Deterministic Minimax Impulse Control

N. El Farouq[*], Guy Barles[†] and P. Bernhard[‡]

March 02, 2009, revised september, 15, and october, 21, 2009

Abstract

We prove the uniqueness of the viscosity solution of an Isaacs quasi-variational inequality arising in an impulse control minimax problem, motivated by an application in mathematical finance.

Keywords: impulse control, robust control, differential games, quasi-variational inequality, viscosity solution.
Mathematical subject classification: 35D40, 49L25, 49N70, 91A23, 91G80

1 Introduction

The concept of viscosity solution has become central to control theory ever since the seminal paper of Crandall & Lions [6] following the volume [11]. For first-order Hamilton-Jacobi Equations and applications to optimal control and differential games, achievements due to this concept can be found noticeably in [2] and [1]. Its application to differential games was pioneered by W. Fleming, before its formalization by P.L. Lions et al., in [10], and in modern form by Lions & Souganidis [12], Evans & Souganidis [9], and Souganidis [14]. It is a relatively simple matter to show that the value function of control problems is a viscosity solution of a relevant Hamilton-Jacobi Equation. The strength of the concept, though, lies in the uniqueness. Early uniqueness proofs were based upon the comparison theorem of Crandall & Lions [6] and its variations. The proof we develop here for the QVI of an impulse minimax problem is based upon the Lipschitz character of the dependence of the solution of a variational inequality upon its data.

Previous results in the same direction were obtained by Yong [15] in the finite horizon case but allowing less general jumps. We want also to mention, in the

[*]University Blaise Pascal, Clermont-Ferrand, France(Elfarouq@i3s.unice.fr).
[†]University François Rabelais, Tours, France(Barles@lmpt.univ-tours.fr).
[‡]INRIA-Sophia Antipolis-Méditerranée, France(Pierre.Bernhard@sophia.inria.fr).

infinite horizon case, the works of Dharmatti and Shaiju [13, 7] and Dharmatti and Ramaswamy [8] involving also hybrid controls. See also [3].

2 The problem

2.1 System

2.1.1 Dynamics

Let a two-player differential game system be defined by the solution of following dynamical equations

$$\begin{cases} \dot{y}(t) & = & f(t, y(t), \tau(t)), \\ y(t_0) & = & x \in \mathbb{R}^n, \\ y(t_k^+) & = & y(t_k^-) + g(t_k, y(t_k^-), \xi_k),\ t_k \geq t_0\ \ , \xi_k \neq 0. \end{cases}$$

Here and below, the time variable belongs to $[t_0, T]$ where $T > t_0 \geq 0$ are given. The state at time t, $y(t)$ lies in $\mathbb{R}^n$.

The system is driven by two controls, a "continuous" control $\tau(t) \in \mathcal{K} \subset \mathbb{R}^\ell$, where $\mathcal{K}$ is a compact set, and an impulsive control defined by a finite sequence of impulse times t_k and the controls $\xi_k \in \mathbb{R}^m$ controlling the jumps in $y(t_k)$. Let $\psi = (\{t_k\}, \{\xi_k\})$, where $k \in \mathbb{N}$.

For any initial condition (t_0, x) and controls $\tau(\cdot)$ and ψ generating a trajectory $y(\cdot)$ of this system, let a pay-off J be defined as

$$J(t_0, x, \psi, \tau(.)) = \int_{t_0}^{T} L(t, y(t), \tau(t))\, \mathrm{d}t + \sum_k C(\xi_k) + G(y(T))\,. \tag{1}$$

2.1.2 Regularity assumptions

In all the paper, we assume the following

1. $f(t, y, \tau)$ and $L(t, y, \tau)$ are continuous with respect to t uniformly in y and τ, Lipschitz continuous with respect to y uniformly in t and τ with constant c_f and c_L respectively, and continuous with respect to τ.

2. $g(t, y, \xi)$ is Lipschitz continuous with respect to t, uniformly in y and ξ, with constant $\tilde{c}_{gg}$ and it is Lipschitz continuous with respect to y uniformly in t and ξ, with constant $\tilde{c}_g$.

3. L, f, and G are bounded.

4. C is continuous and $\inf_\xi C(\xi) = \gamma > 0$.

5. G is Lipschitz continuous with constant c_G.

It follows, inter alia, that there exists a unique solution $y(\cdot)$ and a unique J for any measurable $\tau(\cdot)$ and any sequence ψ, and thus for any nonanticipative strategy Φ as defined hereafter.

2.2 Strategies

2.2.1 Multiple jumps

It may be to the best advantage of the minimizer to make a jump at some time t, immediately followed, at the same time, by another jump, and so on. As any such jump entails a cost not less than γ, a near optimal strategy will never attempt to make an infinite number of jumps. As a matter of fact, we shall show that it follows from the boundedness of L and J that the number of jumps may be restricted, with no loss of generality, to be less than $\bar{K} = 2[(T - t_0)\|L\|_\infty + \|G\|_\infty + 1/2]/\gamma$. To allow for the possibility of several successive but simultaneous jumps, we proceed as follows. Let

$$\Xi = \bigcup_{p=1}^{\bar{K}} (\mathbb{R}^m)^p \,.$$

We extend g and C from $\mathbb{R}^m$ to Ξ in the natural way: let $\xi \in (\mathbb{R}^m)^p$ be a *multiple jump* of *multiplicity* p. Let $(\xi^1, \xi^2, \ldots, \xi^p)$ be its components. For a given $y \in \mathbb{R}^n$, let $z_0 = y$, and for $\ell \in \{1, \ldots, p\}$, $z_\ell = z_{\ell-1} + g(t, z_{\ell-1}, \xi^\ell)$. Then we set

$$g(t, y, \xi) = z_p - y \qquad C(\xi) = \sum_{\ell=1}^{p} C(\xi^\ell)\,.$$

There is no point in considering the possibility of several successive multiple jumps at the same jump time, since a sequence of simultaneous multiple jumps is a multiple jump.

It easily follows that g so extended is still Lipschitz continuous in t and in y, uniformly in the other two variables, with new coefficients c_{gg} and c_g respectively.

From now on, when we refer to jumps, it will always be multiple jumps, unless specifically referred to as *simple jumps*. Simple jumps are the same thing as a multiple jump of multiplicity 1. And of course, in a control $\psi = (\{t_k\}, \{\xi_k\})$ the ξ_k are to be considered as multiple jumps. But the t_k's are always assumed to be distinct.

We also state the following definition:

Definition 2.1 *For any bounded function $V : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}$, let the operator $\mathcal{M}$ be given by*

$$\mathcal{M}V(t,x) = \inf_{\xi \in \Xi} [V(t, x + g(t,x,\xi)) + C(\xi)]$$

It enjoys the following properties:

Lemma 2.1 *The operator $\mathcal{M}$ is continuous for the topology $\| \cdot \|_\infty$. Moreover, if the function V is bounded and Lipschitz continuous, so is the function $\mathcal{M}V$.*

Proof The first statement is elementary. Clearly, if, for all (t,x), $|V(t,x)| \leq \|V\|_\infty$, then, clearly $|\mathcal{M}V(t,x)| \leq \|V\|_\infty + \gamma$.

To prove the regularity of $\mathcal{M}V$, we need to make the following remark, which as akin to the lemma 3.5 below: the range of ξ in the $\inf_\xi$ operator defining $\mathcal{M}V$ may be restricted to jumps of multiplicity no larger than $1 + 2\|V\|_\infty/\gamma$. As a matter of fact, the simple jumps 2 to K will "cost" at least $(K-1)\gamma$ and cannot decrease V by more than $2\|V\|_\infty$. Therefore, if $K - 1 > 2\|V\|_\infty/\gamma$, for any jump of multiplicity K, say $\xi = (\xi_1, \ldots, \xi_K)$,

$$V(t, x + g(t,x,\xi_1)) + C(\xi_1) < V(t, x + g(t,x,\xi)) + C(\xi)\,.$$

Over this retricted range $\widetilde{\Xi}$, $(t,x) \mapsto g(t,x,\xi)$ is Lipschitz continuous uniformly in ξ: there exists a constant $C_g > 0$: for all $(t_1, x_1), (t_2, x_2), \xi \in \widetilde{\Xi}$, $\|g(t_1, x_1, \xi) - g(t_2, x_2, \xi)\| \leq C_g(|t_1 - t_2| + \|x_1 - x_2\|)$.

Assume V is Lipschitz continuous, more precisely that

$$\forall (t_1, x_1), (t_2, x_2)\,, \quad |V(t_1, x_1) - V(t_2, x_2)| \leq \lambda(|t_1 - t_2| + \|x_1 - x_2\|)\,.$$

We need now evaluate $\Delta = \mathcal{M}V(t_1, x_1) - \mathcal{M}V(t_2, x_2)$. Choose an arbitrary $\varepsilon > 0$ and a ξ_ε such that

$$V(t_2, x_2 + g(t_2, x_2, \xi_\varepsilon)) + C(\xi_\varepsilon) \leq \mathcal{M}V(t_2, x_2) + \varepsilon\,.$$

Then

$$\begin{aligned}
&\Delta \leq \mathcal{M}V(t_1, x_1) - V(t_2, x_2 + g(t_2, x_2, \xi_\varepsilon)) - C(\xi_\varepsilon) + \varepsilon \\
&\leq V(t_1, x_1 + g(t_1, x_1, \xi_\varepsilon)) + C(\xi_\varepsilon) - V(t_2, x_2 + g(t_2, x_2, \xi_\varepsilon)) - C(\xi_\varepsilon) + \varepsilon \\
&\leq \lambda(1 + C_g)[|t_1 - t_2| + \|x_1 - x_2\|] + \varepsilon\,.
\end{aligned}$$

Since, on the one hand ε was arbitrary, and on the other hand, (t_1, x_1) and (t_2, x_2) play symmetrical roles, we may conclude that

$$|\mathcal{M}V(t_1, x_1) - \mathcal{M}V(t_2, x_2)| \leq \lambda(1 + C_g)[|t_1 - t_2| + \|x_1 - x_2\|]\,.$$

∎

2.2.2 Admissible strategies

We want to investigate the problem of minimizing $\sup_{\tau(.)\in\Omega} J$ through the impulse control. We mean to allow closed loop strategies for the minimizing control. We remark that, being only interested in the $\inf\sup$ problem, and not a possible saddle point, there is no loss of generality in restricting $\tau(\cdot)$ to open loop controls

$$\tau(\cdot) \in \Omega = \{\text{measurable functions } [t_0, T] \to \mathcal{K}\}$$

We shall sometimes write $\tau \in \Omega$ instead of $\tau(\cdot) \in \Omega$.

Let Ψ be the set of all finite sequences ψ. We now define the admissible closed loop strategies Φ for the minimizing impulse control ψ, as *nonanticipative strategies*. We shall let Π be the set of all such nonanticipative strategies.

Definition 2.2 *A map $\Phi : \Omega \to \Psi$ is called a* nonanticipative strategy *if for any two controls $\tau_1(\cdot)$ and $\tau_2(\cdot)$, and any $t \in [t_0, T]$, the condition on their restrictions to $[t_0, t[$: $\tau_1|_{[t_0,t[} = \tau_2|_{[t_0,t[}$ implies $\Phi(\tau_1)|_{[t_0,t]} = \Phi(\tau_2)|_{[t_0,t]}$.*

3 The value function

We define the value function of the problem $W : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ as

$$W(t_0, x) = \inf_{\Phi\in\Pi} \sup_{\tau(.)\in\Omega} J(t_0, x, \Phi(\tau(\cdot)), \tau(.)) \,. \tag{2}$$

3.1 Dynamic Programming Principle

The solution of the minimax impulse control problem at hand depends on the following basic properties

Proposition 3.1 (Isaacs' Tenet of transition) *For all $t \le t' \in [t_0, T[$, for all $x \in \mathbb{R}^n$,*

$$W(t, x) = \inf_{\Phi\in\Pi} \sup_{\tau\in\Omega} \left[\int_t^{t'} L(s, y(s), \tau(s))\mathrm{d}s + \sum_{t_k<t'} C(\xi_k) + W(t', y(t')) \right]$$

Proof Assume first that for some x, t and $t' > t$,

$$W(t, x) > \inf_{\Phi\in\Pi} \sup_{\tau\in\Omega} \left[\int_t^{t'} L(s, y(s), \tau(s))\,\mathrm{d}s + \sum_{t_k<t'} C(\xi_k) + W(t', y(t')) \right],$$

and let the difference be 3ε. Choose an admissible strategy Φ_1^ε that approximates the infimum in the r.h.s. up to ε. Then, for any $\tau(\cdot)$ defined over $[t,t']$,

$$\int_t^{t'} L(s,y(s),\tau(s))\,\mathrm{d}s + \sum_{t_k<t'} C(\xi_k) + W(t',y(t')) \le W(t,x) - 2\varepsilon\,.$$

Whatever $y(t')$, from t' on, choose a non anticipative strategy Φ_2^ε of the game over $[t',T]$ that approximates the Value $W(t',y(t'))$ again up to ε. The concatenation Φ^ε of Φ_1^ε and Φ_2^ε is a nonanticipative strategy of the game over $[t,T]$. It yields, for all $\tau(\cdot)$,

$$J(t,x,\Phi^\varepsilon,\tau(\cdot)) \le W(t,x) - \varepsilon,$$

a contradiction.

Assume to the contrary that

$$W(t,x) < \inf_{\Phi\in\Pi}\sup_{\tau\in\Omega}\left[\int_t^{t'} L(s,y(s),\tau(s))\,\mathrm{d}s + \sum_{t_k<t'} C(\xi_k) + W(t',y(t'))\right],$$

and let the difference be 2ε. Choose an admissible strategy Φ^ε that approximates $W(t,x)$ up to ε, and denote $(t_k^\varepsilon,\xi_k^\varepsilon)$ the jumps it produces. Then,

$$\begin{aligned}
&\sup_{\tau(\cdot)}\left[\int_t^{t'} L(s,y(s),\tau(s))\,\mathrm{d}s + \sum_{t_k^\varepsilon<t'} C(\xi_k^\varepsilon) + J(t',y(t'),\Phi^\varepsilon,\tau)\right]\\
&\le \inf_{\Phi}\sup_{\tau(\cdot)}\left[\int_t^{t'} L(s,y(s),\tau(s))\,\mathrm{d}s + \sum_{t_k<t'} C(\xi_k) + W(t',y(t'))\right] - \varepsilon.
\end{aligned}$$

The above inequality can also be written in terms of the restrictions τ' and τ'' of $\tau(\cdot)$ to $[t,t']$ and $[t',T]$ respectively, as

$$\begin{aligned}
&\sup_{\tau'}\left[\int_t^{t'} L(s,y(s),\tau'(s))\,\mathrm{d}s + \sum_{t_k^\varepsilon<t'} C(\xi_k^\varepsilon) + \sup_{\tau''} J(t',y(t'),\Phi^\varepsilon,\tau'')\right]\\
&\le \inf_{\Phi}\sup_{\tau'}\left[\int_t^{t'} L(s,y(s),\tau'(s))\,\mathrm{d}s + \sum_{t_k<t'} C(\xi_k) + W(t',y(t'))\right] - \varepsilon.
\end{aligned} \tag{3}$$

Observe that the knowledge of τ' is useless to evaluate $J(t',y(t'),\Phi,\tau'')$ once $y(t')$ is known. Therefore, the restriction of Φ^ε to $[t',T]$ cannot do better than a nonanticipative strategy of the game over $[t',T]$. As a result,

$$W(t',y(t')) \le \sup_{\tau''} J(t',y(t'),\Phi^\varepsilon,\tau'')\,.$$

But placing that in the l.h.s. of inequality (3) leads to a contradiction. ∎

Proposition 3.2 *For all $t \in [t_0, T]$ and $x \in \mathbb{R}^n$,*

$$W(t,x) \le \mathcal{M}W(t,x)\,.$$

Proof Among the admissible Φ's there are those that place a jump at time t. Using the same reasoning as above, minimizing over the jump parameter ξ at time t, one gets the required inequality. ∎

Proposition 3.3 *For any admissible strategy Φ, let $t_1 \in \Phi(\tau(\cdot))$ be the first impulse time after t. Then, for all $x \in \mathbb{R}^n$,*

$$W(t,x) = \inf_{\Phi} \sup_{\tau} \left[\int_t^{t_1} L(s, y(s), \tau(s))\,\mathrm{d}s + \mathcal{M}W(t_1, y(t_1)) \right]$$

Proof Proceed as in proposition 3.1 with t' replaced by t_1 as specified by the strategy Φ^ε chosen, and notice that the $\inf$ over $[t_1, T]$ involves a $\inf_{\xi_1}$. ∎

3.2 Regularity of the value function

In this section, we prove the following theorem

Theorem 3.4 *Under the standing assumptions (paragraph 2.1.2) the value function W is bounded, Lipschitz continuous in x uniformly in t, and Lipschitz continuous in t, uniformly in x.*

The rest of the section is devoted to the proof of this theorem.

3.2.1 Boundedness

Let us first show that thanks to hypothesis 3, W as defined by (2) is bounded. On the one hand, a particular strategy Φ is the one where we have no impulse time. In this case $\sum_k C(\xi_k) = 0$. Since L and G are bounded, it is easy to see that

$$\text{for all } t \in [t_0, T] \text{ and } x \in \mathbb{R}^n, \quad W(t,x) \le (T - t_0)\|L\|_\infty + \|G\|_\infty\,.$$

On the other hand, since the costs $C(\xi_k)$ are positive numbers and again since L and G are bounded, then again it is easy to see that

$$\text{for all } t \in [t_0, T] \text{ and } x \in \mathbb{R}^n, \quad W(t,x) \ge -(T - t_0)\|L\|_\infty - \|G\|_\infty\,.$$

We can then say that

$$\|W\|_\infty \le (T - t_0)\|L\|_\infty + \|G\|_\infty$$

proving the first result.

It follows the important property:

Lemma 3.5 *There exists $\bar{K} \in \mathbb{N}$ depending only on the data of the problem such that, in the definition of W, the $\inf_{\Phi\in\Pi}$ can be replaced by $\inf_{\Phi\in\Pi_{\bar{K}}}$ where $\Pi_{\bar{K}}$ is the set of all nonanticipative strategies who's sums of the multiplicity of their jumps is less or equal to $\bar{K}$.*

Proof of the lemma Let us choose a strategy $\bar{\Phi}$ such that

$$W(t,x) \geq \sup_{\tau} \left[\int_t^T L(s,y(s),\tau)\,\mathrm{d}s + \sum_{k=1}^{K} C(\bar{\xi}_k) + G(y(T)) \right] - 1$$

Since

$$-(T-t)\|L\|_\infty - \|G\|_\infty \leq \sup_{\tau} \left[\int_t^T L(s,y(s),\tau)\,\mathrm{d}s + G(y(T)) \right]$$

then

$$\begin{aligned} \sum_{k=1}^{K} C(\bar{\xi}_k) &\leq (T-t)\|L\|_\infty + \|G\|_\infty + \|W\|_\infty + 1 \\ &\leq 2((T-t_0)\|L\|_\infty + \|G\|_\infty) + 1 \end{aligned}$$

with $\gamma = \inf_\xi C(\xi)$, we obtain then

$$K \leq \frac{1}{\gamma}\left(2((T-t_0)\|L\|_\infty + \|G\|_\infty) + 1\right) .$$

This proves that, for the strategies which are close to optimality, the number of impulses, simple or multiple, is bounded, with a bound depending only on the data of the problem. ∎

3.2.2 Lipschitz continuity in x

Let us now prove that the value function W is Lipschitz continuous in x, uniformly in t. To that aim, we estimate the difference

$$\Delta = W(t,x_1) - W(t,x_2)$$

Fix an arbitrary $\varepsilon > 0$. Let Φ_2 be a strategy such that

$$\sup_{\tau} J(t,x_2,\Phi_2,\tau) \leq \inf_{\Phi} \sup_{\tau} J(t,x_2,\Phi,\tau) + \varepsilon = W(t,x_2) + \varepsilon .$$

Then

$$\begin{aligned} &\Delta \leq W(t,x_1) - \sup_\tau J(t,x_2,\Phi_2,\tau) + \varepsilon \\ &\leq \sup_\tau J(t,x_1,\Phi_2,\tau) - \sup_\tau J(t,x_2,\Phi_2,\tau) + \varepsilon . \end{aligned}$$

Now, pick τ_1 such that

$$J(t, x_1, \Phi_2, \tau_1) \geq \sup_{\tau} J(t, x_1, \Phi_2, \tau) - \varepsilon \, .$$

Then

$$\begin{aligned} \Delta &\leq J(t, x_1, \Phi_2, \tau_1) - \sup_\tau J(t, x_2, \Phi_2, \tau) + 2\varepsilon \\ &\leq J(t, x_1, \Phi_2, \tau_1) - J(t, x_2, \Phi_2, \tau_1) + 2\varepsilon \, . \end{aligned}$$

Consider now the control $\psi_2 = \Phi_2(\tau_1)$, composed of jump instants $t_1, t_2 ... t_K$ in the interval $[t, T]$, with jumps $\xi_1, \xi_2 ... \xi_K$, and let $y_1(\cdot)$ and $y_2(\cdot)$ be the trajectories generated by (Φ_2, τ_1), or equivalently (ψ_2, τ_1), from $y_i(t) = x_i$, $i = 1, 2$.

By Gronwall's lemma, we can say

$$\text{for all } s \in [t, t_1], \; \|y_1(s) - y_2(s)\| \leq \exp(c_f(s - t)) \|x_1 - x_2\| \, .$$

Looking more carefully at the first jump and using the Lipschitz continuity of g, we have

$$\begin{aligned} &\|y_1(t_1^+) - y_2(t_1^+)\| \\ &\quad = \|(y_1(t_1^-) + g(t_1, y_1(t_1^-), \xi_1)) - (y_2(t_1^-) + g(t_1, y_2(t_1^-), \xi_1))\| \\ &\quad \leq (1 + c_g) \|y_1(t_1^-) - y_2(t_1^-)\| \\ &\quad \leq \exp(c_f(t_1 - t))(1 + c_g) \|x_1 - x_2\| \, . \end{aligned}$$

Repeating inductively the same argument, we have, for an impulse time t_k, $k \in \{1, 2 ... K\}$,

$$\|y_1(t_k^+) - y_2(t_k^+)\| \leq \exp(c_f(t_k - t))(1 + c_g)^k \|x_1 - x_2\| \, .$$

Finally

$$\text{for all } s \in [t, T], \; \|y_1(s) - y_2(s))\| \leq \exp(c_f(s - t))(1 + c_g)^K \|x_1 - x_2\| \, ,$$

where we recall that K is the number of impulses in ψ_2. Then,

$$\begin{aligned} & J(t, x_1, \Phi_2, \tau_1) - J(t, x_2, \Phi_2, \tau_1) \\ & \leq \int_t^T c_L \|y_1(s) - y_2(s)\| \, \mathrm{d}s + c_G \|y_1(T) - y_2(T)\| \\ & \leq (1 + c_g)^K \|x_1 - x_2\| \left[\int_t^T c_L \exp(c_f(s - t)) \, \mathrm{d}s + c_G \exp(c_f(T - t)) \right] \\ & \leq (1 + c_g)^K \|x_1 - x_2\| \left[\frac{1}{c_f} (\exp(c_f(T - t_0)) - 1) + c_G \exp(c_f(T - t_0)) \right] , \end{aligned}$$

and we can say now $W(t,x_1) - W(t,x_2) \leq C_1\|x_1 - x_2\| + 2\varepsilon$ and, since ε was arbitrary,

$$W(t,x_1) - W(t,x_2) \leq C_1\|x_1 - x_2\|.$$

where the constant

$$C_1 = (1+c_g)^{\bar{K}} \left[\frac{1}{c_f}(\exp(c_f(T-t_0)) - 1) + c_G \exp(c_f(T-t_0))\right],$$

with $\bar{K}$ the maximum number of impulses according to the lemma.

The roles of x_1, x_2 being symmetrical, we finally have $|W(t,x_1) - W(t,x_2)| \leq C_1\|x_1 - x_2\|$.

3.2.3 Lipschitz continuity in t

Let us examine now the difference $W(s_1,x) - W(s_2,x)$. Assume first that $s_2 > s_1$. By the dynamic programming principle,

$$\begin{aligned} &W(s_1,x) - W(s_2,x) = \\ &\inf_{\Phi}\sup_{\tau}\left[\int_{s_1}^{s_2} L(s,y(s),\tau(s))\,\mathrm{d}s + \sum_{t_k<s_2} C(\xi_k) + W(s_2,y(s_2))\right] \\ &\qquad\qquad\qquad\qquad\qquad\qquad -W(s_2,y(s_1))\end{aligned}$$

We choose the strategy Φ where there is no impulse time between s_1 and s_2, then

$$\begin{aligned} &W(s_1,x) - W(s_2,x) \\ &\leq \sup_{\tau}\left[\int_{s_1}^{s_2} L(s,y(s),\tau(s))\,\mathrm{d}s + W(s_2,y(s_2)) - W(s_2,y(s_1))\right] \\ &\leq \|L\|_\infty|s_2 - s_1| + C_1\|f\|_\infty|s_2 - s_1| \\ &\leq c_2|s_2 - s_1|\end{aligned}$$

with $c_2 = \|L\|_\infty + C_1\|f\|_\infty$

Assume now that $s_1 > s_2$. We denote by $y_2(\cdot)$ a trajectory generated by the controls considered from $y(s_2) = x$. By the dynamic programming principle, we have

$$\begin{aligned} &W(s_1,x) - W(s_2,x) = \\ &W(s_1,x) - \inf_{\Phi}\sup_{\tau}\left[\int_{s_2}^{s_1} L(s,y_2(s),\tau(s))\,\mathrm{d}s + \sum_{k|t_k<s_1} C(\xi_k) + W(s_1,y_2(s_1))\right].\end{aligned}$$

Choose an arbirary $\varepsilon > 0$, and a strategy Φ_ε, such that the $\inf_\Phi$ above is reached within ε. We have

$$
\begin{aligned}
& W(s_1, x) - W(s_2, x) \leq \\
& W(s_1, x) - \sup_\tau \left[\int_{s_2}^{s_1} L(s, y_2(s), \tau(s))\,\mathrm{d}s + \sum_{k | t_k < s_1} C(\xi_k) + W(s_1, y_2(s_1)) \right] + \varepsilon \\
& \leq W(s_1, x) - \int_{s_2}^{s_1} L(s, y_2(s), \tau(s))\,\mathrm{d}s - \sum_{k | t_k < s_1} C(\xi_k) - W(s_1, y_2(s_1)) + \varepsilon
\end{aligned}
$$

for any given $\tau(\cdot)$. Let us, from now on fix a $\tau(\cdot)$, and let $\Phi_\varepsilon(\tau(\cdot))$ exhibit K jumps $(\xi_1, \ldots, \xi_K)$ in the interval $[s_2, s_1[$. And let $y_2(\cdot)$ be the trajectory generated by these controls from $y_2(s_2) = x$.

We now bound $W(s_1, x)$ using this sequence of jumps and Proposition 3.2. First, consider the effect of the multiple jump $\tilde{\xi} = (\xi_1, \ldots, \xi_K)$ at time s_1 on x. Let therefore $z_K = x + g(s_1, x, \tilde{\xi})$, constructed as in the definition of multiple jumps, via $z_0 = x$ and $z_k = z_{k-1} + g(s_1, z_{k-1}, \xi_k)$.[1] Using Proposition 3.2, we have

$$
W(s_1, x) \leq W(s_1, z_K) + \sum_{k=1}^{K} C(\xi_k)
$$

Therefore

$$
\begin{aligned}
& W(s_1, x) - W(s_2, x) \leq \\
& \qquad W(s_1, z_K) - \int_{s_2}^{s_1} L(s, y_2(s), \tau(s))\,\mathrm{d}s - W(s_1, y_2(s_1)) + \varepsilon \\
& \leq \quad \|L\|_\infty |s_1 - s_2| + W(s_1, z_K) - W(s_1, y_2(s_1)) + \varepsilon.
\end{aligned}
$$

We can write, for any $\tau(\cdot)$,

$$
\begin{aligned}
& W(s_1, z_K) - W(s_1, y_2(s_1)) = \\
& \qquad W(s_1, z_K) - W(s_1, y_2(t_K^+)) + W(s_1, y_2(t_K^+)) - W(s_1, y_2(s_1)) \\
& \leq \quad C_1 \|z_K - y_2(t_K^+)\| + C_1 \|f\|_\infty |s_1 - s_2|.
\end{aligned}
$$

Now, for $k \in \{1, .., K\}$, we examine the difference $\Delta_k = \|y_2(t_k^+) - z_k\|$.

$$
\begin{aligned}
\Delta_{k+1} &= \|y_2(t_{k+1}^-) + g(t_{k+1}, y_2(t_{k+1}^-), \xi_{k+1}) - z_k - g(s_1, z_k, \xi_{k+1})\| \\
&\leq (1 + c_g)\Delta_k + (1 + c_g)\|f\|_\infty |t_{k+1} - t_k| + c_{gg} |t_{k+1} - s_1|.
\end{aligned}
$$

[1] as opposed to the construction in subsection 2.2.1, here each ξ^k may be a multiple jump itself.

We introduce $\delta_k = (1+c_g)\|f\|_\infty |t_{k+1}-t_k| + c_{gg}|t_{k+1}-s_1|$, then

$$\frac{\Delta_{k+1}}{(1+c_g)^{k+1}} \leq \frac{\Delta_k}{(1+c_g)^k} + \frac{\delta_k}{(1+c_g)^{k+1}} \tag{4}$$

which can be written as

$$u_{k+1} \leq u_k + \frac{\delta_k}{(1+c_g)^{k+1}}$$

with $u_k = \dfrac{\Delta_k}{(1+c_g)^k}$. The strategy considered here is with K impulse times where K is bounded. So,

$$u_K \leq u_1 + \sum_{k=1}^{k=K-1} \frac{\delta_k}{(1+c_g)^{k+1}}$$

On the one hand, for any $k \in \{1,..K\}$

$$\delta_k \leq ((1+c_g)\|f\|_\infty + c_{gg})\,|s_1 - s_2| = \delta = C_2|s_1-s_2|\,.$$

On the other hand, $u_1 = \dfrac{\Delta_1}{1+c_g}$, with

$$\Delta_1 = \|y_2(t_1^-) + g(t_1, y_2(t_1^-), \xi_1) - y_2(s_2) - g(s_1, y_2(s_2), \xi_1)\| \leq \delta$$

Finally,

$$u_K \leq \delta \sum_{k=0}^{K-1} \frac{1}{(1+c_g)^{k+1}} \leq K\delta \leq \bar{K}\delta\,,$$

and hence

$$\Delta_K \leq (1+c_g)^K K\delta \leq (1+c_g)^{\bar{K}}\bar{K}\delta\,.$$

Therefore,

$$W(s_1,x) - W(s_2,x) \leq C_3|s_1-s_2| + \varepsilon$$

where

$$C_3 = \|L\|_\infty + C_1[\|f\|_\infty + C_2(1+c_g)^{\bar{K}}\bar{K}]\,.$$

We let $\varepsilon \to 0$ to obtain

$$W(s_1,x) - W(s_2,x) \leq C_3|s_1-s_2|.$$

Since the inequality is symmetric in s_1 and s_2, it follows that

$$|W(s_1,x) - W(s_2,x)| \leq C_3|s_1-s_2|\,.$$

We then proved that W is Lipschitz continuous with respect to time t, uniformly in x.

3.3 Terminal value

Because of the possible jumps at the terminal time T, it is easy to see that, in general, $W(t,x)$ does not tend to $G(x)$ as t tends to T. Extend the set of multiple jumps to include jumps of *zero multiplicity*, meaning no jump. Call this extended set Ξ_0, extend trivially the operator $\mathcal{M}$ to a function independant from t, and let

$$\widetilde{G}(x) = \inf_{\xi\in\Xi_0}[G(x+g(T,x,\xi))+C(\xi)] = \min\{G(x), \mathcal{M}G(T,x)\}\,. \qquad (5)$$

We know from lemma 2.1 that $\widetilde{G}(x)$ is Lipschitz continuous. We claim

Lemma 3.6

$$W(t,x)\to\widetilde{G}(x) \quad \textit{as} \quad t\to T\,.$$

Proof Fix (t,x) and a strategy Φ. As in the previous proof, for each $\tau(\cdot)$, gather all jumps of $\psi=\Phi(\tau)$, if any, in a multiple jump $\tilde{\xi}$ at the time T, and let $z = x+g(T,x,\tilde{\xi})$. The same argument as previously shows that there exists a constant C_T such that, for all Φ and τ

$$|J(t,x,\Phi,\tau)-[G(z)+C(\tilde{\xi})]|\le C_T(T-t)\,.$$

or

$$J(t,x,\Phi,\tau) = G(z)+C(\tilde{\xi})+O(T-t)\,.$$

The right hand side above only depends on $\tilde{\xi}$, not on $\tau(\cdot)$ itself. It follows that

$$\inf_{\phi}\sup_{\tau} J(t,x,\Phi,\tau) = \inf_{\xi\in\Xi_0}(G(z)+C(\xi))+O(T-t) = \widetilde{G}(x)+O(T-t)\,.$$

The result follows letting $t\to T$. ∎

Remark 3.1 *Because of the uniform convergence of $W(t,x)$ to $\widetilde{G}(x)$ as $t\uparrow T$, using the lemma 2.1 we have $\mathcal{M}W(t,x)\to\mathcal{M}\widetilde{G}(T,x)$ as $t\uparrow T$, uniformly in $\mathbb{R}^n$, and using proposition 3.2, $\widetilde{G}(x)\le\mathcal{M}\widetilde{G}(T,x)$.*

4 Isaacs'quasi-variational inequality

In this section we proove that the value function W is a viscosity solution of the Hamilton-Jacobi-Isaacs quasi-variational inequality, that we replace by an equivalent QVI easier to investigate.

4.1 "Natural" Isaacs' Quasi-Variational Inequality

In the domain $[t_0, T[\times\mathbb{R}^n$, we consider the QVI

$$\max\left\{\min_{\tau\in\mathcal{K}}\left[-\frac{\partial W}{\partial t}-\frac{\partial W}{\partial x}f(t,x,\tau)-L(t,x,\tau)\right],\right.\\ \left. W(t,x)-\mathcal{M}W(t,x)\right\}=0\,. \tag{6}$$

with the terminal condition: $W(T,x)=\widetilde{G}(x)$ in $\mathbb{R}^n$, where $\widetilde{G}$ is given by (5).

Notice that it follows from hypothesis 1 that the term in square brackets in equation (6) above is continuous with respect to τ so that the minimum in τ over the compact $\mathcal{K}$ exists.

Theorem 4.1 *The function:* $(t,x)\longmapsto W(t,x)$ *is a viscosity solution of the quasi-variational inequality (6).*

Proof The proof will be in two parts

(i) Let $\phi\in C^1([t_0,T[\times\mathbb{R}^n)$ and let $(\bar{t},\bar{x})$ be a local maximum for $W-\phi$. We have to prove that $(\bar{t},\bar{x})$ satisfies

$$\max\left\{\min_{\tau}\left[-\frac{\partial\phi}{\partial t}-\frac{\partial\phi}{\partial x}f(\bar{t},\bar{x},\tau)-L(\bar{t},\bar{x},\tau)\right],\right.\\ \left. W(\bar{t},\bar{x})-\inf_{\xi}\left[W(\bar{t},\bar{x}+g(\bar{t},\bar{x},\xi))+C(\xi)\right]\right\}\le 0\,.$$

(ii) Let $\phi\in C^1([t_0,T[\times\mathbb{R}^n)$ and let $(\underline{t},\underline{x})$ be a local minimum for $W-\phi$. We have to prove that $(\underline{t},\underline{x})$ satisfies

$$\max\left\{\min_{\tau}\left[-\frac{\partial\phi}{\partial t}-\frac{\partial\phi}{\partial x}f(\underline{t},\underline{x},\tau)-L(\underline{t},\underline{x},\tau)\right],\right.\\ \left. W(\underline{t},\underline{x})-\inf_{\xi}\left[W(\underline{t},\underline{x}+g(\underline{t},\underline{x},\xi))+C(\xi)\right]\right\}\ge 0\,.$$

Let us prove (i). By Proposition 3.2, we have

$$W(\bar{t},\bar{x})-\inf_{\xi}\left[W(\bar{t},\bar{x}+g(\bar{t},\bar{x},\xi))+C(\xi)\right]\le 0\,.$$

Next for $t'>\bar{t}$, t' close to $\bar{t}$, we choose a strategy Φ where we have no jumps between $\bar{t}$ and t'. By Proposition 3.1, we have

$$W(\bar{t},\bar{x})\le\sup_{\tau(\cdot)}\left[\int_{\bar{t}}^{t'}L(s,y(s),\tau(s))\,\mathrm{d}s+W(t',y(t'))\right]\,.$$

By hypothesis, there exists a neighborhood $\mathcal{N}$ of $(\bar{t}, \bar{x})$ such that $W - \phi$ is maximum at $(\bar{t}, \bar{x})$ in that neighborhood. For small enough $t^{'} - \bar{t} > 0$, since f is bounded, so is $\dot{y}$ and then we can conclude that $(t', y(t')) \in \mathcal{N}$. Hence we get

$$W(t', y(t')) - W(\bar{t}, \bar{x}) \leq \phi(t', y(t')) - \phi(\bar{t}, \bar{x}) \,,$$

thus

$$0 \leq \sup_{\tau(\cdot)} \left[\int_{\bar{t}}^{t'} L(s, y(s), \tau(s)) \, \mathrm{d}s + \phi(t', y(t')) - \phi(\bar{t}, \bar{x}) \right].$$

This yields

$$0 \geq \inf_{\tau(\cdot)} \left[\int_{\bar{t}}^{t'} \left(-\frac{\partial \phi}{\partial s} - \frac{\partial \phi}{\partial y} f(s, y(s), \tau(s)) - L(s, y(s), \tau(s)) \right) \mathrm{d}s \right]. \qquad (7)$$

If

$$\min_{\tau} \left[-\frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) - \frac{\partial \phi}{\partial y} f(\bar{t}, \bar{x}, \tau) - L(\bar{t}, \bar{x}, \tau) \right] > 0 \,,$$

then, this inequality is also true in a neighborhood of $(\bar{t}, \bar{x})$, for any choice of $\tau \in \mathcal{K}$. Then

$$\int_{\bar{t}}^{t'} \left(-\frac{\partial \phi}{\partial s} - \frac{\partial \phi}{\partial y} f(s, y(s), \tau(s)) - L(s, y(s), \tau(s)) \right) \mathrm{d}s > 0 \,,$$

which is in contradiction with inequality (7). Then, we conclude

$$\min_{\tau} \left[-\frac{\partial \phi}{\partial t}(\bar{t}, \bar{x}) - \frac{\partial \phi}{\partial y} f(\bar{t}, \bar{x}, \tau) - L(\bar{t}, \bar{x}, \tau) \right] \leq 0 \,.$$

Let us now prove (ii). Let $(\underline{t}, \underline{x})$ be a local minimum for $W - \phi$. We have, by Proposition 3.2,

$$W(\underline{t}, \underline{x}) - \inf_{\xi} \left[W(\underline{t}, \underline{x} + g(\underline{t}, \underline{x}, \xi)) + C(\xi) \right] \leq 0 \,.$$

If

$$W(\underline{t}, \underline{x}) - \inf_{\xi} \left[W(\underline{t}, \underline{x} + g(\underline{t}, \underline{x}, \xi)) + C(\xi) \right] = 0 \,,$$

then we are done. Otherwise

$$W(\underline{t}, \underline{x}) - \inf_{\xi} \left[W(\underline{t}, \underline{x} + g(\underline{t}, \underline{x}, \xi)) + C(\xi) \right] < 0 \,. \qquad (8)$$

But we have, by Proposition 3.3,

$$W(\underline{t}, \underline{x}) = \inf_{\Phi} \sup_{\tau} \left[\int_{\underline{t}}^{t_1} L(s, y(s), \tau(s)) \, \mathrm{d}s + \mathcal{M}W(t_1, y(t_1)) \right] \,,$$

t_1 being the first impulse time after $\underline{t}$. For any $\varepsilon > 0$, there exists a strategy Φ^ε where t_1^ε is the first impulse time after $\underline{t}$, such that

$$W(\underline{t}, \underline{x}) \geq \sup_\tau \left[\int_{\underline{t}}^{t_1^\varepsilon} L(s, y(s), \tau(s))\, \mathrm{d}s + \mathcal{M}W(t_1^\varepsilon, y(t_1^\varepsilon)) \right] - \varepsilon\,.$$

We prove in the sequel that there exists $\delta > 0$ such that $t_1^\varepsilon \geq \underline{t} + \delta$ for all ε. Assume to the contrary that there exists a sequence $t_1^{\varepsilon_n}$ that converges towards $\underline{t}$ when ε_n converges towards 0.

Let $\eta > 0$ be any fixed real number. Since L is bounded, for n large enough, it holds that, for any $\tau(\cdot)$,

$$\left| \int_{\underline{t}}^{t_1^{\varepsilon_n}} L(s, y(s), \tau(s))\, \mathrm{d}s \right| \leq \|L\|_\infty |t_1^{\varepsilon_n} - \underline{t}| \leq \eta\,.$$

Hence

$$W(\underline{t}, \underline{x}) \geq \mathcal{M}W(t_1^{\varepsilon_n}, y(t_1^{\varepsilon_n})) - \varepsilon_n - \eta.$$

Now we choose ξ_n such that

$$\mathcal{M}W(t_1^{\varepsilon_n}, y(t_1^{\varepsilon_n})) \geq W(t_1^{\varepsilon_n}, y(t_1^{\varepsilon_n}) + g(t_1^n, y(t_1^{\varepsilon_n}), \xi_n)) + C(\xi_n) - \eta\,.$$

Using the continuity of W and of g, it follows that, again for large enough n,

$$W(t_1^{\varepsilon_n}, y(t_1^{\varepsilon_n}) + g(t_1^n, y(t_1^{\varepsilon_n}), \xi_n)) \geq W(\underline{t}, \underline{x} + g(\underline{t}, \underline{x}, \xi_n)) - \eta\,.$$

Finally, we also have

$$W(\underline{t}, \underline{x} + g(\underline{t}, \underline{x}, \xi_n)) + C(\xi_n) \geq \mathcal{M}W(\underline{t}, \underline{x})\,.$$

Combining the last four inequalities, we get, in the limit in n

$$W(\underline{t}, \underline{x}) \geq \mathcal{M}W(\underline{t}, \underline{x}) - 3\eta.$$

This being true for any positive η, we conclude

$$W(\underline{t}, \underline{x}) \geq \mathcal{M}W(\underline{t}, \underline{x})\,.$$

This inequality is in contradiction with inequality (8). This means that $t_1^\varepsilon \geq \underline{t} + \delta$.

Now we choose $t' \in]\underline{t}, \underline{t} + \delta[$. By Proposition 3.1 , we have

$$\begin{aligned} W(\underline{t}, \underline{x}) &= \inf_\Phi \sup_{\tau \in \Omega} \left[\int_{\underline{t}}^{t'} L(s, y(s), \tau(s))\, \mathrm{d}s + W(t', y(t')) \right] \\ &= \sup_{\tau \in \Omega} \left[\int_{\underline{t}}^{t'} L(s, y(s), \tau(s))\, \mathrm{d}s + W(t', y(t')) \right] \\ &\geq \int_{\underline{t}}^{t'} L(s, y(s), \tau)\, \mathrm{d}s + W(t', y(t'))\,, \end{aligned}$$

for any constant $\tau \in \mathcal{K}$ in the last line above. Since $(\underline{t}, \underline{x})$ is a local minimum of $W - \phi$, then for t' close enough to $\underline{t}$, for the same reasons than in (i), $(t', y(t'))$ is in the neighborhood of $(\underline{t}, \underline{x})$, then we have that

$$0 \geq \int_{\underline{t}}^{t'} L(s, y(s), \tau)\,\mathrm{d}s + \phi(t', y(t')) - \phi(\underline{t}, \underline{y}),$$

Then

$$0 \geq \int_{\underline{t}}^{t'} \left(L(s, y(s), \tau) + \frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial y} f(s, y(s), \tau) \right) \mathrm{d}s \,.$$

Dividing by $t' - t$ and letting t' tend to t, we obtain

$$-\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial y} f(\underline{t}, \underline{x}, \tau) - L(\underline{t}, \underline{x}, \tau) \geq 0 \,.$$

and then

$$\min_{\tau} \left[-\frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial y} f(\underline{t}, \underline{x}, \tau) - L(\underline{t}, \underline{x}, \tau) \right] \geq 0 \,.$$

∎

4.2 An Equivalent Quasi-Variational Inequality

In this section, we consider the new function Γ given by the classical change of variable $\Gamma(t, x) = \exp(t) W(t, x)$, for any $t \in [t_0, T]$ and $x \in \mathbb{R}^n$. Of course, the function Γ is bounded and uniformly continuous with respect to its arguments.

A second property is given by the

Proposition 4.2 *W is a viscosity solution of (6) if and only if Γ is a viscosity solution to the following quasi-variational inequality in $[t_0, T[\times \mathbb{R}^n$,*

$$0 = \max \left\{ \min_{\tau} \left[-\frac{\partial \Gamma}{\partial t} + \Gamma(t, x) - \frac{\partial \Gamma}{\partial x} f(t, x, \tau) - \exp(t) L(t, x, \tau) \right], \; \Gamma(t, x) - \mathcal{N}\Gamma(t, x) \right\} \tag{9}$$

where $\mathcal{N}\Gamma(t, x) = \inf_{\xi} (\Gamma(t, x + g(t, x, \xi)) + \exp(t) C(\xi))$. The terminal condition for Γ is: $\Gamma(T, x) = \exp(T) \widetilde{G}(x)$ in $\mathbb{R}^n$.

Proof It suffices to prove that if W is a viscosity solution to (6), then Γ is a viscosity solution to (9). The relation between W and Γ is symmetric and the proof the other way around would be the same. The proof will be in two parts

(i) Let $\phi \in C^1([t_0, T[\times R^n)$ and let $(\bar{t}, \bar{x})$ be a local maximum for $\Gamma - \phi$. We have to prove that $(\bar{t}, \bar{x})$ satisfies

$$\max\left\{\min_{\tau}\left[-\frac{\partial\phi}{\partial t}(\bar{t},\bar{x}) + \Gamma(\bar{t},\bar{x}) - \frac{\partial\phi}{\partial x}(\bar{t},\bar{x})f(\bar{t},\bar{x},\tau) - \mathrm{e}^{\bar{t}}L(\bar{t},\bar{x},\tau)\right],\right.$$
$$\left.\Gamma(\bar{t},\bar{x}) - \mathcal{N}\Gamma(\bar{t},\bar{x})\right\} \leq 0$$

(ii) Let $\phi \in C^1([t_0, T[\times R^n)$ and let $(\underline{t}, \underline{x})$ be a local minimum for $\Gamma - \phi$. We have to prove that $(\underline{t}, \underline{x})$ satisfies

$$\max\left\{\min_{\tau}\left[-\frac{\partial\phi}{\partial t}(\underline{t},\underline{x}) + \Gamma(\underline{t},\underline{x}) - \frac{\partial\phi}{\partial x}(\underline{t},\underline{x})f(\underline{t},\underline{x},\tau) - \mathrm{e}^{\underline{t}}L(\underline{t},\underline{x},\tau)\right],\right.$$
$$\left.\Gamma(\underline{t},\underline{x}) - \mathcal{N}\Gamma(\underline{t},\underline{x})\right\} \geq 0$$

Let us prove (i). For (t, x) in the neighborhood of $(\bar{t}, \bar{x})$, we have

$$M_1 = (\Gamma - \phi)(\bar{t},\bar{x}) \geq (\Gamma - \phi)(t,x) = \exp(t)W(t,x) - \phi(t,x)$$

then

$$0 \geq (W - \Phi)(t,x)$$

where $\Phi(t, x) = \exp(-t)(\phi(t, x) + M_1)$. Remark that $(\bar{t}, \bar{x})$ is also a local maximum to $W - \Phi$. But since W is a viscosity solution to (6), then

$$\max\left\{\min_{\tau}\left[\exp(-\bar{t})(\phi(\bar{t},\bar{x}) + M_1) - \exp(-\bar{t})\frac{\partial\phi}{\partial t}(\bar{t},\bar{x}) - \right.\right.$$
$$\left.\exp(-\bar{t})\frac{\partial\phi}{\partial x}(\bar{t},\bar{x})f(\bar{t},\bar{x},\tau) - L(\bar{t},\bar{x},\tau)\right],$$
$$\left.\exp(-\bar{t})\Gamma(\bar{t},\bar{x}) - \inf_{\xi}(\exp(-\bar{t})\Gamma(t, x + g(t,x,\xi)) + C(\xi))\right\} \leq 0$$

and then

$$\max\left\{\min_{\tau}\left[\Gamma(\bar{t},\bar{x}) - \frac{\partial\phi}{\partial t}(\bar{t},\bar{x}) - \frac{\partial\phi}{\partial x}(\bar{t},\bar{x})f(\bar{t},\bar{x},\tau) - \exp(\bar{t})L(\bar{t},\bar{x},\tau)\right],\right.$$
$$\left.\Gamma(\bar{t},\bar{x}) - \inf_{\xi}\Gamma(t, \bar{x} + g(t,\bar{x},\xi)) + \exp(\bar{t})C(\xi))\right\} \leq 0$$

This means that Γ is a viscosity subsolution to (9). The idea of the proof is the same for point (ii). We then proved that Γ is a viscosity solution to (9). ∎

5 Uniqueness

We are now going to use the equivalent QVI to prove uniqueness of the viscosity solution. To do so, we introduce a variational inequality which we first study.

5.1 General Obstacle Problem

In this section, we are interested by the following variational inequality where the second term under the max in the quasi-variational inequality (9) $(\Gamma - \mathcal{N}\Gamma)(t,x)$ is replaced by the more general term $(\Gamma - P)(t,x)$ where P is in BUC($[t_0, T[\times\mathbb{R}^n$): for all $(t,x) \in [t_0, T[\times\mathbb{R}^n$,

$$0 = \max\left\{\min_\tau\left[-\frac{\partial\Gamma}{\partial t} + \Gamma(t,x) - \frac{\partial\Gamma}{\partial x}f(t,x,\tau) - \exp(t)L(t,x,\tau)\right],\right.$$
$$\left.(\Gamma - P)(t,x)\right\}, \tag{10}$$

with the terminal condition:

$$\forall x \in \mathbb{R}^n, \Gamma(T,x) = \exp(T)\widetilde{G}(x)\,. \tag{11}$$

For compatibility reasons, in order to have a solution for such obstacle problems, one has to assume that the function P satisfies $P(T,x) \geq \exp(T)\widetilde{G}(x)$; by Remark 3.1, this is indeed the case for $P := \mathcal{N}\Gamma(t,x)$. In all the obstacle problems we consider below, we assume implicitly that such condition is fulfilled.

Let Γ_1 and Γ_2 be respectively viscosity solutions of (10) in BUC, where L is equal to L_1, P is equal to P_1 and $\widetilde{G}$ is equal to G_1, respectively L is equal to L_2, P is equal to P_2 and $\widetilde{G}$ is equal to G_2, that is,

$$\max\left\{\min_\tau\left[-\frac{\partial\Gamma_1}{\partial t} + \Gamma_1(t,x) - \frac{\partial\Gamma_1}{\partial x}f(t,x,\tau) - \exp(t)L_1(t,x,\tau)\right],\right.$$
$$\left.(\Gamma_1 - P_1)(t,x)\right\} = 0\,,$$

with the terminal condition: for all $x \in \mathbb{R}^n, \Gamma_1(T,x) = \exp(T)G_1(x)$, and

$$\max\left\{\min_\tau\left[-\frac{\partial\Gamma_2}{\partial t} + \Gamma_2(t,x) - \frac{\partial\Gamma_2}{\partial x}f(t,x,\tau) - \exp(t)L_2(t,x,\tau)\right],\right.$$
$$\left.(\Gamma_2 - P_2)(t,x)\right\} = 0\,,$$

with the terminal condition: for all $x \in \mathbb{R}^n, \Gamma_2(T,x) = \exp(T)G_2(x)$.

Theorem 5.1 *Assume that both Γ_1 and Γ_2 are bounded and uniformly continuous with respect to their arguments. If f, L_2, and G satisfy the assumptions 1, 3, and 5 then,*

$$\begin{aligned} &\|(\Gamma_1-\Gamma_2)^+\|_\infty \leq \\ &\max\Big\{\mathrm{e}^T\|(L_1-L_2)^+\|_\infty, \|(P_1-P_2)^+\|_\infty, \mathrm{e}^T\|(G_1-G_2)^+\|_\infty\Big\}. \end{aligned} \tag{12}$$

The end of this section is devoted to the proof of that theorem. We build the following test-function: let $t, s \in [t_0, T]$ and $x, z \in \mathbb{R}^n$,

$$\phi_{\alpha,\beta,\gamma}(t,x,s,z) = \Gamma_1(t,x) - \Gamma_2(s,z) - \alpha(\|x\|^2 + \|z\|^2) - \frac{\|x-z\|^2}{\beta^2} - \frac{|t-s|^2}{\gamma^2}\,.$$

Let $(\bar{t}, \bar{x}, \bar{s}, \bar{z})$ be a maximum point of $\phi_{\alpha,\beta,\gamma}$ which exists, since this is a continuous function going to infinity when x or z does, and s and t range over a compact set. We need the following lemma for the proof of Theorem (5.1).

Lemma 5.2 *Assume that $M = \sup_{[t_0,T]\times\mathbb{R}^n}(\Gamma_1-\Gamma_2)(t,x) > 0$, let $(\bar{t}, \bar{x}, \bar{s}, \bar{z})$ be a maximal point of $\phi_{\alpha,\beta,\gamma}$, and $M_{\alpha,\beta,\gamma} = \phi_{\alpha,\beta,\gamma}(\bar{t}, \bar{x}, \bar{s}, \bar{z})$. Then, for any $\varepsilon > 0$, there exists α_0, β_0 and γ_0 such that, for any $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, $\gamma \leq \gamma_0$,*

$$\alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) + \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} + \frac{|\bar{t}-\bar{s}|^2}{\gamma^2} \leq \varepsilon\,. \tag{13}$$

Proof By definition, $(\bar{t}, \bar{x}, \bar{s}, \bar{z})$ satisfies, for all $t, s \in [t_0, T], \forall x, z \in \mathbb{R}^n$

$$\begin{aligned} &\Gamma_1(\bar{t},\bar{x}) - \Gamma_2(\bar{s},\bar{z}) - \alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) - \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} - \frac{|\bar{t}-\bar{s}|^2}{\gamma^2} \geq \\ &\Gamma_1(t,x) - \Gamma_2(s,z) - \alpha(\|x\|^2 + \|z\|^2) - \frac{\|x-z\|^2}{\beta^2} - \frac{|t-s|^2}{\gamma^2}\,. \end{aligned}$$

Let us take $t = s$ and $x = z$, then

$$M_{\alpha,\beta,\gamma} \geq \Gamma_1(t,x) - \Gamma_2(t,x) - 2\alpha\|x\|^2\,.$$

Let (t^*, x^*) be a point where the $\sup(\Gamma_1 - \Gamma_2)$ is reached within δ, then

$$\Gamma_1(t^*, x^*) - \Gamma_2(t^*, x^*) \geq M - \delta\,,$$

where $\delta > 0$ can be chosen arbitrarily small. We choose it and α such that $M - \delta - 2\alpha\|x^*\|^2 > 0$, which is always possible because x^* depends only on δ. We have in particular that

$$\begin{aligned} M_{\alpha,\beta,\gamma} \quad &\geq \Gamma_1(t^*, x^*) - \Gamma_2(t^*, x^*) - 2\alpha\|x^*\|^2 \\ &\geq M - \delta - 2\alpha\|x^*\|^2 > 0\,. \end{aligned} \tag{14}$$

Let $r^2 = \|\Gamma_1\|_\infty + \|\Gamma_2\|_\infty$, then

$$0 < M - \delta - 2\alpha\|x^*\|^2 \le M_{\alpha,\beta,\gamma} \le r^2 - \alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) - \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} - \frac{|\bar{t}-\bar{s}|^2}{\gamma^2}$$

Thus,

$$\|\bar{x}-\bar{z}\| \le r\beta \quad \text{and} \quad |\bar{t}-\bar{s}| \le r\gamma\,. \tag{15}$$

The function Γ_2 is assumed uniformly continuous in t and also in x. Let us therefore introduce the following functions:

$$\begin{aligned} m(u) &= \sup_{t,\|x-z\|\le u} |\Gamma_2(t,x) - \Gamma_2(t,z)|\,, \\ n(v) &= \sup_{|t-s|\le v,x} |\Gamma_2(t,x) - \Gamma_2(s,x)|\,. \end{aligned} \tag{16}$$

Clearly, m and n decrease to 0 with their arguments. With these notations, and using (15), we have

$$\begin{aligned} &\Gamma_1(\bar{t},\bar{x}) - \Gamma_2(\bar{s},\bar{z}) = \\ &\Gamma_1(\bar{t},\bar{x}) - \Gamma_2(\bar{t},\bar{x}) + \Gamma_2(\bar{t},\bar{x}) - \Gamma_2(\bar{t},\bar{z}) + \Gamma_2(\bar{t},\bar{z}) - \Gamma_2(\bar{s},\bar{z}) \\ &\qquad \le M + m(r\beta) + n(r\gamma). \end{aligned}$$

Place this in the definition of $M_{\alpha,\beta,\gamma}$ and use (14):

$$\begin{aligned} M - \delta - 2\alpha\|x^*\|^2 &\le M_{\alpha,\beta,\gamma} \\ &\le M + m(r\beta) + n(r\gamma) - \alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) - \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} - \frac{|\bar{t}-\bar{s}|^2}{\gamma^2} \end{aligned}$$

or equivalently

$$\alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) + \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} + \frac{\|\bar{t}-\bar{s}\|^2}{\gamma^2} \le \delta + 2\alpha\|x^*\|^2 + m(r\beta) + n(r\gamma)\,.$$

Pick $\varepsilon < 5M/3$, take $\delta = \varepsilon/5$, $\alpha_0 = \varepsilon/(5\|x^*\|^2)$ if $\|x^*\| \ne 0$, $\alpha_0 = 1$ if $x^* = 0$, $m(r\beta_0) = n(r\gamma_0) = \varepsilon/5$ to get (13). ∎

Let us now give the proof of Theorem (5.1). Let $(\bar{t},\bar{x},\bar{s},\bar{z})$ be a maximum point of $\phi_{\alpha,\beta,\gamma}$.

Case $\bar{t}$ and $\bar{s}$ different from T Assume first that both $\bar{t}$ and $\bar{s}$ are different from T. Then, for all $x \in \mathbb{R}^n, \forall t \in [t_0,T]$,

$$\begin{aligned} &\Gamma_1(\bar{t},\bar{x}) - \Gamma_2(\bar{s},\bar{z}) - \alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) - \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} - \frac{|\bar{t}-\bar{s}|^2}{\gamma^2} \ge \\ &\Gamma_1(t,x) - \Gamma_2(\bar{s},\bar{z}) - \alpha(\|x\|^2 + \|\bar{z}\|^2) - \frac{\|x-\bar{z}\|^2}{\beta^2} - \frac{|t-\bar{s}|^2}{\gamma^2}\,. \end{aligned}$$

Let $\phi_1 \in C^1([t_0, T[\times \mathbb{R}^n)$ be defined as

$$\phi_1(t,x) = \Gamma_2(\bar{s},\bar{z}) + \alpha(\|x\|^2 + \|\bar{z}\|^2) + \frac{\|x-\bar{z}\|^2}{\beta^2} + \frac{|t-\bar{s}|^2}{\gamma^2}\,.$$

This last inequality means that $(\bar{t},\bar{x})$ is a maximal point of $\Gamma_1(t,x) - \phi_1(t,x)$.

We also have, for all $z \in \mathbb{R}^n, \forall s \in [t_0, T]$

$$\begin{aligned}&\Gamma_1(\bar{t},\bar{x}) - \Gamma_2(\bar{s},\bar{z}) - \alpha(\|\bar{x}\|^2 + \|\bar{z}\|^2) - \frac{\|\bar{x}-\bar{z}\|^2}{\beta^2} - \frac{|\bar{t}-\bar{s}|^2}{\gamma^2} \geq \\ &\Gamma_1(\bar{t},\bar{x}) - \Gamma_2(s,z) - \alpha(\|\bar{x}\|^2 + \|z\|^2) - \frac{\|\bar{x}-z\|^2}{\beta^2} - \frac{|\bar{t}-s|^2}{\gamma^2}\,.\end{aligned}$$

Let also $\phi_2 \in C^1([t_0, T[\times \mathbb{R}^n)$ be defined as

$$\phi_2(s,z) = \Gamma_1(\bar{t},\bar{x}) - \alpha(\|\bar{x}\|^2 + \|z\|^2) - \frac{\|\bar{x}-z\|^2}{\beta^2} - \frac{|\bar{t}-s|^2}{\gamma^2}.$$

This inequality means that $(\bar{s},\bar{z})$ is a minimal point of $\Gamma_2(s,z) - \phi_2(s,z)$. Then,

$$\begin{aligned}\max\Big\{&\min_\tau\Big[-\frac{\partial\phi_1}{\partial t}(\bar{t},\bar{x}) + \Gamma_1(\bar{t},\bar{x}) - \frac{\partial\phi_1}{\partial x}(\bar{t},\bar{x})f(\bar{t},\bar{x},\tau) - \mathrm{e}^{\bar{t}}L_1(\bar{t},\bar{x},\tau)\Big]\,, \\ &\Gamma_1(\bar{t},\bar{x}) - P_1(\bar{t},\bar{x})\Big\} \leq 0\,,\end{aligned} \tag{17}$$

and

$$\begin{aligned}\max\Big\{&\min_\tau\Big[-\frac{\partial\phi_2}{\partial s}(\bar{s},\bar{z}) + \Gamma_2(\bar{s},\bar{z}) - \frac{\partial\phi_2}{\partial z}(\bar{s},\bar{z})f(\bar{s},\bar{z},\tau) - \mathrm{e}^{\bar{s}}L_2(\bar{s},\bar{z},\tau)\Big]\,, \\ &\Gamma_2(\bar{s},\bar{z}) - P_2(\bar{s},\bar{z})\Big\} \geq 0\,.\end{aligned} \tag{18}$$

Inequality (17) can be decomposed into two inequalities (19), where we have set $p_\beta = 2(\bar{x}-\bar{z})/\beta^2$, and (20):

$$\min_\tau\left[\frac{-2(\bar{t}-\bar{s})}{\gamma^2} + \Gamma_1(\bar{t},\bar{x}) - \langle p_\beta + 2\alpha\bar{x}, f(\bar{t},\bar{x},\tau)\rangle - \mathrm{e}^{\bar{t}}L_1(\bar{t},\bar{x},\tau)\right] \leq 0, \tag{19}$$

$$\Gamma_1(\bar{t},\bar{x}) - P_1(\bar{t},\bar{x}) \leq 0, \tag{20}$$

Inequality (18) is equivalent to either inequality (21), or inequality (22):

$$\min_\tau\left[\frac{-2(\bar{t}-\bar{s})}{\gamma^2} + \Gamma_2(\bar{s},\bar{z}) - \langle p_\beta - 2\alpha\bar{z}, f(\bar{s},\bar{z},\tau)\rangle - \mathrm{e}^{\bar{s}}L_2(\bar{s},\bar{z},\tau)\right] \geq 0, \tag{21}$$

$$\Gamma_2(\bar{s}, \bar{z}) - P_2(\bar{s}, \bar{z}) \geq 0\,. \tag{22}$$

In (21), $\min_\tau$ may be replaced by "for all τ". Whenever it holds,

$$\begin{aligned}
\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \quad &\leq \quad \sup_\tau \Big[\langle p_\beta, f(\bar{t}, \bar{x}, \tau) - f(\bar{s}, \bar{z}, \tau) \rangle \\
&\qquad + 2\alpha \langle \bar{x}, f(\bar{t}, \bar{x}, \tau) \rangle + 2\alpha \langle \bar{z}, f(\bar{s}, \bar{z}, \tau) \rangle \\
&\qquad + \mathrm{e}^{\bar{t}} L_1(\bar{t}, \bar{x}, \tau) - \mathrm{e}^{\bar{s}} L_2(\bar{s}, \bar{z}, \tau) \Big] \\
&\leq \quad \sup_\tau [A + B + D + E + F + G]
\end{aligned}$$

where

$$\begin{aligned}
A &= \langle p_\beta, f(\bar{t}, \bar{x}, \tau) - f(\bar{t}, \bar{z}, \tau) \rangle\,, \\
B &= \langle p_\beta, f(\bar{t}, \bar{z}, \tau) - f(\bar{s}, \bar{z}, \tau) \rangle\,, \\
D &= 2\alpha \|f\|_\infty (\|\bar{x}\| + \|\bar{z}\|)\,, \\
E &= \mathrm{e}^{\bar{t}} (L_1(\bar{t}, \bar{x}, \tau) - L_2(\bar{t}, \bar{x}, \tau))\,, \\
F &= \mathrm{e}^{\bar{t}} (L_2(\bar{t}, \bar{x}, \tau) - L_2(\bar{s}, \bar{z}, \tau))\,, \\
G &= (\mathrm{e}^{\bar{t}} - \mathrm{e}^{\bar{s}}) L_2(\bar{s}, \bar{z}, \tau)\,.
\end{aligned}$$

It follows from the proof of Lemma 5.2 that, for $\alpha \leq \alpha_0$, $|\bar{t} - \bar{s}| \leq \gamma\sqrt{\varepsilon}$ and $\|\bar{x} - \bar{z}\| \leq \beta\sqrt{\varepsilon}$. L_2 is continuous in t, uniformly in (x, τ), and Lipschitz continuous in x with constant c_L uniform in (t, τ). Choose $\gamma_1 \leq \gamma_0$ such that, for $|t - s| \leq \gamma_1\sqrt{\varepsilon}$, $\mathrm{e}^T |L_2(t, x, \tau) - L_2(s, x, \tau)| \leq \varepsilon$ for all $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, and $\gamma \leq \gamma_1$, and $\beta_1 = \min\{\beta_0, \mathrm{e}^{-T}\sqrt{\varepsilon}/c_L\}$. This insures that, for all $\alpha \leq \alpha_0$, $\beta \leq \beta_1$, $\gamma \leq \gamma_1$, $F \leq 2\varepsilon$. Finally, pick $\gamma_2 = \min\{\gamma_1, \mathrm{e}^{-T}\sqrt{\varepsilon}/\|L_2\|_\infty\}$, so that for $\gamma \leq \gamma_2$, we furthermore insure that $G \leq \varepsilon$.

We have $|A| \leq 2c_f \|\bar{x} - \bar{z}\|^2/\beta^2 \leq 2c_f\varepsilon$ for all $\alpha \leq \alpha_0$, $\beta \leq \beta_0$ and $\gamma \leq \gamma_0$ and a fortiori if $\beta \leq \beta_1$ and $\gamma \leq \gamma_2$.

Let us turn to B. Clearly,

$$|B| \leq \|p_\beta\| \|f(\bar{t}, \bar{z}, \tau) - f(\bar{s}, \bar{z}, \tau)\|\,.$$

Fix $\beta = \beta_1$. Then, for small α and γ, $\|p_\beta\| \leq 2\sqrt{\varepsilon}/\beta_1$. Since f is continuous in t, uniformly in (x, τ), using the bound $|\bar{t} - \bar{s}| \leq \gamma\sqrt{\varepsilon}$, it is possible to find $\gamma_3 \leq \gamma_2$ such that, for $\gamma \leq \gamma_3$, $\|f(\bar{t}, \bar{z}, \tau) - f(\bar{s}, \bar{z}, \tau)\| \leq \sqrt{\varepsilon}\beta_1$, hence $|B| \leq 2\varepsilon$.

Finally, it follows also from the lemma 5.2 that, for α, β, and γ as in the lemma, $\alpha\|\bar{x}\|^2 \leq \varepsilon$, i.e. $\alpha\|\bar{x}\| \leq \sqrt{\alpha\varepsilon}$, and similarily for $\alpha\|\bar{z}\|$. Hence, keeping $\beta = \beta_1$, $\gamma = \gamma_3$, we may choose $\alpha_1 = \min\{\alpha_0, \varepsilon\|f\|_\infty^{-2}\}$ to insure that $D \leq 4\varepsilon$.

Hence in the case under investigation, there exists a triple $(\alpha_1, \beta_2, \gamma_3)$ such that for this triple,

$$\begin{aligned}\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) &\leq \mathrm{e}^{\bar{t}} \sup_{\tau} [L_1(\bar{t}, \bar{x}, \tau) - L_2(\bar{t}, \bar{x}, \tau)] + a\varepsilon \\ &\leq \mathrm{e}^{T} \|(L_1 - L_2)^+\|_\infty + a\varepsilon\,. \end{aligned} \tag{23}$$

where $a := 9 + 2c_f$ does not depend on ε nor (α, β, γ).

Let us assume now that (22) happened. Substracting (22) from (20), we obtain

$$\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq P_1(\bar{t}, \bar{x}) - P_2(\bar{s}, \bar{z})\,.$$

We use the bounds on $|\bar{t} - \bar{s}|$ and $\|\bar{x} - \bar{z}\|$ arising from the lemma and the uniform continuity of P_1 (or of P_2) to conclude that (α, β, γ) can be chosen small enough to insure

$$\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq \|(P_1 - P_2)^+\|_\infty + a\varepsilon\,.$$

In every cases, if $\bar{t}$ and $\bar{s} \neq T$, we have either (23) or the above inequality, thus, there always exist (α, β, γ) such that

$$\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq \max\Big\{\mathrm{e}^T \|(L_1 - L_2)^+\|_\infty, \|(P_1 - P_2)^+\|_\infty\Big\} + a\varepsilon\,. \tag{24}$$

Case $\bar{t}$ or $\bar{s}$ equal to T Now, we shall examine the case where $\bar{t}$ or $\bar{s}$ are equal to T. Then, (10) is not available. Assume that $\bar{t} = T$. Provided that $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, and $\gamma \leq \gamma_0$, we may rewrite the definition (16) using the bounds (15) on $|\bar{t} - \bar{s}|$ and $\|\bar{x} - \bar{z}\|$ as

$$\Gamma_2(T, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq m(r\beta) + n(r\gamma).$$

From the boundary condition (11), we get

$$\Gamma_1(T, \bar{x}) = \mathrm{e}^T G_1(\bar{x}) \quad \text{and} \quad \Gamma_2(T, \bar{x}) = \mathrm{e}^T G_2(\bar{x}).$$

Hence we obtain

$$\Gamma_1(T, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq \exp(T)(G_1(\bar{x}) - G_2(\bar{x})) + m(r\beta) + n(r\gamma)\,.$$

The same reasoning holds for the case where $\bar{s} = T$. Therefore, whenever $\bar{t}$ or $\bar{s} = T$, we may again choose $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, and $\gamma \leq \gamma_0$ such that

$$\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq \mathrm{e}^T \|(G_1 - G_2)^+\|_\infty + a\varepsilon\,. \tag{25}$$

And clearly, this also holds if both $\bar{s} = \bar{t} = T$, since then, using the continuity of G_2 and the bound (15), again if $\alpha \leq \alpha_0$, $\beta \leq \beta_0$, and $\gamma \leq \gamma_0$,

$$\Gamma_1(T, \bar{x}) - \Gamma_2(T, \bar{z}) = \mathrm{e}^T (G_1(\bar{x}) - G_2(\bar{z})) \leq \mathrm{e}^T (G_1(\bar{x}) - G_2(\bar{x})) + e^T c_G r\beta,$$

which is less than $\mathrm{e}^T \|(G_1 - G_2)^+\|_\infty + a\varepsilon$ provided that $\beta \leq \min\{\beta_0, a\varepsilon/(e^T c_G r)\}$.

Synthesis Using (25) if either $\bar{t} = T$ or $\bar{s} = T$ (or both), and (24) otherwise, we may conclude that there always exist (α, β, γ) such that

$$\begin{aligned} &\Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z}) \leq \\ &\max\Big\{ \mathrm{e}^T \|(L_1 - L_2)^+\|_\infty, \|(P_1 - P_2)^+\|_\infty, \mathrm{e}^T \|(G_1 - G_2)^+\|_\infty \Big\} + a\varepsilon. \end{aligned} \tag{26}$$

Finally, it follows from (14) that, for all (α, β, γ) used to get (26), $M - \varepsilon \leq M - \delta - 2\alpha\|x^*\|^2 \leq M_{\alpha,\beta,\gamma} \leq \Gamma_1(\bar{t}, \bar{x}) - \Gamma_2(\bar{s}, \bar{z})$. Hence, for all $(t, x) \in [t_0, T] \times \mathbb{R}^n$, using (26) we have

$$\begin{aligned} &\Gamma_1(t, x) - \Gamma_2(t, x) \leq M \leq \\ &\max\Big\{ \mathrm{e}^T \|(L_1 - L_2)^+\|_\infty, \|(P_1 - P_2)^+\|_\infty, \mathrm{e}^T \|(G_1 - G_2)^+\|_\infty \Big\} + (a + 1)\varepsilon. \end{aligned}$$

And as ε was arbitrary, it follows (12), proving theorem 5.1.

5.2 Uniqueness of the Viscosity Solution of the New Quasi-variational Inequality

In this section, we give the uniqueness result for the viscosity solution of the Hamilton-Jacobi-Isaacs quasi-variational inequality (9).

Theorem 5.3 *The quasi-variational inequality (9) has a unique bounded uniformly continuous viscosity solution.*

Proof Assume that (9) has two bounded uniformly continuous viscosity solutions Γ_1 and Γ_2. Let us first remark that for $0 < \mu < 1$, $\Theta = \mu\Gamma_1$ is a viscosity solution to the following quasi-variational inequality

$$\begin{aligned} 0 = \max \Bigg\{ &\min_\tau \left[-\frac{\partial \Theta}{\partial t} + \Theta(t, x) - \frac{\partial \Theta}{\partial x} f(t, x, \tau) - \mathrm{e}^t \mu L(t, x, \tau) \right], \\ &\Theta(t, x) - \mathcal{K}\Theta(t, x) \Bigg\}, \end{aligned} \tag{27}$$

where $\mathcal{K}\Theta(t, x) = \inf_\xi (\Theta(t, x + g(t, x, \xi)) + \mu \mathrm{e}^t C(\xi))$, with the terminal condition: $\Theta(T, x) = \mu \exp(T) \widetilde{G}(x)$ in $\mathbb{R}^n$.

We then apply the result (12) obtained in the previous section, with $\Theta = \mu\Gamma_1$ is the viscosity solution to the variational inequality (27) where the obstacle $P_1 = \mathcal{K}\Theta$ and Γ_2 is a viscosity solution of (9) with the obstacle $P_2 = \mathcal{N}\Gamma_2$. So, we have

$$\begin{aligned} \|(\Theta - \Gamma_2)^+\|_\infty \quad \leq \quad &\max \left(\mathrm{e}^T \|((\mu - 1)L)^+\|_\infty, \|\mathcal{K}\Theta - \mathcal{N}\Gamma_2)^+\|_\infty, \right. \\ &\left. \mathrm{e}^T \|((\mu - 1)\widetilde{G})^+\|_\infty \right) \end{aligned} \tag{28}$$

We have that, for any $(t,x) \in [t_0,T] \times \mathbb{R}^n$

$$(\mathcal{K}\Theta - \mathcal{N}\Gamma_2)(t,x) \quad \leq \quad \sup_\xi(\mu\Gamma_1(t,x+g(t,x,\xi)) - \Gamma_2(t,x+g(t,x,\xi)) + \sup_\xi((\mu-1)\exp(t)C(\xi))$$

We recall that $C(\xi) \geq \inf_\xi C(\xi) = \gamma > 0$: since $\mu < 1$, this yields

$$\sup_\xi((\mu-1)\exp(t)C(\xi)) = (\mu-1)\exp(t)\gamma < \gamma^* < 0\,.$$

Hence

$$(\mathcal{K}\Theta - \mathcal{N}\Gamma_2)(t,x) \leq \sup_\xi(\mu\Gamma_1(t,x+g(t,x,\xi)) - \Gamma_2(t,x+g(t,x,\xi))) + \gamma^*$$

and then

$$\|(\mathcal{K}\Theta - \mathcal{N}\Gamma_2))^+\|_\infty < \|(\mu\Gamma_1 - \Gamma_2)^+\|_\infty\,.$$

Equation (28) and the last inequality together imply

$$\|(\mu\Gamma_1 - \Gamma_2)^+\|_\infty \leq \max\left(\mathrm{e}^T\|((\mu-1)L)^+\|_\infty, \mathrm{e}^T\|((\mu-1)\widetilde{G})^+\|_\infty\right).$$

Now, let $\mu \to 1$, then since L and $\widetilde{G}$ are bounded, we finally obtain

$$\|(\Gamma_1 - \Gamma_2)^+\|_\infty \leq 0\,.$$

This clearly implies that, for any $(t,x) \in [t_0,T] \times \mathbb{R}^n$, $\Gamma_1(t,x) - \Gamma_2(t,x) \leq 0$.

If Γ_1, Γ_2 are solutions of (9), we can exchange their role and obtain as well $\Gamma_2(t,x) \leq \Gamma_1(t,x)$, for all $(t,x) \in [t_0,T] \times \mathbb{R}^n$. Finally $\Gamma_1(t,x) = \Gamma_2(t,x)$ for any $(t,x) \in [t_0,T] \times \mathbb{R}^n$. ∎

6 Conclusion

We can now give the result of this paper.

Corollary 6.1 *Under the assumptions of paragraph 2.1.2, the value function W is the unique bounded and uniformly continuous viscosity solution of the quasi-variational inequality (6).*

As an example of a use of this resut, one may consider the option pricing problem of references [4, 5]. If the piecewise linear transaction costs are replaced by a more realistic piecewise affine cost, i.e. a fixed cost is charged for any transaction in addition to a variable part, then the problem at hand is exactly that considered here. This was actually the motivation for the present analysis. The problem with no fixed cost, investigated by other means in these references, leads to a more difficult problem in terms of uniqueness of the viscosity solution, since it corresponds to the case $\gamma = 0$ in this paper. As far as we know, the uniqueness of the bounded uniformly continuous viscosity solution in that case is still an open problem.

References

[1] Bardi M, Capuzzo-Dolcetta I (1997) Optimal and viscosity solutions of Hamilton-Jacobi-Bellman equations. Birkhaüser. Boston, Basel, Berlin

[2] Barles G (1994) Solutions de viscosité des équations de Hamilton-Jacobi. Springer-Verlag, Mathématiques & Applications. Berlin, Heidelberg, New York

[3] Barles G (1985) Deterministic impulse control problems. SIAM Journal on Control and Optimization 23:419–432

[4] Bernhard P (2005) A robust control approach to option pricing including transaction costs. Annals of the ISDG 7:391–416. Birkhaüser

[5] Bernhard P, El Farouq N, Thiery S (2006) An Impulsive Differential Game Arising in Finance with Interesting Singularities. Annals of the ISDG 8:335–363. Birkhaüser

[6] Crandall MG, Lions PL (1983) Viscosity solutions of Hamilton Jacobi equations. Transactions of the American Mathematical Society 177:1–42

[7] Dharmatti S, Shaiju AJ (2007) Infinite dimensional differential games with hybrid controls. Proceedings of Indian Academy of Sciences, Mathematics 117:233–257

[8] Dharmatti S, Ramaswamy M (2006) Zero-sum differential games involving hybrid controls. Journal of Optimization Theory and Applications 128:75–102

[9] Evans LC, Souganidis PE (1984) Differential games and representation formulas for the solution of Hamilton-Jacobi-Isaacs equations. Indiana University Journal of Mathematics 33:773–797

[10] Fleming WH (1964) The convergence problem for differential games, 2. Annals of Mathematical Study 52:195–210

[11] Lions PL (1982) Generalized solutions of Hamilton-Jacobi equations. Pitman, Boston

[12] Lions PL, Souganidis PE (1985) Differential games, optimal control and directional derivatives of viscosity solutions of Bellman's and Isaacs' Equations. SIAM Journal on Control and Optimization 23:566–583

[13] Shaiju AJ and S. Dharmatti S (2005) Differential games with continuous, switching and impulse controls. Nonlinear Analysis 63:23–41

[14] Souganidis PE (1985) Max-min representations and product formulas for the viscosity solutions of Hamilton-Jacobi equations with applications to differential games. Nonlinear Analysis, Theory, Methods and Applications 9:217–257

[15] Yong JM (1994) Zero-sum differential games involving impulse controls. Applied Mathematics and Optimization 29:243–261