Pierre Bernhard,
Jacob Engwerda,
Berend Roorda, Hans Schumacher,
Vassili Kolokoltsov,
Patrick Saint-Pierre, Jean-Pierre Aubin.

The Interval Market Model in Mathematical Finance: Game Theoretic Methods

July 18, 2012

Springer

Preface

Mathematical finance was probably initiated by Louis Bachelier in 1900, [19]. In his thesis and succeeding contributions, he constructed a stochastic model of stock price processes, essentially inventing the random walk or brownian motion. But this was five years before Einstein investigated the brownian motion, long before Kolmogorov refounded probabilities on sound mathematical grounds, and some basic probabilistic tools were missing. His contribution was labeled un-rigorous, and was consequently not recognized to its true pioneering value.

In contrast, few works in mathematical finance enjoyed the fame, and had the impact, of Black and Scholes'seminal paper [46]. In a bold move, it took the subjective concept of risk aversion out of the rationale for pricing financial derivatives, grounding such pricing on purely objective considerations.

Objective, though, does not mean that no arbitrariness remains. In line with Bachelier, Black and Scholes theory is based upon an arbitrary choice of mathematical, stochastic model for the underlying stock price, that we shall call "Samuelson's model", although some authors trace it back to earlier works.

Samuelson's model is called a "geometric diffusion", or "lognormal distribution". In that model, the price process $S(t)$ is assumed to obey the following Itô stochastic equation:

$$\frac{\mathrm{d}S}{S} = \mu \mathrm{d}t + \sigma \mathrm{d}B\,, \tag{0.1}$$

where μ and σ are known, deterministic parameters or time functions, called "drift" and "volatility" respectively, and $B(\cdot)$ a standard Brownian motion (or Wiener process).

Following these prestigious forerunners, most of the literature in mathematical finance relies on Samuelson's model, although notable exceptions have existed ever since, *e.g.* [117, 56, 57, 76, 134, 86, 66, 109, 123, 125].

The aim of this volume is to report several accomplishments using another class of models, that we call, after [133], "interval models". In these models, if n stocks are considered, it is assumed that a compact convex set of $\mathbb{R}^n$ is known, which always contains the vector of relative stock price velocities (in a continuous time setting) or the one step relative changes in prices (discrete time setting). In the scalar case, corresponding to the classical Black & Scholes problem, and in discrete time, this means that we know two constants $\mathbf{d} < 1$ and $\mathbf{u} > 1$ —notations used here in reference to [57]— such that, for a given $\delta t > 0$, and for all possible price trajectories

$$S(t+\delta t) \in [\mathbf{d}S(t), \mathbf{u}S(t)] \,,$$

a line segment. In contrast, Cox, Ross, and Rubinstein [57] assume that

$$S(t+\delta t) \in \{\mathbf{d}S(t), \mathbf{u}S(t)\} \,,$$

the end points of a line segment. Of course a huge difference in terms of realism, and also of mathematics, even if in some cases, we shall recover some of their results. More generally, in higher dimensional problems, whether discrete or continuous time, this results in a tube of possible tajectories, or "trajectory tube model".

These "interval models" were introduced independently, and almost simultaneously, by the authors of this volume. We only quote here some early papers as a historical record. A common feature is that, remote from the main stream finance literature, they suffered long delays between their original form and their eventual publication, usually not in finance journals. Beyond Roorda, Engwerda and Schumacher [133] already quoted, whose preprint dates back to 2000, let us mention a 1998 paper by Vassili Kolokoltsov [95] and a paper of 2003 only appeared in 2007 [85], a thesis supervised by Jean-Pierre Aubin defended in 2000 [129] —but a published version [17] waited until 2005— and a conference paper by Pierre Bernhard also in 2000 [37], an earlier form of which [35] only appeared in print in 2003.

If probabilities are the *lingua franca* of classical mathematical finance, it could be said that, although probabilities are not ruled out of course, the most pervasive tool of the theories developed in this volume is dynamic game theory, in some form. Most developments to be reported here belong to the realm of robust control, *i.e.* minimax approaches to decision in the presence of uncertainties. These take several forms: the discrete Isaacs' equation, Isaacs and Breakwell's geometric analysis of extremal fields, Aubin's viability approach, Crandall and Lions' viscosity solutions as extended to differential games by Evans and Souganidis, Bardi and others, Frankowska's non smooth analysis approach to viscosity solutions, and geometric properties of risk neutral probability laws and positively complete sets.

As a consequence, we shall not attempt to give here a general introduction to dynamic game theory, as different parts of the book use different approaches. We shall attempt to make each part self contained. Nor did we try to unify the notation, although some of these works deal with closely related topics. As a matter of fact, the developments we report here have evolved, relatively independently, over more than a decade. As a result, they have developed independent consistent notation systems. Merging them at this late stage was close to impossible. We shall provide a short "dictionary" between the notations of the parts 2 to 5.

Part I is just an introductory one, aiming to recall, for the sake of reference, two of the most classical results of dynamic portfolio management: Merton's optimal portfolio and Black and Scholes' pricing theory, each with a flavor more typical of this volume than classical textbooks. The Cox, Ross and Rubinstein model will be presented in detail in part 2, together with the interval model.

Parts II and III mostly deal with the classical problem of hedging one option with one underlying asset. Part II tackles the problem of incompleteness of the interval model, introducing the fair price interval, and an original problem of maximizing the best case profit with a bound on worst case loss. Part III only deals with the seller's price —the upper bound of the fair price interval—, but adding transaction costs, continuous and discrete trading schemes, and the convergence of the latter to the former, for both Vanilla and Digital options. Both parts deal in some respect with the robustness of the interval model to errors in the estimation of the price volatility. Both use a detailed mathematical analysis of the problems at hand: portfolio optimization under a robust risk constraint in part II, the classical option pricing in part III, to provide a "fast algorithm" solving with two recursions on functions of one variable a problem whose natural dynamic programming algorithm would deal with one function of two variables.

It is known that in the approach of Cox, Ross and Rubinstein, the risk neutral probability associated to the option pricing problem spontaneously appears in a rather implicit fashion. Part IV elucidates the deep links between the minimax approach and the risk neutral probability, and exploits this relationship to solve the problem of pricing so called rainbow options and credit derivatives such as C.D.S.

Part V uses the tools of viability theory, and more specifically the guaranteed capture basin algorithm, to solve the pricing problem for complex options. The remarkable fact is that, as opposed to the fast algorithm of part 3, which is specifically tailored to the problem of pricing a classical option, the algorithm used here is general enough that, with some variations, it solves this large set of problems.

There obviously is no claim of unconditional superiority of one model over the other one, or of our theories over the classical ones. Yet, we claim that these theories do bring new insight into the problems investigated. On the one hand, they are less isolated now than they used to be in the early 2000's, as a large body of literature has appeared since then applying robust control methods to various fields including finance, a strong hint that each may have a niche where it is better suited than more entrenched approaches. On the other hand, and more importantly, we share the belief that uniform thinking is not amicable to good science. In some sense, two different —sensible— approaches of the same problem are more than twice better than one, as they may enlighten each other, be it by their similarities or by their contradictions.

Notation dictionary

II	III	IV	V	Part number
T	T	T	T	Exercise time
X	$\mathscr{K}$	K	K	Exercise price
F	M	f	U	Terminal payment
0	$C^{\pm}, c^{\pm}$	β	δ	Transaction costs rates
	$S_0 = RS_0(T)$		S_0	Riskless bonds price
		$j \in \{1, \dots, J\}$		Asset (upper) index
	Continuous time			
	Constants			
	μ_0		r_0	Riskless return rate
	$\tau^- + \mu_0$		$r^\flat$	Min risky asset return
	$\tau^+ + \mu_0$		$r^\sharp$	Max risky asset return
	Time functions			
$t \in [0,T]$	$t \in [0,T]$	$t \in [0,T]$	$t \in [0,T]$	Current time
	$R = S_0/S_0(T)$			End time discount rate
S	$S = Ru$	S	S	Risky asset price
	$\tau + \mu_0$		r	Risky asset return rate
	$XS = Rv$		E	Portfolio exposure
	$v = \varphi^\star(t,u)$		$E = E^\heartsuit(t,W)$	Optimal hedging strategy
	Y		p_0	# of bonds in portfolio
	X		p	# of risky shares in —
	Rw		W	Portfolio worth
	RW		$W^\heartsuit$	Optimal portfolio worth
	(Control)	Impulses	(Triggered)	
	t_k		t^n	Impulse times
	ξ_k		$\psi(x) - x$	Impulse amplitudes
	Discrete time			
	Constants			
h	h	τ	ρ	Time step
n	K	n	N	Total number of steps
	$e^{\mu_0 h}$	$\rho = 1 + r\tau$	$1 + \rho r_0$	One step riskless ratio
d	$1 + \tau_h^-$	d^j	$1 + \rho r_d$	Min one step S ratio
u	$1 + \tau_h^+$	u^j	$1 + \rho r_u$	Max one step S ratio
	Time functions			
$t_j = jh$	$t_k = kh$	m	$t_n = n\rho$	Current time
S_j	$S_k = R_k u_k$	S_m^j	S^n	Risky asset price
v	$1 + \tau_k$	ξ^j	$1 + r_\rho^n$	One step S ratio
γ_j	X_k	γ_m^j		Risky shares in portfolio
$\gamma_j = g_j(S_j)$	$v_k = \varphi_k(u_k)$			Hedging strategy
	$R_k w_k$	X_m	W^n	Portfolio worth

Contents