About the resolution of discrete pursuit games and its applications to naval warfare

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Introduction

Tactical analysis of naval situations is a crucial and difficult problem. Threat evaluation must be done with more or less delayed and incomplete information: for instance, actual kinematics of each actor are only estimated by his opponent. This aspect of missing or ill-defined information is a difficult task for a ship officer, and much work has been devoted to the design of useful real time tools for decision aids, based on Artificial Intelligence : Expert Systems can be applied, for example, to the analysis of the opponent intents [Rai89].

As, at Thomson Sintra ASM, we are involved in modelling for submarine control and control system evaluation [LBN90], we think that complementary tools can be founded upon Games Theory : more, a time-space discrete representation allows a realistic modelling yielding the application of methods based on discrete game theory. This discretization takes into account the fact that, for naval situations, there is uncertainty in the available informations and that these informations are only obtained at discrete times. For example, if a ship officer wants to know position and speed of his opponent, he has to manoeuver in order to obtain these informations with his sonars, and, during this manoeuver, no available information can be provided by some of the sonars. More, a ship officer always makes approximations about kinematics informations of his opponent : indeed, he assumes values of maximum speed and of turn radius which cannot be the actual values of the opponent ship. That is why discrete games seem to be a very interesting tool because it is possible to play a set of games in "real time" with various assumptions (like kinematics, position, speed, opponent intents, ...), and, from this set, the ship officer can take his own decision.

In this paper, we first propose our mathematical modelling giving a method of resolution of discrete pursuit games, then its implementations on a workstation (SUN), finally the possible improvments of this approach.

1 Mathematical modelling

1.1 Preliminaries

1.1.1 Available lattices

Firstly, we want to find out the geometrical framework of the game. Indeed, we choose a geometrical description of the space and of the available controls so that this space description is always available after the motions given by the controls. In our description, a control can be modelled as both a translation and a rotation, and the geometry of the space is chosen so that, after each control, i.e. for each translation and for each rotation, for each player, space representation and controls are the same as previously. There is a famous modelling of such a space with the help of cristallography : indeed, physicists have described all the spaces allowing sorts of couple "translation-rotation" by the famous description of the "Bravais lattices" (see for example [Kit71]). With the help of these works, we know that, in bidimensionnal spaces, there are only five kinds of lattices, owning either twofold, or fourfold, or sixfold rotation axes (let us note that, in three dimensionnal spaces, there are only 14 kinds of lattices, yielding 230 space groups).

For bringing such space description to games applications, we emphasize the fact that the choice of controls is rather restrictive because rotations can only be a multiple of either $\pi/3$ or $\pi/2$. Yet, in our discrete games and for our naval applications, this rather rough approximation of controls seems to suit well with the uncertainty of the situation description.

In order to obtain a realistic description of a naval 2-D situation, we have chosen an hexagonal lattice (so that motion of each player is described as translation on this lattice, and that there are six possibilities for the final speed direction of each player) —as in [Isa65]—.

1.1.2 Solution of the game

We shall see two different ways of solving the problem : the first one, inspired by Pontryagin's paper [Pon68], is based on the building of capture zones W_n in which the evader is caught within n steps, and the second one consists in solving Isaacs' equation using the dynamic programming method. At first, we shall study the game in which the players play alternatively, with the minimizer playing first, which can be solved with pure strategies. Then we shall solve the game in which both players play simultaneously, which requires mixed strategies.

1.2 Alternating game

1.2.1 Notations

- We call x_t the relative coordinates at instant t, $t \in \mathbb{N}$.
- As usual, we call $u_t \in \mathcal{U}$ and $v_t \in \mathcal{V}$ the controls of the minimizer and of the maximizer at instant t. We suppose that $\mathcal{U}$ and $\mathcal{V}$ are both finite sets. The dynamics of the game are described by :

  $$x_{t+1} = h(g(x_t, u_t), v_t)$$

  which gives a proper model for this game where the pursuer (P) plays first and the evader (E) plays second, knowing P's control.
- When we don't need to distinguish the minimizer's controls from the maximizer's ones, we will more simply write the classical equation :

  $$x_{t+1} = f(x_t, u_t, v_t)$$

  with a given initial position x_0.
- We call C the Capture set.
- The performance index J we want to optimize is the duration of the game .
- We also assume the following conditions on g and h :

  $$\begin{cases} x \in C \Rightarrow \forall u \; g(x,u) = x \\ y \in C \Rightarrow \forall v \; h(y,v) = y \end{cases}$$

  which will allow us to consider the finite duration games as infinite duration ones with a stationary trajectory. Let's notice that :

  $$x \in C \Rightarrow \forall u \forall v \; f(x,u,v) = x$$

Our purpose is to find the value $w(x)$ on the whole game space. We shall prove that a method based on the building of capture zones W_n in which the evader is caught within n steps is equivalent to the classical dynamic programming method that is used to solve Isaacs' equation, and quite faster.

1.2.2 Calculation of w: link between the Capture Zones and Dynamic Programming

In the theory of zones, we first define $W_0(\equiv C)$, which is the target, then we build the capture zones W_n by induction :

$$W_{n+1} = \{x \ : \ \exists u \, \forall v \; f(x,u,v) \ \in \ W_n\}.$$

Remark 1 *if $x \notin W_0$ and if $\exists u^* \ g(x,u^*) \in W_0$, then $x \in W_1$.*

Proof : $\forall v \ \ f(x,u^*,v) = h(g(x,u^*),v) = g(x,u^*) \in W_0$.

Remark 2 *One can easily check : $\forall n \in \mathbf{N} \ \ W_n \subset W_{n+1}$.*

In the theory of dynamic programming, we initialize by giving a nil value to all the points of C, and infinite values elsewhere :

$$\begin{cases} w_0(x) = 0 & \text{if} \quad x \in C \\ w_0(x) = \infty & \text{if} \quad x \notin C \,, \end{cases}$$

and we iterate the following process :

$$w_{n+1}(x) \ = \ \min_u \max_v \ w_n \ \circ f(x,u,v) \ + \ c(x) \,,$$

where

$$\begin{cases} c(x) = 1 & \text{if} \quad x \notin C \\ c(x) = 0 & \text{if} \quad x \in C \,. \end{cases}$$

Remark 3 *If $x \in C$, $f(x,u,v) = x$ and $c(x) = 0$ so $\forall n \ \ w_n(x) = w_0(x) = 0$.*

Remark 4 *If $x \notin C$ and if there exists u^* so that $g(x,u^*) \in C$ — i.e. the Pursuer P plays and brings the Evader E into C before E has played—, then $w_1(x) = 1$.*

Proof : We have : $\forall v \ h(g(x,u^*),v) = g(x,u^*) = f(x,u^*,v)$.
Since $g(x,u^*) \in C$, we can also write :

$$0 \ \le \ \min_u \max_v \ w_0 \circ f(x,u,v) \ \le \ \max_v \ w_0 \circ f(x,u^*,v) \ = \ w_0 \circ g(x,u^*) \ = \ 0$$

yielding : $w_1(x) = 1$.

We want to prove the equivalence of the two approaches, formalized in the next Theorem 1. For this purpose, we need a few preliminary results : lemma 1, 2 and 3.

Lemma 1 *For a given x, $(w_n(x))_{n\in \mathbf{N}}$ is decreasing.*

Proof : If $x \in C$, remark 3 proves that $w_0(x) \ge w_1(x)$.
If $x \notin C$, then $w_0(x) = \infty \ge w_1(x)$. So $\forall x \ \ w_0(x) \ \ge \ w_1(x)$.
Now, assuming that $\forall y \ w_n(y) \ge w_{n+1}(y)$, let's take x in the game space. We have :

$$w_{n+2}(x) = \min_u \max_v w_{n+1} \circ f(x,u,v) + c(x) \ \le \ \min_u \max_v w_n \circ f(x,u,v) + c(x) = w_{n+1}(x).$$

This proves the lemma.

Lemma 2 *for a given n :*

$$\begin{cases} i) & x \in W_0 & \Rightarrow & w_{n+1}(x) = w_0(x) = 0 \\ ii) & x \in W_{n+1} - W_n & \Rightarrow & w_{n+1}(x) = n+1 \\ iii) & x \notin W_{n+1} & \Rightarrow & w_{n+1}(x) = \infty \,. \end{cases}$$

Proof : by induction.
Let us call A_n the assertion of the lemma. A_0 is true. Now let us assume A_n is true and let's prove A_{n+1}.
Let $x \in W_{n+2} - W_{n+1}$:

$$x \in W_{n+2} \Rightarrow \exists u^* \forall v \; f(x,u^*,v) \in W_{n+1}.$$

Let $\mathcal{U}^* = \{u^* : \forall v \; f(x,u^*,v) \in W_{n+1}\}$, and let $u^* \in \mathcal{U}^*$:

$$x \in W_{n+2} - W_{n+1} \Rightarrow x \notin W_{n+1} \Rightarrow \exists v^* \; f(x,u^*,v^*) \notin W_n,$$

thus $\quad f(x,u^*,v^*) \in W_{n+1} - W_n,$

that implies $\quad w_{n+1} \circ f(x,u^*,v^*) = n+1 \quad$ (due to assertion A_n).

Furthemore, $\forall y \in W_{n+1}$, $w_{n+1}(y) \leq n+1$ (because of remark 2, lemma 1 and assertion A_n), so:

$$\max_v w_{n+1} \circ f(x,u^*,v) \leq n+1 = w_{n+1} \circ f(x,u^*,v^*).$$

This is true for all $u^* \in \mathcal{U}^*$, yielding :

$$\min_{u \in \mathcal{U}^*} \max_v w_{n+1} \circ f(x,u,v) = n+1.$$

Now let $\overline{u} \notin \mathcal{U}^*$:

$$\overline{u} \notin \mathcal{U}^* \Rightarrow \exists \overline{v} \; f(x,\overline{u},\overline{v}) \notin W_{n+1},$$

then, using assertion A_n, $\quad w_{n+1} \circ f(x,\overline{u},\overline{v}) = \infty,$

so $\quad \max_v w_{n+1} \circ f(x,\overline{u},v) = \infty,$

and finally $\quad \min_{u \notin \mathcal{U}^*} \max_v w_{n+1} \circ f(x,u,v) = \infty.$

In conclusion, we have :

$$\min_u \max_v w_{n+1} \circ f(x,u,v) = \min\left\{\min_{u \in \mathcal{U}^*} \max_v w_{n+1} \circ f(x,u,v), \min_{u \notin \mathcal{U}^*} \max_v w_{n+1} \circ f(x,u,v)\right\} = \min(n+1,\infty) = n+1,$$

so using $c(x) = 1$ (since $x \notin W_0$) , $w_{n+2}(x) = n+2$ which proves *ii)*.
i) has already proven in remark 3 and it is easy to check *iii)*.

Lemma 3 *if* $x \in W_n$, *then* $w_{n+1}(x) = w_n(x)$.

Proof : by induction.
Let us call B_0 the assertion of the lemma. Assertion B_0 is true. Let's assume B_n is true and let's prove B_{n+1}.
If $x \in W_0$, B_{n+1} is true. Let's take $x \in W_{n+1}$ and $x \notin W_0$:

$$x \in W_{n+1} \; and \; x \notin W_0 \Rightarrow w_{n+2}(x) = \min_u \max_v w_{n+1} \circ f(x,u,v) + 1 = \max_v w_{n+1} \circ f(x,u_*,v) + 1 .$$

(Such a u^* exists since we supposed that $\mathcal{U}$ is a finite set). Let :

$$\mathcal{U}_* = \{u_* : \min_u \max_v w_{n+1} \circ f(x,u,v) = \max_v w_{n+1} \circ f(x,u_*,v)\} .$$

Let $u_* \in \mathcal{U}_*$. Using : $\forall x \in W_{n+1} \; w_{n+2}(x) \leq w_{n+1}(x) \leq n+1$, we can deduce :

$$\max_v w_{n+1} \circ f(x,u_*,v) \leq n,$$

then $\quad \forall v f(x,u_*,v) \notin W_{n+1} - W_n,$

moreover $\quad \forall v \; f(x,u_*,v) \in W_{n+1}$ (otherwise because of the lemma, $\max_v w_{n+1} \circ f(x,u_*,v) = \infty$),

that implies, since $W_n \subset W_{n+1} \quad \forall v f(x,u_*,v) \in W_n,$

and using assertion B_n, we have :

$$\forall v \;\; w_{n+1} \circ f(x, u_*, v) = w_n \circ f(x, u_*, v).$$

This is true for all $u_* \in \mathcal{U}_*$. We can deduce :

$$w_{n+1}(x) = \min_u \max_v w_n \circ f(x, u, v) + 1$$

$$\leq \min_{u \in \mathcal{U}_*} \max_v w_n \circ f(x, u, v) + 1 \; = \; \min_{u \in \mathcal{U}_*} \max_v w_{n+1} \circ f(x, u, v) + 1 \; = \; w_{n+2}(x),$$

yielding

$$w_{n+1}(x) \leq w_{n+2}(x).$$

As we already had $w_{n+2}(x) \leq w_{n+1}(x)$ (due to the decreasing of function w), we obtain :

$$x \in W_{n+1} \Rightarrow w_{n+2}(x) = w_{n+1}(x).$$

This prove the lemma.

Now, in conclusion, we can assert the following theorem :

Theorem 1
Let's take x in the game space.

- *If $x \notin \bigcup W_k$: it is not capturable and $w_n(x) = \infty$.*
- *If $x \in \bigcup W_k$, let n be the smallest integer such that $x \in W_n$; then the first k verifying $w_k(x) \neq \infty$ is n and, $\forall p \in \mathbb{N}$, $n = w_n(x) = w_{n+1}(x) = \ldots = w_{n+p}(x)$.*

Proof : it is a direct consequence of lemma 3 and remark 1.

Then let's call $w(x)$ the limit : $w(x) = \lim_{n \to \infty} w_n(x)$.

$$\begin{cases} w(x) = \infty & \text{if} \;\; x \notin \bigcup W_k \\ w(x) = n+1 & \text{if} \;\; x \in W_{n+1} - W_n \\ w(x) = 0 & \text{if} \;\; x \in W_0 \,. \end{cases}$$

$w(x)$ verifies the following equation:

$$w(x) = \min_u \max_v w \circ f(x, u, v) + c(x) \,. \tag{1}$$

$$\text{with} \begin{cases} c(x) = 0 & \text{if } x \in C \\ c(x) = 1 & \text{elsewhere.} \end{cases}$$

Equation (1) is exactly Isaacs' stationary equation.

1.2.3 Optimal strategy

Now, we can easily see that we have obtained the solution of the game where the players play alternatively. Since the minimizer plays first and the maximizer second, making a decision that depends on the control which the minimizer has just chosen, the notion of strategy is slightly different from the classical concept in differential games. We therefore call strategies ϕ and ψ functions of the type:

$$\begin{cases} u_t = \phi(x_t) \\ v_t = \psi(x_t, u_t). \end{cases}$$

One recognizes the definition of what is usually called upper strategy. A pair of strategies and an initial point induce a unique trajectory, and we can write the dynamics of the game:

$$\begin{cases} x_{t+1} = f(x_t, \phi, \psi) \\ x_{t=o} = x_0 \end{cases}$$

with no ambiguity. Now let's call ϕ^* and ψ^* the strategies defined as follows ($\mathcal{U}$ and $\mathcal{V}$ are finite):

$$\phi^* \text{ consists in playing } \phi^*(x_t) = u_t^* \text{ corresponding to } \min_u \max_v w \circ f(x_t, u, v)$$

$$\psi^* \text{consists in playing } \psi^*(x_t, u_t) = v_t^* \text{ corresponding to } \max_v w \circ f(x_t, u_t, v).$$

Actually, these definitions are ambiguous since they don't define a unique control at each step of the game, but it has no consequence as far as we are concerned, and anyway, we could get round this problem by, for instance, imposing a rule for the choice. We can now assert:

Theorem 2 *With the previous notations, if the game starts on a finite $w(x_0)$, then the duration is finite for all pairs of strategies (ϕ^*, ψ); the pair (ϕ^*, ψ^*) is optimal and $w(x_0)$ is the value of the game.*

Proof : let's place ourselves in a game, at instant t, and position x_t. Then, by definition: u_t^* and v_t^* verify the following inequality:

$$\forall (u, v), \quad w \circ f(x_t, u_t^*, v) \leq w \circ f(x_t, u_t^*, v_t^*) \leq \max_v w \circ f(x_t, u, v),$$

which, in terms of strategies, means:

$$\forall (u, v), \quad w \circ f(x_t, \phi^*, v) \leq w \circ f(x_t, \phi^*, \psi^*) \leq w \circ f(x_t, u, \psi^*),$$

which allows us to write :

$$\forall (u, v)$$
$$w \circ f(x_t, \phi^*, v) - w(x_t) + c(x_t) \leq w \circ f(x_t, \phi^*, \psi^*) - w(x_t) + c(x_t) = 0 \leq w \circ f(x_t, u, \psi^*) - w(x_t) + c(x_t), \tag{2}$$

which is exactly Isaac's equation. The *minmax* equation in terms of controls is a saddle-point equation in terms of strategies, and ϕ^* and ψ^* are the optimal strategies. Indeed, with obvious notations, with a game starting in x_0, we have, using equations (1) and (2) :

$$\forall (\phi, \psi) : \quad w \circ f(x_0, \phi^*, \psi) + c(x_0) \leq w(x_0) \leq w \circ f(x_0, \phi, \psi^*) + c(x_0),$$

which can be written:

$$w(x_1^{\phi^*, \psi}) + c(x_0) \leq w(x_0) \leq w(x_1^{\phi, \psi^*}) + c(x_0).$$

In the same manner, we have

$$w(x_1^{\phi, \psi^*}) \leq c(x_1^{\phi, \psi^*}) + w(x_2^{\phi, \psi^*})$$

and

$$c(x_1^{\phi^*, \psi}) + w(x_2^{\phi^*, \psi}) \leq w(x_1^{\phi^*, \psi}),$$

yielding

$$c(x_0) + c(x_1^{\phi^*, \psi}) + w(x_2^{\phi^*, \psi}) \leq w(x_0) \leq c(x_0) + c(x_1^{\phi, \psi^*}) + w(x_2^{\phi, \psi^*}).$$

And we can deduce by induction the following relation :

$$\forall t$$
$$c(x_0) + c(x_1^{\phi^*, \psi}) + \ldots + c(x_t^{\phi^*, \psi}) + w(x_{t+1}^{\phi^*, \psi}) \leq w(x_0) \leq c(x_0) + c(x_1^{\phi, \psi^*}) + \ldots + c(x_t^{\phi, \psi^*}) + w(x_{t+1}^{\phi, \psi^*}),$$

which proves the result.

Remark 5 *There are two interesting results here:*

- *the algorithm of "iteration on values" converges in this case, in a very simple manner since it is stationary, and allows us to solve Isaacs' stationary equation.*
- *this classical algorithm, where one iterates calculations on the whole game space, can be replaced by the building of these capture zones which we interpreted before: it is a faster algorithm, since one only has to do erosion and dilatation like operations on zones that have a small size at the beginning ($W_0 \equiv C$) and that don't grow fast.*

As a matter of fact, the "n" of W_n is the non linear counterpart of what Pontryagin called the "estimating function" [Pon68] ; it means that the evader will be caught within at most n steps; the previous paragraph proves that, in this case, "n" is also the classical value of the game.

Yet the problem is that we have only found a solution of the game in which the players play one after the other, and our model of incomplete information leads us to solve the game in which both play at the same time. This kind of game requires mixed strategies but we will see that here again, the "iteration on values" method converges and gives the solution of the game.

1.3 Simultaneous game

1.3.1 Calculation of v

Our method consists in initializing the whole game space with the previous $\min_u \max_v$ algorithm that gave us a value function $w(x)$ and in building a new value function $v(x)$ by iterating the process described in theorem 3. Now U and V are considered as random variables of distributions Y and Z. The sets $\mathcal{U}$ and $\mathcal{V}$ of admissible values for u and v are still supposed to be finite, and if we call p and q the number of elements they contain, Y and Z are elements of the p and q dimensional simplices.

Theorem 3 *If* $v_0(x) = w(x)$ *and* $v_{n+1}(x) = \min_Y \max_Z \; E_{Y,Z}\; v_n \circ f(x, U, V) \; + \; c(x)$, *then* $(v_n(x))_{n\in\mathbb{N}}$ *converges for all* x.

Proof : In this case, the *minmax* is the saddle point of the function $E_{Y,Z}\; v_n \circ f(x, u, v))$; it is smaller than or equal to the $\min_u \max_v$:

$$\forall n \in \mathbb{N} \;\; \min_Y \max_Z \; E_{Y,Z}\; v_n \;\circ f(x, U, V) \;\leq\; \min_u \max_v \; v_n \;\circ f(x, u, v),$$

so: $\forall x \;\; v_1(x) \leq v_0(x)$,

and, by induction, we find $\forall x \;\; v_{n+1}(x) \leq v_n(x)$.

Using $\forall k \in \mathbb{N} \; v_k(x) \geq 0$, since we always calculate saddle points of positive matrices, we prove that the method converges.

We obtain a function $v(x)$ verifying the following equation:

$$v(x) \;=\; \min_Y \max_Z \;\; E_{Y,Z}\; v \circ f(x, U, V) \;+\; c(x)\,, \tag{3}$$

with the limit condition: $v(x) \;=\; v_0(x) \;=\; 0$ if $x \in C$.

1.3.2 Optimal Strategy

Now, let's place ourselves in the context of a game. We shall use mixed strategies : the strategy ϕ for P is a function $x_t \mapsto \phi(x_t) \;=\; Y_t$, distribution of the random variable u_t, and the strategy ψ for E is a function $x_t \mapsto \psi(x_t) \;=\; Z_t$, distribution of the random variable v_t. A pair (ϕ, ψ) of strategies therefore defines a unique random process, and x_t becomes a random variable that we shall write X_t, of which we know how it is distributed, for given initial conditions.

We shall call $\Omega(x_0, \phi, \psi)$ the set of events ω induced by an initial condition x_0 and the pair of strategies (ϕ, ψ). We shall write with no ambiguity:

$$E_{\phi,\psi}^{|X_k=x_k}\, v(X_t)\,,\; (k \;\leq\; t)$$

instead of

$$E_{\phi,\psi}^{|X_k=x_k}\, v(X_t^{\phi,\psi})$$

the conditional mean in the process induced by (ϕ, ψ).
The duration of the game is a random variable $D(\omega)_{\omega\in\Omega(x_0,\phi,\psi)}$ and the performance index we want to optimize is M, the mean of D:

$$M(x_0, \phi, \psi) = \int_{\omega\in\Omega(x_0,\phi,\psi)} D(\omega)d\mu(\omega) = E^{|X_0=x_0}_{\phi,\psi} D.$$

Let's call ϕ^* and ψ^* the strategies defined as follows: we place ourselves at time t, with a position x_t ($X_t = x_t$); then, by construction of v,

$$v(x_t) = \min_Y \max_Z E_{Y,Z}\ v \circ f(x_t, U, V) + c(x_t) = E_{Y^*,Z^*}\ v \circ f(x_t, U, V) + c(x_t).$$

We shall define:

$$\begin{cases} \phi^*(x_t) = Y^* \\ \psi^*(x_t) = Z^*. \end{cases}$$

Theorem 4 *With the previous notations, if the game starts on a finite $v(x_0)$, then the probability for the duration to be infinite is nil for all pair of strategies (ϕ^*, ψ); the pair (ϕ^*, ψ^*) is optimal for the performance index $M(x_0, \phi, \psi)$, and the value of the game is $v(x_0)$.*

Proof : let ϕ and ψ be two other strategies; we have the following relations :

$$\begin{cases} E^{|X_t=x_t}_{\phi^*,\psi^*} v \circ f(X_t, \phi^*, \psi^*) &= E_{Y^*,Z^*}\ v \circ f(x_t, U, V) = v(x_t) - c(x_t) \text{ (using(3))} \\ E^{|X_t=x_t}_{\phi,\psi^*} v \circ f(X_t, \phi, \psi^*) &= E_{\phi(x_t),Z^*}\ v \circ f(x_t, U, V) \\ E^{|X_t=x_t}_{\phi^*,\psi} v \circ f(X_t, \phi^*, \psi) &= E_{Y^*,\psi(x_t)}\ v \circ f(x_t, U, V). \end{cases}$$

The property of v becomes:

$$\begin{aligned} &E^{|X_t=x_t}_{\phi^*,\psi}(v \circ f(X_t, \phi^*, \psi) + c(X_t)) \\ &\quad \le E^{|X_t=x_t}_{\phi^*,\psi^*}(v \circ f(X_t, \phi^*, \psi^*) + c(X_t)) = v(x_t) \\ &\quad\quad \le E^{|X_t=x_t}_{\phi,\psi^*}(v \circ f(X_t, \phi, \psi^*) + c(X_t)). \end{aligned} \tag{4}$$

This equation proves that ϕ^* and ψ^* give the solution of the game. Indeed, let x_0 be an initial condition : we can write, using (4):

$$E^{|X_0=x_0}_{\phi^*,\psi}(v(X_1) + c(X_0)) \le v(x_0) \le E^{|X_0=x_0}_{\phi,\psi^*}(v(X_1) + c(X_0)),$$

and, in the same manner,

$$v(x_1) \le E^{|X_1=x_1}_{\phi,\psi^*}(v(X_2) + c(X_1)),$$

or, in terms of random variables,

$$v(X_1) \le E^{|X_1}_{\phi,\psi^*}(v(X_2) + c(X_1)),$$

which yields

$$E^{|X_0=x_0}_{\phi,\psi^*} v(X_1) \le E^{|X_0=x_0}_{\phi,\psi^*}(E^{|X_1}_{\phi,\psi^*}(v(X_2) + c(X_1)))$$

The process is a Markov process, so we can write :

$$E^{|X_1}_{\phi,\psi^*}(v(X_2) + c(X_1)) = E^{|X_1,X_0=x_0}_{\phi,\psi^*}(v(X_2) + c(X_1))$$

and

$$E^{|X_0=x_0}_{\phi,\psi^*}(E^{|X_1}_{\phi,\psi^*}(v(X_2) + c(X_1))) = E^{|X_0=x_0}_{\phi,\psi^*}(E^{|X_1,X_0=x_0}_{\phi,\psi^*}(v(X_2) + c(X_1)))$$

and because of the total probability theorem applied to the probability space $\Omega(x_0, \varphi, \psi^*)$,

$$E^{|X_0=x_0}_{\phi,\psi^*}(E^{|X_1,X_0=x_0}_{\phi,\psi^*}(v(X_2) + c(X_1))) = E^{|X_0=x_0}_{\phi,\psi^*}(v(X_2) + c(X_1)),$$

yielding :

$$v(x_0) \leq E_{\phi,\psi^*}^{|X_0=x_0}(v(X_2)+c(X_1)+c(X_0)).$$

We have the same inequality on the other side, and by induction,
$\forall t$

$$E_{\phi^*,\psi}^{|X_0=x_0}(v(X_{t+1})+c(X_t)+...+c(X_1)+c(X_0)) \leq v(x_0) \leq E_{\phi,\psi^*}^{|X_0=x_0}(v(X_{t+1})+c(X_t)+...+c(X_1)+c(X_0)),$$

which proves that we optimized the mean of a performance index which is interpreted as the duration of the game if it is finite, since in this case $c(x_t)$ becomes equal to zero for t greater than a certain t_1.

We shall now prove that if we take an initial point x_0 and if $v(x_0)$ is finite, then the probability for the game to have an infinite duration is zero, assuming that P plays ϕ^*. Indeed,

$$\forall\psi\ \forall t\ ,E_{\phi^*,\psi}^{|X_0=x_0}(v(X_{t+1})+c(X_t)+...+c(X_1)+c(X_0)) \leq v(x_0) < \infty.$$

Let's call $\Omega_\infty(x_0,\phi,\psi)$ the set of infinite games starting from x_0 with the strategies ϕ and ψ. Using the fact that $\Omega_\infty(x_0,\phi,\psi)$ is a subset of $\Omega(x_0,\phi,\psi)$, and that for a game of infinite duration,
$c(x_0)+c(x_1)+...+c(x_t)+v(x_{t+1}) \geq t,$
we can write:

$$\begin{aligned}\infty > v(x_0) &\geq E_{\phi^*,\psi}^{|X_0=x_0}(v(X_{t+1})+c(X_t)+...+c(X_1)+c(X_0)) \\ &\geq \int_{\Omega_\infty(x_0,\phi^*,\psi)}(v(X_{t+1})+c(X_t)+...+c(X_1)+c(X_0))(\omega)d\mu(\omega) \\ &\geq t*\mu(\Omega_\infty(x_0,\phi^*,\psi)),\end{aligned}$$

which is possible for all t only if

$$\mu(\Omega_\infty(x_0,\phi^*,\psi)) = 0,$$

which proves the assertion.

2 Implementation

As we already mentionned, we use a hexagonal (or triangular) lattice. Our programs are written in C language, and we run them on "Sun" workstations. For the alternate game, the method of dynamic programming (theorem 3) is easy to compute. The method of zones (theorem 4) requires more subtle programming; we use C memory allocation "malloc" to put the previously calculated zone in memory, and we calculate the following one using only these data. As we expected, the capture zone program is quite faster than the first one: about 5 to 6 times faster.

For the simultaneous game, we use the previous maps (which are actually matrices) for initializing the process. Then we iterate the resolution of the simplex problem on each point of the game space, using the linear programming algorithm.

3 Improvments

Two important ways of improvments which seem promising for naval applications are extensions to higher dimensionnal spaces and to other kinds of games.

3.1 higher dimensional spaces

The previously chosen lattices are 2-D lattices : the two players are moving on a plane, with either rectangular or hexagonal lattices. This kind of game can be restritive for actual applications which, in the case of naval games. can involve submarines. Yet, a submarine warfare application does not need a true 3-D space : indeed. available controls of a submarine are restricted to variation of the depth and have nothing to do with true 3-D controls as for airfights. By the same analogy with cristallography, we propose a graphite-like lattice for discrete game modelling in order to deal with the motions of submarines. In this case, there is an hexagonal subspace, like the previously described one, and the motions available between these subspaces can be restricted to simple variations of depth. By this way, we can apply the mathematical descriptions we propose in section 1. and algorithms giving charts can easily deduced from the 2-D ones.

If we want to deal with a true 3-D space —i.e. we consider that the space is isotropic— a model can be constructed with a 3-D lattice. A first one, similar to the rectangular lattice of the 2-D space, is a cubic lattice : indeed, in this case, each control can be splitted into a translation and a rotation which belong to the space group of this lattice. This case is rather restrictive as final speed is actually parallel to one of the cristal vector. An interesting but more complex lattice can be a diamond-like one. In this case, there is much possibilities for the final speed; indeed, representation in such a space can become easy only for a specialist of cristallography or space groups...

3.2 Surveillance games

Naval applications required good models of survey games : indeed, a ship can be viewed as a threat if she is too near. Matching of our model to survey games can easily deduced from the previous tools. Nevertheless, we have firstly to estimate if the minimizer, who wishes a survey situation, can win (it is easy to imagine a situation where the maximizer can always evade or attack). In this case, a chart of the space, with which optimal controls can be deduced, can be computed by the same way as previously.

4 Conclusions

These methods for the resolution of discrete pursuit games provide useful tools for naval applications : indeed, they take into account ill-defined data and provide interesting informations about possible strategies. It seems that a possible application for decision aids can be an interactive system displaying several charts to the ship officer, each chart being based on given assumptions about the opponent. As our modelling allows a "real-time" computation of these charts, the ship officer can introduce all kinds of values deduced from ill defined informations and all kinds of opponent intent and, with the help of these charts, he can deduce his own decision and his own strategy.

Finally, we want to emphasize the fact that, on one hand, the discretization of the space is not so much of a problem as far as undersea pursuits are concerned, since ships are rather big, slow, and since specific problems of underwater acoustics yield imprecision in detection and measurements, and on the other hand, our methods, based on this discretization of the space, do not seem to be so appropriate for missiles launching and airfights, that other models of Pursuit Evasion game match well. So naval warfare has to suggest us new game models for which we have to define a pragmatic approach to find suitable solutions.

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