NONLINEAR ROBUST CONTROL AND MINIMAX TEAM PROBLEMS

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Abstract

The problem of robust stabilization of a linear system leads to the classical $\mathcal{H}_\infty$ control problem. The same analysis applied to a nonlinear system leads to the problem of insuring via output feedback that a nonlinear operator be Lipshitz continuous, with a prescribed Lipshitz modulus. We show that, in the same way as the $\mathcal{H}_\infty$ control problem is equivalent to a minimax control problem, the Lipshitz modulus control problem can be approached via a minimax team decision problem. This motivates us to re-visit a class of so-called "static" team decision problems for nonlinear dynamical control systems. Because of the "static" character, signaling plays no role in that case, which is important for the equivalence with the Lipshitz modulus control problem. We show that under some conditions, a certainty equivalence principle applies that yields a practical solution to the team problem at hand. To reach that conclusion we must first investigate a "partial team" problem where one of the team members has all the information.

1 Introduction

During the past 17 years or so, a rather powerful theory of robust control of linear systems has been developed under the now classical name of $\mathcal{H}_\infty$-optimal control. The development of that theory may be summarized as follows.

In the early 80's, Zames and others [12, 31] showed the relationship between a problem of robust control loop design and the problem of minimizing the $\mathcal{H}_\infty$ norm of the so called "complementary sensitivity" transfer function of the control system. Hence, a main technical problem appeared, of minimizing the $\mathcal{H}_\infty$ norm of a linear system via dynamic output feedback. This problem was first tackled and essentially solved via function theory. This culminated in several books among which [13].

It was not until 1988, after [11] used a realization of the system to perform the required inner-outer operator factorization, ending up in a simple looking state variable solution, that it was understood that the main technical problem could be cast into one of min-max control [2, 27]. That approach was followed by several authors [3, 22, 26] and led to a powerful theory, solving the minimax control problem for a wide class of problems, both stationary and non stationary.

The problem of min-max control was then extended to a nonlinear setup, leading to interesting results for the robust control of nonlinear systems [19, 28, 20, 6].

The point in the above historical sketch is to stress the fact that three problems are considered here: one of robust control, one of minimization of an $\mathcal{H}_\infty$ norm, one of minimax control. The first and third one have natural extensions to nonlinear systems. The second one also if one replaces "$\mathcal{H}_\infty$ norm" by the so called L^2 gain. And then the nonlinear minimax problem is indeed equivalent to that one.

However, we claim that the most natural nonlinear extension of Zames' analysis, equating a robust control problem with one of $\mathcal{H}_\infty$ norm minimization, is *not* the natural nonlinear extension of the minimax control problem. Instead, it leads to the problem of minimizing (or keeping small) a Lipshitz continuity modulus, and not a L^2 gain. Two problems which are *not* equivalent in the nonlinear case.

That problem has received less attention in the literature than the L^2 gain problem, although it was mentioned as early as [30], and was also considered in the context of robust control, in [18, 14] for instance. (Sometimes using the unfortunate name of "incremental norm".) Most papers follow the approach of [29], which uses the equivalence between the Lipshitz property and a uniform bound on the norm of the Freychet derivative of the operator (if it is differentiable). We shall propose here a more direct approach.

Exactly as a minimax problem is associated with the problem of controlling the L^2 gain of a system, the problem of controlling the Lipshitz continuity modulus is associated with a minimax team problem. Hence we are led to the investigation of a (simple) class of team problems, "static" in the sense of Marschak and Radner [23], although they are dynamic control problems. As a matter of fact, the class we consider is slightly more general than needed for our purpose, only because this does not make that analysis any more complicated. In a restricted case, where a certainty equivalence principle is showed to hold, we provide a complete solution of the team problem. The technique used borrows its ideas from our previous works on certainty equivalence in minimax control problems [3, 6, 7, 9].

As compared to the classical literature [25, 16, 21, 4] on team theory, we need to have a minimax treatment of the disturbances instead of a stochastic treatment. But this is not a major difference as recent work shows [7, 8]. A deeper difference is that the classical literature uses necessary conditions, and exhibits situations where these necessary conditions have a simple solution. In keeping with our work on certainty equivalence (that considers a partial information problem where a *single* controller, with a *single* information flow, chooses the minimizing controls), we exhibit cases where a team strategy inspired by a certainty equivalence principle in the case of *complete decentralization* does as well as the full information optimal strategy, and is thus optimal. To achieve this, we *do* use a necessary condition, but for a maximization problem, not a game, or team problem. This is important because, as is well known, there is no such thing as a "two sided Pontryagin minimum principle" that would serve as a necessary condition for dynamic game problems.[1]

There does not seem to exist much literature on minimax teams. In fact, this is a particular case of a Nash equilibrium, where the team members share the same performance index, while the disturbance has the opposite. However, dynamic Nash equilibria seem difficult to compute, even in the linear quadratic case [15]. Usually, the players must share more information to lead to a computable Nash equilibrium. (See, e.g. [1, 4]).

[1]Isaacs' adjoint equations have often been mistaken for a necessary condition of optimality in two-person zero-sum games. Such a mistake has led to the publication of many a false result.

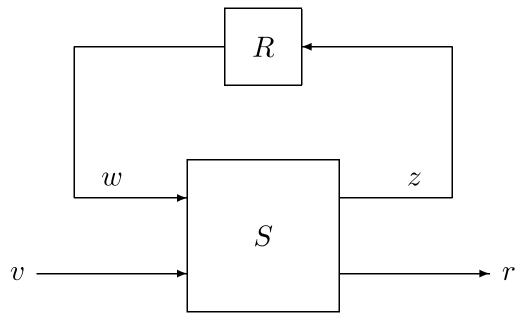

Figure 1: A partially unknown system

2 Nonlinear robust control

2.1 The classical approach

Let a partially unknown system be represented by the feedback connection of an unknown part with a known plant, as in figure 1. However, here nothing is assumed linear.

In $L^2(0,\infty)$ space, we have the known plant

$$r = P(v,w)\,, \tag{1}$$
$$z = Q(v,w)\,, \tag{2}$$

where r is the regulated output, z an auxiliary output lumping all signals that enter into an unknown part, v is any external input, w the signals entering the plant coming from the unknown part, and

$$w = R(z) \tag{3}$$

that unknown part. Assume further that P sends $L^2 \times L^2$ into H^1. (Which is the case if it is a stable state variable system, with dynamics having linear growth at infinity.)

Assume the only thing we know about R is that it is Lipshitz continuous with Lipshitz modulus less or equal to a given δ:

$$\forall z_1, z_2 \in L^2 \times L^2, \quad \|R(z_1) - R(z_2)\| \leq \delta \|z_1 - z_2\|\,.$$

The classical result is as follows

Theorem 1 ("Small gain theorem"). *If for all $v \in L^2$, the partial function $w \mapsto Q(v,w)$ is Lipshitz continuous with modulus less or equal to γ, and $\delta\gamma < 1$, then the overall system is stable.*

Proof The proof is elementary. The system equations (2)(3) constitute a fixed point mapping

$$z = Q(v, R(z)).$$

By Banach's fixed point theorem, a sufficient condition for the existence of a solution is that $z \mapsto Q(v, R(z))$ be a contraction, i.e. Lipshitz continuous with modulus less than one, which is insured by the hypothesis that $\gamma\delta < 1$. (Moreover, Banach's theorem insures unicity of the solution and bounds its norm.) Therefore, for any v, there exists a unique z, and hence a unique w, in L^2 solution of the system equations, and therefore a unique r, which by our hypothesis on P is in $H^1(0,\infty)$. Hence $r(t) \to 0$ as $t \to \infty$, and the system is stable.

Remark If we insist that for stability, all signals should go to zero at infinity, we must make a similar assumption on Q as we did on P, and assume also that R sends H^1 into H^1.

Notice that the property needed on Q is *not* that it have L^2 gain less or equal to γ. This is not sufficient. It should have Lipshitz continuity modulus less or equal to γ. For linear systems, these two properties coincide, and it is why the standard problem of $\mathcal{H}_\infty$-optimal control is that of controlling the L^2 gain of the linear system, i.e. the $\mathcal{H}_\infty$ norm of its transfer function. That these two properties do not coincide in nonlinear functions results from the simple following counter-example.

Let $q : \mathbb{R} \to \mathbb{R}$ be given by

$$q(z) = \frac{z^3}{1+z^2}.$$

It is straightforward to check that indeed

$$|q(z)| = \frac{z^2}{1+z^2}|z| < |z|,$$

(hence q has "gain" no greater than one,) but that the Lipshitz modulus of q is only bounded by 9/8, and, for instance, $q(2) - q(1) = 1.1 > 1$.

2.2 Equivalence with a team problem

We now assume that the known part itself is a control system, with input-output map of the form

$$z = T(u,w)\,, \tag{4}$$
$$y = S(u,w)\,. \tag{5}$$

(S and T may further depend on an "exogenous" input v, that we ignore for the time being. Everything should hold for every fixed v.) Here, y is an observed output. The problem at hand is therefore as follows.

Standard Problem Given a positive number γ, does there exist an admissible (causal) control law $u = \varphi(y)$ such that under that control law, the system is stable and Lipshitz continuous from w to z with Lipshitz modulus no greater than γ ? If yes, find one.

Admissible means that φ is causal and that the fixed point equation $y = S(\varphi(y), w)$ has a unique solution in L^2 for every w in L^2. We shall write $z = T(\varphi, w)$ to mean the corresponding z output.

Rephrased in equations, the standard problem is to find an admissible φ such that

$$\forall w_1, w_2 \in L^2 \times L^2, \quad \|T(\varphi, w_1) - T(\varphi, w_2)\| \le \gamma \|w_1 - w_2\|\,.$$

Consider the composite system made of two copies of the original one:

$$\begin{aligned} z_1 &= T(u_1, w_1)\,, \\ z_2 &= T(u_2, w_2)\,, \\ y_1 &= S(u_1, w_1)\,, \\ y_2 &= S(u_2, w_2)\,. \end{aligned}$$

Let $Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, and, with transparent notations,

$$\begin{aligned} Z &= \mathbb{T}(U, W)\,, \\ Y &= \mathbb{S}(U, W)\,. \end{aligned}$$

Introduce the linear operator $\Delta = [I \ \ -I]$, and define a performance index associated with the composite system as

$$J(U, W) = \|\Delta Z\|^2 - \gamma^2 \|\Delta W\|^2\,, \tag{6}$$

where all norms are L^2 norms.

The standard problem is to find an admissible control law $U = \Phi(Y)$ such that

$$\sup_W J(\Phi, W) \leq 0\,,$$

which is possible if and only if (assuming the min exists)

$$\min_\Phi \sup_W J(\Phi, W) \leq 0\,.$$

The crucial point now is that *admissible* control laws must be made of two copies of the same, decentralized, control law: $u_i = \varphi(y_i)$, $i = 1, 2$, independant of the other trajectory.

Consider the team control problem where we only impose decentralized information structure, i.e. $u_i = \varphi_i(y_i)$. The systems 1 and 2 above are completely decoupled. Hence for any disturbance $W = (w_1, w_2)$, each output y_i only depends on the corresponding w_i. No dependance on w_j, $j \neq i$ can be induced through the controls either. Because of the symmetry inherent in that team problem, the optimal solutions $(\varphi_1^*, \varphi_2^*)$ will automatically satisfy the added requirement that for any output history y, $\varphi_1^*(y) = \varphi_2^*(y)$.

Hence φ_1^* and φ_2^* are the same control law, defined on the (identical) isolated systems 1 and 2, as desired.

The conclusion is that if one can solve the minimax team problem, this indeed answers the question of whether there exists a control law $u = \varphi(y)$ that makes the system (4)(5) Lipshitz continuous from w to z with a Lipshitz modulus less or equal to γ. If the team optimal solution leads to a $\sup_W J$ which is positive, the problem has no solution. If, to the contrary, this optimal strategy leads to a nonpositive $\sup_W J(\Phi^*, W)$, then the problem has a solution, and the optimal team strategy is made of two copies of a solution of that problem.

3 Minimax Team Problems

3.1 The system considered

Because it may be interesting in its own sake, and it does not complicate the analysis, we consider a slightly more general team problem.

Consider a team of two decision makers, whom we call *players* for short, each controlling different actions and having access to different informations. There is a common pay-off for both players, which has to be minimized by them. We consider the special case, when their dynamics are completely separated with respect to all variables. To be more precise let the variables $x_1, x_2, u_1, u_2, w_1, w_2$ denote correspondingly the state, control and disturbance variables for each of the players, in terms of which the dynamic equations in the nonlinear general setup can be presented as:

$$\begin{cases} \dot{X} = F(t, X, U, W)\,, \\ X(t_0) = X_0\,, \end{cases} \tag{7}$$

where $X = (x_1, x_2)$, $F = (f_1, f_2)$, $U = (u_1, u_2)$, $W = (w_1, w_2)$. The disturbance variables w_1, w_2 are treated as control variables of "an opposite player", leading to a formulation in terms of a dynamic game problem.

Otherwise the system (7) may be presented by the so-called "augmented system"

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, u_1, w_1)\,, \\ \dot{x}_2 = f_2(t, x_2, u_2, w_2)\,, \\ x_1^0 = x_1(t_0)\,, \\ x_2^0 = x_2(t_0)\,, \end{cases}$$

where $t \in [t_0, +\infty), x_i(t) \in \mathbb{R}^{n_i}, u_i \in \mathbb{R}^{m_i}, w_i \in \mathbb{R}^{l_i}, i = 1, 2$. The control parameters of the players and the disturbances obey the following restrictions:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathsf{U} := \mathsf{U}_1 \times \mathsf{U}_2, \quad \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathsf{W} := \mathsf{W}_1 \times \mathsf{W}_2,$$

where the U_1, U_2, W_1, W_2 are compact convex sets in appropriate spaces. The sets $\mathcal{U}_i$ and $\mathcal{W}_i$ of admissible open-loop controls $u_i(.)$ and $w_i(.)$ will contain all measurable functions from $[t_0, \infty)$ into U_i and W_i respectively.

Under the necessary regularity assumptions (specified below) we shall denote for a given initial time $t_0 \in \mathbb{R}$ by $X(.) = S(t_0, X_0, U(.), W(.))$ the unique (Cauchy) solution of the system (7). By $x_i^{(s)}(.) = S_i(t_0, x_i^0, u_i(.), w_i(.))$ we shall denote correspondingly the components of that solution for each of the players separately.

We shall consider the following performance index, where the coupling between the two players resides:

$$J = M(T, x_1(T), x_2(T)) + \int_{t_0}^{T} L(t, x_1, x_2, u_1, u_2, w_1, w_2)\, dt + N(x_1^0, x_2^0). \tag{8}$$

where L, M and N are given differentiable functions from the appropriate spaces into $\mathbb{R}$.

The problem considered here is more general than needed for our purpose as stated in section 2 in two respects. On the one hand we allow differing dynamics for both players, on the other hand the above payoff is more general than (6)

The precise formulation of the problems depends upon information structure and will be given below in the following sections.

Let us introduce the standard problem in perfect information, that is with admissible strategies in state feedback, i.e. of the form:

$$u_i = \varphi_i(t, x_1, x_2),\ i = 1, 2$$

and recall the classical Hamilton-Jacobi-Isaacs solution [17, 5].

The classical game problem formulation: Given the initial time and state (t_0, x_1^0, x_2^0) determine, if it exists, the Isaacs' value function:

$$V(t_0, x_1^0, x_2^0) = \min_{u_1} \min_{u_2} \max_{w_1} \max_{w_2} J^*\,, \tag{9}$$

where $J^* = J - N(x_1^0, x_2^0)$.

Proposition 1. *If there exists a C^1 function $V : [t_0, T] \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, solution of the partial differential equation*

$$-\partial V/\partial t = \min_{u_1} \min_{u_2} \max_{w_1} \max_{w_2} H(t, x_1, x_2, \frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, u_1, u_2, w_1, w_2) \tag{10}$$

with boundary condition:

$$\forall x_1, x_2,\ V(T, x_1, x_2) = M(T, x_1, x_2),$$

where

$$H(t, x_1, x_2, \mu_1, \mu_2, u_1, u_2, w_1, w_2) = L + < \mu_1, f_1 > + < \mu_2, f_2 >,$$

is the Hamiltonian of the system (the angled brackets $< \cdot, \cdot >$ denote the scalar product in $\mathbb{R}^{n_i}$), then the value of the game (9) is $V(t_0, x_1^0, x_2^0)$. Moreover, if the Hamiltonian has a saddle point in (U, W) for all (x, μ_1, μ_2), and if there exist admissible strategies

$$U = \Phi^*(t, X) = \begin{pmatrix} \varphi_1^*(t, x_1, x_2) \\ \varphi_2^*(t, x_1, x_2) \end{pmatrix},\ W = \Psi^*(t, X) = \begin{pmatrix} \psi_1^*(t, x_1, x_2) \\ \psi_2^*(t, x_1, x_2) \end{pmatrix} \tag{11}$$

which are a saddle point of $H(t, x_1, x_2, \partial V/\partial x_1, \partial V/\partial x_2, u_1, u_2, w_1, w_2)$, then they are optimal.

Φ^* and Ψ^*, together with V, will be referred to as the *Isaacs solution.*

3.2 The state feedback partial team problem

3.2.1 Statement of the problem

In the problem investigated in this section the players have different informations about the evolution of the system over time: we shall suppose that the first player (indicated by subindex 1) only has the knowledge of its own state x_1, while the second one (indicated by subindex 2) has access to both states, hence the admissible strategies are:

$$u_1 = \varphi_1(t, x_1)\,, \quad u_2 = \varphi_2(t, x_1, x_2). \tag{12}$$

For arbitrary initial conditions (t_0, x_i^0) call disturbances the pairs $\omega_i := (x_i^0, w_i) \in \Omega_i := \mathbb{R}^{n_i} \times \mathcal{W}_i$, $i = 1, 2$, and define $X^0 := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We shall consider several information structures beyond (12), where x_i^0 is not known to the players. This is why we have added the "initial cost" $N(x_1^0, x_2^0)$ in (8).

The state feedback partial team problem is the following: *Under the information structure (12) find optimal controls for the minimizing players, guaranteeing*

$$\min_{\varphi_1} \min_{\varphi_2} \max_{\omega_1} \max_{\omega_2} J(t_0, x_1^0, x_2^0, \varphi_1, \varphi_2, w_1, w_2)$$

J being given by (8).

For any function $a(\cdot) : t \to a(t)$, we shall use the notation a^τ for its restriction to $[t_0, \tau]$. Notice, that with a mild abuse of notations we may write causality of S as $S_i^\tau(x_i^0, u_i, w_i) = S_i^\tau(x_i^0, u_i^\tau, w_i^\tau)$.

3.2.2 Information

Denote by

$$\Omega_1^\tau(u_1^\tau, x_1^\tau) = \{\omega_1 \in \Omega_1 \,|\, S_1^\tau(x_1^0, u_1^\tau, w_1^\tau) = x_1^\tau\} \tag{13}$$

the set of ω_1's which are compatible with the past observations of the first player and by $\Omega_1^{\tau\tau}$ the set of restrictions to $[0, \tau]$ of the elements of Ω_1^τ. It is clear that $\forall t, \Omega_1^t \in \Omega_1$. We do not introduce such set (depending upon a time parameter) of disturbances of the second player (available for the first player), since the first player has no information about the second player in general. We do not introduce as well similar set(s) (depending upon a time parameter) for the second player since the latter has complete information about the system's evolution at every moment of time.

Main assumptions. We shall make the following main assumptions which will allow us to construct an optimal control for the above problem [9]:

1. Regularity assumptions.

   We shall suppose that the functions f_i, L are of class C^1 and a growth condition holds on f_i, that guarantees the existence of unique solution S to (7) over $[t_0, T]$ for any $(U, W) \in (\mathcal{U}, \mathcal{W})$.

2. Existence of solution of the perfect-state information case.

   We shall suppose that the corresponding zero-sum differential game with perfect state information, stated in Section 3.1, has a unique state feedback saddle-point solution.

We notice that the observation process satisfies the following three important properties [9]:

a) *it is consistent*

$$\forall u_1,\, \forall \omega_1,\, \forall t \quad \omega_1 \in \Omega_1^t(u_1^t, S_1^t(x_1^0, u_1^t, w_1^t)), \tag{14}$$

b) *it is perfect recall*

$$\forall u_1, \forall \omega_1 \quad t' \geq t \Rightarrow \Omega_1^{t'} \subset \Omega_1^t, \tag{15}$$

c) *it is non-anticipative*

$$\forall t, \quad \omega_1 \in \Omega_1^t \Leftrightarrow \omega_1^t \in \Omega_1^{tt}. \tag{16}$$

3.2.3 Auxiliary problem

Under the hypothesis that the perfect-state information problem has a solution we define for all admissible $(u_1, \omega_1) \in \mathcal{U}_1 \times \Omega_1$, $(u_2, \omega_2) \in \mathcal{U}_2 \times \Omega_2$ and for all $t \in [t_0, T]$:

$$\begin{aligned} G(\tau, u_1^\tau, \omega_1, \omega_2) = {} & V(\tau, x_1^{(s)}(\tau), x_2^{(s)}(\tau)) \\ & + \int_{t_0}^{\tau} L(t, x_1^{(s)}(t), x_2^{(s)}(t), u_1, \varphi_2^*(t, x_1^{(s)}, x_2^{(s)}), w_1, w_2)\, dt \\ & + N(x_1^0, x_2^0), \end{aligned} \tag{17}$$

where the upper index (s) indicates the above mentioned Cauchy solution $S_i(t_0, x_i^0, u_i, w_i)$, the index (*) denotes the optimal solution of the perfect information case, stated in Section 3.1.

We define the *Auxiliary problem*:
Does there exist

$$g(\tau) = \max_{\omega_1 \in \Omega_1^\tau} \max_{\omega_2 \in \Omega_2} G(\tau, u_1^\tau, \omega_1, \omega_2)\,? \tag{18}$$

Notice that (18) defines a set of optimization problems indexed by time.

Remark 1. *Notice, that the Ω_1^t, introduced by (13), and $G(\tau, u_1^\tau, \omega_1, \omega_2)$ depend upon the past values of u_1. That is why the result of the maximization, the function $g(\cdot)$, depends only upon the time parameter, and this problem is well posed for player 1.*

When it exists we shall write:

$$\hat{\Omega}_2^t = \arg\max_{\omega_2 \in \Omega_2} \big[\max_{\omega_1 \in \Omega_1^t} G(t, u_1, \omega_1, \omega_2) \big],$$

and

$$\hat{X}_2(t) = \{ \hat{x}_2(t) \,|\, \hat{x}_2(\cdot) = S_2(t_0, \hat{x}_2^0, \varphi_2^*, \hat{w}_2) \quad \text{and} \quad (\hat{x}_2^0, \hat{w}_2) \in \hat{\Omega}_2^t \}.$$

Remark 2. *Notice that the set $\hat{\Omega}_2^t$, being a subset of Ω_2, defines the set of worst disturbances from the viewpoint of the first player (and not the second), under the condition that the first player has no information about the second player's actions. Technical treatment of matters here supposes to introduce also some $\hat{\Omega}_1$ and $\hat{X}_1$ (because maximization operation in (18) consists of two maximums), but since the first player has complete information about his or her "past" values (the partial team problem is being solved for him or her), then these sets are not needed here. Below, in the case of noise-corrupted information for the first player, this matter will be discussed in details.*

3.2.4 Main results for the partial team problem

Crucial Assumption. Assume that, for all pairs (ω_i, u_i) and for all $t \in [t_0, T]$, $\hat{X}_2$ is a singleton [9].

Remark 3. *Notice that $\hat{X}_2(t)$ is never empty, and denote $\hat{x}_2(t)$ its unique member. This doesn't necessarily imply the unicity of $\hat{\omega}_2$.*

Theorem 2. *Under the Crucial Assumption above, the pair of optimal controls $\varphi_1^*(t, x_1(t), \hat{x}_2(t))$, $\varphi_2^*(t, x_1(t), x_2(t))$ ($\varphi_i^*(t, x_1(t), x_2(t))$ being defined by (11)) solves the partial team problem. Moreover*

$$\max_{\omega_1 \in \Omega_1} \max_{\omega_2 \in \Omega_2} J(t_0, \varphi_1^*(t, x_1(t), \hat{x}_2(t)), \varphi_2^*(t, x_1(t), x_2(t)), \omega_1, \omega_2) = \max_{X_0 \in X^0} [V(t_0, x_1^0, x_2^0) + N(x_1^0, x_2^0)]. \tag{19}$$

Proof. The proof of the theorem strongly relies upon the following fact:

Lemma 1. *If the first team member uses the control*

$$\hat{u}_1(t) = \varphi_1^*(t, x_1(t), \hat{x}_2(t)),$$

then the function $g(\tau)$ is non-increasing.

Proof of the lemma. Notice that the function (18) can be presented as

$$g(\tau) = \max_{(\omega_1,\omega_2)} G(\tau, u_1^\tau, \omega_1, \omega_2),$$

where the vector $(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2$, and the set $\Omega_1 \times \Omega_2$ satisfies the conditions (14)-(16).

Write the following system:

$$\dot{X} = \tilde{F}(t, X, u_1, W), \tag{20}$$

defined in the coordinates of the "augmented system" as:

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, u_1, w_1)\,, \\ \dot{x}_2 = f_2(t, x_2, \varphi_2^*(t, x_1, x_2), w_2). \end{cases}$$

Notice that φ_1^* is the optimal state feedback for the game problem $\min\limits_{u_1} \max\limits_{\omega} J$ under the dynamics $\tilde{F}$. If we write the auxiliary problem from [3], p.195, for the system (20) and present it in coordinates of the "augmented system", then we shall have exactly our auxiliary problem (18). The proof of the lemma then proceeds from the proof of Lemma 5.1 of [3], p.197, for the system (20).

Then according to the Theorem 5.1 of [3] $\hat{U} := (\varphi_1^*(t, x_1, \hat{x}_2), \varphi_2^*(t, x_1, x_2))$ will be the optimal control, solving the incomplete information problem for the system (20). Thus Theorem 2 holds.

Hereafter we shall call the strategy $\hat{u}_1(t) = \varphi_1^*(t, x_1(t), \hat{x}_2(t))$ partial team strategy of the first player.

Corollary 1. *If for all $u_1 \in \mathcal{U}$ there exists $t^* \in [t_0, T]$, such that for $\tau > t^*$ the auxiliary problem (18) fails to have a solution and exhibits an infinite supremum, then, if for some other pair of strategies (φ_1, φ_2) there exists a finite $\sup_{\omega_1,\omega_2} J(\varphi_1, \varphi_2, \omega_1, \omega_2)$, it is larger than $\max_{X_0}[V + N]$.*

Remark 4. *Notice that the value (19) of the partial team problem is equal to the Isaacs value, corresponding to the full information case, and (on the base of uniqueness of $\hat{x}_2(t)$) can be presented as:*

$$\max_{\omega_1\in\Omega_1}\max_{\omega_2\in\Omega_2} J(t_0,\varphi_1^*(t,x_1,\hat{x}_2),\varphi_2^*(t,x_1,x_2),\omega_1,\omega_2) = \max_{x_1}[V(t_0,x_1,\hat{x}_2^0)+N(x_1,\hat{x}_2^0)]. \tag{21}$$

Remark 5. *It is clear that a similar result can be proved for the second player, supposing that the first player has complete knowledge of the system's evolution in time, whereas the second one only has the knowledge of his or her actions. The partial team strategy of the second player, solving his or her partial team problem, will be:*

$$\hat{u}_2(t) = \varphi_2^*(t,\hat{x}_1(t),x_2(t))\,,$$

$\hat{x}_1$ being generated by a similar auxiliary problem, well posed for player 2. The value of the game in this case is also equal to Isaacs' value and can be presented as:

$$\max_{\omega_1\in\Omega_1}\max_{\omega_2\in\Omega_2} J(t_0,\varphi_1^*(t,x_1,x_2),\varphi_2^*(t,\hat{x}_1,x_2),\omega_1,\omega_2) = \max_{x_2}[V(t_0,\hat{x}_1^0,x_2)+N(\hat{x}_1^0,x_2)]. \tag{22}$$

One can conclude on the base of (19), (21), (22), that

$$(\hat{x}_1^0,\hat{x}_2^0) = \arg\max_{(x_1,x_2)}[V(t_0,x_1,x_2)+N(x_1,x_2)].$$

3.2.5 The noise-corrupted information case for the first player

The above result can be extended to the case when the information available to the player 1 is not the "pure history" of the past values of his or her state, but is a disturbance-corrupted function of these values. Assume that an output y_1 is defined by a map:

$$y_1 = h_1(t,x_1,w_1)\,, \tag{23}$$

and that the admissible strategies are of the form:

$$u_1(t) = \varphi_1(t,y_1^t).$$

Precise formulation supposes to include the equation (23) in the dynamics of the system (7), which we do not write down here once more. We are still assuming that the second player has complete information about the system states over time. (We nevertheless write y_1 and not y to recall its non-symmetric role.)

The set (13) will be modified in the following way:

$$\Omega_1^\tau(u_1^\tau, y_1^\tau) = \{\omega_1 \in \Omega_1 \,|\, h_1^\tau(t, x_1^\tau, w_1) = y_1^\tau\}. \tag{24}$$

The *Main assumptions* above are supposed to hold here as well.

Let us write down the auxiliary problem for this case: Does there exist

$$g(\tau) = \max_{\omega_1 \in \Omega_1^\tau} \max_{\omega_2 \in \Omega_2} G(\tau, u_1^\tau, \omega_1, \omega_2)\,? \tag{25}$$

where Ω_1^τ is modified by (24), the function $G(\tau)$ is given by (17).

When it exists we shall write:

$$\hat{\Omega}^t = \arg \max_{\omega_1 \in \Omega_1^t} \max_{\omega_2 \in \Omega_2} G(t, u_1, \omega_1, \omega_2)\,,$$

$$\begin{aligned} \hat{X}_1(t) = \{(\check{x}_{11}(t), \check{x}_{12}(t)) \,|\, & \check{x}_{11}(\cdot) = S_1(t_0, \check{x}_{11}^0, u_1^t, \check{w}_{11}), \\ & \check{x}_{12}(\cdot) = S_2(t_0, \check{x}_{12}^0, \varphi_2^*, \check{w}_{12})\}\,, \end{aligned}$$

where $\left(\begin{pmatrix} \check{x}_{11}^0 \\ \check{x}_{12}^0 \end{pmatrix}, \begin{pmatrix} \check{w}_{11}^0 \\ \check{w}_{12}^0 \end{pmatrix} \right) \in \hat{\Omega}^t$, and the first subindex at $\check{x}$, $\check{w}$ denotes that the worst trajectory is being computed by the first player, the second subindex denotes the variable.

Notice that $\check{x}_{1i}$ and $\hat{x}_i$ (see Remark (5)) are different. As well the $\check{w}_{1i}$ and $\check{\omega}_{1i}$, obtained here, differ from the ones that the second player computes in his or her partial team problem.

Remark 6. *Notice that the components of the vector $(\check{\omega}_{11}, \check{\omega}_{12})$ are not "symmetric" by their physical sense; that is $\check{\omega}_{11}$ is the worst disturbance of the first player based upon his or her information given by $y_1(t)$, whereas $\check{\omega}_{12}$ is the supposed worst disturbance of the second player under the condition that the first player has no information about it. Thus $\hat{\Omega}^t$ defines the set of worst disturbances of the whole system for the first player based upon his or her information.*

Crucial Assumption. For all pairs (ω_i, u_i) and for all $t \in [t_0, T]$, suppose that $\hat{X}_1$ is singleton [9].

The Remark 3 here holds in the corresponding sense.

Theorem 3. *Under the Crucial Assumption above the pair of optimal controllers* $\varphi_1^*(t, \check{x}_{11}, \check{x}_{12}), \varphi_2^*(t, x_1, x_2)$ *solves the partial team problem with noise-corrupted output for the first player. Moreover*

$$\max_{\omega_1 \in \Omega_1} \max_{\omega_2 \in \Omega_2} J(t_0, \varphi_1^*(t, \check{x}_{11}, \check{x}_{12}), \varphi_2^*(t, x_1, x_2), \omega_1, \omega_2) = \max_{X_0 \in X^0} [V(t_0, x_1^0, x_2^0) + N(x_1^0, x_2^0)].$$

Proof. The proof of the theorem strongly relies upon the following fact:

Lemma 2. *: If player 1 uses the control*

$$\check{u}_1(t) = \varphi_1^*(t, \check{x}_{11}(t), \check{x}_{12}(t)),$$

then the function $g(\tau)$ *(25) is non-increasing.*

As previously, the proof is being reduced to the Theorem 5.1 of [3].

Corollary 2. *If for all* $u_1 \in \mathcal{U}$ *there exists* $t^* \in [t_0, T]$*, such that for* $\tau > t^*$ *the problem (25) fails to have a solution and exhibits an infinite supremum, then, if for some other pair of strategies* (φ_1, φ_2) *there exists a finite* $\sup_{\omega_1, \omega_2} J(\varphi_1, \varphi_2, \omega_1, \omega_2)$*, it is larger than* $\max_{X_0}[V + N]$.

Remark 7. *The strategy of the second player, solving similar problem for him or her, will be*

$$\check{u}_2(t) = \varphi_2^*(t, \check{x}_{21}(t), \check{x}_{22}(t)).$$

3.3 The state feedback team problem

3.3.1 Uncorrupted state measurement

Now consider the case when each of the players only has the knowledge of his or her own state, that is the admissible strategies may be:

$$u_1 = \varphi_1(t, x_1), \; u_2 = \varphi_2(t, x_2).$$

(Hence, in the application to robust control, this is the case where the output y is the state variable itself.)

Write, for short, $\hat{\varphi}_1$ for $\varphi_1^*(t, x_1, \hat{x}_2)$ and $\hat{\varphi}_2$ for $\varphi_2^*(t, \hat{x}_1, x_2)$.

The state feedback team problem is the following: *Under this information structure find optimal controls for players-minimizers, guaranteeing*

$$\min_{\varphi_1} \min_{\varphi_2} \max_{\omega_1} \max_{\omega_2} J(t_0, x_1^0, x_2^0, \varphi_1, \varphi_2, w_1, w_2)$$

J being defined by (8).

We need:

Crucial Assumption. The Crucial Assumptions of the partial team problems for players 1 and 2 both hold, and in addition, the functional $(\omega_1, \omega_2) \to J(\hat{\varphi}_1, \hat{\varphi}_2, \omega_1, \omega_2)$ has a unique local and global minimum (e.g., it is quasi-concave).

Technical Assumptions. The Value function in (10) is C^1 and the optimal strategies Φ^* and Ψ^* are continuously differentiable with respect to x_1 and x_2.

Theorem 4. *Under the above two assumptions the pair of control laws* $\hat{\varphi}_1 = \varphi_1^*(t, x_1, \hat{x}_2)$, $\hat{\varphi}_2 = \varphi_2^*(t, \hat{x}_1, x_2)$ *solves the state feedback team problem. Moreover*

$$\max_{\omega_1 \in \Omega_1} \max_{\omega_2 \in \Omega_2} J(t_0, \hat{\varphi}_1, \hat{\varphi}_2, \omega_1, \omega_2) = \max_{X_0 \in X^0} [V(t_0, x_1^0, x_2^0) + N(x_1^0, x_2^0)] \,. \tag{26}$$

Proof Notice first, that following [10], and because in the partial team problem the player with incomplete information has no information whatsoever on the the other's controls or state, then the worst state variables $\hat{x}_i(\cdot)$ are generated by the following equations and initial conditions:

$$\begin{aligned} \dot{\hat{x}}_1 &= f_1(t, \hat{x}_1, \varphi_1^*(t, \hat{x}_1, x_2), \psi_1^*(t, \hat{x}_1, x_2)), \\ \dot{\hat{x}}_2 &= f_2(t, \hat{x}_2, \varphi_2^*(t, x_1, \hat{x}_2), \psi_2^*(t, x_1, \hat{x}_2)), \end{aligned}$$

$$\begin{aligned} \hat{x}_1^0 &= \arg\max\nolimits_{x_1} [V(t_0, x_1, x_2^0) + N(x_1, x_2^0)], \\ \hat{x}_2^0 &= \arg\max\nolimits_{x_2} [V(t_0, x_1^0, x_2) + N(x_1^0, x_2)], \end{aligned} \tag{27}$$

where the $\psi_i^*(\cdot, x_1(\cdot), x_2(\cdot))$, $i = 1, 2$ are the optimal Isaacs policies of disturbances (opposite players) in the case of full information (11).

Consider the following system of differential equations:

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, \varphi_1^*(t, x_1, \hat{x}_2), w_1) \,, \\ \dot{x}_2 = f_2(t, x_2, \varphi_2^*(t, \hat{x}_1, x_2), w_2) \,, \\ \dot{\hat{x}}_1 = f_1(t, \hat{x}_1, \varphi_1^*(t, \hat{x}_1, x_2), \psi_1^*(t, \hat{x}_1, x_2)) =: \hat{f}_1 \,, \\ \dot{\hat{x}}_2 = f_2(t, \hat{x}_2, \varphi_2^*(t, x_1, \hat{x}_2), \psi_2^*(t, x_1, \hat{x}_2)) =: \hat{f}_2 \,, \end{cases} \tag{28}$$

with $\hat{x}_1^0$, $\hat{x}_2^0$ being given by (27), the following performance index:

$$\begin{aligned} J(t_0, \omega_1, \omega_2) &= M(T, x_1(T), x_2(T)) \\ &\quad + \int_{t_0}^{T} L(t, x_1(t), x_2(t), \hat{\varphi}_1, \hat{\varphi}_2, w_1, w_2)\, dt + N(x_1^0, x_2^0) \end{aligned} \tag{29}$$

and the goal

$$\max_{(\omega_1,\omega_2)} J(t_0,\omega_1,\omega_2). \tag{30}$$

The system (28), (29), (30) presents a classical optimal control problem with respect to control variables (w_1, w_2). Under the Technical Assumption it can be investigated via Pontryagin's maximum principle [24]. We need some more notations. Beyond the notations

$$\hat{\varphi}_1 = \varphi_1^*(t, x_1, \hat{x}_2)\,, \quad \hat{\varphi}_2 = \varphi_2^*(t, \hat{x}_1, x_2)$$

already introduced, we shall also let

$$\check{\varphi}_1 = \varphi_1^*(t, \hat{x}_1, x_2)\,, \quad \check{\varphi}_2 = \varphi_2^*(t, x_1, \hat{x}_2)$$

so that the following will be understood:

$$\begin{array}{ll} \frac{\partial\hat{\varphi}_1}{\partial x_1} := \frac{\partial\varphi_1^*}{\partial x_1}(x_1,\hat{x}_2)\,, & \frac{\partial\hat{\varphi}_1}{\partial\hat{x}_2} := \frac{\partial\varphi_1^*}{\partial x_2}(x_1,\hat{x}_2)\,, \\ \frac{\partial\check{\varphi}_1}{\partial\hat{x}_1} := \frac{\partial\varphi_1^*}{\partial x_1}(\hat{x}_1,x_2)\,, & \frac{\partial\check{\varphi}_1}{\partial x_2} := \frac{\partial\varphi_1^*}{\partial x_2}(\hat{x}_1,x_2)\,, \end{array}$$

and similarly, mutatis mutandis, for the partial derivatives of $\hat{\varphi}_2$ and $\check{\varphi}_2$.

Let us write the necessary conditions. They involve the hamiltonian:

$$H(x_1,x_2,\hat{x}_1,\hat{x}_2,\lambda_1,\lambda_2,\hat{\lambda}_1,\hat{\lambda}_2,w_1,w_2) = L + \lambda_1^t f_1 + \lambda_2^t f_2 + \hat{\lambda}_1^t \hat{f}_1 + \hat{\lambda}_2^t \hat{f}_2\,,$$

where $\lambda_i, \hat{\lambda}_i$, $i = 1,2$ are the corresponding adjoint variables for $x_i, \hat{x}_i$, $i = 1,2$. We write the adjoint equations for the maximization problem (30):

$$\left\{\begin{array}{l} -\dot{\lambda}_1^t = \frac{\partial L}{\partial x_1} + \frac{\partial L}{\partial u_1}\frac{\partial\hat{\varphi}_1}{\partial x_1} + \lambda_1^t\frac{\partial f_1}{\partial x_1} + \lambda_1^t\frac{\partial f_1}{\partial u_1}\frac{\partial\hat{\varphi}_1}{\partial x_1} + \hat{\lambda}_2^t\frac{\partial\hat{f}_2}{\partial u_2}\frac{\partial\check{\varphi}_2}{\partial x_1} + \hat{\lambda}_2^t\frac{\partial\hat{f}_2}{\partial w_2}\frac{\partial\psi_2^*}{\partial x_1}\,, \\ -\dot{\hat{\lambda}}_1^t = \frac{\partial L}{\partial u_2}\frac{\partial\hat{\varphi}_2}{\partial\hat{x}_1} + \lambda_2^t\frac{\partial f_2}{\partial u_2}\frac{\partial\hat{\varphi}_2}{\partial\hat{x}_1} + \hat{\lambda}_1^t\frac{\partial\hat{f}_1}{\partial\hat{x}_1} + \hat{\lambda}_1^t\frac{\partial\hat{f}_1}{\partial u_1}\frac{\partial\check{\varphi}_1}{\partial\hat{x}_1} + \hat{\lambda}_1^t\frac{\partial\hat{f}_1}{\partial w_1}\frac{\partial\psi_1^*}{\partial\hat{x}_1}\,, \\ -\dot{\lambda}_2^t = \frac{\partial L}{\partial x_2} + \frac{\partial L}{\partial u_2}\frac{\partial\hat{\varphi}_2}{\partial x_2} + \lambda_2^t\frac{\partial f_2}{\partial x_2} + \lambda_2^t\frac{\partial f_2}{\partial u_2}\frac{\partial\hat{\varphi}_2}{\partial x_2} + \hat{\lambda}_1^t\frac{\partial\hat{f}_1}{\partial u_1}\frac{\partial\check{\varphi}_1}{\partial x_2} + \hat{\lambda}_1^t\frac{\partial\hat{f}_1}{\partial w_1}\frac{\partial\psi_1^*}{\partial x_2}\,, \\ -\dot{\hat{\lambda}}_2^t = \frac{\partial L}{\partial u_1}\frac{\partial\hat{\varphi}_1}{\partial\hat{x}_2} + \lambda_1^t\frac{\partial f_1}{\partial u_1}\frac{\partial\hat{\varphi}_1}{\partial\hat{x}_2} + \hat{\lambda}_2^t\frac{\partial\hat{f}_2}{\partial\hat{x}_2} + \hat{\lambda}_2^t\frac{\partial\hat{f}_2}{\partial u_2}\frac{\partial\check{\varphi}_2}{\partial\hat{x}_2} + \hat{\lambda}_2^t\frac{\partial\hat{f}_2}{\partial w_2}\frac{\partial\psi_2^*}{\partial\hat{x}_2}\,. \end{array}\right. \tag{31}$$

and the following initial and boundary conditions:

$$\left\{\begin{array}{l} \lambda_i^t(t_0) + \frac{\partial N}{\partial x_i}(x_1^0, x_2^0) = 0\,, \\ \lambda_i^t(T) - \frac{\partial M}{\partial x_i}(x_1(T), x_2(T)) = 0\,, \\ \hat{\lambda}_i(T) = 0. \end{array}\right.$$

Pontryagin's principle [24] claims, that the pair of optimal controls $(\hat{w}_1, \hat{w}_2)$, solving the problem (28), (29) (30) yields:

$$\max_{(w_1,w_2)} H(t, x_1, x_2, \hat{x}_1, \hat{x}_2, \lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2, w_1, w_2) = \\ H(t, x_1, x_2, \hat{x}_1, \hat{x}_2, \lambda_1, \lambda_2, \hat{\lambda}_1, \hat{\lambda}_2, \hat{w}_1, \hat{w}_2). \tag{32}$$

Consider the necessary conditions, satisfied according to Pontryagin's principle, to define the maximizing policies of disturbances in the case of complete information. Denote by μ_1, μ_2 the adjoint variables, corresponding to x_1, x_2 in the following system:

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, \varphi_1^*(t, x_1, x_2), w_1)\,, \\ \dot{x}_2 = f_2(t, x_2, \varphi_2^*(t, x_1, x_2), w_2)\,, \end{cases}$$

with the criterion

$$\max_{(\omega_1,\omega_2)} J(t_0, \varphi_1^*(t, x_1, x_2), \varphi_2^*(t, x_1, x_2), \omega_1, \omega_2).$$

From (10) one can conclude that along an optimal trajectory $\mu_1 = \partial V/\partial x_1$, $\mu_2 = \partial V/\partial x_2$. They satisfy the conditions:

$$\begin{cases} -\dot{\mu}_1^t = \frac{\partial L}{\partial x_1} + \frac{\partial L}{\partial u_1}\frac{\partial \varphi_1^*}{\partial x_1} + \mu_1^t \frac{\partial f_1}{\partial x_1} + \mu_1^t \frac{\partial f_1}{\partial u_1}\frac{\partial \varphi_1^*}{\partial x_1}\,, \\ -\dot{\mu}_2^t = \frac{\partial L}{\partial x_2} + \frac{\partial L}{\partial u_2}\frac{\partial \varphi_2^*}{\partial x_2} + \mu_2^t \frac{\partial f_2}{\partial x_2} + \mu_2^t \frac{\partial f_2}{\partial u_2}\frac{\partial \varphi_2^*}{\partial x_2}\,, \end{cases} \tag{33}$$

$$\begin{cases} \mu_i^t(t_0) + \frac{\partial N}{\partial x_i}(x_1^0, x_2^0) = 0\,, \\ \mu_i^t(T) - \frac{\partial M}{\partial x_i}(x_1(T), x_2(T)) = 0\,. \end{cases}$$

Denote also by $x_1^*(\cdot), x_2^*(\cdot)$ the corresponding optimal trajectories. Then we claim the following fact, that proves the theorem:

Proposition 2. *The following solves Pontryagin's necessary conditions (31) to (32):*

$$\begin{array}{ll} x_1(t) = \hat{x}_1(t) = x_1^*(t)\,, & x_2(t) = \hat{x}_2(t) = x_2^*(t)\,, \\ \lambda_1(t) = \mu_1(t), & \lambda_2(t) = \mu_2(t), \\ \hat{\lambda}_1(t) = 0, & \hat{\lambda}_2(t) = 0\,, \\ \hat{w}_1(t) = \psi_1^*(t, x_1^*(t), x_2^*(t))\,, & \hat{w}_2(t) = \psi_2^*(t, x_1^*(t), x_2^*(t))\,. \end{array}$$

Proof It suffices to notice, that with the proposed solution,

$$\frac{\partial L}{\partial u_1} + \lambda_1^t \frac{\partial f_1}{\partial u_1} = 0\,, \qquad \frac{\partial L}{\partial u_2} + \lambda_2^t \frac{\partial f_2}{\partial u_2} = 0\,, \tag{34}$$

so that the differential equations for $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are homogeneous. And their boundary condition being 0, $\hat{\lambda}_i \equiv 0$ is indeed a solution. The rest follows easily: equations for λ_i coincide with those for μ_i, and the proposed w_i indeed maximize the Hamiltonian.

Corollary 3. *If any of the partial team problems fails to have a solution, then the state feedback team problem has no solution, the* $\sup J$ *in (26) being infinite in* (ω_1, ω_2).

Proof From (21), (22), (26) it follows, that

$$\max_{\omega_1\in\Omega_1}\max_{\omega_2\in\Omega_2} J(t_0, \hat{\varphi}_1, \varphi_2^*, \omega_1, \omega_2) = \max_{\omega_1\in\Omega_1}\max_{\omega_2\in\Omega_2} J(t_0, \varphi_1^*, \hat{\varphi}_2, \omega_1, \omega_2) = \max_{\omega_1\in\Omega_1}\max_{\omega_2\in\Omega_2} J(t_0, \hat{\varphi}_1, \hat{\varphi}_2, \omega_1, \omega_2)\,,$$

which proves the claim.

3.3.2 Noise-corrupted information for both players

The above results can be generalized for the case, where each of the players has noise-corrupted information about his or her past states and no information about the other player.

Theorem 5. *Under the same Assumptions as for Theorem 4 the pair of controls* $\check{u}_1(t) = \varphi_1^*(t, \check{x}_{11}(t), \check{x}_{12}(t))$, $\check{u}_2(t) = \varphi_2^*(t, \check{x}_{21}(t), \check{x}_{22}(t))$ *solves the above formulated state feedback team problem, that is:*

$$\max_{\omega_1\in\Omega_1}\max_{\omega_2\in\Omega_2} J(t_0, \varphi_1^*(t, \check{x}_{11}(t), \check{x}_{12}(t)), \varphi_2^*(t, \check{x}_{21}(t), \check{x}_{22}(t)), \omega_1, \omega_2) = \max_{X_0\in X^0}[V(t_0, x_1^0, x_2^0) + N(x_1^0, x_2^0)].$$

Proof According to [10] the $\check{x}_{ij}(t)$ trajectories in this case do not obey the natural equations of dynamics and are presented by more general differential equations. But this turns out not to be crucial, since, in any case, the equations for $\hat{\lambda}_i$ appear to be homogeneous thanks to property (34) (which holds in the case of complete information, and thus, is independent of information structure), and the boundary conditions preserve their form. This crucial argument (that $\hat{\lambda}_i \equiv 0$) finalizes the proof exactly in the same way as in Theorem 4.

Remark 8. *The Corollary 3 is being generalized in the corresponding sense.*

Remark 9. *Notice, that the results obtained in this paper easily carry over to a team of* n *players,* $n > 2$.

4 Conclusion

Comparing the results obtained with the formulation of the robust control problem in section 2, we can conclude that, when the crucial assumptions are satisfied, we have solved the robust stabilization problem in the sense that

- either the Isaacs value (9) in the symmetric team problem with criterion (6) is nonpositive, then the proposed ("certainty equivalent") strategies insure robust stability,
- or this value is positive, and the goal attempted cannot be met, one may try again a less demanding problem, with a larger γ coefficient.

Whether in practical applications this makes any big difference with the L^2-gain minimization usually sought in the litterature on "non linear $\mathcal{H}_\infty$-control" is yet to be investigated. First of all, one has to exhibit practical applications of that classical litterature. As it stands, L^2 gain control does not allow one to achieve guaranteed robust stabilization of a (family of) nonlinear system(s) the way one does with the linear theory. (This is the main point of section 2.)

We do have small nonlinear models of interest in the domain of bioreactors for instance. The available models are fondamentally non-linear (of the Lotka Volterra type, or with Monod's growth law, say). Because the "physics" of the process are ill known, there is no point in writing many equations : one hardly knows what quantities should enter into the model beyond a few fundamental ones. Thus the models remain of small dimension, and there is some hope of being able to exploit the above theory in spite of its relative complexity. But for the same reason, of course, robust control is very important.

One of the authors is currently working on the control of such systems. It is for them that the theory has been developed. The first results we have exploit the same type of ideas as above, although not yet the full fledge theory.

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