Basic System Theory for Singular Implicit Systems

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Abstract. The phrase *implicit systems* refers to systems defined by an implicit, linear differential or difference equation, i.e. of the form $E\dot{x} = Fx + Gu$ or $Ex_{t+1} = Fx_t + Gu_t$, together with an ordinary output equation, say of the form $y = Hx + Ju$. The adjective *singular* in this paper refers to the matrix pencil (E, F) which will be assumed to be singular. For such systems, the state equation may have no solutions, or an infinity, depending on the matrix G and on the control function $u(\cdot)$. Most of the early literature on implicit systems was restricted to regular systems. This paper is a synthesis of our own work, as it relates to what we call basic system theory. We shall therefore omit some results more specialized like system inversion and universal expansion of the singularity in the neighborhood of a singular E matrix, as it appeared in [2] and [4], or some representation results of [1] and [2]. This work was in its major part published in the references [1] to [8]. The latest part is as yet unpublished.

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1. Introduction.

We shall all along consider systems governed by a dynamic equation of the form

$$Ex_{t+1} = Fx_t + Gu_t, \tag{1}$$

or

$$E\dot{x} = Fx + Gu, \tag{2}$$

together with an output equation

$$y = Hx + Gu. \tag{3}$$

Here, $t \in \mathbf{Z}$ for discrete time systems (1)(3), or $t \in \mathbf{R}$ for continuous time systems (2)(3). We shall refer to $x(t)$ or x_t as to the *state* at time t, (although this terminology has been criticized with valid arguments), and we shall have $x \in \mathbf{R}^n$, Ex and $Fx \in \mathbf{R}^r$ (hence E and F are of type $r \times n$), $u \in \mathbf{R}^m$, $y \in \mathbf{R}^p$. Therefore, (1) or (2) is a set of r implict difference or differential equations on the n variables $x(t)$.

Although several of the following results can be extended to time varying systems, we shall assume all along that E, F, G, H, and J are constant real matrices.

It can, with no loss of generality be assumed that the lines of the composite matrix $[E \quad F \quad G]$ are independant. Otherwise, one, at least, of the implicit equations is a linear combination of the others, and can therefore be removed. We shall call r the *rank* of the system, while n will be its *dimension*. We do not assume that $r = n$, nor a fortiori that $(zE - F)$ has a rational inverse. The major consequence, present throughout the paper, is that neither the existence nor the unicity of the solution of (1) or (2) is granted.

Since our investigation is mostly algebraic, it applies to both discrete and continuous time systems, except the last part which was developed for the discrete time only.

1. The causal case.

This part is only concerned with the triple $(E\,F\,G)$.

1.1 Pencil of matrices.

The classical mathematical literature has considered the differential equation $E\dot{x} = Fx$. Clearly, this equation is basically unchanged in the transformation

$$(E, F) \mapsto (PEQ, PFQ)$$

where P and Q are arbitrary nonsingular matrices of appropriate size. The investigation of invariants under this group of transformations has led to the theory of the Kronecker canonical form. We shall not review it in detail here. Let us only recall that it decomposes the (transformed) matrices E and F in blocks of four different types, each associated with important system theoretic properties.

- row Kronecker indices:(blocks η) possible non existence of the solution,
- column Kronecker indices: (blocks ϵ) non unicity,
- infinite invariant factors: (blocks μ) non causal or impulsive behavior,
- finite invariant factors: (blocks λ) classical system behavior.

In the presence of blocks η, existence of a solution requires that the sequence of the PGu_t restricted to the lines in those blocks satisfies some recurence relationship. This will be systematically dealt with in the second part. Otherwise, existence is guaranteed for all $u(\cdot)$ if and only if the lines of PG in the blocks η are all zero.

Furthermore, in the context of system theory, some problems and properties fit naturally with the assumption of causality. Such is the problem of filtering, but also the concept of stability, since it priviledges a direction of time evolution. According to the above decomposition, we will have a causal behaviour if and only if PG also has zero rows in the blocks μ.

This is the situation we analyze now in a more system theoretic language.

1.2 Causality.

Let us begin with the concept of *solution* of (1) or (2) that we need.

DEFINITION 1. A *correspondance of solutions* of (1) or (2) is a point to set mapping from the set $\mathcal{U}$ of input functions to (subsets of) the set $\mathcal{X}$ of state trajectories which to each $u(\cdot) \in \mathcal{U}$ associates a set $S\big(u(\cdot)\big) \subset \mathcal{X}$ such that, $\forall x(\cdot) \in S\big(u(\cdot)\big)$ the pair $\big(x(\cdot), u(\cdot)\big)$ satisfies the implicit dynamic equation.

We shall denote by $\bar{S}\big(u(\cdot)\big)$ the maximal correspondance of solutions, which always exist. Of course it may be empty if the implicit equation has no solution.

Let now $S_\tau\big(u(\cdot)\big)$ be the set of restrictions to $[0,\tau]$ of the trajectories of $S\big(u(\cdot)\big)$. We define causal solutions in the classical way:

DEFINITION 2. Acorrespondance of solutions is said to be *causal* if

$$u_1(t) = u_2(t) \quad \forall t \leq \tau \Rightarrow S_\tau\big(u_1(\cdot)\big) = S_\tau\big(u_2(\cdot)\big).$$

it is said to be *strictly causal* if, in the discrete case

$$u_1(t) = u_2(t) \quad \forall t < \tau \Rightarrow S_\tau\big(u_1(\cdot)\big) = S_\tau\big(u_2(\cdot)\big).$$

and in the continuous case if furthermore, to measurable $u(\cdot)$ corresponds absolutely continuous trajectories.

PROPOSITION 1. There exists a largest causal correspondance of solutions of (1) or (2), and likewise for strictly causal solutions.

Introduce the following natural geometrical object.

DEFINITION 3. We call *characteristic subspace* of the pair (E, F) the largest subspace $\mathcal{V}^*$ of $\mathbf{R}^n$ satisfying the relation

$$F\mathcal{V}^* \subset E\mathcal{V}^*. \tag{4}$$

The space $\mathcal{V}^*$ is akin to (F, G) invariant subspaces of the classical theory and can be characterized via the same type of recursion. We can now state the first theorem which is a system theoretic version of the condition stated in the previous subsection.

THEOREM 1. System (1) or (2) admits strictly causal solutions for all input functions $u(\cdot)$ if and only if

$$\mathrm{Im}G \subset E\mathcal{V}^*, \tag{5}$$

and

$$x_0 \in \mathcal{V}^*. \tag{6}$$

It admits causal solutions if and only if

$$\mathrm{Im}G \subset E\mathcal{V}^* + F\mathrm{Ker}E, \tag{7}$$

$$x_0 \in \mathcal{V}^* + \mathrm{Ker}E. \tag{8}$$

Moreover, in these cases, the maximal causal correspondance of solutions is just $S^* = \bar{S} \cap \mathcal{V}^*$.

1.3. Unicity

Let us now turn to the question of unicity. We have two alternate ways of answering it, one geometric and one algebraic. It turns out to be a property of the sole pair (E, F). Introduce the following definitions.

DEFINITION 4. We call *characteristic kernel* of the pair (E, F) the subspace

$$\mathcal{N} = \mathcal{V}^* \cap \mathrm{Ker}E. \tag{9}$$

DEFINITION 5. We call

• *generalized eigenvalue* any complex λ such that there exists a nonzero complex vector $\xi \in \mathbf{C}^n$ such that

$$(\lambda E - F)\xi = 0. \tag{10}$$

• *essential eigenvalue* any complex λ such that the rank of $(\lambda E - F)$ is less than the generic rank of $(zE - F)$ over $\mathbf{C}$.

These concepts are related in the following way:

PROPOSITION 2. Let $q = \mathrm{dim}\mathcal{N}$, then,

$$\forall z \in \mathbf{C}, \quad \mathrm{rank}(zE - F) \leq n - q,$$

with equality for all z in $\mathbf{C}$ except a finite number of essential eigenvalues.

Hence, the generalized spectrum of the pair (E, F) is either $\mathbf{C}$, if $\mathcal{N} \neq \{0\}$, or finite if $\mathcal{N} = \{0\}$.

DEFINITION 6. The pair (E, F) is said *column regular* (or c-regular) if $\mathcal{N} = \{0\}$.

According to proposition 2, we can also check this property by looking at determinants. (ie without determining $\mathcal{N}$).

PROPOSITION 3. The pair (E, F) is c-regular iff $r \geq n$ and the matrix $(zE - F)$ has at least one nonzero $n \times n$ determinant.

We can now state the result on unicity.

THEOREM 2. The solution of the dynamic equation (1) or (2), if it exists, is unique iff the pair (E, F) is c-regular.

This theorem is the same as the classical one for the implicit equation without the forcing term Gu. As a matter of fact, this follows in a classical way from standard superposition arguments.

We can summarize the above results in terms of determinants.

COROLLARY 1. If $r < n$, $\mathcal{V}^*$ is never trivial and the system never regular.
If $r = n$, $\mathcal{V}^*$ is never trivial, the system is c-regular iff $\det(zE - F) \not\equiv 0$.
If $r > n$, $\mathcal{V}^*$ is non trivial iff the matrix $(zE - F)$ is reducible, i.e. has a common root to all its $n \times n$ determinants. The system is c-regular iff one of these $n \times n$ determinants is not identically zero.

1.4 State space representation.

We have the following representation of all trajectories of such a system, which is usefull in the investigation of further properties. Since we necessarily have $x_k \in \mathcal{V}^*$, we parametrize $\mathcal{V}^*$ by a set of parameters (ξ, v) of appropriate dimension, where v parametrizes $\mathcal{N}$. We then have:

THEOREM 3. For a causal system, there exist matrices A, B, C, D, M, and N such that the following is a representation of all trajectories:

$$\begin{aligned} \xi_{k+1} &= A\xi_k + Bu_k + Cu_k, \\ x_k &= M\xi_k + Nv_k + Du_k. \end{aligned} \tag{11}$$

or its continuous time equivalent. Moreover, if the system is strictly causal, $D = 0$.

Theorem 3 above does not state that the set of matrices A, B, C, D, M, N is unique. It is clearly not. However, this nonunicity admits a simple representation and interesting invariants.

THEOREM 4. The triple (A, B, C) is uniquely defined up to a change of basis in its state space and a feedback on v. The pair (A, C) is entirely characterizd by the Kronecker invariants of the pencil $(zE - F)$ in the following way: its control invariants are the minimal column indices of the pencil and the invariant factors of its uncontollable part are the finite invariant factors of the pencil.

This representation reduces that particular class of implicit systems to perturbed (or two player) control systems. It was suggested in [1] that one might use game-like theory to derive results for these systems. An instance of this possibility is the following fact deduced from capturability theory, and, as far as we know, knew.

A consequence of theorem 3 of [1] is that whatever G and the control used, the nonunicity of the solution of (1) or (2) extends to at least the whole subspace $\mathcal{N}$. We further have;

COROLLARY 2. There exists a state feedback that restricts the nonunicity of the trajectory to the charasteristic kernel $\mathcal{N}$ if and only if $\mathrm{Im}G \supset F\mathcal{N}$.

1.5 Transfer function representation.

From the theorem 3 above follows immediately the necessary part of the following fact.

THEOREM 5. The system (E, F, G) admits (strictly) causal solutions if and only if there exists a (strictly) proper rationnal matrix $K(z)$ such that

$$(zE - F)K(z) = G. \tag{12}$$

Then, all causal solutions are given in laplace transform by

$$X(z) = K(z)U(z) + L(z)V(z), \tag{13}$$

where $L(z)$ is the proper rational matrix of lowest degree such that

$$(zE - F)L(z), \tag{14}$$

and $V(z)$ is an arbitrary power series in z^{-1}.

Conclusion of part 1.

These results are, in essence, taken from [1]. Since then, the geometric theory has been widely developped, noticeably by Ozçaldiran, Banaszuk and others. However, the link between state space and external representations has attracted less attention. This topic is hinted at in the previous section, and is the topic of the next part, without the restriction (5),(6).

2. Realization theory.

2.1 Internal and external representations.

Notice first the following simple fact.

PROPOSITION 4. By an appropriate renaming of variables, a system of the form

$$\sum_{i=0}^{j} A_i x_i + \sum_{i=0}^{k} B_i y_i + \sum_{i=0}^{l} C_i u_i = 0$$

can be cast into the fundamental form (1), and similarly for continuous time systems. Therefore, (1) is a fairly general type of implicit system.

In the same fashion, we shall use the following definition of an implicit system in external form:

DEFINITION 7. An implicit system in external form is given by three rational matrices $\mathcal{K}(z)$, $\mathcal{L}(z)$, $\mathcal{M}(z)$ and the formulas

$$\mathcal{M}U = 0, \tag{15a}$$

$$Y - \mathcal{K}U \in \mathrm{Im}\mathcal{L}, \tag{15b}$$

where $U(z)$ and $Y(z)$ are the Laplace transforms of the vector time functions $u(\cdot)$ and $y(\cdot)$, and the Image operator is to be understood in the space of vector formal power series in z^{-1}.

Again, standard manipulations of matrices ove the field of formal power series yield the following fact:

PROPOSITION 5. Any system of the form

$$\mathcal{A}(z)\begin{pmatrix} U(z) \\ Y(z) \end{pmatrix} = 0$$

where $\mathcal{A}(z)$ is any rational matrix, can be cast into the form (15).

Although this is not qite proper, (see [2]), we shall here call *transfer function* a triple ($\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$). Now, we borrow from Rosenbrock the following definitions.

DEFINITION 8. We call *system matrix* of (1), (3) the matrix

$$S(z) = \begin{pmatrix} zE - F & -G \\ H & J \end{pmatrix} \tag{16}$$

and

DEFINITION 9. Two systems in internal form are said to be *strongly equivalent* if there system matrices are related to each other through a transformation of the form

$$S_2(z) = \begin{pmatrix} U & 0 \\ N & I \end{pmatrix} S_1(z) \begin{pmatrix} V & M \\ 0 & I \end{pmatrix}$$

where U, V, M, and N are constant matrices, the first two square and regular.

2.2. Controllability, observability, and canonicity.

DEFINITION 10. A system of the form (1)(3) or (2)(3) is said to be canonical if the following three conditions hold:

$$F\mathrm{Ker}E \subset \mathrm{Im}E, \tag{17a}$$

$$\forall(\lambda,\mu) \in \mathbf{C}\times\mathbf{C},\ \text{with } (\lambda,\mu) \neq (0,0),$$
$$[\lambda E - \mu F\ G] \text{ is surjective}, \tag{17b}$$

$$\forall(\lambda,\mu) \in \mathbf{C}\times\mathbf{C},\ \text{with } (\lambda,\mu) \neq (0,0),$$
$$\begin{bmatrix} \lambda E - \mu F \\ H \end{bmatrix} \text{ is injective}. \tag{17c}$$

Restriction (17a) seems to be necessary to construct a meaningfull theory. In effect it eliminates *nondynamic variables*, (or blocks of type μ of size one in the Kronecker form).

Properties (17b) and (17c) closely resemble the classical Hautus controllability and observability conditions, and are exactly that, as stated by the following theorem now well known. (As far as we know, it first appear in Grimm [2]).

THEOREM 6. Under hypothesis (17a)

i) if (17b) is satisfied, for any $x_1 \in \mathbf{R}^n$, there exists a control function $u(\cdot)$ and a state trajectory satisfying (1) (or (2)) such that $x(0) = 0$, $x(t_1) = x_1$,

ii) if (17c) is satisfied, if a trajectory satisfies $u(t) = 0$, $y(t) = 0$, $\forall t$, then $x(t) = 0$ along the trajectory.

We must notice the following fact:

PROPOSITION 6. Strong equivalence preserves canonicity.

2.3. Realization.

Two systems are called *equivalent* if they define the same input-output relation. The following fact is easy to see:

PROPOSITION 7. Every system in internal form admits an equivalent representation in external form.

The aim of realization theory is to make precise the converse assertion. An internal form equivalent to a system given in external form will be called a *realization* of the later. We emphasize the following definition:

DEFINITION 11. A realization (or a system in internal form) will be called *minimal* if both the number of rows (the rank) and of columns (the dimension) of E and F are minimal among all realizations of its transfer function.

Contrary to the situation for classical systems where these matrices are square, it is not at all obvious that such a minimal realization should exist. The main theorem of realization theory, (and, we feel, as such of all this basic theory) is the following.

THEOREM 7. Every implicit transfer function admits a minimal realization, which is canonical, and unique up to a strong equivalence. Conversely, every canonical system is minimal.

An immediate but important corollary is as follows:

COROLLARY 3. Every non canonical implicit system admits an equivalent canonical system of lower rank and/or dimension.

References [2] and [4] give explicit reduction procedures.

2.4. Reduced form.

The main tool in [4] is the following reduced form, which is of interest of its own.

THEOREM 8. Up to changes of basis in the input and output spaces, every implicit system in internal form is equivalent to a system in the following *reduced form* (we write it in the continuous time case for simplicity.)

$$\begin{aligned} \dot{x}_1 &= F_{11}x_1 + F_{12}x_2 + G_1u_2, \\ 0 &= F_{21}x_1 + u_1, \\ y_1 &= H_1x_1 + J_1u_2, \\ y_2 &= x_2. \end{aligned} \tag{18}$$

with

$$\left(\begin{pmatrix} F_{21} \\ H_1 \end{pmatrix}, F_{11}\right)$$

completely observable in the classical sense, and

$$\big(F_{11},\ [F_{12},\ G_1]\big)$$

completely reachable in the classical sense again. Moreover, the sizes of the subvectors u_1, u_2, y_1, and y_2 are uniquely determined.

This suggests considering y_2 as an input and u_1 as an output. It naturally leads to the following external reduced form.

COROLLARY 3. Every implicit system admits the following reduced external form, up to a change of basis in input and output spaces:

$$\begin{pmatrix} U_1 \\ Y_1 \end{pmatrix} = \begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{R} & \mathcal{S} \end{pmatrix} \begin{pmatrix} U_2 \\ Y_2 \end{pmatrix} \tag{19}$$

where $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{S}$ are strictly proper rational matrices, and $\mathcal{R}$ is a proper rational matrix.

Clearly, these two forms are related through the formula

$$\begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{R} & \mathcal{S} \end{pmatrix} = \begin{pmatrix} -F_{21} \\ H_1 \end{pmatrix} \left(zI - F_{11}\right)^{-1} \left(G_1 \ F_{12}\right) + \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}. \tag{20}$$

This also strongly suggests that Willems' theory, that does not distinguish a priori input variables from output variables may be the good one to describe implicit systems. Anyway, the present reduced form gives clear answers and a nice parametrization in the existence and unicity issues.

3. Optimal control and estimation theory.

This part of the paper is the only one where discrete and continuous time problems are very different. We deal only with discrete time systems. The extension to time varying systems, though, would be absolutely straightforward, except for the infinite time problems of course.

3.1. Quadratic control.

To system (1), we associate a quadratic function of the control and state sequences:

$$J = x'_N S x_N + \sum_{k=0}^{N-1} \left(x'_k Q x_k + u'_k R u_k\right) \tag{21}$$

where Q and S are symmetric positive semidefinite matrices, and R is symmetric positive definite.

One must be carefull in stating the control problem, since to a given control sequence $\{u_k\}$ may correspond, depending on the system, no state sequence, or an infinity. We therefore use the following formulation (which has been used for ill posed problems in distributed systems)

PROBLEM P. Among all pairs of control and state sequences $\{u_k\}$,$\{x_k\}$ satisfying (1) with x_0 given, find one that minimizes J as given in (21).

We need the following assumption.

HYPOTHESIS H.

$$(E \;\; G) \quad \text{is surjective}$$

$$\begin{pmatrix} E \\ Q \end{pmatrix} \text{ and } \begin{pmatrix} E \\ S \end{pmatrix} \quad \text{are injective}$$

Notice that it is strictly weaker than controllability of (1) and observability with the output Qx and Sx.

The solution always use a version of dynamic programming adapted to implicit control problems, that can be found, together with the other results on quadratic control quoted here, in [5] and [6].

When S is positive definite, this problem admits a solution that resembles very much the classical Riccati equation approach. When S is only semidefinite, we can still give a solution, but it is somewhat more complicated. We deal now with these two cases. We shall after look at the infinite time, so called regulator, problem.

3.2. Finite time case.

Introduce the following *implicit Riccati equation* on the matrix P_k

$$P_k = F'\big(EP_{k+1}^{-1}E' + GR^{-1}G'\big)^{-1}F + Q, \qquad (22a)$$
$$P_N = S. \qquad (22b)$$

Notice that in the case where $E = I$, this equation reduces to an alternate form of the classical discrete Riccati equation. We have the following result.

THEOREM 9. If, beyond hypothesis H, the matrix S is positive definite, equation (22) admits a solution, which is positive definite. Then, Problem P admits a *unique* solution, given by

$$u_k = -R^{-1}G'\left(EP_{k+1}^{-1}E' + GR^{-1}G'\right)^{-1}Fx_k,$$

$$x_{k+1} = P_{k+1}^{-1}E'\left(EP_{k+1}^{-1}E' + GR^{-1}G'\right)^{-1}Fx_k.$$

The optimal value of the performance index is $x_0'P_0x_0$.

If we want to avoid the hypothesis that S is positive definite, we must introduce the following generalized Riccati equation.

$$M_{k+1} = \begin{pmatrix} P_{k+1} & E' \\ E & -GR^{-1}G' \end{pmatrix}, \tag{23a}$$

$$P_k = (O\ F)M_{k+1}^{-1}\begin{pmatrix} P_{k+1} & 0 \\ 0 & GR^{-1}G' \end{pmatrix} M_{k+1}^{-1}\begin{pmatrix} 0 \\ F \end{pmatrix} + Q, \tag{23b}$$

$$P_N = S. \tag{23c}$$

We have in this case the following equivalent to theorem 9 above.

THEOREM 1O. Under hypothesis H, the above equations (23) have a solution, with P_k positive semidefinite (and M_k invertible). Problem P admits a unique solution given by

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = M_{k+1}^{-1}\begin{pmatrix} 0 \\ F \end{pmatrix} x_k, \tag{24a}$$

$$u_k = R^{-1}G'\lambda_{k+1}. \tag{24b}$$

Checking that this solution coincides with the classical one when $E = I$ is slightly less simple than in the previous case. Obviously it is still true, though.

Our hypothesis can be further weakened by raising the restriction that $(Q'\ S')'$ be injective. In that case, we still have a solution, unique except at final time, where several state x_N may be possible. We do not detail this case here.

3.3. The infinite time regulator problem.

We consider now the same cost functional, but summing up to infinity, and thus without the final term in S.

We need an additional definition.

DEFINITION 11. The system (1) is called *stablizable* if,

$$\forall(\lambda,\mu) \in \mathbf{C}^2\backslash(0,0) \text{ such that } |\lambda| \geq |\mu| \qquad (\lambda E - \mu F\ G) \text{ is surjective.}$$

The system (1) with the output $y = Qx$ is called *detectable* if,

$$\forall(\lambda,\mu) \in \mathbf{C}^2\backslash(0,0) \text{ such that } |\lambda| \geq |\mu| \qquad \begin{pmatrix} \lambda E - \mu F \\ Q \end{pmatrix} \text{ is injective.}$$

One can show that these conditions imply stablizability and detectability in the classical sense. We can now state the theorem, which is similar to the classical one, and proved in a similar way.

THEOREM 11. If the system (1) is stablizable and detectable with Q, then the solution of equation (23a)(23b) initialized by $P_0 = Q$ converges as $k \to -\infty$ to a symmetric positive semidefinite $\bar{P}$ solution of the same equations where it is substituted to P_k and P_{k+1}. Substituting $\bar{P}$ in the equations (24) yields the unique solution of the regulator problem.

3.4. The filtering problem.

We consider now a perturbed implicit system, with no control to make things simpler.

$$Ex_{k+1} = Fx_k + v_k, \tag{25}$$

$$y_k = Hx_k + w_k. \tag{26}$$

The initial state x_0 is assumed to be a gaussian random variable of mean $\hat{x}_0$ and of covariance matrix Σ_0, $\{v_k\}$ and $\{w_k\}$ are white gaussian sequences with covariances Q and R respectively.

We are interested in recovering the conditional mean $\hat{x}_k$ of x_k given the measurements y_i up to $i = k$. The results reported here are taken from [5] but otherwise unpublished.

We need the equivalent of hypothesis H above.

HYPOTHESIS H'. $(E\,Q)$ is surjective and $\binom{E}{H}$ is injective.

We also introduce the dual generalized Riccati equation:

$$M_k = \begin{pmatrix} P_k & E \\ E' & -H'R^{-1}H \end{pmatrix}, \tag{27a}$$

$$P_{k+1} = (0\ F)M_k^{-1} \begin{pmatrix} P_k & 0 \\ 0 & H'R_{-1}H \end{pmatrix} M_k^{-1} \begin{pmatrix} 0 \\ F' \end{pmatrix} + Q. \tag{27b}$$

Let x_0 be a gaussian random variable with mean $\hat{x}_0$ and covariance P_0.We have the following form for the optimal filter.

THEOREM 12. Equations (27) initialized with a positive semidefinite P_0 admit a solution with P_k positive semidefinite, that converges to a positive semidefinite $\bar{P}$ as

$k \to \infty$. The optimal estimate (conditional mean) obeys the equations

$$E\hat{x}_{k+1} = (0\ F)M_k^{-1}\begin{pmatrix} E\hat{x}_k \\ -H'R^{-1}y_k \end{pmatrix}$$

initialized at $\hat{x}_0$, with P_k and M_k given by equations (27) initialized at P_0, and the first step replaced by

$$E\hat{x}_1 = F\hat{x}_0 + FP_0\big(R + HP_0H'\big)^{-1}(y_0 - H\hat{x}_0).$$

Again, some work is needed to check that, as they should, the above equations yield the classical Kalman filter equation when $E = I$.

3.5. The complete filter.

The above result was obtained by a classical duality technique, and then transforming the non recursive formulas it gives to the above recursive form. However, it does not allow one to compute the estimate $\hat{x}$. We shall give here recursive formulas that do that. But it seems clear that such formulas have litle chance to exist if the system does not have causal solutions. We shall therefore assume that conditions (5) and (6) of the first part are met. We then have:

PROPOSITION 8. Under conditions H', (5) and (6), the system (25)(26) can be, by an appropriate change of variables, cast into the form

$$x_1(k+1) = F_1x(k) + Vv(k), \tag{28a}$$

$$y_1(k) = H_1x_1(k) + w_1(k) \tag{28b}$$

$$y_2(k) = H_2x_1(k) + x_2(k) + w_2(k) \tag{28c}$$

With this decomposition, introduce the following matrices, which are actually time varying. We omitt the index k for simplicity.

$$\bar{F} = \begin{pmatrix} I \\ H_2 \end{pmatrix} F_1 \qquad \bar{H} = \begin{pmatrix} H_1 \\ O \end{pmatrix} \qquad \bar{K}_k = \begin{pmatrix} K_k & 0 \\ H_2 K_k & I \end{pmatrix}$$

and also

$$\bar{A} = \begin{pmatrix} I \\ -H_2 \end{pmatrix} (I - K_{k+1} H) F_1,$$

$$\bar{B} = \begin{pmatrix} I \\ -H_2 \end{pmatrix} (I - K_{k+1} H) V,$$

$$\bar{C} = \left(-\begin{pmatrix} I \\ H_2 \end{pmatrix} K_{k+1} \quad \begin{pmatrix} 0 \\ -I \end{pmatrix} \right).$$

We can now state the last theorem.

THEOREM 13. The optimal (conditional mean) estimate $\hat{x}$ of the state of system (25)(26) is given, together with the error covariance matrix Σ, by the following equations, initialized at $\hat{x}_0$ and Σ_0 respectively:

$$\hat{x}_{k+1} = \bar{F}\hat{x}_k + \bar{K}_{k+1}\big(y_{k+1} - \bar{H}\hat{x}_k\big),$$

$$\Sigma_{k+1} = \bar{A}\Sigma_k \bar{A}' + \bar{B}Q\bar{B}' + \bar{C}R\bar{C}',$$

where

$$K_{k+1} =$$
$$\big(F_1\Sigma_k F_1' + VQV'\big)H_1'\Big(H_1\big(F_1\Sigma_k F_1' + VQV'\big)H_1' + R_1\Big)^{-1}.$$

Of course, once one has equations for the filter, it is possible, via classical techniques of state augmentation, to solve for the various smoothers one may want.

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