Pursuit evasion game with costly information

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Abstract

In this paper we study a pursuit evasion game in which information is costly. The pursuer has to pay, i.e. lose some time, whenever he wants an information on the evader's position. Therefore the capture will be done in successive stages. The pursuer gets an information, then moves using an open loop control, and so on. We characterize the set C_1 of the initial states that the pursuer can capture in one stage whatever the evader does. This set is taken as a new target for an other stage. In this way we characterize the set C_n of the initial states the pursuer can capture in n stages in the worst case. We also give a pursuer's strategy that minimizes the total duration of the game, as opposed to the number of stages.

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1 Introduction.

It is well known that perfect information covers a very small part of reality, that makes sense of any attempt to understand situations with non perfect information. In this paper we intend to study a class of dynamic games in which information is costly. More precisely, we take an interest in “pursuit-evasion” games, in which the pursuer is blind and has to pay whenever he wants to have information concerning the evader’s position. Since the payoff of a pursuit evasion game is in terms of time, we may consider that the pursuer loses time for each piece of information he wants. For example, let us imagine the following game. It involves two players, Paul the pursuer, and Elise the evader. Paul has a headband on his eyes and want to catch Elise. Whenever he wants to know where she is, he has to stand still in order to unknot his headband and see her position. This kind of situation allows us to study the compromise between spending a lot in order to have good information, or not increasing the cost but playing with poor information.

The basic idea of this work consists of researching successive capture areas C_n. The capture will be done in sucessive stages. In the interior of each stage the pursuer does not receive information. He has then to move with an open loop control, and this control must be available whatever the evader does. The set C_n will be the set of all initial states capturable in n stages. This idea has been developed, without costly information notions, by Pierre Bernhard and Gregory Tomski in [1].

The game we investigate shares many aspects of search games since Paul wants to capture Elise independantly of what she does. One does not have to assume that she actually attempts to escape. (see, e.g. [3], [4])

2 Game Rules.

Let’s call Paul (P) the blind pursuer and Elise, (E), the evader. We note y_P and y_E which are their respective positions at each time, and

$$x(t) = y_E(t) - y_P(t),$$

Elise’s position in Paul’s coordinate system. We consider that both Paul and Elise can have an infinite acceleration, and that their motions are only restricted by a maximum speed. For the sake of being a litle less restrictive, we add a constant drift term w, which we arbitrarily put in Paul’s dynamics. We get thus

$$\begin{array}{l} \dot{y}_P = u + w \text{ with } \parallel u \parallel \leq a, \\ \dot{y}_E = v \text{ with } \parallel v \parallel \leq b, \end{array}$$

where w is a given fixed speed such that : $\parallel w \parallel \leq a$, and obviously $a > b$.

In Paul’s coordinate system, this dynamics become :

$$\dot{x}(t) = v - u - w, \text{ with } \left\{ \begin{array}{l} \parallel u \parallel \leq a, \\ \parallel w \parallel \leq a, \\ \parallel v \parallel \leq b, \end{array} \right. \tag{1}$$

u and v being the players respective controls.

The game is a “pursuit-evasion game ”, so the cost is expressed in terms of time. We will consider that for each piece of information he wants, Paul loses the time δ. During this lapse of time, he has to stay still ($\dot{y}_P(t) = 0$). It is in this sense that we consider that the information is costly.

A stage of this game is then defined as being a period made up of a motion period and an information period (of duration δ). We notice that this notion of stage is only connected with the pursuer.

We also define a target set. We will say that capture occurs, as soon as $x(t) \in C_0$, for t the end of a stage. For the sake of making computations easy, we will let :

$$C_0 = B(0, R_0), \quad R_0 \in I\!R. \tag{2}$$

(For $A \in I\!R^2$ and $a \in I\!R$, $B(A, a)$ stands for the sphere of radius a and of center A).

Let x_0 be the initial state of the system, and τ the length of the first stage. At the end of this stage we have :

$$x(\tau) = x_0 + \int_0^\tau (v(s) - u(s) - w)ds,$$

that is, taking into account the fact that P keeps still during the interval $[\tau - \delta, \tau]$, (u is then $-w$).

$$x(\tau) = x_0 + \int_0^\tau v(s)ds - \int_0^{\tau-\delta} u(s)ds - w(\tau - \delta).$$

We will state

$$Q_\tau = \{\int_0^\tau v(s)ds, \ \ \| v \| < b\},$$

$$P_\tau = \{\int_0^{\tau-\delta} u(s)ds, \ \ \| u \| < a\}.$$

Q_τ and P_τ stand for all Elise and Paul's respective possible movements during a perod τ. With these notations, $x(\tau)$ may be written again as :

$$x(\tau) = x_0 + q - p - \Omega_1(\tau),$$

where

$$q_\tau \in Q_\tau, \quad p_\tau \in P_\tau,$$
and
$$\Omega_1(\tau) = w(\tau - \delta) \in I\!R^2.$$

In the sequel, P_τ, Q_τ, and C_0 will be balls, and Ω_1 has the simple form given above. It is clear, however, that a large part of the derivation does not use these special forms, and carries over to more general dynamics and target set.

3 One stage capturability.

In this section we intend to find out all the possible initial states x_0, that allow Paul to catch Elise at the end of a stage of length $\tau = \tau(x_0)$, whatever Elise does during this stage. In a broad terms we will speak of a state that P can bring into the set C_0. We will also find the controls that Paul has to use for doing so.

3.1 Geometric difference.

First of all let us define the geometric difference (also called Minkovski's difference, or erosion) between two sets A and B (noted $A \overset{*}{-} B$. See [2]

$$A \overset{*}{-} B = C \Rightarrow B + C \subset A$$
$$B + C \subset A \Rightarrow C \subset A \overset{*}{-} B$$

where

$$B + C = \{b + c \mid \quad b \in B, \ \ c \in C\},$$

or similarily

$$A \overset{*}{-} B = \{a \in A \text{ such that } \forall b \in B, \qquad a + b \in A\}.$$

We can easily prove the main relation we will use :

$$B(A, a) \overset{*}{-} B(B, b) = \begin{cases} B(A - B, a - b) \text{ if } a \leq b, \\ \emptyset \text{ otherwise.} \end{cases}$$

3.2 Capturability.

Let x_0 be the initial state. Elise will be caught at the end of a first stage, if a stage duration $\tau = \tau(x_0) \geq \delta$ and a control $p = p_\tau(x_0)$ of Paul exist, such that for all controls $q \in Q_\tau$, we have :

$$x(\tau) \in C_0,$$

that is, in expanded form

$$\forall q \in Q_\tau, \quad x_0 + q - p - \Omega_1(\tau) \in C_0,$$

that is

$$x_0 - p \in (C_0 \overset{*}{-} Q_\tau) + \Omega_1(\tau). \tag{3}$$

This last equation is meaningful if the erosion $C_0 \overset{*}{-} Q_\tau$ is not equal to the empty set. Finally such a p can be found, and thus the initial state x_0 can be captured, if a duration $\tau = \tau(x_0) \leq \delta$ exists, such that :

$$x_0 \in (C_0 \overset{*}{-} Q_\tau) + \Omega_1(\tau) + P_\tau.$$

Let us note $C_1(\tau)$ this last set, $C_1(\tau)$ is then the set of all the initial states that are capturable in one stage of duration τ. Therefore the set of all the initial states that can be caught in one stage is :

$$C_1 = \bigcup_{\tau > \delta} C_1(\tau).$$

As a matter of fact, we will see that $C_1(\tau)$ is non void only for $\tau \in [\tau_1^m, \tau_1^s]$, so that we may write :

$$C_1 = \bigcup_{\tau = \tau_1^m}^{\tau_1^s} C_1(\tau),$$

that is

$$C_1 = \bigcup_{\tau = \tau_1^m}^{\tau_1^s} (C_0 \overset{*}{-} Q_\tau) + \Omega_1(\tau) + P_\tau,$$

The interval $[\tau_1^m, \tau_1^s]$ will be described more precisely later.

This set still needs to be computed explicitly, taking into account the specific geometry of the involved sets. The set C_1 may be written again :

$$C_1 = \bigcup_{\tau = \tau_1^m}^{\tau_1^s} (B(0, R_0) \overset{*}{-} B(0, b\tau)) + \Omega_1(\tau) + B(0, a(\tau - \delta)),$$

that is finally

$$C_1 = \bigcup_{\tau=\tau_1^m}^{\tau_1^s} B(\Omega_1(\tau)\ ,\ (a-b)\tau - a\delta + R_0) \tag{4}$$

In order to define τ_1^m and τ_1^s, we have to take into account

– the erosion condition noticed above :

$$B(0,R_0) \stackrel{*}{-} B(0,b\tau) \neq \emptyset \text{ that implies that } \tau < \frac{R_0}{b} = \tau_1^s,$$

– the fact that each stage has a duration greater than the information cost δ :

$$\tau \geq \delta \text{ that implies that } \ \tau_1^m = \delta.$$

Furthermore we need to have

$$\frac{R_0}{b} = \tau_1^s \geq \tau_1^m = \delta,$$

that implies that C_1 is not the empty set if the condition

$$R_0 \geq b\delta \tag{5}$$

is satisfied.

– Lastly, we have to make sure that all the sphere $B_1(\tau), \quad \tau \in [\tau_1^m, \tau_1^s]$ are non empty. According to the last remark (5), $R_1(\delta) = R_0 - b\delta$ is positive, so $R_1(\tau)$ is positive to, since $\tau \geq \delta$, and then $B_1(\tau), \quad \tau \in [\tau_1^m, \tau_1^s]$, is non empty.

To summarize :

$$C_1 = \bigcup_{\tau=\tau_1^m}^{\tau_1^s} B(\Omega_1(\tau)\ ,\ R_1(\tau)) \tag{6}$$

where

$$\begin{cases} \Omega_1(\tau) = w(\tau - \delta), \\ R_1(\tau) = (a-b)\tau - a\delta + R_0, \\ \tau_1^s = \dfrac{R_0}{b}, \\ \tau_1^m = \delta, \end{cases} \tag{7}$$

with the existence condition :

$$R_0 \geq b\delta. \tag{8}$$

Remark : Let us take the x axis aligned with the vector w. On the one hand we have :

$$\Omega_1(\tau) \begin{pmatrix} \omega(\tau) \\ 0 \end{pmatrix}, \quad \omega(\tau) = \parallel w \parallel (\tau - \delta),$$

and on the other hand we have :

$$R_1(\tau) = (a-b)\tau - a\delta + R_0.$$

It follows that

$$R_1(\tau) = \frac{a-b}{w}\omega(\tau) - b\delta + R_0.$$

The radius of the sphere is therefor linear in $\omega(\tau)$, and the coefficient is $\frac{a-b}{w}$. In cases where this coefficient is greater than 1, the set C_1 is reduced to the sphere

$$C_1 = B(\Omega_1(\tau_1^s), R_1(\tau_1^s)).$$

We still need to make sure that C_1 is a non trivial set, that is that C_1 is not included in the capture set C_0. We have to make sure that $\tau \in [\tau_1^m, \tau_1^s]$ exists such that either the condition

$$\| \Omega_1(\tau) \| + R_1(\tau) > R_0,$$

or the condition

$$\| \Omega_1(\tau) \| - R_1(\tau) < -R_0$$

is satisfied. Given that $\| \Omega_1(\tau) \|$ and $R_1(\tau)$ are increasing, it appears that it is enough to make sure that one of the two following conditions is true :

$$\begin{array}{l} (a) \quad \| \Omega_1(\tau_1^s) \| + R_1(\tau_1^s) > R_0, \\ \text{or} \\ \ \text{if } a-b > w : \\ (b) \quad \| \Omega_1(\tau_1^s) \| - R_1(\tau_1^s) < -R_0, \\ \ \text{if } a-b < w : \\ (b') \quad \| \Omega_1(\tau_1^m) \| - R_1(\tau_1^m) < -R_0. \end{array}$$

One can easily prove that (b') is always false, that (b) implies that

$$\delta \leq R_0(\frac{1}{b} - \frac{1}{a-w}),$$

and that (a) is equivalent to the weakest condition, that will be taken then :

$$\delta \leq R_0(\frac{1}{b} - \frac{1}{a+w}). \tag{9}$$

Conclusion :

For any initial state belonging to the set C_1, Paul has an open loop control that makes him sure to catch Elise whatever she does. Accordingly to (3), if $x_0 \in C_1$ is the initial state, a duration $\tau(x_0) = \tau$ and a control $p_\tau(x_0)$ belonging to P_τ exist such that :

$$-p_\tau(x_0) \in (C_0 \overset{*}{-} Q_\tau) + \Omega_1(\tau) - x_0 \cap op(P_\tau) = op(\mathcal{P}_\tau(x_0)).$$

(For a set A, we note $op(A) = \{a \mid -a \in A)$.

Paul has to chose a duration τ such that the previous set, $\mathcal{P}_\tau(x_0)$ is non empty, and then he can play according to any control in it. Nevertheless in order to minimize the duration of this stage, he must chose the minimum τ such that the set $\mathcal{P}_\tau(x_0)$ is non empty. With the dynamics we have chosen it is enought to choose τ such the following set

$$B(x_0 - \Omega_1(\tau)\ ,\ R_0 - b\tau) \cap B(0, a(\tau - \delta)$$

is reduced to a singleton, which contains, then, the right control.

4 n stage capturability.

In a reccurcive way we define C_n as the set of all initial states that Paul can bring into the set C_{n-1} whatever Elise does. In other words, C_n is the set of initial states that Paul can capture, whatever Elise does in no more than n stages.

We shall prove the following fact :

Proposition :

$$C_n = \bigcup_{t=t_n^m}^{t_n^s} B(\Omega_n(t), R_n(t)),$$

with

$$\begin{cases} \Omega_n(t) = w(t - n\delta) \\ R_n(t) = (a-b)t - na\delta + R_0 \\ t_n^s = \dfrac{R_{n-1}^s}{b} \\ t_n^m = \sup\left(t_{n-1}^m + \delta, \dfrac{na\delta - R_0}{a-b}\right), \end{cases} \tag{10}$$

under the existence condition :

$$R_{n-1}^s > b\delta.$$

From now on, $B(\Omega_n(t), R_n(t)$ will alternatively be denoted as $B_n(t)$.

Proof :

Let us suppose the proposition true at the rank n, and let us prove it at the rank $n+1$.

Let x_0 be a state that belongs to the set C_{n+1}. Accordingly to the definition of C_{n+1}, a duration $\tau = \tau(x_0) > \delta$ and a control $p_\tau(x_0) = p \in P_\tau$ must exist, such that,

$$\forall q \in Q_\tau, \quad x_0 - p + q \in C_n,$$

that is as previously

$$x_0 - p \in (C_n \overset{*}{-} Q_\tau) + \Omega_1(\tau), \tag{11}$$

or

$$x_0 \in (C_n \overset{*}{-} Q_\tau) + \Omega_1(\tau) + P_\tau.$$

That allows us to write :

$$C_{n+1} = \bigcup_{\tau=\tau_{n+1}^m}^{\tau_{n+1}^s} \left(\quad (C_n \overset{*}{-} Q_\tau) + \Omega_1(\tau) + P_\tau \quad \right),$$

where τ_{n+1}^m and τ_{n+1}^s are precisely defined further. We have then to compute this set taking into account the geometry of the sets involved in it.

$$C_n \overset{*}{-} Q_\tau = \{\xi_\tau \mid \ \forall q_\tau \in Q_\tau, \ \xi_\tau + q_\tau \in C_n\},$$

that is

$$C_n \overset{*}{-} Q_\tau = \left\{ \xi_\tau \mid \ \forall q_\tau \in Q_\tau, \ \xi_\tau + q_\tau \in \bigcup_{t_n^m}^{t_n^s} B(\Omega_n(t), R_n(t)), \right\}$$

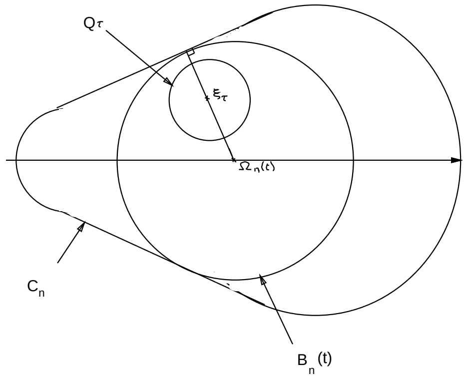

Figure 1:

Due to the shape of the sets, it is equivalent to write (see figure 1 bellow) :

$$C_n \overset{*}{-} Q_\tau = \{\xi_\tau \mid \ \ \forall q_\tau \in Q_\tau, \ \ \exists t \in [t_n^m, t_n^s], \quad \xi_\tau + q_\tau \in B(\Omega_n(t), R_n(t))\}$$

or still

$$\begin{aligned} C_n \overset{*}{-} Q_\tau &= \bigcup_{t=t_n^m}^{t_n^s} \left(B(\Omega_n(t), R_n(t)) \overset{*}{-} Q_\tau \right), \\ &= \bigcup_{t=t_n^m}^{t_n^s} B(\Omega_n(t), R_n(t) - b\tau). \end{aligned}$$

We still need to make sure that this last erosion is allowed, and that the result is not the empty set. As previously that is equivalent to take :

$$\begin{array}{l} \tau_n^s = \frac{R_n^s}{b}, \\ \text{and} \\ \tau > \tau_n^m = \delta, \end{array} \tag{12}$$

and the inequality $\tau_n^s \geq \tau_n^m$ produces the existence condition :

$$R_n^s > b\delta.$$

For the union, we only keep the $t \in [t_n^m, t_n^s]$ that make the set $B(\Omega(t), R_n(t) - b\tau)$ non empty. We can take $t \in [\bar{t}_n^m(\tau), t_n^s]$ with :

$$\bar{t}_n^m(\tau) = \sup(\frac{na\delta - R_0}{a-b} + \frac{b\tau}{a-b}, t_n^m).$$

Hence we have

$$
\begin{aligned}
C_{n+1} &= \bigcup_{\tau=\tau_n^m}^{\tau_n^s} \left\{ \left(C_n \overset{*}{-} Q_\tau \right) + \Omega_1(\tau) + P_\tau \right\} \\
&= \bigcup_{\tau=\tau_n^m}^{\tau_n^s} \left\{ \bigcup_{t=\bar{t}_n^m(\tau)}^{t_n^s} B\left(\Omega_n(t) + \Omega_1(\tau)\ ,\ R_n(t) - b\tau + a(\tau-\delta)\right) \right\} \\
&= \bigcup_{\tau=\tau_n^m}^{\tau_n^s} \left\{ \bigcup_{t=\bar{t}_n^m(\tau)}^{t_n^s} B\left(\Omega_n(t) + \Omega_1(\tau)\ ,\ (a-b)(t+\tau) - (n+1)a\delta + R_0\right) \right\} \\
&= \bigcup_{t+\tau=\tau_n^m+\bar{t}_n^m(\tau_n^m)}^{\tau_n^s+t_n^s} B\left(w(t+\tau-(n+1)\delta)\ ,\ (a-b)(t+\tau) - (n+1)a\delta + R_0\right)
\end{aligned}
$$

and finally, using (12) and the reccurence hypothesis (10)

$$
C_{n+1} = \bigcup_{t=t_{n+1}^m}^{t_{n+1}^s} B(\Omega_{n+1}(t), R_{n+1}(t)), \tag{13}
$$

with

$$
\begin{cases}
t_{n+1}^m = \sup\left(t_n^m + \delta, \dfrac{(n+1)a\delta - R_0}{a-b}\right) \ ;\ t_0^m = 0, \\
t_{n+1}^s = t_n^s + \dfrac{R_n^s}{b}, \quad R_n^s = R_n(t_n^s)\ ;\ t_0^s = 0, \\
\Omega_{n+1}(t) = w(t-(n+1)\delta) \\
R_{n+1}(t) = (a-b)t - (n+1)a\delta + R_0
\end{cases} \tag{14}
$$

with the existence condition.

$$
R_n^s > b\delta. \tag{15}
$$

As previously we can prove that the radius $R_{n+1}(t)$ is a linear function of $\omega_{n+1}(t)$ (the absicissa of the center $\Omega_{n+1}(t)$). The coefficient is still $\frac{a-b}{w}$. When this coefficient is greater than 1, the set C_{n+1} is simply the sphere :

$$
C_{n+1} = B(\Omega_{n+1}(t_n^s), R_{n+1}(t_n^s)),
$$

We still need to make sure that this set is not a subset of C_n. As previously we have to verify that one of the following conditions is true :

(a) $\| \Omega_{n+1}(\tau_{n+1}^s) \| + R_{n+1}(\tau_{n+1}^s) > \| \Omega_n(\tau_n^s) \| + R_n(\tau_n^s)$
or if $a-b > w$
(b') $\| \Omega_{n+1}(\tau_{n+1}^s) \| - R_{n+1}(\tau_{n+1}^s) < \| \Omega_n(\tau_n^s) \| - R_n(\tau_n^s)$
and if $a-b < w$
(b') $\| \Omega_{n+1}(\tau_{n+1}^m) \| - R_{n+1}(\tau_{n+1}^m) < \| \Omega_n(\tau_n^m) \| - R_n(\tau_n^m)$

That is equivalent to imposing the condition :

$$\delta < R_n^s \left(\frac{1}{b} - \frac{1}{a+w} \right) . \tag{16}$$

$\diamondsuit\diamondsuit$

We should point out the rather unexpected fact that $C_n(t)$ is expressed in terms of the total capture time t, and not of the detailed sequence $\tau_1, \tau_2, \ldots, \tau_n$.

As previously the equation (11) gives Paul his control; For each initial state x_0 belonging to C_n, he first have to choose the duration $\tau(x_0) = \tau$ such that the set

$$\mathcal{P}_\tau(x_0) = op((C_n \stackrel{*}{-} Q_\tau)) + \Omega_1(\tau) - x_0 \quad \cap \quad P_\tau$$

is non empty, and then he can play according to any control in it. In order to minimize the duration of the stage he chooses τ minimum such that $\mathcal{P}_\tau(x_0)$ is non empty, which is equivalent equivalent, with our hypothesis, to choose τ such that $\mathcal{P}_\tau(x_0)$ is a singleton.

5 Capturability sets.

Now a question naturally arises : "Are all initial states capturable in a finite number of stages ?". We have chosen not to develop the calculus : it is rather cumbersome and does not have any interest in itself. We will just expose the results.

First of all we can prove that :

$$t_n^s = \frac{R_0}{b} \left(\frac{1-\alpha^n}{1-\alpha} \right) - \frac{\alpha\delta}{1-\alpha} \left(n - \frac{1-\alpha^n}{1-\alpha} \right)$$

and

$$\begin{aligned} R_n^s &= \alpha R_{n-1}^s - a\delta \\ &= \alpha^n R_0 - a\delta \frac{1-\alpha^n}{1-\alpha} \end{aligned} \tag{17}$$

with $\alpha = \frac{a}{b}$.

The analysis of the variation of these quantities allows us to study the variations of the bound of the sets C_n. The results are summed up bellow.

The fact that the set of capturability

$$C = \bigcup_n C_n$$

is finite or not, depend on the radius R_0 of the target. In fact, it is quite natural to think that if the target is very small, only few initial states will be caught. It has to be noted that the fact that $a - b$ is greater or smaller than w does not modify the conclusions.

Before giving the results, we first define the radius R_c^w by :

$$\frac{R_c^w}{b\delta} = \frac{a+b}{a+w-b}.$$

We can verify easily that R_c^0 is a fixed point for the sequence R_t^s given by (17).

$\underline{R_0 \leq R_c^w}$

The inequality (9) is not satisfied. We conclude that, for each initial state $x_0 \notin C_0$, Elise might always escape.

$$C = C_0.$$

$\underline{R_c^w < R_0 < R_c^0}$

A positif integer n_0 exists such that C_{n_0+1} is a subset of C_{n_0}. The set of capturability is then finit :

$$C = \bigcup_{n=0}^{n_0} C_n.$$

For each initial state in C the capture is done with at more n_0 stages.

$\underline{R_0 = R_c^0}$

R_0 is a fixed point of the sequence R_n^s. The set C is made up of union of an infinite number of sets which have the same size.

$$C = \bigcup_n B(\Omega_n(\tau_n^s), R_0).$$

and

$$d(\Omega_n^s, \Omega_{n+1}^s) = cste = w(\frac{R_0}{b} - \delta).$$

$\underline{R_0 > R_c^0}$

we have :

$$\lim_{n \to \infty} C_n = I\!R^2.$$

Each initial state can be captured in a finite number of stages.

6 Minimized time for capture.

Sections 3,4 and 5 give us a description of capturability sets as well as Paul's controls. For each state of C_n we have given a control that allows him, in the worst case, to bring the state into the set C_{n-1}. Furthermore, we have exhibited a control that makes it in minimized time. Nevertheless the problem of capture in minimax time is not yet solved, since there is no reason to beleive that minimizing the length of the stage to come is optimal. This has consequences on the length of the later stages. Moreover, it may happen that in some case it be better to make two short stages than one longer. In this section we show that, with our hypothesis ((1) and (2)), this never happens..

6.1 With fixed number of stages.

In this subsection we compute the minimized capture time with fixed number of stages. We will note $T_n(x)$ the minimum duration to capture Elise, in the worth case, with exactly n stage. By a classical dynamic programming argument, $T_n(x)$ is then defined by the following recurrence equations.

$$\begin{array}{ll} T_0(x) = \left\{ \begin{array}{l} 0 \text{ if } x \in C_0, \\ +\infty \text{ otherwise,} \end{array} \right. \\ \text{and} \\ T_n(x) = & \min\limits_{\tau \in J_n} \left(\min\limits_{p_\tau \in P_\tau} \max\limits_{q_\tau \in Q_\tau} (\tau + T_{n-1}(x - \Omega_1(\tau) - p_\tau + q_\tau)) \right) \\ & J_n = [\tau_n^m, \tau_n^s]. \end{array} \tag{18}$$

We then have the two following propositions :

Proposition 1 :

$$T_n(x) = +\infty \text{ if and only if } x \notin C_n.$$

Proposition 2 :
Let x be an element of C_n we have :

$$T_n(x) = \min \left\{ t \in [t_n^m, t_n^s] \text{ such that } x \in C_n(t) \right\}. \tag{19}$$

with

$$C_n(t) = B(\Omega_n(t), R_n(t)).$$

Proof
The proof of the first proposition is obvious, we only give the proof of the second proposition.

Let us prove the proposition at the rank 1. Let x be an element of C_1, the definition (18) of T_0 gives us

$$T_1(x) = \min_{\tau \in J_n} \left(\min_{p_\tau} \max_{q_\tau} (\ \tau + T_0(x - \Omega_1(\tau) - p_\tau + q_\tau) \ ) \right),$$

and accordingly to the definition (18) of T_0,

$$T_1(x) = \min \left\{ \tau \in J_n \text{ such that } \exists p_\tau \text{ such that } \max_{q_\tau} \ T_0(x - \Omega_1(\tau) - p_\tau + q_\tau) = 0 \right\},$$

that is

$$T_1(x) = \min \left\{ \tau \in J_n \text{ such that } \exists p_\tau \text{ such that } \forall q_\tau \ \ x - \Omega_1(\tau) - p_\tau + q_\tau \in C_0 \right\},$$

or still, using the development of section 3.2 (see p 3)

$$T_1(x) = \min \left\{ \tau \in J_n \text{ such that } x \in B(\Omega_1(\tau), R_1(\tau)) \right\}.$$

Let us suppose now the proposition true until n and let us prove it at the rank $n+1$. Let x be a state that belongs to the set C_{n+1}, let us compute $T_{n+1}(x)$. (For $x \notin C_{n+1}$ we know that $T_{n+1}(x) = +\infty$). Accordingly to (18) we have

$$T_{n+1}(x) = \min_{\tau \in J_n} \min_{p_\tau} \max_{q_\tau} \left(\tau + T_n(x - \Omega_1(\tau) - p_\tau + q_\tau) \right).$$

As x belongs to the set C_{n+1} we know that a set that we will note $I_\tau(x)$ exists, such that for each of its element τ, a control of Paul exists that captures Elise whatever she does.

Let us fixe the $\tau \in I_\tau(x)$ and let us compute the following quantity :

$$A_\tau = \min_{p_\tau} \max_{q_\tau} \left(T_n(x - \Omega_1(\tau) - p_\tau + q_\tau) \right).$$

Let us note that if $\tau \notin I_\tau(x)$ then obviously $A_\tau = +\infty$. Accordingly to the reccurence hypothesis (19) we have :

$$A_\tau = \min_{p_\tau} \max_{q_\tau} \left(\min \left\{ t \in [t_n^m, t_n^s] = I_n \text{ such that } x - \Omega_1(\tau) - p_\tau + q_\tau \in C_n(t) \right\} \right),$$

Let us fixe $p_\tau \in \mathcal{P}_\tau(x_0)$, and let us have a look at the quantity

$$\max_{q_\tau} (\min\{t \in I_n, x - \Omega_1(\tau) - p_\tau + q_\tau \in C_n(t)\}).$$

Let $\hat{q}_\tau$ and $\hat{t}(q_\tau)$ the arguments of the maximum and minimum above. We have

$$x - \Omega_1(\tau) - p_\tau + \hat{q}_\tau \in C_n(\hat{t}) \subset \bigcup_{t \geq t_n^m}^{\hat{t}} C_n(t).$$

Let q_τ be any E's control such that $q_\tau \neq \hat{q}_\tau$. Then

$$\min\{t, x - \Omega_1(\tau) - p_\tau + q_\tau \in C_n(t)\} \leq \hat{t},$$

and then for any $q_\tau \neq \hat{q}_\tau$,

$$x - \Omega_1(\tau) - p_\tau + q_\tau \in \bigcup_{t \geq t_n^m}^{\hat{t}} C_n(t).$$

we therefore have

$$\max_{q_\tau} \min\{t \in I_\tau, \quad x - \Omega_1(\tau) - p_\tau + q_\tau \in C_n(t)\}$$

$$= \min\{\hat{t} \in I_n, \ \ \forall q_\tau, x - \Omega_1(\tau) - p_\tau + q_\tau \in \bigcup_{t > \delta}^{\hat{t}} C_n(t)\}.$$

So we can write,

$$A_\tau = \min_{p_\tau} \left\{ \min\{\hat{t} \in I_n \text{ such that } \forall q_\tau \ \ , x - \Omega_1(\tau) - p_\tau + q_\tau \in \bigcup_{t > t_n^m}^{\hat{t}} C_n(t)\} \right\},$$

or

$$A_\tau = \min_{p_\tau} \left\{ \min\{\hat{t} \in I_n \text{ such that } x - p_\tau \in (\bigcup_{t > t_n^m}^{\hat{t}} C_n(t)) \stackrel{*}{-} Q_\tau + \Omega_1(\tau)\} \right\},$$

As the value of the inner min above depends on p_τ, only through its existing, we have

$$\begin{aligned} A_\tau \quad &= \min\{\hat{t} \in I_n \text{ such that } \exists p_\tau \text{ such that } x - p_\tau \in (\bigcup_{t > t_n^m}^{\hat{t}} C_n(t)) \stackrel{*}{-} Q_\tau + \Omega_1(\tau), \} \\ &= \min\{\hat{t} \in I_n \text{ such that } x - P_\tau \bigcap (\ (\bigcup_{t > t_n^m}^{\hat{t}} C_n(t)) \stackrel{*}{-} Q_\tau + \Omega_1(\tau) \ \ ) \neq \emptyset\}. \end{aligned}$$

And finally using calculations made before we obtain

$$A_\tau = \min\{\hat{t} \in I_n \text{ such that } x \in B(\Omega_{n+1}(\hat{t} + \tau), R_{n+1}(\hat{t} + \tau))\}.$$

We are now able to compute $T_{n+1}(x)$:

$$T_{n+1}(x) = \min_{\tau \in J_n} (A_\tau + \tau)$$

that is

$$\begin{aligned} T_{n+1}(x) &= \min_{\tau \in J_n} (\tau + \min\{ \ \hat{t} \in I_n \text{ such that } x \in B(\Omega_{n+1}(\hat{t}+\tau), R_{n+1}(\hat{t}+\tau)) \ \} \ \} \\ &= \min\{t \in I_n + [\tau_n^m, \tau_n^s], \text{ such that } x \in B(\Omega_{n+1}(t), R_{n+1}(t))\}. \end{aligned}$$

$$\begin{aligned} I_n + J_n &= [t_n^m, t_n^s] + [\tau_n^m, \tau_n^s] \\ &= [t_{n+1}^m, t_{n+1}^s] \end{aligned}$$

Finally we obtain the proposition for the stage $n+1$, that is :

$$T_{n+1}(x) = \min\{t \in I_{n+1}, \text{ such that } x \in B(\Omega_{n+1}(t), R_{n+1}(t))\}.$$

$\diamondsuit\diamondsuit$

6.2 Minimized capture time.

In this last subsection we intend to prove that Paul has to minimize the number of capture stages in order to minimize the capture duration.

Proposition 3 :

If x belongs to the intersection $C_n \bigcap C_{n+1}$, then :

$$T_n(x) \leq T_{n+1}(x).$$

Proof :

Let us start by a lemma . We will note $\imath(A)$ the interior of a set A and ∂A its boundary. We have :

$$\imath(A) = A \setminus \partial A.$$

Lemma :

If $x \in \imath(C_n(t) - C_n(t_n^m))$, then a real t' exists such that $t_n^m \leq t' < t$ and $x \in \partial C_n(t')$.

Proof :

An easy proof is given comparing the two functions $d(x, \Omega_n(t))$ and $R_n(t)$.

$\diamondsuit\diamondsuit$

Let us come back to the proof of the proposition. Let x be an element of $C_n \cap C_{n+1}$. Let us note $T_{n+1}(x) = t_{n+1} < +\infty$. On one hand we have :

$$\Omega_{n+1}(t_{n+1}) = \Omega_n(t_{n+1} - \delta),$$

and on the other hand :

$$R_{n+1}(t_{n+1}) = R_n(t_{n+1} - \delta) - b\delta < R_n(t_{n+1} - \delta).$$

That allows us to conclude :

$$\partial B(\Omega_{n+1}(t_{n+1}), R_{n+1}(t_{n+1})) \subset B(\Omega_n(t_n, R_n(t_n)),$$

where

$$t_n = t_{n+1} - \delta < t_{n+1}.$$

Hence $x \in \imath(\Omega_n(t_n), R_n(t_n))$, and that allows us to write :

$$T_n(x) \leq t_n < t_{n+1} = T_{n+1}(x),$$

which ends the proof.

$\diamondsuit\diamondsuit$

7 The limit free information case.

Consider what happens when the cost of information decreases. It can be shown that the bounds R^0_{-w}, R^0_c and R^w_c are linear in δ. Therefore, the capturability set C increases when δ decreases. In the limit, when $\delta = 0$, information is free, and as expected, each initial state is in C.

Nevertheless, when δ goes to zero, the strategies found in this paper are still piecewise closed-loop, which is different from the classical, closed loop, optimal pursuer strategy. Indeed, our limiting strategy is still an optimal closed loop strategy in the following sense.

Let ϕ_p denote the optimal piecewise closed loop strategy of the pursuer, when $\delta = 0$, and let (ϕ^*, ψ^*) be a pair of optimal state feedback strategies of the pursuer and the evader respectively, in the classical sense. Let also T denote capture time. Then it can be shown that :

$$T(\phi, \psi^*) \leq T(\phi_p, \psi^*) = T(\phi^*, \psi^*) \leq T(\phi^*, \psi),$$

for any strategies ϕ and ψ.

However, if information is actually free, ϕ^* is a better strategy in the following sense:

$$T(\phi^*, \psi) \leq T(\phi_p, \psi),$$

for any strategy ψ of the evader.

The interpretation of the last inequality is that ϕ^* takes better advantage of the evader's "mistakes" (deviations from ψ^*) than ϕ_p, since they are sensed, and thus exploited, immediately and continuously.

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