

TECHNICAL NOTE

**A Simple Game with a Singular Focal Line**<sup>1</sup>

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**Abstract.** We propose a simple game the solution of which contains a singular focal line, i.e. a focal line reached by optimal trajectories in a non tangential fashion. We also provide a discussion of how the optimal “discriminating” strategy can be approximated by a pure feedback.

**Keywords.** Isaacs’equation, focal lines, viscosity solution.

## 1. Introduction

The investigation of singular surfaces in differential games has a rather long history. It was all started by Isaacs (Ref. 1) who discovered “universal lines”, the equivalent of the more classical singular arcs, and “equivocal lines”, with no equivalent in classical optimal control theory, since on such a singularity, the “adjoint variables” have a discontinuity, in essence violating Weierstrass’ condition. The first author discovered (Ref. 2,3) that in the case where the vectorgrams of the players are strictly convex, the vanishing of a switch function upon arrival at the singular surface must be replaced by a condition of tangential approach of that surface by the optimal trajectories. This led to the discovery of “switch envelopes”, as counterparts of “equivocal lines”. Later on, “focal lines” were discovered by Merz (Ref. 4), where the singular surface is reached tangentially on both sides. A somewhat unifying theory of singularities was proposed by the second author (Ref. 5,6), according to which there should exist “singular focal lines” approached in a nontangential manner on both sides. This, however, as was *not* recognised in the above two references, requires that the vectorgram of one of the players have two separate “flat” parts. A first (unpublished) example of such a singular focal line was arrived at by the authors by perturbing the game of the obstacle tag chase. The present example, however, although quite artificial, is much simpler, leading to closed form integration of much of the solution.

## 2. Problem

The state is two-dimensional, and obeys the following dynamics:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \begin{pmatrix} \phi \\ |\phi| - 1 \end{pmatrix} + 2\exp(-y) \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \quad (1)$$

$\phi \in [-1, 1]$  is the minimizer’s control,

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The playing space is  $y > 0$ , with terminal condition  $y_f = 0$ . The payoff for an initial state  $x, y$  is

$$V(x, y) = \max_{\psi} \min_{\phi} [t_f + k|x_f|]. \quad (2)$$

$k$  is a positive constant.

The velocity  $(\dot{x} \dot{y})$  has been expressed as the sum of three terms. The first is independent of the controls. It insures that in the total  $\dot{y} \leq -1$  so that termination is assured in a time no longer than  $y(t_0)$ . The other two depend on the controls of minimizer and maximizer respectively. The “vectorgrams” corresponding to these two velocity contributions are shown in figs. 1a and 1b.

### 3. Solution for $1/2 < k < 1$ .

The hamiltonian of the problem

$$H = 1 - 2V_y + (V_x + V_y \text{sign } \phi)\phi + 2V_x \exp(-y)\psi$$

is minimaximized by

$$\bar{\phi} = \begin{cases} -\text{sign } V_x & \text{if } V_y < |V_x|, \\ 0 & \text{if } V_y \geq |V_x|, \end{cases} \quad (3a)$$

$$\bar{\psi} = \text{sign } V_x. \quad (3b)$$

The adjoint equations and transversality conditions are

$$\dot{V}_x = 0, \quad V_x(t_f) = k \text{sign } x_f, \quad (4a)$$

$$\dot{V}_y = 2V_x \exp(-y)\bar{\psi} = |2V_x \exp(-y)|, \quad V_y(t_f) = k + \frac{1}{2}. \quad (4b)$$

According to (3b) and (4b),  $\phi_f = 0$ , (rather than  $-\text{sign } x_f$ ). Backwards integration of the “primaries” is straightforward and shows that we get  $V_y = |V_x|$  when  $y$  reaches the switch value

$$y_s = \text{Log} \left( \frac{1}{1 - \frac{1}{2k}} \right).$$

The optimal paths and the contours of constant payoff are sketched in figure 2, along with the optimal controls  $\phi, \psi$ . The  $y$ -axis, for  $0 \leq y \leq y_s$ , is a dispersal line (D.L.) on which the maximizer chooses  $\psi = \pm 1$  while the minimizer chooses  $\phi = 0$ . Here,  $|V_x| = k$  and  $V_y = \frac{1}{2} + k \exp(-y) \geq |V_x|$ .

The remainder of the  $y$ -axis is a singular focal line (S.F.L.), i.e. a locus of discontinuity of  $V_x$ . The value surface exhibits a “valley” along that locus, with an edge at the bottom.

The vanishing of the minimizer's switch function along this valley implies that  $|V_x| = V_y$ , so that both the incoming control  $\phi = \pm 1$  and the “focal” control

$$\phi = -2\exp(-y)\psi, \tag{5}$$

devised to insure  $\dot{x} = 0$ , can make the hamiltonian minimum. The adjoint  $V_y$  still satisfies the adjoint equation (4b), but is most easily calculated using the first integral  $H = 0$ , which, together with the above condition yields

$$V_y = |V_x| = \frac{1}{2(1 - \exp(-y))}$$

Along this S.F.L., the minimizer faces a “perpetuated dilemma”. The maximizer may switch at will between  $\psi = \pm 1$ , and the minimizer must respond accordingly with his focal control (5), which assumes instantaneous knowledge of his opponent's control. We shall see, in section 6 below, how and to what extent this can be approximated by a state feedback control.

#### 4. Proof of Optimality.

Let

$$H^*(y, V_x, V_y) = \min_{\phi} \max_{\psi} H = 1 + 2|V_x| \exp(-y) - 2V_y + \frac{1}{2}(V_y - |V_x| - |V_y - |V_x||).$$

A *sufficient* condition for optimality (in addition to the satisfaction everywhere of Isaacs' Main Equation  $H^* = 0$ ) is the “viscosity condition” (Ref. 7):  $H^*(y, W_x, W_y) \leq 0$  at any minimum of  $V - W$  where  $W$  is any smooth test function. (See figure 3). Since the only points where  $V$  is not smooth are along the  $y$ -axis, we may adopt  $W = V$  along that axis, and  $|W_x| \leq |V_x|$ , so that  $V - W \geq 0$  in a neighborhood, making the axis a local minimum. Since  $V_y = |V_x|$  on the S.F.L., and  $V_y \geq |V_x|$  on the D.L., we immediately see that  $H^*(y, W_x, W_y) \leq H^*(y, V_x, V_y) = 0$ , and the viscosity condition is met.

#### 5. Values of $k$ Outside the Range (1/2, 1).

As  $k \rightarrow 1/2$  from above,  $y_s \rightarrow \infty$ , and the solution for  $k \leq 1/2$  is almost trivial.

For  $k > 1$ , we have

$$y_s = \text{Log} \left( \frac{1}{1 - \frac{1}{2k}} \right) < \text{Log} 2,$$

and that part of the  $y$ -axis for which  $y_s < y < \ln 2$  constitutes a D.L. on which again the maximizer chooses  $\psi = \pm 1$ , to which the minimizer must now respond with  $\phi = -\psi$ . (See fig. 3). Here,  $|V_x| = k$  and  $V_y = 1 + (2 \exp(-y) - 1)k \in [1, k]$ . Therefore,  $V_y \leq |V_x|$

Using the same family of test functions as previously, it is clear that the viscosity condition is satisfied for any  $W$  such that  $V_y \leq |W_x| \leq |V_x|$ , since  $2 \exp(-y) - 1 \geq 0$ , and for  $|W_x| \leq V_y$  as well,  $H^*$  being a monotonic increasing function of  $|W_x|$  throughout.

## 6. Feedback Approximation of the Focal Strategy.

We pointed out that the strategy (5) is not within the usual “rules” of a differential game, and will most often not be implementable as such, since it involves instantaneous knowledge of the opponent’s control. This situation during a perpetuated dilemma has already been investigated in the case of a “regular” focal line, where the incoming optimal paths are tangent. See (Ref. 9) for instance. It is useful to quickly recall how the argument goes.

Assume the responder, the minimizer here, responds to a switch by the leader after a time  $\epsilon$ . Since after switching to his optimal strategy the path will come back to the focal line tangentially, this will take a time of the order of  $\sqrt{\epsilon}$ . Therefore the total number of such deviations from the focal line can only be of the order of  $1/\sqrt{\epsilon}$ . Since the time spent playing non optimally, i.e. with  $H > 0$ , is  $O(\epsilon)$  each time, the total time spent is of order  $\epsilon/\sqrt{\epsilon} = \sqrt{\epsilon}$ , and thus goes to zero with  $\epsilon$ . Therefore the optimal cost can be approximated arbitrarily well with this type of feedback (with a delay).

Now this does not hold in the current setting where the optimal trajectories come back to the focal line in a non tangential manner, therefore in a time  $O(\epsilon)$  only. Notice also that even though the deviation of the state from the focal line remains small, say  $O(\epsilon)$ , the deviation of the hamiltonian from zero is large, so that the trajectory thus generated is really non optimal, leading to a payoff much larger than  $V(x_0, y_0)$ .

Another way of looking at the situation is the following. Assume the maximizer tries to drive the state off the focal line as often as he can, and that the minimizer responds by his best approximation of his optimal strategy. The minimizer will end up chattering between

$\phi = \pm 1$ , as will the maximizer, and since, by hypothesis the resulting average speed must be with  $\dot{x} = 0$ , by simple inspection of the vectorgram, it will be also with  $\dot{y} = -1$  (the part of  $\dot{y}$  independent of the controls, not shown in fig. 1), instead of  $-2(1 - \exp(-y)) < -1$ . This is clearly bad for the minimizer since elapsed time is part of the payoff.

However, we have pointed out that the game always terminates in a short time if the state is in a neighborhood of the terminal manifold. Therefore the condition required by Friedman (Ref. 10) to insure existence of a value is satisfied, and therefore the upper value and lower value, in the sense of Fleming, or Friedman, must coincide. Furthermore, we know from (Ref. 7,8) that this value is the *unique* viscosity solution of Isaac's equation, thus the one we have computed. It is therefore interesting to examine Fleming's piecewise open loop upper strategies.

Assume that from  $x = 0$ , Minimizer is to choose his control value for the time  $\epsilon$  to come. Assume he chooses  $\phi \geq 0$ . Then Maximizer can insure that  $\dot{x} > 0$ , thus  $x > 0$ . It is readily seen that he maximizes the hamiltonian by choosing  $\psi = 1$  (he tries to "climb" the side of the valley as fast as possible). We thus have  $V_x = V_y > 0$  and

$$H = 1 - 2V_y(1 - \exp(-y)) + 2V_y\phi.$$

Thus Minimizer achieves the min max  $H$  by choosing  $\phi = 0$ . The situation is symmetrical if we start with  $\phi \leq 0$ , still ending up with the conclusion  $\phi = 0$ . After a duration  $\epsilon$  of play with these controls, both players know that  $x > 0$ . There grad  $V$  is continuous, the hamiltonian has a saddle point, thus both choose their optimal controls according to our optimal feedback strategies, leading to  $\phi = -1$ , and  $\psi = 1$ .

We conclude that the approximating strategy for the minimizer should be, for some small positive  $\eta$

$$\phi = \begin{cases} 0 & \text{if } y \leq y_s \text{ or } y > y_s \text{ and } |x| < \eta, \\ -\text{sign } x & \text{if } y > y_s \text{ and } |x| \geq \eta. \end{cases} \quad (6)$$

It is interesting to examine the possible resulting chatter. If a chatter occurs, it can only be along the lines  $x = \pm\eta$ . Let us look at the case  $x = \eta$  for instance. The limit cycle will occur with  $\psi = -1$  and  $\phi$  chattering between 0 and +1. Since the resulting average velocity is by hypothesis with  $\dot{x} = 0$ , figure 4 shows that it will actually be the "focal" velocity.

We can now conclude that a strategy such as (6), with a small delay to allow for a limit cycle along the two switching lines, allows the minimizer to guarantee himself a value arbitrarily close to the optimal one. As a matter of fact, we have seen that he can obtain an average velocity equal to the focal velocity, along a line  $x = \pm\eta$ . Now,  $\text{grad } V$  is continuous in both regions  $x > 0$  and  $x < 0$ , so that  $V_y$  is close to the focal  $V_y$  on those lines, the difference being  $O(\eta)$ . Since the time spent is the same as on the optimal play along the S.F.L. (within  $O(\eta)$ ), the optimal payoff will be approximated within  $O(\eta)$ .

## 7. Conclusions.

The game we have investigated here provides a very simple (and the first published?) example of an S.F.L., well suited to study various aspects of those singular lines. We have focused here on the proof of optimality and the feedback approximation of the focal “discriminating” control during the perpetuated dilemma. Other questions that one might want to look at include the following perturbation schemes.

**Strictly Convex Vectorgrams.** In the fashion of (Ref. 12), one may look at the situation where the vectorgrams of the players are made slightly strictly convex, as depicted in figure 5 a, b. So perturbing the maximizer’s vectorgram does not change the solution of the game. However, modifying the minimizer’s vectorgram will turn the S.F.L. into a regular focal line, modifying the slope of the optimal trajectories in a small boundary layer along the  $y$ -axis, very much in the same fashion as was shown for other types of junctions in (Ref. 12).

**Additive Small Noise.** In the fashion of Fleming (Ref. 11), and followers, one may add a small white noise of spectral density  $\sigma^2 I$  to the dynamics (1). By the general theory, the resulting Value function  $V^\eta$  will be close to  $V$ , but the bottom of the valley along the  $y$ -axis will be smoothed.(Fig 6). On this axis, the optimal controls will be  $\phi = 0$  and, assuming we have further made the maximizer’s vectorgram slightly convex as above, (“raising” it slightly to make it tangent to the  $\dot{x}$ -axis at the origin),  $\psi = 0$ . The resulting Main Equation is then

$$\frac{1}{2}\sigma^2\nabla^2V^\eta + 1 - 2V_y^\eta = 0.$$

The solution will yield  $V_y^\eta \cong V_y > \frac{1}{2}$ .

So, provided it corresponds to an existing noise, some dispersion cancels the advantage the minimizer had in being able to force the state in the bottom of a valley.

### **Bibliography.**

1. ISAACS, R., *Differential games*, John Wiley, New York, New York, 1965.
2. BREAKWELL, J. V., *Some Differential Games with Interesting Discontinuities*, SUDAAR, Stanford University, 1973.
3. BREAKWELL, J. V., and MERZ, A. W., *Toward a Complete Solution of the Homicidal Chauffeur Game*, Proceedings of the First International Conference on the Theory and Applications of Differential Games, University of Massachusetts, Amherst, Massachusetts, 1969.
4. MERZ, A. W., *The Homicidal Chauffeur – a Differential Game*, PhD Dissertation, Stanford University, 1971.
5. BERNHARD, P., *Singular Surfaces in Differential Games, an Introduction*, in *Differential Games and Applications*, edited by P. Hagedorn et al. Springer lecture notes on Control and Information Sciences, Vol. **3**, Springer, Berlin, Germany, pp 1–33, 1977.
6. BERNHARD, P., *Contribution à l'étude des jeux différentiels à deux joueurs, somme nulle et information parfaite*, thèse d'état, Université Paris 6, Paris, France, 1979.
7. LIONS, P. L., *Generalized Solutions of Hamilton Jacobi Equations*, Pitman, London, Great Britain, 1982.
8. SOUGANIDIS, P. E., *Max Min Representations and Product Formulas for the Viscosity Solution of Hamilton Jacobi Equations, with Applications to Differential Games*, Journal of Nonlinear Analysis, TMA, Vol. **9**, pp 217–257, 1985.
9. BREAKWELL, J. V., *Time Optimal Pursuit Inside a Circle*, in *Differential Games and Applications*, edited by T. Başar and P. Bernhard, Springer lecture notes on Control and Information Sciences, Springer, Berlin, Germany, pp 72–85, 1989.
10. FRIEDMAN, A., *Existence of Value and of Saddle Points for Differential Games of Survival*, Journal of Differential Equations, Vol. **7**, pp 111–125, 1970.
11. FLEMING, W. H., *The Convergence Problem for Differential Games, II*, Advances

in Game Theory, Annals of Mathematical Studies Vol. **52**, pp 195–210, Princeton University Press, Princeton, New Jersey, 1964.

12. DELGADO, M., *Sobre las Perturbaciones de las Superficies Singulares de Ciertos Juegos Diferenciales*, Memoria de Doctor en Ciencias, Universidad de Sevilla, Sevilla, Spain, 1986.

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fig. 1a: Pursuer vectorgram

fig. 1b: Evader vectorgram

fig. 2 : Optimal field,  $1/2 < k < 1$

fig. 3 : Optimal field,  $k > 1$

fig. 4 : Vectorgram along the focal line

fig. 5a: Perturbed Pursuer vectorgram

fig. 5b: Perturbed Evader vectorgram

fig. 6 : Value function for the game with small noise added.