# Min-Max Certainty Equivalence Principle and Differential Games

Pierre Bernhard and Alain Rapaport

INRIA Sophia-Antipolis

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**Abstract**

This paper presents a version of the Certainty Equivalence Principle, usable for nonlinear, variable end-time, partial observation Zero-Sum Differential Games, which states that under the unicity of the solution to the auxiliary problem, optimal controllers can be derived from the solution of the related perfect observation game. An example is provided where in one region, the new extended result holds, giving an optimal control, and in another region, the unicity condition is not met, leading indeed to a non-certainty equivalent optimal controller.

## 1 Introduction

It has already been pointed out by Isaacs in 1965 (chapter 12 of [Isa65]) that differential games with incomplete information are worth being studied, but a general theory seemed to be out of the scope of the simple calculus of variations. For players perfectly knowing the state, the optimal controllers exhibited by the Isaacs/Breakwell theory are pure state-feedbacks. When at least one of the players is denied the complete knowledge of the state, the history of past observations and controls is extremely relevant and, as a consequence, the structure of optimal controllers may be radically different.

If we think of the maximizing control $v(\cdot)$, or may be the pair $(x_0, v(\cdot))$, as a *disturbance*, then min-max control may be seen as an alternative to stochastic control. However, until recently, the former has been much less developed than the latter. A notable exception being the case of linear systems and convex information sets. See [KS77, chapt. XV]

The certainty equivalence principle, originally discovered in 1990 (see [BB91]), was motivated by $H^\infty$-optimal control of linear systems. In the state space approach, the game theoretic view point leads to an easy determination of the equations of the $H^\infty$-optimal controllers via the well-known Riccati equations, considering the disturbance as another player's control ('worst-case design'). For this purpose, the principle is used with linear quadratic fixed terminal time problems, and [BB91] was the first book completely devoted to this approach. It has already been pointed out by Bernhard that the principle holds for non-linear problems [Ber90a]. Nevertheless, two difficulties were met for using it with Isaacs' differential games. The proof needed that for all $(t, x)$, the output map $v \mapsto h(t, x, v)$ be onto, which does not allow simple observed outputs such as $y = h(x)$. Didinsky, Başar and Bernhard [DBB93] have proposed an extension where the unicity of the worst disturbance is replaced by the unicity of the worst state, under the condition that the set of compatible states be open, and some extra regularity properties of the "cost to come" function be met. Terminal time should again be fixed, and so here also, we cannot use these results in the scope of Isaacs' differential games. That version was still with fixed terminal time, and therefore did not apply to, say, pursuit evasion games.

The present paper proposes a generalization of the principle overcoming these difficulties. It can be used for a larger class of classical zero-sum differential games —where terminal time is defined by the first entrance inside a target—, where one of the players has a partial knowledge of the state, even if the set of compatible states is not open, but on the condition that no barrier exists in the related perfect information game. Nevertheless, we show how to solve a qualitative game (or *game of kind* in Isaacs'terminology) via the study of an alternate game to which the certainty equivalence principle might apply.

# 2 The problems

## 2.1 The standard problem

Let us consider a zero-sum differential game, as in [Isa65] defined by the following components:

- A dynamical system : $x(.) = S(t_0, x_0, u(.), v(.))$ is the unique solution of the system :

$$\begin{cases} \dfrac{dx}{dt} &= f(t,x,u,v) \\ x(t_0) &= x_0 \end{cases}$$

where $t \in [t_0, +\infty)$, $x(t) \in X = \mathbb{R}^n$, $x_0 \in X_0 \subset X$, $u(t) \in \mathsf{U}$ and $v(t) \in \mathsf{V}$ ($\mathsf{U}$ and $\mathsf{V}$ are closed subsets respectively in $\mathbb{R}^p$ and $\mathbb{R}^q$). The sets $\mathcal{U}$ and $\mathcal{V}$ of admissible open loop controls $u(.)$ and $v(.)$ will be measurable functions from $[t_0, +\infty)$ into $\mathsf{U}$ and $\mathsf{V}$ respectively. When this causes no ambiguity, we shall denote them $u \in \mathcal{U}$ and $v \in \mathcal{V}$. We assume that adequate regularity and growth assumptions hold on $f$ such that any $(x_0, u, v) \in X_0 \times \mathcal{U} \times \mathcal{V}$ generates a unique trajectory $x(.) = S(t_0, x_0, u(.), v(.))$ over $[t_0, T]$ where $T$ is a positive number that appears in the next item.

- A capture set : $\mathcal{T}$ is an open subset of $\mathbb{R} \times X$ and the capture time $t_f$ is defined as the first instant when the state enters $\mathcal{T}$ :

  $$t_f(t_0, x_0, u, v) = \inf_{t \geq t_0} \{\, t \,|\, (t, x(t)) \in \mathcal{T}\}\,.$$

  The system will be assumed to be capturable over $(t_0, X_0)$ in the very strong sense that:

  $$\exists T < +\infty,\ \exists u \in \mathcal{U},\ \forall x_0 \in X_0,\ \forall v \in \mathcal{V},\ t_f(t_0, x_0, u, v) < T(t_0, X_0).$$

  This is satisfied if, for instance, $\exists T$ such that $[T, +\infty) \times X \subset \mathcal{T}$ (A classical assumption in the theory of capture-evasion games [Fri71]).

- A criterion : We shall consider the case where $x_0$ is not known of the controller who has to choose $u$, and we shall include it in the unknown disturbance or opponent's control (see section 2.2). Thus we add an "initial cost" $N(x_0)$ to the classical performance index, leading to :

  $$J(t_0, x_0, u, v) \ = \ M(t_f, x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t), v(t))dt + N(x_0).$$

  However, when dealing with perfect information, we shall need to consider the criterion $J^* = J - N$:

  $$J^*(t_0, x_0, u, v) \ = \ M(t_f, x(t_f)) + \int_{t_0}^{t_f} L(t, x(t), u(t), v(t))dt$$

- Sets of admissible controllers : We will consider classes $\Phi$ and $\Psi$ of admissible feedback strategies $u(t) = \phi(t, x(t))$ and $v(t) = \psi(t, x(t), u(t))$ satisfying the following properties:
  - Open-loop controls are admissible: $\mathcal{U} \subset \Phi$ and $\mathcal{V} \subset \Psi$.
  - $\Phi$ and $\Psi$ are closed by concatenation.
  - $\forall(\phi, \psi) \in \Phi \times \Psi$, the system (S) admits an unique solution $x(.)$, leading to admissible controls $u(.) = \phi(., x(.)) \in \mathcal{U}$ and $v(.) = \psi(., x(.), u(.)) \in \mathcal{V}$.

  Such a $\psi(.)$ is called in the game theoretic literature a *discriminating feedback* [Bre86]. It is known, though, that if "Isaacs' condition" is met, the optimal $\psi^*$ is in fact a simple state feedback. These properties do not uniquely specify the pair $(\Phi, \Psi)$, but we know that such pairs exist, and the above assumptions suffice to justify Isaacs' equation (see [Ber87]).

Let us now introduce the standard problem in perfect information :

**Problem $\mathcal{P}^*(\mathbf{t_0}, \mathbf{X_0})$** If the initial condition $(t_0, x_0)$ belongs to $(\{t_0\}X_0)\backslash\mathcal{T}$, does a Isaacs' value function:

$$V(t_0, x_0) = \min_{\phi \in \Phi} \max_{\psi \in \Psi} J^*(t_0, x_0, \phi, \psi)$$

exist?

Let us also recall the classical Hamilton-Jacobi-Isaacs theorem [Isa65] [Ber87] :

**Proposition 1** *If there exists a $C^1$ function $V$ over $([t_0, T(t_0, X_0)] \times X)\backslash\mathcal{T}$, solution of the partial differential equation:*

$$\begin{cases} -\dfrac{\partial V}{\partial t} & = & \min\limits_{u \in \mathsf{U}} \max\limits_{v \in \mathsf{V}} H(t, x, \dfrac{\partial V}{\partial x}, u, v) \\ V(t, x) & = & M(t, x) \quad \text{if } (t, x) \in \partial\mathcal{T} \end{cases}$$

*where $H(t, x, \lambda, u, v) = L(t, x, u, v) + \lambda^t f(t, x, u, v)$ is the Hamiltonian of the system, then $V$ is the solution of the problem $\mathcal{P}^*(t_0, X_0)$. Moreover, any admissible $(\phi^*, \psi^*)$ such that:*

$$\begin{cases} \phi^*(t, x) & \in & \arg\min\limits_{u \in U} H(t, x, \dfrac{\partial V}{\partial x}, u, \psi^*(t, x, u)) \\ \psi^*(t, x, u) & \in & \arg\max\limits_{v \in V} H(t, x, \dfrac{\partial V}{\partial x}, u, v) \end{cases}$$

*are optimal state feedbacks.*

### 2.2 An imperfect information structure

For a given pair $(t_0, X_0)$, let us write $T = T(t_0, X_0), \mathcal{U} = \mathcal{U}|_{[t_0,T]}, \mathcal{V} = \mathcal{V}|_{[t_0,T]}$ and consider disturbances as pairs $\omega = (x_0, v) \in \Omega = X_0 \times \mathcal{V}$, and so write $S(t_0, u, \omega) = S(t_0, x_0, u, v),\ J(t_0, u, \omega) = J(t_0, x_0, u, v)$. Instead of a perfect knowledge of the state, we shall consider the following observation scheme for the player u :

$$\mathcal{O} \ : \ \left\{ \begin{array}{ccc} [t_0, T] \times \mathcal{U} \times \Omega & \longrightarrow & \mathcal{Y} \subset 2^{\Omega} \\ (t, u, \omega) & \longmapsto & \mathcal{O}_t(u, \omega) \in \mathcal{Y}_t \end{array} \right.$$

meaning that u knows, at time t, that $x_0$ and the control $v(.)$ of its opponent belong to a certain subset $\mathcal{O}_t(u, \omega)$ of $\Omega$ (We shall give more concrete examples shortly).

Given a pair $(u, v) \in \mathcal{U} \times \mathcal{V}$, define for all $t \in [t_0, T]$ their restrictions to $[t_0, t]$ as :

$$\left\{ \begin{array}{l} u^t \ : \left( \begin{array}{ccc} [t_0, t] & \longrightarrow & \mathsf{U} \\ \tau & \longmapsto & u(\tau) \end{array} \right), \\ v^t \ : \left( \begin{array}{ccc} [t_0, t] & \longrightarrow & \mathsf{V} \\ \tau & \longmapsto & v(\tau) \end{array} \right), \end{array} \right.$$

and $\omega^t = (x_0, v^t)$. Let us write $\Omega_t = \mathcal{O}_t(u, \omega)$ and define :

$$\Omega_t^t = \left\{ \omega^t \mid \omega \in \Omega_t \right\}.$$

We shall request the following assumptions for the observation scheme $\mathcal{O}$ :

**Hypotheses H1 :** $\forall (u, \omega) \in \mathcal{U} \times \Omega,\ \forall t \in [t_0, T],\ \Omega_t = \mathcal{O}_t(u, \omega)$ satisfies:

**Hypothesis H1a** The process $\mathcal{O}$ is *consistent* :

$$\omega \in \Omega_t.$$

**Hypothesis H1b** The process $\mathcal{O}$ is *perfect recall* :

$$\forall t' > t,\ \Omega_{t'} \subset \Omega_t.$$

**Hypothesis H1c** The process $\mathcal{O}$ is *nonanticipative* :

$$\omega^t \in \Omega_t^t \Longleftrightarrow \omega \in \Omega_t.$$

Typically, an observed output: $y(t) = h(t, x(t), v(t))$ leads to such a process :

$$\Omega_t = \{(x_0, v) \in \Omega \mid \text{for } x(.) = S(t_0, x_0, u, v),\ \forall \tau \le t,\ y(\tau) = h(\tau, x(\tau), v(\tau))\}$$

is the equivalence class of all disturbances $\omega$ that together with $u^t$ lead to the same observed output history up to time t : $y^t = \{\, y(\tau) \,|\, \tau \le t\,\}$.

We shall also need the following *restricted observation sets* for the problem $\mathcal{P}$ decribed in the next section :

$$\widetilde{\Omega}_t = \Omega_t \cap \{\omega \mid \forall \tau \le t,\ x(\tau) \notin \mathcal{T}\}\,.$$

**Proposition 2** $\widetilde{\Omega}_t$ *satisfies the hypotheses H1 up to* $t_f$*.*

**Proof**

Consistence up to $t_f$ is by definition, the decreasing character is preserved by intersection of two decreasing subset families, and the same holds true for the non anticipative character. ∎

As a matter of fact, in the sequel we shall only use $\widetilde{\Omega}_t$ and thus omit the $\tilde{}$, meaning that we have included the restriction into the definition of $\Omega_t$.

Finally, let us define a class $\mathcal{M}_{[t_0,T]}$ of admissible controllers :

$$\mathcal{M}_{[t_0,T]} = \left\{ \begin{array}{cccc} \mu \ : & [t_0, T] \times \mathcal{Y} & \longrightarrow & \mathsf{U} \\ & (t, \Omega_t) & \longmapsto & \mu_t(\Omega_t) \end{array} \right\}$$

such that $\forall \omega \in \Omega$, the pair $(\mu, \omega)$ generates a well defined unique trajectory. Remark that due to the definition of $\Omega_t$, it is indeed a class of nonanticipative controllers.[1]

### 2.3 The imperfect information problem

**Problem** $\mathcal{P}(\mathbf{t_0}, \mathbf{X_0})$ If for all $x_0 \in X_0$, $(t_0, x_0) \notin \mathcal{T}$, does there exist a controller $\mu$ guaranteing the value :

$$\min_{\mu \in \mathcal{M}_{[t_0,T]}} \max_{\omega \in \Omega} J(t_0, \mu, \omega)$$

and if yes, how to build one?

[1] In fact, the notation above is not quite satisfactory as $\Omega_t$ has to belong to $\mathcal{Y}_t$ and not simply to $\mathcal{Y}$. Hence, $\mu$ is a function from the bundle $\bigcup\{t, \mathcal{Y}_t\}$ into $\mathsf{U}$.

## 3 The min-max certainty equivalence principle

Thanks to a certain auxiliary problem, we shall build a controller $\widehat{\mu}$ and then prove its optimality for the problem $\mathcal{P}$ under some hypotheses.

### 3.1 The auxiliary problem

Let us introduce an *auxiliary problem* defined by :

- **The auxiliary function G :** Under the hypothesis :

  **Hypothesis H2** The problem $\mathcal{P}(t_0, X_0)$ admits a upper saddle-point $(\phi^*, \psi^*)$ and a upper value function $V$ of class $C^1$,

  we define for all admissible $(u, \omega) \in \mathcal{U} \times \Omega$ and for all $t \in [t_0, T]$ :

  $$G_t(u, \omega) = G_t(u^t, \omega^t) = V(t, x(t)) + \int_{t_0}^{t} L(\tau, x(\tau), u(\tau), v(\tau))d\tau + N(x_0)$$

  where $x(.) = S(t_0, x_0, u, v)$.

- **The problem $\mathbf{Q_t(u^t, \Omega_t^t)}$ :** Does there exists

  $$\max_{\omega^t \in \Omega_t^t} G_t(u^t, \omega^t) \ \ ?$$

  When it exists, we shall write :

  $$\begin{aligned} \widehat{\Omega_t^t} &= \left\{ \widehat{\omega_t} = (\widehat{x_{0t}}, \widehat{v_t}) \,|\, \widehat{\omega_t} \in \arg\max_{\omega^t \in \Omega_t^t} G_t(u^t, \omega^t) \right\} \\ \widehat{X_t} &= \left\{ \widehat{x_t}(t) \,|\, \widehat{x_t}|_{[t_0,t]}(.) = S(t_0, \widehat{x_{0t}}, u^t, \widehat{v_t}) \text{ et } (\widehat{x_{0t}}, \widehat{v_t}) \in \Omega_t^t \right\} \end{aligned}$$

  for the sets of maximizing disturbances and corresponding *conditional worst states* at time $t$.

### 3.2 The controller $\widehat{\mu}$

Let us introduce the crucial assumption :

**Hypothesis H3a** For all pair $(u(.), \Omega_{(.)}) \in \mathcal{U} \times \mathcal{Y}$, there exists a solution to the problem $Q_t(u^t, \Omega^t)$ and for all $t \in [t_0, T]$, $\widehat{X_t}$ is a singleton.

Under that assumption, we define an application $\mathcal{A} : \mathcal{U} \longrightarrow \mathcal{U}$ in the following way : for all $t \in [t_0, T]$,

$$\mathcal{A}_t \ : \begin{cases} \mathcal{U}^t \times \mathcal{Y}_t & \longrightarrow & \mathsf{U} \\ (u^t, \Omega_t) & \longmapsto & \bar{u}(t) = \phi^*(t, \hat{x}(t)) \end{cases}$$

where $\phi^*$ is the unique $u$-optimal feedback for the problem $\mathcal{P}^*(t_0, X_0)$, thanks to hypothesis H2, and $\widehat{x}(t)$ the unique member of $\widehat{X_t}$ thanks to hypothesis H3a.
**Remark :** $\widehat{X_t}$ is never empty and the unicity of $\widehat{x}_t(t)$, which does not necessarily imply the unicity of a maximizing $\widehat{\omega_t}$, allows one to consider it as the *worst state* compatible with past observations, simply written $\widehat{x}(t)$.

Let us then define the controller $\widehat{\mu}$ as a fixed point of the application $\mathcal{A}$, under the following hypothesis :

**Hypothesis H3b** For all pair $\omega = (x_0, v(.)) \in \Omega$, the application $\mathcal{A}$ admits a unique fixed point $\widehat{\mu}$, and as a consequence, the system $S(t_0, \bar{u}, \omega)$ driven by $\bar{u}(t) = \phi^*(t, \widehat{x}(t))$ admits the unique solution $\widehat{x}(.)$. ($\widehat{\mu}$ is defined up to time $t_f$ such that $(t_f, \widehat{x}(t_f)) \in \mathcal{T}$).

**Comments :**

- In practice, when the solution of the game with perfect information is known, and we receive the information $\Omega_{(.)}$ over the time $t$, the determination of the control $\bar{u}(.)$ consists in solving at each instant of time the auxiliary problem $\mathcal{Q}_t(\bar{u}^t, \Omega_t^t)$, which provides then the value $\bar{u}$ at time $t$, under the condition that the unicity of the worst state is met. Choosing the controller $\mu$ amounts to playing as if we knew that the real state of the system were $\widehat{x}(t)$. This is the reason why we call it a *certainty equivalent* controller.

- The definition of $\mu$ in terms of a fixed point is no more implicit that any feedback control. As a matter of fact, the dynamics and observation process can be seen as an evolution equation on $\Omega_t$ driven by $u$, and $\mu$ as a "$\Omega$-feedback". We do not attempt here the feat of exhibiting general conditions on the data that would guarantee existence of a unique solution. The exemple of the section 6 will show how things may go in practice.

- Even when the hypotheses H2 and H3 are satisfied, nothing guarantees *a priori* that the set of positions $\widehat{x}(t)$ constitutes a trajectory solution of $S(t_0, \widehat{x_0}, \bar{u}(.), \widehat{v}(.))$ for a certain disturbance $\widehat{\omega} = (\widehat{x_0}, \widehat{v}) \in \Omega$, which is always true for linear-quadratic games (see [BB91]). For this, it would be necessary to have:

$$\forall t_2 > t_1 \quad \exists (\widehat{\omega_{t_1}}, \widehat{\omega_{t_2}}) \in \widehat{\Omega_{t_1}^{t_1}} \times \widehat{\Omega_{t_2}^{t_2}} \mid \widehat{\omega_{t_2}}|_{[t_0,t_1]} = \widehat{\omega_{t_1}}.$$

## 3.3 The principle

**Proposition 3** *Under the hypotheses H1-H3, $\widehat{\mu}$ is an optimal controller that guarantees:*

$$\max_{\omega \in \Omega} J(t_0, \widehat{\mu}, \omega) = \max_{x_0 \in X_0} [V(t_0, x_0) + N(x_0)] = g(t_0).$$

**Proof**

For a given $\overline{\omega} = (\overline{x}_0, \overline{v}) \in \Omega$, let us call $\overline{u}(.)$ the open-loop representation of the controller $\widehat{\mu}$, $t_f$ the corresponding final time, and $\Omega_{(.)}$ the information process. Any other $\omega \in \Omega_t$ would produce the same $\overline{u}^t$. Then, let us define, for any t in $[t_0, t_f]$,

$$G(t, \omega) = G_t(\overline{u}^t, \omega^t).$$

Notice that although G is defined as a function from $[t_0, t_f] \times \Omega$ into $I\!R$, it actually only depends upon the restriction $\omega^t$ of $\omega$ to $[t_0, t]$. Thus, using also hypothesis H1c, we get

$$g(t) = \max_{\omega \in \Omega_t} G(t, \omega) = \max_{\omega^t \in \Omega_t^t} G_t(\overline{u}^t, \omega^t).$$

Let us also notice that $g$ is well defined, because $\Omega_t$ is never empty, thanks to the hypothesis H1a.

We are now going to prove that the function $g(.)$ is nonincreasing over $[t_0, t_f]$. To each time instant t, associate the function:

$$g_t(\tau) : \tau \longmapsto \max_{\omega \in \Omega_t} G(\tau, \omega).$$

where the constraint bearing upon $\omega$ has been frozen at $\omega \in \Omega_t$. For t in $(t_0, t_f)$ and $h > 0$ such that $t + h \leq t_f$ and $t - h \geq t_0$, we have, according to the hypothesis H1b:

$$\begin{cases} g(t-h) & \geq & g_t(t-h), \\ g(t) & = & g_t(t), \\ g(t+h) & \leq & g_t(t+h). \end{cases}$$

So, the upper Dini derivative of $g$ satisfies:

$$D^+g(t) = \limsup_{t' \to t} \frac{g(t') - g(t)}{t' - t} \le \limsup_{t' \to t} \frac{g_t(t') - g_t(t)}{t' - t}.$$

Thanks to Danskin's theorem (for instance Corollary 2, p. 87 of [Cla83]), $g_t(.)$ has left and right derivatives $g_t^l(.)$ and $g_t^r(.)$ :

$$g_t^l(\tau) = \min_{\widehat{\omega} \in \widehat{\Omega}_t} \frac{\partial G(\tau, \widehat{\omega})}{\partial \tau}, \; g_t^r(\tau) = \max_{\widehat{\omega} \in \widehat{\Omega}_t} \frac{\partial G(\tau, \widehat{\omega})}{\partial \tau}.$$

When $\tau = t$, these last expressions depend only on the restriction of $\widehat{\Omega}_t$ to the interval $[t_0, t]$. For each $\widehat{\omega}^t = (\widehat{x}_0, \widehat{v}^t)$ realizing the maximum of the problem $Q_t(\overline{u}^t, \Omega_t^t)$, the generated state $\widehat{x}(.) \in S(t_0, \widehat{x_0}, u, \widehat{v})$ at time t is the same, thanks to the hypothesis H3a. Let us call it $\widehat{x}(t)$. For any maximizing $\widehat{\omega}^t$, we have:

$$\begin{aligned} \frac{\partial G(t, \widehat{\omega})}{\partial t} &= \frac{\partial V}{\partial t}(t, \widehat{x}(t)) + \frac{\partial V}{\partial x}(t, \widehat{x}(t)).f(t, \widehat{x}(t), \overline{u}(t), \widehat{v}(t)) + L(t, \widehat{x}(t), \overline{u}(t), \widehat{v}(t)) \\ &= \frac{\partial V}{\partial t}(t, \widehat{x}(t)) + H\left(t, \widehat{x}(t), \frac{\partial V}{\partial x}(t, \widehat{x}(t)), \overline{u}(t), \widehat{v}(t)\right). \end{aligned}$$

According to Isaacs' equation of problem $\mathcal{P}^*$ and the choice of $\widehat{\mu}$, this last expression is nonpositive. One concludes that the Dini derivative of $g$ is nonpositive, and therefore $g$ is decreasing (see Appendix). So, $\overline{x}$ being the solution of $S(t_0, \overline{u}, \overline{\omega})$, and by H1a, $\overline{\omega} \in \Omega_{(.)}$, we have :

$$J(t_0, \widehat{\mu}, \overline{\omega}) = M(t_f, \overline{x}(t_f)) + \int_{t_0}^{t_f} L(\tau, \overline{x}(\tau), \overline{u}(\tau), \overline{v}(\tau) d\tau) + N(x_0) \le g(t_f) \le g(t_0).$$

This bound being uniform in $\overline{\omega}$, we get:

$$\max_{\omega \in \Omega} J(t_0, \widehat{\mu}, \omega) \le g(t_0).$$

Conversely, for any controller $\mu(.)$, a possible $\omega$ is

$$\omega^* = \left( x_0^* = \arg \max_{x_0 \in X_0} V(t_0, x_0) \;,\; v^*(.) = \psi^*(., x^*(.), \mu(.)) \right)$$

along the trajectory $x^*(.)$ and then, according to the other saddle-point inequality of Isaacs' equation:

$$\max_{\omega \in \Omega} J(t_0, \mu, \omega) \ge J(t_0, \mu, \omega^*) = V(t_0, x_0^*) + N(x_0^*) = g(t_0).$$

Hence, $\widehat{\mu}$ is a min-max controller, and even a saddle point controller, and the theorem is proved. ∎

**Remark :** The hypothesis H3a requires the unicity of the worst state for all open-loops $u(.) \in \mathcal{U}$, even though the proof only needs to check it for all possible open-loop representations of the controller $\widehat{\mu}$. This is due to the difficulty to write properly this hypothesis for a controller, defined only implicitly as a feedback.

We recall also the following result:

**Proposition 4** *If* $\forall (u, \Omega_{(.)}) \in \mathcal{U} \times \mathcal{Y}$*, there exists* $\tau^* \in (t_0, T)$ *such that* $\sup_{\omega \in \Omega_{\tau^*}} G_{\tau^*}(u, \omega) = +\infty$*, then* $\forall \mu \in \mathcal{M}, \sup_{\omega \in \Omega} J(u, \omega) = +\infty$*.*

**Proof**

For the proof, we refer to [Ber90a], [BB91]. ∎

This is important to notice because it is to get this result that we constrain $\omega$ to $\Omega_t$.

## 4 An alternate criterion

Instead of a capture time $t_f$, we shall consider the following terminal time :

$$t_f(t_0, x_0, u, v) = \arg\min_{t \geq t_0} \left\{ M(t, x(t)) + \int_{t_0}^{t} L(\tau, x(\tau), u(\tau), v(\tau)) d\tau \right\}.$$

Then the problem $\mathcal{P}^*$ reads :

$$V(t_0, x_0) = \min_{\phi \in \Phi} \max_{\psi \in \Psi} \min_{t \geq t_0} \left[ M(t, x(t)) + \int_{t_0}^{t} L(\tau, x(\tau), u(\tau), v(\tau)) d\tau \right].$$

Typically, in pursuit-evasion games, when the capturability assumption is not met, one may be interested in the qualitative problem, what Isaacs calls the *game of kind* : does there exist initial conditions from which capture never occurs ? If the targer is described via a function $C \in C^1(\mathbb{R} \times X, \mathbb{R})$ as :

$$\mathcal{T} = \{ (t, x) \in \mathbb{R} \times X \mid C(t, x) < 0 \}$$

the sign of the criterion associated with $M = C$ and $L = 0$ provides the qualitative result : capture or evasion.

The strong capturability hypothesis is here replaced by :

$$\exists T < +\infty, \forall x_0 \in X_0, \forall (u, v) \in \mathcal{U} \times \mathcal{V}, t_f(t_0, x_0, u, v) < T.$$

An equivalent form of the Isaacs equation for this particular criterion is the following variationnal inequality :

**Proposition 5** *If there exists a $C^1$ function $V$ over $[t_0, +\infty) \times X$ such that : $\forall (t, x) \in [t_0, +\infty) \times X$,*

$$
\begin{aligned}
&i) \quad V(t,x) \leq M(t,x),\\
&ii) \quad \min_{u \in \mathsf{U}} \max_{v \in \mathsf{V}} \left\{ \frac{\partial V}{\partial t} + H(t, x, \frac{\partial V}{\partial x}, u, v) \right\} \geq 0,\\
&iii) \quad V(t,x) \neq M(t,x) \Longrightarrow \min_{u \in \mathsf{U}} \max_{v \in \mathsf{V}} \left\{ \frac{\partial V}{\partial t} + H(t, x, \frac{\partial V}{\partial x}, u, v) \right\} = 0,\\
&iv) \quad \forall x_0 \in X_0, \forall (\phi, \psi) \in \Phi \times \Psi, \exists \widehat{t} \in [t_0, +\infty) \,|\, V(\widehat{t}, x(\widehat{t})) = M(\widehat{t}, x(\widehat{t}))\\
&\qquad \textit{when } x(.) = S(t_0, x_0, \phi, \psi)
\end{aligned}
$$

*where $H(t, x, \lambda, u, v) = L(t, x, u, v) + \lambda^t f(t, x, u, v)$ is the Hamiltonian of the system, then $V$ is the solution of the problem $\mathcal{P}^*(t_0, X_0)$. Moreover, any admissible $(\phi^*, \psi^*)$ such that:*

$$
\left\{
\begin{array}{lcl}
\phi^*(t,x) & \in & \arg \min\limits_{u \in U} H(t, x, \frac{\partial V}{\partial x}, u, \psi^*(t, x, u))\\
\psi^*(t,x,u) & \in & \arg \max\limits_{v \in V} H(t, x, \frac{\partial V}{\partial x}, u, v)
\end{array}
\right.
$$

*are optimal state feedbacks, and :*

$$t^* = \min\{t \geq t_0 \,|\, V(t, x^*(t)) = M(t, x^*(t))\}$$

*when $x^*(.) = S(t_0, x_0, \phi^*, \psi^*)$, is an optimal stopping time.*

Then, we introduce an alternate notion of target :

$$\mathcal{T} = \{(t, x) \in \mathbb{R} \times X \,|\, V(t, x) = M(t, x)\}$$

from which we shall restrict the observation sets for the problem $\mathcal{P}$ :

$$\widetilde{\Omega}_t = \Omega_t \cap \{\omega \,|\, \forall \tau \leq t, V(\tau, x(\tau)) < M(\tau, x(\tau))\}$$

With the same hypotheses on the the value function of the game with perfect information and the observation process, one obtains the same proposition 3.

## 5 When the principle cannot be applied

When the auxiliary problem admits an unique solution and the principle holds, the value of the game is the same as in the perfect information case, and so satisfies the same Isaacs' equation. For the general case, it seems to be reasonable to look after a kind of dynamic programming equation, where the maximization in $v$ may be replaced by a maximization in $\omega$. Unfortunately, only one inequality can be proved, and we give further an example that shows that we may not expect anything better : when the certainty equivalence principle does not hold, none of the trajectories compatible with the observation may be an extremal of the problem in perfect information.

Assume there exists an optimal controller $\widehat{\mu}$ solution of the problem $\mathcal{P}(t_0, X_0)$ when $N(.) = 0$. We shall call $U(t_0, X_0)$ its value function. For any $\omega \in \Omega$, let $\Omega_{(.)}$ be the information process observed when u plays $u_\omega(.) = \widehat{\mu}_{(.)}(\Omega_{(.)})$. Then, let us write $x_\omega(.)$ for the solution of the system $S(t_0, u_\omega, \omega)$. We claim the following property :

**Proposition 6** *If $\forall t \in [t_0, T]$, $\forall \omega^t \in \Omega^t$, the problem $\mathcal{P}(t, \{x_{\omega^t}(t)\})$ admits a solution $U(t, \{x_{\omega^t}(t)\})$, then:*

$$\forall t \in (t_0, T), \quad U(t_0, X_0) \geq \max_{\omega^t \in \Omega^t} \left\{ \int_{t_0}^{t} L(\tau, x_{\omega^t}(\tau), u_{\omega^t}(\tau), v_{\omega^t}(\tau)) d\tau + U(t, \{x_{\omega^t}(t)\}) \right\}$$

**Proof**

Let $\overline{\omega}^t = (x_0, \overline{v}^t)$ be an element of $\Omega^t$, for any $v \in \mathcal{V}|_{[t,T]}$, define $\omega = \overline{\omega}^t.v = (x_0, \overline{v}^t.v)$ (where $\overline{v}.v$ defines the concatenation of $\overline{v}$ and $v$).

$$\begin{aligned} U(t_0, X_0) &\geq \max_{v \in \mathcal{V}|_{[t,T]}} J(t_0, \widehat{\mu}, \overline{\omega}^t.v) \\ &= \max_{v \in \mathcal{V}|_{[t,T]}} \left[ \int_{t_0}^{t} L(\tau, \bar{x}^t(\tau), \bar{u}^t(\tau), v(\tau)) d\tau + J(t, \bar{x}^t(t), \widehat{\mu}, v) \right]. \end{aligned}$$

Thanks to the hypothesis H1c of the nonanticipative behavior of the observation process $\Omega_{(.)}$, the first term does no longer depend on the future disturbance. So,

$$\begin{aligned} U(t_0, X_0) &\geq \int_{t_0}^{t} L(\tau, \bar{x}^t(\tau), \bar{u}^t(\tau), \bar{v}^t(\tau)) d\tau + \max_{v \in \mathcal{V}|_{[t,T]}} J(t, \bar{x}^t(t), \widehat{\mu}, v) \\ &\geq \int_{t_0}^{t} L(\tau, \bar{x}^t(\tau), \bar{u}^t(\tau), \bar{v}^t(\tau)) d\tau + U(t, \{\bar{x}^t(t)\}). \end{aligned}$$

∎

# 6 An example

Let us consider the following problem in the plane:

$$\begin{cases} \dot{x} = -cu + b\cos v & a > b > c > 0 \\ \dot{y} = -a + b\sin v & u \in [-1,1] \quad , \quad v \in [0, 2\pi) \end{cases}$$

with the end-point condition:

$$y(0) > 0 \quad , \quad y(t_f) = 0,$$

and the objective:

$$\min_{u \in \mathcal{U}} \max_{v \in \mathcal{V}} |x(t_f)|.$$

### 6.0.1 The perfect information solution

When $\Phi$ and $\Psi$ are admissible state feedback classes, one can easily derive a saddle point solution thanks to Isaacs' differential game theory.
Let us write the Isaacs' equation:

$$(I) \begin{cases} b\sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2} - c\left|\frac{\partial V}{\partial x}\right| - a\frac{\partial V}{\partial y} = 0 \\ V(x,0) = |x|, \end{cases}$$

and check that:

$$V(x,y) = |x| + r^+ y,$$

where

$$r^+ = \frac{ac + \sqrt{a^2c^2 + (a^2 - b^2)(b^2 - c^2)}}{a^2 - b^2}$$

is the positive root of

$$b\sqrt{1 + r^2} - c - ar = 0\,,$$

is a solution of (I) in the viscosity sense (see [Lio82]). Thanks to the results on viscosity solutions for Hamilton-Jacobi-Isaacs equations, we know that the solution is unique, and furthermore that it is the value function of the game [Sor93].

It is immediate to check that V is a $C^1$ solution, except on the $y$ axis, where Fréchet sub and superdifferentials are:

$$\begin{cases} \partial^+ V(0,y) &= \emptyset \\ \partial^- V(0,y) &= [-1,1] \times \{r^+\} \end{cases}$$

and we have to check only one viscosity inequality:

$$\forall \rho \in [0,1] \quad , \quad H^*(\rho, r^+) = b\sqrt{\rho^2 + (r^+)^2} - c\rho - ar^+ \leq 0$$

We see that

$$\begin{cases} H(0,r^+) &= (b-a)r^+ & < 0 \\ H(\rho, r^+) &= \rho \left[ b\sqrt{1 + \left(\frac{r^+}{\rho}\right)^2} - c - a\frac{r^+}{\rho} \right] & \leq 0 \end{cases}$$

So,

$$V(x,y) = |x| + \frac{-ac + \sqrt{(ac)^2 + (a^2-b^2)(b^2-c^2)}}{a^2 - b^2} y$$

is the value function of the problem.

Notice that when $x_0 \neq 0$, there exist optimal (saddle-point) open loop strategies:

- If $x_0 > 0$,

$$\begin{cases} u^* &= +1 \\ (\sin v^*, \cos v^*) &= \left( \frac{r^+}{\sqrt{1+(r^+)^2}}, \frac{1}{\sqrt{1+(r^+)^2}} \right) \end{cases}$$

- If $x_0 < 0$,

$$\begin{cases} u^* &= -1 \\ (\sin v^*, \cos v^*) &= \left( \frac{r^+}{\sqrt{1+(r^+)^2}}, \frac{-1}{\sqrt{1+(r^+)^2}} \right) \end{cases}$$

If the initial state stands at $x_0 = 0$, two symmetric trajectories are possible. Whatever is the choice of u, v can force the state to definitely go to one side of the y axis: It is a dispersal line for the maximizer (the maximizer is called the *overrider* with a terminology proposed by Breakwell [Bre86]).

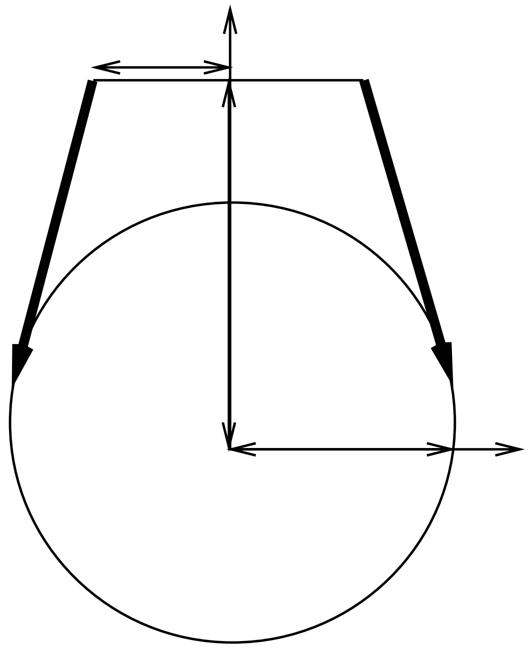

Figure 1: Vectogram and semi-permeable directions

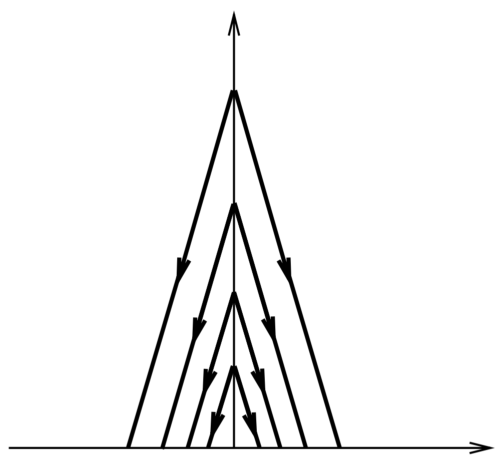

Figure 2: Extremals field

### 6.0.2 An imperfect information scheme

Assume now that the player $u$ only has the knowledge of $x(0) = x_0$ and $\{y(.)\}_{[t_0,t)}$ at time t. Here, the information scheme is:

$$\left\{\begin{array}{lcl} X_0 & = & \{(x_0, y_0)\} \\ \Omega_t & = & \left\{((x_0, y_0), v(.)) \mid \forall \tau \in [0,t),\ \sin v(\tau) = \dfrac{\dot{y}(\tau) + a}{b}\right\} \end{array}\right.$$

We are looking for an optimal controller $\mu(x_0, \Omega_{(.)})$ against the worst $v(.)$.

**Proposition 7** *The open-loop controller:*

$$\left\{\begin{array}{lcll} u^*(t) & = & \mathrm{sgn}(x_0) & 0 \leq t \leq \dfrac{|x_0|}{c} \\ u^*(t) & = & 0 & t > \dfrac{|x_0|}{c} \end{array}\right.$$

*is optimal.*

**Proof**

Let $v^*$ be a maximizing disturbance against $u^*$. Then the final time is defined as:

$$\int_0^{t_f} (a - b \sin v^*(\tau))\, d\tau = y_0.$$

$|x_0|/c$ is exactly the time at which player $u$ can make the system reach the $y$-axis. So, depending on $t_f$, two cases have to be considered :

- $t_f < \dfrac{|x_0|}{c}$

  Then,

$$\begin{array}{rcl} x(t_f) & = & x_0 - \mathrm{sgn}(x_0) c t_f + b \displaystyle\int_0^{t_f} \cos v^*(\tau) d\tau \\ & = & \mathrm{sgn}(x_0) \left[|x_0| - c t_f\right] + b \displaystyle\int_0^{t_f} \cos v^*(\tau) d\tau. \end{array}$$

  So,

$$\max_{v(.)} |x(t_f)| = \max_{v(.)} \mathrm{sgn}(x_0) x(t_f).$$

  The control $v^*$ is necessarily solution of this last optimal control problem. Thanks to the open-loop saddle point solution of the game in perfect information: $(u^*_{PI}, v^*_{PI})$, and the fact that $u^*_{PI} = u^*|_{[0,t_f]}$, $v^* = v^*_{PI}$

is optimal.

So, $J(u^*, v^*) = J(u^*_{PI}, v^*_{PI})$ which is optimal, corroborating the certainty equivalence principle.

Notice that the current theory accounts for that fact, not any previous one: beyond the variable end time, which is of little consequence here, the theory of [Ber90a] does not apply because the observation function is not surjective in v, and that of [DBB93] does not either because $\widehat{X}$ is not open.

- $t_f \geq \dfrac{|x_0|}{c}$

  Let $\mu(.)$ be a controller, and consider the open-loop control $\overline{v^*}(.) = \pi - v^*(.)$. It leads to the same final time $t_f$ and the same function of time $y(.)$ than $v^*(.)$, when it is played against the same $\mu(.)$ (i.e. it always belong to the same set $\Omega_t$ as $v^*$). Let us write $u(.)$ the open-loop representation of $\mu(.)$ played along with $v^*(.)$ or $\overline{v^*}(.)$.

$$x(t_f) = x_0 - c\int_0^{t_f} u(\tau)d\tau + b\int_0^{t_f} \cos v(\tau)d\tau.$$

  where $v$ may be either $v^*$ or $\overline{v}^*$. So,

$$\begin{aligned}
&\max_{v(\cdot)} J(\mu, v(\cdot)) \geq \\
&\max\left\{\left|x_0 - c\int_0^{t_f} u(\tau)d\tau + b\int_0^{t_f} \cos v^*(\tau)d\tau\right|, \left|x_0 - c\int_0^{t_f} u(\tau)d\tau - b\int_0^{t_f} \cos v^*(\tau)d\tau\right|\right\} \\
&\quad = \left|x_0 - c\int_0^{t_f} u(\tau)d\tau\right| + b\left|\int_0^{t_f} \cos v^*(\tau)d\tau\right| \\
&\quad \geq b\left|\int_0^{t_f} \cos v^*(\tau)d\tau\right| = J(u^*, v^*) = \max_v J(\widehat{\mu}, v)
\end{aligned}$$

  and the result is proved.

∎

### 6.0.3 Calculation of the value

Two cases are possible:

- $u^*(.)$ is constant until $t_f$

  Then, the value is the same as in the perfect information scheme:

$$J_1(x_0, y_0) = |x_0| + \frac{-ac + \sqrt{(ac)^2 + (a^2 - b^2)(b^2 - c^2)}}{a^2 - b^2} y_0$$

  on the condition that:

$$t_{f_1}(y_0) = \frac{y_0}{a - b\sin v_1^*} = \frac{y_0}{a - b\frac{r^+}{\sqrt{1+(r^+)^2}}} < \frac{|x_0|}{c}.$$

- $u^*(.)$ switches to 0 before $t_f$

  Determining the worst $v(.)$ against $u^*(.)$ is a one-player optimal control problem, with a non-autonomous non-continuous differential system. Let us transform it into an autonomous one, introducing the variable $m(.)$:

$$m(0) = x_0, \ \dot{m} = -cu.$$

  The system can then be re-written, for a positive $x_0$:

$$f_- : \begin{cases} \dot{x} &= b\cos v - c \\ \dot{y} &= b\sin v - a \\ \dot{m} &= -c \end{cases} \qquad f_+ : \begin{cases} \dot{x} &= b\cos v \\ \dot{y} &= b\sin v - a \\ \dot{m} &= 0 \end{cases}$$

  with the commutation surface $\mathcal{S} : \{m = 0\}$ from $f_-$ towards $f_+$. Classically,

$$\begin{aligned} H_-^* &= b\sqrt{\lambda_x^2 + \lambda_y^2} - c\lambda_x - a\lambda_y - c\lambda_m, \\ H_+^* &= b\sqrt{\lambda_x^2 + \lambda_y^2} - a\lambda_y \end{aligned}$$

  The transversality condition at the crossing of $\mathcal{S}$ imposes only a discontinuity of $\lambda_m$. So, the optimal $v_2^*$:

$$\begin{cases} \cos v_2^* &= \dfrac{\lambda_x}{\sqrt{\lambda_x^2 + \lambda_y^2}} \\ \sin v_2^* &= \dfrac{\lambda_y}{\sqrt{\lambda_x^2 + \lambda_y^2}} \end{cases}$$

is constant along an extremal. The final condition on the target imposing $|\lambda_x| = 1$, $\lambda_y = \dfrac{b}{\sqrt{a^2+b^2}}$ is the solution of $H_+^* = 0$:

$$\begin{cases} |\cos v_2^*| &= \dfrac{\sqrt{a^2-b^2}}{a} \\ \sin v_2^* &= \dfrac{b}{a} \end{cases}$$

from which one can deduce:

$$\begin{cases} t_f &= \dfrac{y_0}{a - b\sin v_2^*} = \dfrac{a}{a^2-b^2} y_0 \\ J_2(x_0,y_0) &= b\left|\displaystyle\int_0^{t_f} \cos v_2^* d\tau\right| = \dfrac{b}{\sqrt{a^2-b^2}} y_0 \end{cases}$$

on the condition that:

$$t_{f_2}(y_0) = \frac{y_0}{a - b\sin v_2^*} \geq \frac{|x_0|}{c}.$$

It is easy to check that $\sin v_1^* < \sin v_2^*$, and so $t_{f_1} < t_{f_2}$. When $|x_0| \in [c\,t_{f_1}, c\,t_{f_2}]$, both hypotheses are valid. The worst $v(.)$ is the one which leads to the greater cost. The switching line is defined by $J_1(x_0,y_0) = J_2(x_0,y_0)$. As a conclusion, the value function satisfies the:

**Proposition 8**

$$\begin{cases} U(x_0,y_0) &= |x_0| + \dfrac{-ac + \sqrt{(ac)^2 + (a^2-b^2)(b^2-c^2)}}{a^2-b^2} y_0 & \text{if } y_0 < \left(\dfrac{b}{\sqrt{a^2-b^2}} - r^+\right)|x_0| \\ U(x_0,y_0) &= \dfrac{b}{\sqrt{a^2-b^2}} y_0 & \text{otherwise} \end{cases}$$

**Proof**

We just have to proof that the union of these two fields covers the whole state space : whatever are $a > b > c > 0$, there exists an initial condition $(x_0, y_0)$ such that $J_1(x_0,y_0) = J_2(x_0,y_0)$. Necessarily, we shall have :

$$t_{f_1}(y_0) < \frac{|x_0|}{c} < t_{f_2}(y_0).$$

- $J_1(x_0,y_0)$ cannot be strictly greater than $J_2(x_0,y_0)$ when the condition $t_{f_1}(y_0) = \dfrac{|x_0|}{c}$ is met. Otherwise, $J_1$ and $J_2$ being continuous with respect to $y_0$, there should exist an $\epsilon > 0$ such that $J_1(x_0, y_0+\epsilon) > J_2(x_0, y_0+\epsilon)$, but $(x_0, y_0+\epsilon)$ belongs to an area where the strategy 2 is valid.

- When the condition $t_{f_2}(y_0) = \dfrac{|x_0|}{c}$ is met, maximizing trajectories associated with the strategy 2 present a switching point, which is obviously less efficient for the overrider (here the maximizer) than a straight line, generated by the strategy 1. So $J_2(x_0, y_0)$ cannot be greater than $J_1(x_0, y_0)$.

$\blacksquare$

This example shows that in the region where the certainty equivalence principle does not hold, the value function does not follow a classical dynamic programming equation. Inside the non-certainty equivalent area of the state space, $U$ only depends on $y$, but whatever are the choices of the players, $y$ is strictly decreasing. So, $U$ is never invariant along any of the trajectories compatible with the observation.

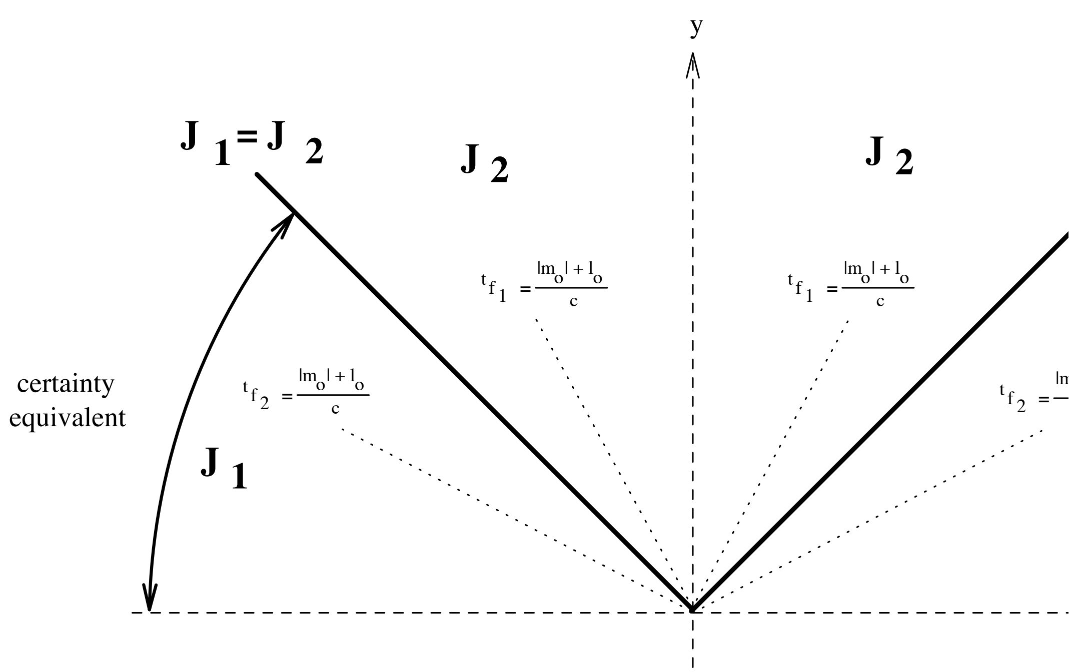


Figure 3: Partition of the domain of the value function

## Appendix

**Lemma 1** *If $f$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that:*

$$\forall t \in \mathbb{R},\ D^+f(t) = \limsup_{h \to 0} \frac{f(t+h) - f(t)}{h} < 0$$

*then $f$ is strictly decreasing.*

**Proof**

Let us choose $a < b$, then

$$\begin{aligned} \forall t \in [a,b],\ \exists \epsilon_t > 0 \mid \quad & \forall t' \in (t - \epsilon_t, t + \epsilon_t), \\ & t' > t \Rightarrow f(t') < f(t) \\ & t' < t \Rightarrow f(t') > f(t). \end{aligned}$$

Otherwise the hypothesis of the lemma would not hold. The compact set $[a,b]$ is so covered by open sets:

$$[a,b] \subset \bigcup_t (t - \epsilon_t, t + \epsilon_t)$$

from which one can extract a finite covering $\{(t_i - \epsilon_{t_i}, t_i + \epsilon_{t_i})\}_{i \in I}$ (the $t_i$ are ordered in an increasing sequence). Then there exists a sequence $(a_i)$ such that:

$$\begin{cases} a_0 & = & a \\ a_n & = & b \\ a_i & \in & (t_i, t_i + \epsilon_{t_i}) \cap (t_{i+1} - \epsilon_{t_{i+1}}, t_{i+1}) \\ a_i & < & a_{i+1} \\ f(a_i) & < & f(t_{i+1}) < f(a_{i+1}) \end{cases}$$

Thus $f(a) < f(a_1) < ... < f(a_{n-1}) < f(b)$.

∎

**Lemma 2** *If $f$ is a function from $\mathbb{R}$ TO $\mathbb{R}$ SUCH THAT:*

$$\forall t \in \mathbb{R},\ D^+f(t) = \limsup_{h \to 0} \frac{f(t+h) - f(t)}{h} \leq 0$$

*then $f$ is nonincreasing.*

**Proof**

If f strictly increases from a to $b > a$, let us define:

$$g(t) = f(t) - (t-a)\frac{f(b) - f(a)}{b-a}$$

over $[a, b]$. Then,

$$D^+g(t) = D^+f(t) - \frac{f(b) - f(a)}{b-a} < 0$$

and so, according to the previous lemma, g is strictly decreasing. But, $g(a) = g(b) = f(a)$, a contradiction.

∎

# References


[Ba90] T. Başar. Game theory and H$^\infty$-optimal control. In *4th international conference on Differential Games Approach*. Springer-Verlag, August 1990.

[BB91] T. Başar and P. Bernhard. *H$^\infty$-Optimal control and Related Minimax Design Problems: A Dynamic Game Approach*. Birkhäuser, Boston, 1991.

[Ber87] P. Bernhard. Differential games: Isaacs' equation. In Madan Singh, editor, *Encyclopedia of Systems and Control*. Pergamon Press, 1987.

[Ber90a] P. Bernhard. A certainty equivalence principle and its applications to continuous-time, sampled data, and discrete time H$^\infty$ optimal control. Technical Report 1347, INRIA, August 1990.

[Bre86] J.V. Breakwell. Sufficient conditions in zero-sum differential games. In *Proceedings IFAC Workshop on Modeling, Decision and Games with Applications to Social Phenomena*, Beiging, China, Aug. 1986.

[Cla83] F.H. Clarke. *Optimization and Nonsmooth Analysis*. Wiley 1983, reprint SIAM *Classics in Applied Mathematics* 1990.

[DBB93] G. Didinsky T. Başar and P. Bernhard. Structural properties of minimax controllers for a class of differential games arising in nonlinear H$^\infty$-control. *Syst. Contr. Lett.*, vol. 21, n. 6, 1993, pp. 433-441.

[Fri71] A. Friedman. *Differential Games.* Wiley 1971.

[Isa65] R. Isaacs. *Differential Games.* Wiley, 1965.

[KS77] N. Krassovski et A. Soubbotine. *Jeux différentiels.* Editions MIR, 1977.

[Lio82] P.L. Lions. *Generalized solutions of Hamilton-Jacobi equations.* Pitman, 1982.

[Sor93] P. Soravia. Pursuit-evasion problems and viscosity solutions of Isaacs equations. *SIAM J. Control and Optimization*, 31(3):604-623, May 1993.