On Finite Approximation of a Game Solution With Mixed Strategies

P. BERNHARD[1] and J. SHINAR[2]

[1]INRIA-Sophia Antipolis, Valbonne, France.
[2]Faculty of Aerospace Engineering, Technion Israel Institute of Technology Haifa, Israel, visiting scientist at INRIA-Sophia Antipolis.

Abstract. Motivated by a pursuit evasion differential game, we investigate an abstract two-person zero-sum game: $P : \Omega_1 \times \Omega_2 \to \mathbb{R}$ where the Ω_i, $i = 1, 2$, are compact metric spaces, so that the game always has a solution in mixed strategies. We show that the value of the game can be approximated arbitrary well by each player randomizing only over a finite set of pure strategies.

1. Introduction

We give here a brief account of the motivation for this study, which evolved from the investigation of an imperfect information pursuit evasion game [1].

The game is governed by a differential equation

$$\dot{x} = f(x, u_1, u_2), \quad x \in X,\ u_1 \in U_1,\ u_2 \in U_2.$$

where U_1 and U_2 are bounded subsets of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ respectively, X is a domain of $\mathbb{R}^n$. Standard regularity (Lipshitz continuity) and growth conditions on f insure existence of the solution of the differential equation over $(0, \infty)$ for every pair of measurable $u_i(\cdot)$ from $(0, \infty)$ to U_i.

The players have access to noisy partial information

$$y_i = h_i(x, w), \quad i = 1, 2,$$

where w is a noise that we do not intend to characterize precisely here, the h_i's being globally Lipshitz over X. They are restricted to using feedback strategies

$$u_i(t) = \delta_i\big(y_i(t)\big)$$

Lipshitz continuous *with a prescribed modulus.* It follows from the Ascoli Arzela theorem, that the set of strategies is compact in the topology of the uniform convergence.

We assume that the noise model and the solution concept of the differential equation are such that the payoff as formulated in [1], i.e. the expected value of a continuous function of closest approach, is a continuous function of the strategies for the topology of uniform convergence. Notice that this assumption is satisfied in the case where there is no noise.

The visit of the second author was sponsored by the French Ministry of Research and Technology, Delegation of international affairs.

2. The formal game

Let a game be given by a pair of compact metric strategy spaces Ω_1 and Ω_2, and a continuous payoff function P from $\Omega_1 \times \Omega_2$ into $\mathbb{R}$. We shall also consider closed subsets Δ_1 and Δ_2 of Ω_1 and Ω_2 respectively, and the subgame defined by the restriction of P to $\Delta_1 \times \Delta_2$. The Δ_i's will always be endowed with the topology induced by that of the Ω_i's. They will therefore be compact spaces.

For any two Δ_i's, the associated subgame has a value in mixed strategies (see [2]). Let $\Pi(\Delta)$ be the set of probability measures over Δ, we set

$$J(\pi_1, \pi_2) = \int_{\Delta_1} \int_{\Delta_2} P(\delta_1, \delta_2)\, d\pi_1(\delta_1)\, d\pi_2(\delta_2),$$

and

$$V(\Delta_1, \Delta_2) = \min_{\pi_1 \in \Pi(\Delta_1)} \max_{\pi_2 \in \Pi(\Delta_2)} J(\pi_1, \pi_2).$$

We know that optimal mixed strategies $\hat{\pi}_1$ and $\hat{\pi}_2$ exist and form a saddle point over $\Delta_1 \times \Delta_2$. (They obviously depend on this pair of admissible pure strategies.)

We shall write Π_1 and Π_2 for $\Pi(\Omega_1)$ and $\Pi(\Omega_2)$ respectively, V for $V(\Omega_1, \Omega_2)$, and we shall call (π_1^*, π_2^*) a saddle point over $\Omega_1 \times \Omega_2$. Finally, let Δ_1^* and Δ_2^* be the support of π_1^* and π_2^* respectively, we have the following simple facts:

Proposition.

$$\forall (\Delta_1, \Delta_2), \quad V(\Delta_1^*, \Delta_2) = V(\Omega_1, \Delta_2) \leq V(\Delta_1, \Delta_2)$$

and

$$V(\Delta_1^*, \Delta_2) \leq V(\Delta_1^*, \Delta_2^*) \leq V(\Delta_1, \Delta_2^*).$$

Proof: The inequality in the first claim simply follows from the fact that $\Omega_1 \supseteq \Delta_1$, and the equality from the fact that $\pi_1^* \in \Pi(\Delta_1^*)$. Placing $\Delta_2 = \Delta_2^*$ in the inequality yields the second one of the second claim. The first one is obtained symmetrically.

3.Continuity of $V(\cdot,\cdot)$

Let $\mathcal{F}(\Omega)$ be the set of closed subsets of Ω, endowed with the topology of the Hausdorf distance. Let $H(\Delta, \Delta')$ be the Hausdorf distance from Δ to Δ'. Let also $\mathcal{F}_i$ stand for $\mathcal{F}(\Omega_i)$, $i = 1, 2$. We can now state the main result of this paper.

Theorem. *$V(\cdot,\cdot)$ is a continuous function from $\mathcal{F}_1 \times \mathcal{F}_2$ into $\mathbb{R}$.*

Proof: We shall proceed with two lemmas.

Lemma 1. *Let Ω be a compact metric space, Δ and Δ' in $\mathcal{F}(\Omega)$, such that $H(\Delta, \Delta') \leq \epsilon/2$. Then there exists an application $\phi : \Delta' \to \Delta$ such that ϕ is a staircase function (constant over each subset of a finite Borel partition of Δ'), satisfying $\forall \delta' \in \Delta'$, $d(\delta', \phi(\delta')) < \epsilon$.*

Proof of lemma 1: To each δ' in Δ' we can make correspond its closest element δ in Δ. Since $H(\Delta, \Delta') < \epsilon/2$ then $d(\delta', \delta) < \epsilon/2$. There also exists an open neighborhood of δ' such that for any δ" in that neighborhood, $d(\delta", \delta) < \epsilon$. All these neighborhoods form an open covering of Δ'. By extracting a finite covering, it is straightforward to conclude the proof.

LEMMA 2. *Let Δ_1 and Δ_2 be given in $\mathcal{F}_1$ and $\mathcal{F}_2$. Let ϵ be a given positive number. There exists a positive number η such that if $H(\Delta_1, \Delta_1') < \eta$, to each $\pi_1' \in \Pi(\Delta_1')$, we can associate a $\pi_1 \in \Pi(\Delta_1)$ such that,*

$$\forall \delta_2 \in \Delta_2, \quad \left| \int_{\Delta_1} P(\delta_1, \delta_2)\, d\pi_1(\delta_1) - \int_{\Delta_1'} P(\delta_1', \delta_2)\, d\pi_1'(\delta_1') \right| < \epsilon.$$

PROOF OF LEMMA 2: Since P is continuous, it is uniformly continuous over $\Omega_1 \times \Omega_2$. Therefore there exists η such that

$$d(\delta_1, \delta_1') < 2\eta \Rightarrow \forall \delta_2 \in \Delta_2,\ |P(\delta_1, \delta_2) - P(\delta_1', \delta_2)| < \epsilon.$$

Let Δ_1' be such that $H(\Delta_1, \Delta_1') < \eta$, $\pi_1' \in \Pi(\Delta_1')$ given, and for any continuous function Q: $\Delta_1 \to I\!R$, set

$$\int_{\Delta_1} Q(\delta_1)\, d\pi_1(\delta_1) = \int_{\Delta_1'} Q\big(\phi(\delta_1')\big)\, d\pi_1'(\delta_1'),$$

where ϕ is the function whose existence is asserted by lemma 1. It is measurable, thus the integral of the right hand side is well defined. Moreover, the above defined π_1 is clearly positive and of total mass one, i.e. in $\Pi(\Delta_1)$. Now we have

$$\left| \int_{\Delta_1} P(\delta_1, \delta_2)\, d\pi_1(\delta_1) - \int_{\Delta_1'} P(\delta_1', \delta_2)\, d\pi_1'(\delta_1') \right| = \left| \int_{\Delta_1'} \Big[P(\phi(\delta_1'), \delta_2) - P(\delta_1', \delta_2) \Big]\, d\pi_1'(\delta_1') \right| \le \epsilon$$

because $d\big(\phi(\delta_1'), \delta_1'\big) < 2\eta$, and as a consequence, the integrand is smaller than ϵ.

We now proceed to prove the theorem. We have

$$V(\Delta_1', \Delta_2) = \max_{\delta_2 \in \Delta_2} \int_{\Delta_1'} P(\delta_1', \delta_2)\, d\hat{\pi}_1'(\delta_1'),$$

where $\hat{\pi}_1'$ is an optimal mixed strategy for the game over (Δ_1', Δ_2). We also associate to $\hat{\pi}_1'$ another mixed strategy $\tilde{\pi}_1 \in \Pi(\Delta_1)$ as in the above lemma, and define $\tilde{\delta}_2$ by

$$\int_{\Delta_1} P(\delta_1, \tilde{\delta}_2)\, d\tilde{\pi}_1(\delta_1) = \max_{\delta_2 \in \Delta_2} \int_{\Delta_1} P(\delta_1, \delta_2)\, d\tilde{\pi}_1(\delta_1).$$

The continuity of P and compacity of Δ_2 insures the existence of $\tilde{\delta}_2$. We now have

$$V(\Delta_1', \Delta_2) = \max_{\delta_2 \in \Delta_2} \int_{\Delta_1'} P(\delta_1', \delta_2)\, d\hat{\pi}_1'(\delta_1') \ge \int_{\Delta_1'} P(\delta_1', \tilde{\delta}_2)\, d\hat{\pi}_1'(\delta_1').$$

Thus, using lemma 2, it comes

$$V(\Delta_1', \Delta_2) \ge \int_{\Delta_1} P(\delta_1, \tilde{\delta}_2)\, d\tilde{\pi}_1(\delta_1) - \epsilon$$

and finally from the definition of $\tilde{\delta}_2$,

$$V(\Delta_1', \Delta_2) \ge \max_{\delta_2 \in \Delta_2} \int_{\Delta_1} P(\delta_1, \delta_2)\, d\tilde{\pi}_1(\delta_1) - \epsilon \ge \min_{\pi_1 \in \Pi(\Delta_1)} \max_{\delta_2 \in \Delta_2} \int_{\Delta_1} P(\delta_1, \delta_2)\, d\pi_1(\delta_1) - \epsilon.$$

So that we have showed that $V(\Delta_1', \Delta_2) \ge V(\Delta_1, \Delta_2) - \epsilon$. Of course, we would show in the same way that $V(\Delta_1, \Delta_2) \ge V(\Delta_1', \Delta_2) - \epsilon$. We have therefore proved that

$$\text{if} \quad H(\Delta_1', \Delta_1) < \eta, \quad \text{then} \quad |V(\Delta_1, \Delta_2) - V(\Delta_1', \Delta_2)| \le \epsilon.$$

The theorem follows by doing similarily for $\Delta_2 \mapsto V(\Delta_1, \Delta_2)$

4. Finite subsets and finite approximations

We shall now consider *finite* subsets Δ_1 andΔ_2. Let $\mathcal{D}_i^{(M)}$ be the set of all finite subsets of Ω_i with M elements. $\mathcal{D}_i^{(M)}$ is isomorphic to Ω_i^M, the Mth cartesian power of Ω_i. Moreover, in this isomorphism, the distance over Ω_i induces a distance over $\mathcal{D}_i^{(M)}$ which, for small distances, is identical to the Hausdorf distance. As a consequence, using the above theorem, the restriction of V to $\mathcal{D}_1^{(M_1)} \times \mathcal{D}_2^{(M_2)}$ is a continuous function from $\Omega_1^{M_1} \times \Omega_2^{M_2}$ into $I\!R$.

By Tychonov's theorem, $\mathcal{D}_1^{(M_1)}$ and $\mathcal{D}_2^{(M_2)}$ are compact. Therefore, for fixed M_1 or M_2, $V(\Delta_1, \Omega_2)$ has a minimum, or, respectively, $V(\Omega_1, \Delta_2)$ has a maximum. If player 1, say, is restricted to randomizing his strategy over a finite number M of pure strategies, he should chose the set $\Delta_1^{(M)}$, which always exist, insuring

$$V(\Delta_1^{(M)}, \Omega_2) = \min_{\Delta_1 \in \mathcal{D}_1^{(M)}} V(\Delta_1, \Omega_2) = V_1^{(M)}.$$

Let now Δ_2 be fixed, and known by both players. (Δ_2 may be Ω_2, or a strict subset, possibly finite.) Let

$$V_1^{(M)}(\Delta_2) = \min_{\Delta_1 \in \mathcal{D}_1^{(M)}} V(\Delta_1, \Delta_2).$$

We can show the following fact:

Corollary.

$$V_1^{(M)}(\Delta_2) \to V(\Omega_1, \Delta_2) \quad \textit{as} \quad M \to \infty.$$

Proof: In fact, we prove the following lemma:

Lemma 3. *For any $\epsilon > 0$, one can find an integer M and $\Delta_1 \in \mathcal{D}_1^{(M)}$ such that*

$$|V(\Delta_1, \Delta_2) - V(\Omega_1, \Delta_2)| < \epsilon.$$

Proof of lemma 3: Since Ω_1 is compact, for any $\eta > 0$, there exists a finite subset $\bar{\Delta}_1$ of Ω_1 such that,

$$\forall \delta_1 \in \Omega_1,\ \exists \bar{\delta}_1 \in \bar{\Delta}_1 \ :\ d(\delta_1, \bar{\delta}_1) < \eta.$$

Therefore, $H(\Omega_1, \bar{\Delta}_1) < \eta$. Since $\Delta_1 \mapsto V(\Delta_1, \Delta_2)$ is continuous by the previous theorem, the lemma follows.

From the lemma follows that the limit of $V_1^{(M)}(\Delta_2)$ as $M \to \infty$ will not be larger than $V(\Omega_1, \Delta_2)$. But from the proposition, we know that $V(\Delta_1^{(M)}(\Delta_2), \Delta_2) \geq V(\Omega_1, \Delta_2)$. Therefore its limit will not be smaller than $V(\Omega_1, \Delta_2)$. The corollary is proved.

Of course, by similar arguments, Δ_2 can be chosen finite, such that $V(\Omega_1, \Delta_2)$ be arbitrarily close to V itself. So that our paper shows that, in some sense, any game over compact sets can be approximated by a matrix game.

References

1. J. Shinar and I Forte, *On the optimal pure strategy sets for a missile guidance law synthesis*, 25th IEEE Conference on Decision and Control (1986), Athens, Greece.
2. I. Ekeland, "La théorie des jeux et ses applications à l'économie mathématique," PUF, Paris, France, 1974.