\alpha_2-\delta$. Hence a sexual selection issue.