Robust control approach to option pricing : a representation theorem and fast algorithm

(Robust option pricing: a representation theorem)

Pierre Bernhard
Pierre.Bernhard@essi.fr
tel: +33 (0)492 965 152

Naïma El Farouq
ElFarouq@i3s.unice.fr
tel: +33 (0)476 405 220

Stéphane Thiery
Thiery@i3s.unice.fr
tel: +33 (0)492 942 710

I3S
University of Nice-Sophia Antipolis and CNRS
Les Algorithmes, Euclide
2000 Route des Lucioles
PO Box 121
06903 Sophia Antipolis Cedex
France

March 1, 2005

Abstract

The so-called “interval model” for security prices, together with a robust control approach, allows one to construct a consistent theory of option pricing, including discrete time trading and arbitrary transaction costs. In this context, a new representation theorem for the viscosity solution of the relevant Isaacs (differential) quasivariational inequality leads to simple formulas and fast numerical algorithms to compute a hedging portfolio. We argue that in spite of a less satisfactory market model, the overall theory is not much less realistic than the classical Black and Scholes theory, but rather, it only shifts from the portfolio model to the market model the place where the model is violated when sudden large price changes occur on the market.
Keywords: options pricing, hedging, transaction costs, robust control.
MSC: 49L25, 49N25, 49N70, 91B28

1 Introduction

1.1 The robust control approach to option pricing

In [6, 8], and [10][1], we introduced a robust control approach to option pricing, and more specifically to the design of a hedging portfolio and managment strategy, using the so called "interval model" for the market.

The main claims of that new approach are on the one hand the possibility of constructing a consistent theory of hedging portfolios with either continuous or discrete time trading paradigms, the former being the limit of the later for vanishing time steps, with one and the same market model, and on the other hand to accomodate transaction costs in a natural way, with a non trivial hedging portfolio.

We postpone until the last section the discussion of the drawbacks of the "interval model" as compared to the classical Samuelson geometric diffusion. But we dispell at once one criticism, that it does not make use of probabilistic knowledge on the price trajectories. Indeed, we have shown [6] that the theory of Black and Scholes actually does not need it either, the volatility appearing only as a measure of the (non)regularity of the admissible trajectories.

Here, after summarizing the previous results, we show a new representation of the solution of the problem at hand —and thus of the pricing function— in terms of the solution of a pair of simple coupled first order linear PDEs in two variables. This yields a fast algorithm to compute both the equilibrium price and the hedging strategy, thus alleviating the computational complexity that could heretofore be considered a drawback of that approach.

1.2 Related contributions

Among previous attempts at using this type of model, let us quote the following.

McEneaney [22] attempts to replace the stochastic framework with a robust control approach. He derives the so called "stop loss" strategy for bounded variation trajectories. He also attempts to recover the Black and Scholes theory, but this is done at the price of artificially modifying the portfolio model with no other justification than recovering the Ito calculus and Black and Scholes'PDE.

In [6], we recover both the stop-loss strategy and the complete Black and Scholes theory without any probability in the model, without having to artificially modify the portfolio model, simply by choosing carefully the set of admissible underlying stock price trajectories, and using a weak version of a lemma of Föllmer [19].

Aubin and co-workers [25, 3] have also adopted the robust control approach (they call it "tychastic" approach), with a market model which is a more general version of our model. Saint-Pierre [28] has done efficient implementations of that theory with exactly the interval model that we use below. A similar approach was used by Olsder

[1] a much revised version of that conference paper is currently submitted for publicaton

[24]. And very similar ideas have been developped by Dupire [18] in the context of what he calls “dominance” theory.

The phrase “interval model” we took from Roorda, Engwerda and Schumacher [26, 27] where the authors adopt a viewpoint close to that of robust control.

1.3 Paper outline

In the next section, we present the interval model, both in the continuous trading formalism and in its discretized form, and the portfolio model we adopt, which includes transaction costs and closing costs at will, the final closing being made either in kind or in cash.

Section 3 is devoted to the continuous trading problem. We recall the main results we have obtained so far stressing the case of simple call and put. Next we show a new representation theorem of the solution of the pricing problem. The complete proof of this theorem, and its main use, relies on results of the next section. We also investigate the optimal trading strategies, which have a simple form.

In section 4, we investigate the discrete trading theory. We provide a new very fast algorithm to compute the equilibrium price. And as we have a convergence theorem of the discrete trading equilibrium price toward the continuous trading one as the time step vanishes [10], this is also a discretization algorithm for the continuous problem.

Finally, having displayed what can be achieved with this new model, we discuss in the final section its relative strengths and weaknesses compared with the classical Black and Scholes theory.

2 Interval model

2.0 Riskless interest rate

We asume a fixed, known, riskless interest rate ρ characteristic of that economy. In a classical fashion, all monetary values will be assumed to be expressed in end-time value computed at that fixed riskless rate, so that, without loss of generality, the riskless rate can be taken as (seemingly) zero. (It reappears in the theory of American options, but we have not covered it here for lack of space.)

2.1 Market

We share with Roorda et al. [26, 27] the view that a market model is a set Ω of possible price trajectories. Our model is defined by two real numbers $\tau^- < 0$ and $\tau^+ > 0$, and Ω is the set of all absolutely continuous functions $u(\cdot)$ such that for any two time instants t_1 and t_2,

$$\mathrm{e}^{\tau^-(t_2-t_1)} \leq \frac{u(t_2)}{u(t_1)} \leq \mathrm{e}^{\tau^+(t_2-t_1)} \,. \tag{1}$$

The notation τ^ε will be used to handle both τ^+ and τ^- at a time. Hence, in that notation, it is understood that $\varepsilon \in \{-,+\}$, sometimes identified to $\{-1,+1\}$.

In the continuous trading theory, we shall use the equivalent characterization

$$\dot{u} = \tau u\,, \quad \tau \in [\tau^-, \tau^+]\,. \tag{2}$$

In that formulation, $\tau(\cdot)$ is a measurable function, which plays the role of the "control" of the market. We shall let Ψ denote the set of measurable functions from $[0, T]$ into $[\tau^-, \tau^+]$. It is equivalent to specify a $u(\cdot) \in \Omega$ or a $(u(0), \tau(\cdot)) \in \mathbb{R}^+ \times \Psi$. This is an a priori unknown time function. The concept of non anticipative strategies embodies that fact.

In the discrete trading theory, we shall call h our time step with $T = Kh$, K an integer.[2] The hypothesis (1) translates into[3]

$$u(t+h) \in [\mathrm{e}^{\tau^- h} u(t), \mathrm{e}^{\tau^+ h} u(t)]\,.$$

For convenience, we let

$$u(t+h) = (1 + \tau(t))u(t)\,, \quad \tau(t) \in [\tau_h^-, \tau_h^+] \tag{3}$$

with

$$\tau_h^\varepsilon = \mathrm{e}^{\tau^\varepsilon h} - 1\,, \quad \varepsilon = \pm\,. \tag{4}$$

Alternatively, we shall write, for any integer k, $u(kh) = u_k$, so that (3) also reads

$$u_{k+1} = (1 + \tau_k)u_k\,, \quad \tau_k \in [\tau_h^-, \tau_h^+]\,. \tag{5}$$

and we let Ψ denote the set of such sequences $\{\tau_k\}$.

The case where h goes to zero will be of interest also. But, contrary to the classical limit process in the Cox Ross Rubinstein theory, we keep the underlying continuous time model, hence here τ^+ and τ^-, fixed. Then τ_h^ε behaves as $h\tau^\varepsilon$.

2.2 Portfolio

A (hedging) portfolio will be composed of an amount v (in end-time value) of underlying stock, and an amount y of riskless *bonds*, for a total worth of $w = v + y$. In the normalized (or end-value) representation, the bonds are seemingly with zero interest.

2.2.1 buying and selling

We let $\xi(t)$ be the buying rate (a sale if $\xi(t) < 0$), which is the trader's control. Therefore we have, in continuous time

$$\dot{v} = \tau v + \xi\,. \tag{6}$$

However, there is no reason to restrict the buying/selling rate, so that there is no bound on ξ. To avoid mathematical ill posedness, we explicitly admit "infinite" buying/selling

[2]This K should not be mistaken with the strike, the confusion does not seem possible.

[3]and *not* $u(t+h) \in \{\mathrm{e}^{\tau^- h} u(t), \mathrm{e}^{\tau^+ h} u(t)\}$ as in the Cox Ross Rubinstein theory.

rate in the form of instantaneous block buy or sale of a finite amount of stock, at time instants chosen by the trader together with the amount. Thus the control of the trader also involves the choice of finitely many time instants t_k and trading amounts ξ_k, and the model must be augmented with

$$v(t_k^+) = v(t_k) + \xi_k \,, \tag{7}$$

meaning that $v(\cdot)$ has a jump discontinuity of size ξ_k at time t_k. Equivalently, we may keep formula (6) but allow for impulses $\xi_k \delta(t - t_k)$ in $\xi(\cdot)$.

We shall therefore let $\xi(\cdot) \in \Xi$, the set of real time functions (or rather distributions) defined over $[0, T]$ which are the sum of a measurable function $\xi_c(\cdot)$ and a finite number of weighted translated Dirac impulses $\xi_k \delta(t - t_k)$.

2.2.2 Transaction costs

We assume that there are transaction costs. In this paper, we assume that they are proportional to the transaction amount. But we allow for different proportionality coefficients for a buy or a sale of underlying stock. Hence let C^+ be the cost coefficient for a buy, and $-C^-$ for a sale, so that the cost of a transaction of amount ξ is $C^\varepsilon \xi$ with $\varepsilon = \mathrm{sign}(\xi)$. We have chosen C^- negative, so that, as it should, that formula always gives a positive cost.

We shall use the convention that when we write $C^\varepsilon(expression)$, and except if otherwise specified, the symbol ε in C^ε stands for the sign of the $expression$.

Our portfolio will allways be assumed *self financed*, i.e. the sale of one of the comodities, underlying stock or riskless bonds, must exactly pay for the buy of the other one *and* the transaction costs. It is a simple matter to see that the worth w of the portfolio then obeys

$$\dot{w} = \tau v - C^\varepsilon \xi \,, \tag{8}$$

and at jump instants,

$$w(t_k^+) = w(t_k^-) - C^{\varepsilon_k} \xi_k \tag{9}$$

This is equivalent to

$$w(t) = w(0) + \int_0^t (\tau(s) v(s) - C^\varepsilon \xi(s)) \, \mathrm{d}s - \sum_{k | t_k < t} C^{\varepsilon_k} \xi_k \,. \tag{10}$$

2.2.3 Discrete trading

The discrete trading case can be seen as a sequence of jumps at prescribed time instants $t_k = kh$, $k \in \mathbb{N}$, and leads to (writing u_k, v_k, w_k for $u(kh), v(kh), w(kh)$)

$$\begin{aligned} v_{k+1} &= (1 + \tau_k)(v_k + \xi_k) \,, &(11)\\ w_{k+1} &= w_k + \tau_k (v_k + \xi_k) - C^{\varepsilon_k} \xi_k \,. &(12) \end{aligned}$$

We shall use the explicit form

$$w_n = w_0 + \sum_{k=0}^{n-1} (\tau_k(v_k + \xi_k) - C^{\varepsilon_k}\xi_k)\,. \tag{13}$$

A *dynamic portfolio* will be a pair of time functions $(v(\cdot), w(\cdot))$, whether time is continuous or discrete, also denoted $(\{v_k\}, \{w_k\})_{k\in\mathbb{N}}$ in the later case.

2.3 Hedging

2.3.1 Strategies

Let us assume for simplicity that we always consider that $v(0) = 0$. Then, formally, admissible hedging strategies will be functions $\varphi : \Omega \to \Xi$ which enjoy the property of being nonanticipative :

$$\forall (u_1(\cdot), u_2(\cdot)) \in \Omega \times \Omega, \quad [u_1|_{(0,t)} = u_2|_{(0,t)}] \Rightarrow [\varphi(u_1(\cdot))|_{[0,t]} = \varphi(u_2(\cdot))|_{[0,t]}]$$

(It is understood here that the restriction of $\delta(t - t_k)$ to a closed interval not containing t_k is 0, and its restriction to a closed interval containing t_k is an impulse.)

In practice, we shall find optimal hedging strategies made of a jump at initial time, followed by a state feedback law $\xi(t) = \phi(t, u(t), v(t))$.

In discrete time, the situation is much simpler. We only need a nonanticipative strategy $\varphi : \Omega \to \mathbb{R}^T$ giving $\xi_k = \varphi_k(u_0, u_1, \ldots, u_k)$. Again, we shall find it in the form of a state feedback $\xi_k = \phi_k(u_k, v_k)$.

Yet, these are only nonanticipative laws, the equivalent of stochastic adapted strategies. We have shown in [10] how to handle *strictly* non anticipative strategies, the equivalent of the stochastic predictable strategies.

We shall call Φ the set of admissible trading strategies.

2.3.2 Closing costs

The idea of a hedging portfolio is that at exercize time, the writer is going to close off its position after abiding by its contract, buying or selling some of the underlying stock according to the necessity. We assume that it sustains proportional costs on this final transaction. We allow for the case where these costs would be different from the running transaction costs because compensation effects might lower them, and also to allow for the case without closing costs just by making their rate 0. Let therefore $c^+ \le C^+$ and $-c^- \le -C^-$ be these rates.

It is a simple matter to see that, in order to cover both cases where the buyer does or does not exercize its option, the portfolio worth at final time should be $N(u, v)$ given for a call and a closure in kind by

$$N(u, v) = \max\{c^\varepsilon(-v)\,,\; u - K + c^\varepsilon(u - v)\}\,, \tag{14}$$

where the notation convention for $c^\varepsilon(expression)$ holds. (And we expect that on a typical optimum hedging portfolio for a call, $0 \le v(T) \le u(T)$.)

In the case of a put, we need to replace the above expression by

$$N(u,v) = \max\{c^\varepsilon(-v)\,,\ K - u + c^\varepsilon(-u-v)\}\,, \tag{15}$$

with $-u(T) \le v(T) \le 0$.

The case of a closure in cash is similar, but leads to less appealing mathematical formulas in later developments. The details can be found in [9].

2.3.3 Hedging portfolio

An initial portfolio $(v(0), w(0))$ and an admissible trading strategy φ together with a price history $u(\cdot)$ generate a dynamic portfolio. We set :

Definition 2.1 *An initial portfolio* $(v(0) = 0, w(0) = w_0)$ *and a trading strategy* φ *constitute a* hedge *at* u_0 *if for any* $u(\cdot) \in \Omega$ *such that* $u(0) = u_0$*, (equivalently, for any admissible* $\tau(\cdot)$*), the dynamic portfolio thus generated satisfies*

$$w(T) \ge N(u(T), v(T))\,. \tag{16}$$

Now, we may use (10) at time T to rewrite this

$$\forall \tau(\cdot) \in \Psi\,,\ N(u(T),v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \xi(t)\Big)\,\mathrm{d}t - \sum_k C^{\varepsilon_k}\xi_k - w_0 \le 0\,.$$

This in turn is clearly equivalent to

$$w_0 \ge \sup_{\tau(\cdot)\in\Psi}\left[N(u(T),v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \xi(t)\Big)\,\mathrm{d}t - \sum_k C^{\varepsilon_k}\xi_k\right]\,.$$

We further set

Definition 2.2 *The* equilibrium price *of the option at* u_0 *is the worth of the cheapest hedging portfolio at* u_0*.*

The equlibrium price at u_0 is therefore

$$\inf_{\varphi\in\Phi}\sup_{\tau(\cdot)\in\Psi}\left[N(u(T),v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \xi(t)\Big)\,\mathrm{d}t - \sum_k C^{\varepsilon_k}\xi_k\right], \tag{17}$$

where it is understood that $v(0) = 0$, and that $\xi(\cdot) = \varphi(u_0, \tau(\cdot))$.

In the case of discrete trading, we get similarly as the equilibrium price at u_0

$$\min_{\varphi\in\Phi}\sup_{\{\tau_k\}\in\Psi}\left[N(u_K, v_K) + \sum_{k=0}^{K-1}\Big(-\tau_k(v_k + \xi_k) - C^{\varepsilon_k}\xi_k\Big)\right]\,. \tag{18}$$

3 Continuous trading

3.1 The differential game

We are therefore led to the investigation of the impulse control differential game whose dynamics are given by (2,6,7), and the criterion by (17). In a classical fashion we introduce its Isaacs Value function

$$W(t,u,v) = \inf_{\varphi\in\Phi}\sup_{\tau(\cdot)\in\Psi}\left[N(u(T),v(T))+\int_t^T\Big(-\tau(s)v(s)+C^{\varepsilon}\xi(s)\Big)\mathrm{d}s+\sum_{k|t_k\geq t}C^{\varepsilon_k}\xi_k\right] \tag{19}$$

where the dynamics are integrated from $u(t) = u$, $v(t) = v$. Hence the equilibrium price is $W(0, u(0), 0)$.

There are new features in that game, in that, on the one hand, impulse controls are allowed, hence a Isaacs quasi-variational inequality (or QVI, see Bensoussan and Lions (1982)) should be at work, but on the other hand, impulse costs have a zero infimum. As a consequence, that QVI is degenerate, and no general result is available. In Bernhard et al. (2002), we introduce the so-called “Joshua transformation” that let us show the following fact:

Theorem 3.1 *The function W defined by (19) is a continuous viscosity solution of the following “differential quasi-variational inequality”:*

$$\begin{aligned}
0 = \min\Bigg\{&\frac{\partial W}{\partial t}+\max_{\tau\in[\tau^-,\tau^+]}\tau\left[\frac{\partial W}{\partial u}u+\left(\frac{\partial W}{\partial v}-1\right)v\right],\\
&\frac{\partial W}{\partial v}+C^+\,,\quad -\left(\frac{\partial W}{\partial v}+C^-\right)\Bigg\},\\
W(T,u,v) = {}& N(u,v)\,.
\end{aligned} \tag{20}$$

This PDE in turn lends itself to an analysis, either along the lines of the Isaacs-Breakwell theory through the construction of a field of characteristics (see [10]), or using the theory of viscosity solutions and the representation theorem as outlined hereafter. The solution we seek is further charcaterized by its behaviour at infinity. Yet its uniqueness does not derive from the classical results on viscosity solutions. We must therefore rely on the fact that the viscosity solution we exhibit has the necessary regularity for the Isaacs Breakwell theory to apply, charctarizing it as the value of the game.

3.2 Simple call or put

3.2.1 Equilibrium price

We give here a new theory of that equation (20). We introduce two functions $\check{v}(t,u)$, a representation of the singular manifold, and $\check{w}(t,u)$, the restriction of W to that manifold, handled jointly as

$$\mathcal{V}(t,u) = \begin{pmatrix}\check{v}(t,u)\\ \check{w}(t,u)\end{pmatrix}.$$

That pair of functions is entirely defined by a linear partial differential equation that involves the following two matrices (q^- and q^+ are defined hereafter (22)):

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\,, \text{ and } \mathcal{T} = \frac{1}{q^+ - q^-}\begin{pmatrix} \tau^+ q^+ - \tau^- q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-)q^+ q^- & \tau^- q^+ - \tau^+ q^- \end{pmatrix},$$

and which seems to play a very important role in the overall theory. Namely:

$$\mathcal{V}_t + \mathcal{T}(\mathcal{V}_u u - \mathcal{S}\mathcal{V}) = 0\,. \tag{21}$$

The definitions of q^+ and q^-, as well as the terminal conditions at T for (21), depend on the type of option considered. For a simple call or put, and a closure in kind, we have

$$\begin{aligned} q^-(t) &= \max\{(1+c^-)\exp[\tau^-(T-t)] - 1\,,\ C^-\}\,, \\ q^+(t) &= \min\{(1+c^+)\exp[\tau^+(T-t)] - 1\,,\ C^+\}\,. \end{aligned} \tag{22}$$

Notice that $q^\varepsilon = C^\varepsilon$ for $t \geq t_\varepsilon$, with

$$T - t_\varepsilon = \frac{1}{\tau^\varepsilon}\ln\frac{1+C^\varepsilon}{1+c^\varepsilon}\,, \tag{23}$$

The terminal conditions are given, for a call, by

$$\mathcal{V}^t(T,u) = \begin{cases} (0 \quad 0) & \text{if } u < \dfrac{K}{1+c^+}\,, \\ \dfrac{(1+c^+)u - K}{c^+ - c^-}(1 \quad -c^-) & \text{if } \dfrac{K}{1+c^+} \leq u < \dfrac{K}{1+c^-}\,, \\ (u \quad u-K) & \text{if } u \geq \dfrac{K}{1+c^-}\,. \end{cases} \tag{24}$$

and symmetric formulas for a put. (All combinations Call/Put, closure in kind/in cash are detailed in [9])

We claim the following fact (first conjectured in [7])

Theorem 3.2 *The function W defined by (19) is given by*

$$W(t,u,v) = \check{w}(t,u) + q^\varepsilon(\check{v}(t,u) - v)\,, \quad \varepsilon = \operatorname{sign}(\check{v} - v)\,, \tag{25}$$

where q^ε is given by formula (22) (for a simple call or put), and $(\check{v}\ \check{w}) = \mathcal{V}^t$ is given by (21) and the terminal conditions (24) for a call (and symmetrical formulas for a put).

The proof is done by checking that the function (25) is indeed the viscosity solution of (20). The complete proof is rather lengthy as it involves checking the viscosity condition on many manifolds where ∇W may be discontinuous. The main ingredients of the proof are given in appendix.

It can also be shown that the solution of (21) is non trivial only in the region where the option may end either in the money or out of the money, i.e. the region

$$K\mathrm{e}^{-\tau^+(T-t)} \leq u \leq K\mathrm{e}^{-\tau^-(T-t)}\,. \tag{26}$$

Outside of this region, it keeps the form of the terminal condition.

Let us add that numerical integration supports that claim with great accuracy.

Corollary 3.2.1 *The equilibrium price of a call is $\check{w}(0,u_0) + q^+(0)\check{v}(0,u_0)$, with $\check{v}$ and $\check{w}$ initialized as in (24). (And symmetrically for a put.)*

The general appearance of the equilibrium price as a function of $u(0)$ is very similar to that of the cassical Black and Scholes theory (a theorem of the next section will make that clearer), but of course slightly larger because of the transaction costs. Some curves are published in [6]

3.2.2 Optimal trading strategy

The optimal trading strategy is $\xi = 0$ (do no trading) as long as $w \geq W(t,u,v)$. When $w = W(t,u,v)$, it is defined in terms of $\varepsilon = \mathrm{sign}(\check{v}(t,u) - v)$ and is $\xi = 0$ if $t \geq t_\varepsilon$, a positive jump towards $\check{v}$ if $\varepsilon = +1$ and $t < t_+$, a negative jump towards $\check{v}$ if $\varepsilon = -1$ and $t < t_-$. On the manifold $v = \check{v}$, we have shown that there is a control, depending on τ, that keeps $w(t)$ on or "above" the graph of W.

The dependence of the control $\xi(t)$ on the instantaneous rate $\tau(t)$ is undesirable. It is not implementable as such and is not an admissible causal strategy. (Accepting such strategies would create arbitrage opportunities.) However, the convergence theorem of [10], recalled in the next section provides a practical solution: use the discrete time theory with whatever time step is feasible. It gives an exact (within our model) admissible hedging strategy for a portfolio value close to the optimum one.

4 Discrete trading

4.1 The multistage game

In the case of discrete trading, we have to investigate the game whose dynamics are given by equations (5),(11) and the criterion by (18). This is a completely classical dynamic game. Let $W^h(kh,u,v) = W^h_k(u,v)$ be its Isaacs Value function. We immediatly obtain its Isaacs equation and the theorem

Theorem 4.1 *The Value function W^h is given by the recursion*

$$\begin{aligned}
&\forall k < K, \forall (u,v)\,, \quad W^h_k(u,v) = \\
&\min_{\xi} \max_{\tau \in [\tau_h^-, \tau_h^+]} [W^h_{k+1}((1+\tau)u, (1+\tau)(v+\xi)) - \tau(v+\xi) + C^\varepsilon \xi], \\
&\forall (u,v)\,, \quad W^h_K(u,v) = N(u,v)\,.
\end{aligned} \tag{27}$$

Finally, the main theorem of [10], and a central result in that theory, is the following convergence theorem. Let $W^h(t,u,v)$ be the function obtained by linear interpolation in time between $W^h_k(u,v)$ and $W^h_{k+1}(u,v)$ with $t \in [kh, (k+1)h]$.

Theorem 4.2 *The functions W^h converge uniformly on every compact towards the function W (of the continuous trading theory) when the step h goes to zero (in a dyadic fashion : $h = T/(2^n)$, $n \to \infty$).*

Optimal hedging strategy An important consequence of this theorem is that, even if we are almost in a "continuous trading" situation, the optimal portfolio and trading strategy can be approached by a discrete trading strategy. However, the optimal discrete trading strategy does *not* make use of τ_k to compute ξ_k. Thus alleviating the problem of the dependence of the optimal strategy on τ in the continuous time theory.

As a matter of fact, one computes a sequence of $\check{v}_k^h(u)$ (see next paragraph), and let $\varepsilon = \operatorname{sign}(\check{v}_k^h(u_k) - v_k)$. The optimal discrete time hedging strategy is just to do nothing if $t_k \geq t_\varepsilon$, (see (23)) —but for most realistic value of the parameters, this is immaterial because $T - t_\varepsilon < h$—, and for all other discrete time instants, jump to $v = \check{v}_k^h(u_k)$, which therefore plays the role of an optimum portfolio composition.

4.2 A fast algorithm

We propose here a new fast algorithm to compute the solution of (27), which, in view of the above theorem (4.2), also yields a fast algorithm to approximate a solution of the continuous trading problem. It can be viewed as a particular difference scheme for (21).

Define the following recursion,

$$\begin{array}{rcl} q_K^\varepsilon & = & c^\varepsilon\,, \\ q_{k+\frac{1}{2}}^\varepsilon & = & (1+\tau_h^\varepsilon) q_{k+1}^\varepsilon + \tau_h^\varepsilon\,, \\ q_{k+1}^\varepsilon & = & \varepsilon \min\{\varepsilon q_{k+\frac{1}{2}}^\varepsilon\,, \varepsilon C^\varepsilon\}\,, \end{array} \tag{28}$$

and let, for every integer ℓ:

$$Q_\ell^\varepsilon = \begin{pmatrix} q_\ell^\varepsilon & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{V}_\ell^h(u) = \begin{pmatrix} \check{v}_\ell^h(u) \\ \check{w}_\ell^h(u) \end{pmatrix}. \tag{29}$$

Take $\check{v}_K^h(u) = \check{v}(T,u)$, $\check{w}_K^h(u) = \check{w}(T,u)$ as given by (24) for a call (symmetrically for a put) and

$$\mathcal{V}_k^h(u) = \frac{1}{q_{k+\frac{1}{2}}^+ - q_{k+\frac{1}{2}}^-} \begin{pmatrix} 1 & -1 \\ -q_{k+\frac{1}{2}}^- & q_{k+\frac{1}{2}}^+ \end{pmatrix} \begin{pmatrix} Q_{k+1}^+ \mathcal{V}_{k+1}^h((1+\tau^+)u) \\ Q_{k+1}^- \mathcal{V}_{k+1}^h((1+\tau^-)u) \end{pmatrix}. \tag{30}$$

We leave to the reader the tedious, but straightforward, task to check that this is indeed a consistent finite difference scheme for (21).

We claim :

Theorem 4.3 *The solution of (27) is given by (28),(29),(30), and (24) for a call, as*

$$W_k^h(u,v) = \check{w}_k^h(u) + q_k^\varepsilon(\check{v}_k^h(u) - v) = Q_k^\varepsilon \mathcal{V}_k^h(u) - q_k^\varepsilon v, \quad \varepsilon = \operatorname{sign}(\check{v}_k^h(u) - v).$$

The proof is given in appendix, together with that of the equivalent "continuous" theorem 3.2.

Corollary 4.3.1 *The equilibrium price of a call is $Q_0^+ \mathcal{V}_0^h(u_0)$ with $\mathcal{V}_K^h$ initialized as in (24), (A symmetric form holds for a put.)*

The important fact, of course, is that we now have two sequences of functions of one variable to compute, $\{\check{v}_k^h(\cdot)\}$ and $\{\check{w}_k^h(\cdot)\}$, instead of one sequence of functions of two variables $\{W_k^h(\cdot,\cdot)\}$. A major saving in computer time and memory. We have typically discretized u and v with 300 to 3000 points each. Therefore the saving is in a ratio of 1:100 to 1:1000. This algorithm has been programmed[4]. The results were indeed identical to those obtained with the straightforward discretization of Isaacs'equation, but much faster and with the above reduction in memory space.

5 Discussion

5.1 Interval vs Samuelson's model

We wish to discuss here the strengths and weaknesses of this new theory as compared to the classical Black and Scholes theory [11] and related work.

Clearly, a major weakness of our model is that it rules out from the start very fast price variations in the market. If we try to take τ^- and τ^+ so large that the model be (essentially) always satisfied, then we will end up with too large a price. This is a classical fact that because our market model is incomplete we have to resort to super-replication, potentially ending up with an unrealistically large price. A way around that drawback is to choose a market model $[\tau^-,\tau^+]$ not too large, but then it will be violated from time to time. If this does not happen too often, the loss may be compensated by the gains accrued each time τ falls in the admissible interval. One needs to reintroduce probabilities to investigate that tradeoff, which we are in the process of doing.

Now, the Black and Scholes theory has its own shortcomings. On the one hand, it fundamentally assumes that trading is continuous and with no delay. It is impossible, within Samuelson's model, to achieve hedging if the trading is not done continuously, except with the trivial —and too expensive— portfolio $v = u$. On the other hand, within Samuelson's model, "there is no non trivial hedging portfolio for option pricing with transaction costs" ([30]). The first problem arises from the fact that Samuelson's model may display arbitrarily large variations in any finite time, the second from the closely related fact that it has almost surely trajectories of unbounded total variation.

Let us concentrate on the continuous versus discrete trading issue. Real trading has to be discrete, forcing a discrepancy between real trading and the Black and Scholes theory. This is of little consequence as long as the underlying's price does not change too quickly. But when it does, then that discrepancy becomes potentially fatal.

Hence both theories fail under the same circumstances : when there are unusually fast variations of the price of the underlying stock on the market. In our theory the market model is violated, in Black and Scholes', it is the portfolio model which fails.

It is impossible to reconcile a model that allows for arbitrarily large stock price variations within one time step with discrete time hedging. Hence a mathematical theory has to give up one of the two features. Black and Scholes gives up the ability to do discrete trading. We wanted to develop a theory of discrete trading, the discrete time market model being consistent with (i.e. the time sampling of) a continuous time underlying market model, kept fixed as the time step is decreased. Thus we had to give

[4]by Laurent Fischer and Loïc Maitrehut, students at ESSI whose contribution we acknowledge

up a model that would allow for arbitrarily large price variations in one step of time. Yet we wanted a model less idealized than that of Cox, Ross and Rubinstein, —and not dependant on the time step—, at the price of giving up market completeness. Thus we were forced to invent the interval model.

The fact that it gives rise to a rather nice mathematical theory, and as a result to very fast algorithms, was an unforeseen property.

The coincidence with the Cox Ross Rubinstein theory in the absence of transaction costs guarantees a reasonable premium curve. However, the ability of the new theory to handle transaction costs, even in the limit as the time step vanishes, is a distinct advantage. Let us now turn to that point.

5.2 Transaction costs

We get the ability to add transaction costs as an added benefit, because having no probability law on our space of trajectories that the trader could exploit, we may keep price trajectories with bounded total variation without creating arbitrage opportunities. (This may not be a very desirable possibility in itself : despite their artificial character, unbounded total variation price trajectories have some appeal as metaphors of real price histories.)

Indeed, authors have been able to deal with vanishing transaction costs, thanks to the concept of diffusion limits as in, e.g., [1]. This allowed them to treat small transaction costs in continuous as well as discrete trading. But other than vanishingly small transaction costs have been known to be incompatible with a non trivial hedging portfolio since the paper [30]. This result had been conjectured for some time. Later papers such as [15, 16] gave simpler and more general proofs, but that do not get around the basic fact proclaimed in the title of [30].

In [4], a paradigm different from Merton's replicating portfolio is used to define the equilibrium premium of a contingent claim. This also leads to super-replication, but a more serious drawback then is that the premium computed depends on the composition of the writer's overall portfolio. This last approach is very similar to the related topic of portfolio optimization and consumption (rather than hedging), for which more is known about the role of transaction costs. See, e.g. [17, 29, 20].

In our model, transaction costs fit naturally. As a matter of fact, they are in some sense a necessary ingredient, as the problem trivializes without them, leading to the naive "stop loss" hedging strategy, the equilibrium price being then the parity price. This, in some sense, may point to a weakness of our theory: only the presence of tansaction costs keeps it away from overly simplistic solutions.

5.3 Conclusion

A carefull analysis shows that it is rather natural to resort to such "interval models", and this explains why several authors developed that idea independently. To this remark, we add that for the strict problem of hedging a contingent claim, the robust control approach, also used by several of these authors, whether explicitly or implicitly, allows us to proceed without endowing the set of admissible stock price trajectories with a probability law. This is so since what is sought is a hedge for *every* possible trajectory.

(And this remark carries over to the Black and Scholes theory if one carefully picks the set of admissible trajectories, as shown in [6].)

The resulting theory has a nice mathematical structure, that can be exploited to get semi-explicit formulas via a very fast algorithm for equilibrum prices in the presence of transaction costs, whether in discrete trading or continuous trading. The later is the limit of the former with vanishing step size, this, we stress, keeping *the same* continuous time model for the underlying price trajectories. Thus the discrete trading strategy, which is very simple to implement, is a good approximation of the theoretical continuous strategy.

The equilibrium prices computed qualitatively and quantitatively resemble the Black and Scholes prices, although the presence of transaction costs makes them larger.

References

[1] H. AHN, M. DAYAL, E. GRANNAN, AND G. SWINDLE (1998): "Option replication with transaction costs: general diffusion limits" *Annals of Applied Probability* **8** pp. 676-707.

[2] P. ARTZNER, F. DELBAEN, J-M. EBER, AND D. HEATH (1999): "Coherent Measures of Risk", *Mathematical Finance* **9**, pp 203–228.

[3] J-P. AUBIN, D. PUJAL, AND P. SAINT-PIERRE (2001): *Dynamic Management of Portfolios with Transaction Costs under Tychastic Uncertainty*, preprint.

[4] G. BARLES AND H.M. SONER (1998): "Option Pricing with transaction Costs and a Nonlinear Black and Scholes Equation", *Finance and Stochastics* **2**, pp 369–397.

[5] A. BENSOUSSAN ET J.L. LIONS (1982): *"Contrôle impulsionnel et inéquations quasi variationnelles*, Dunod, Paris.

[6] P. BERNHARD (2003 a): "Robust Control Approach to Option Pricing", in M. Salmon ed.: *Applications of Robust Decision Theory and Ambiguity in Finance* City University Press, London.

[7] P. BERNHARD (2003 b): "An Explicit Solution of Quasi-variational Inequality Arising in Finances", 4th ISDG workshop, Goslar, Germany, May, 19–21.

[8] P. BERNHARD (2004): "A Robust Control Approach to Option Pricing Including Transaction Costs", in A. Nowak ed.: *Annals of the ISDG* **7**, Birkhauser.

[9] P. BERNHARD (2005): "The robust control approach to option pricing and interval models: an overview", in G. Zaccour ed. : GERAD's anniversary volume, to appear.

[10] P. BERNHARD, N. EL-FAROUQ AND S. THIERY (2002): "An Impulsive Differential Game Arising in Finance with Interesting Singularities", 10th ISDG international symposium on Dynamic Games and Applications, St Petersburg, Russia, July 8–11.

[11] F. BLACK AND M. SCHOLES (1973): "The Pricing of Options and Corporate Liabilities", *Journal of Political Economy*, **81**, pp 637–659.

[12] I. CAPUZZO DOLCETTA (1983): "On a Discrete Approximation of the Hamilton Jacobi Equation of Dynamic Programming", *Applied Mathematics and Optimization* **10**, pp 367–377.

[13] P. CARDALIAGUET AND S. PLASKACZ (2000): "Invariant Solutions of Differential Games and Hamilton-Jacobi Equations for Time-measurable Hamiltonians". *SIAM J. Control and Optimization* **38**, pp: 1501-1520.

[14] J.C. COX, S.A. ROSS, & M. RUBINSTEIN (1979): "Option pricing : a simplified approach", *Journal of Financial Economics*, **7**, pp 229–263.

[15] J. CVITANIĆ AND I. KARATZAS (1996): "Hedging and Portfolio Optimization under Transaction Costs: a Martingale Approach", *Mathematical Finance*, **6**, pp 133–165.

[16] J. CVITANIĆ, H. PHAM, AND N. TOUZI (1999): "A Closed Form Solution to the Problem of Super-replication under Transaction Costs", *Finance and Stochastics* **3**, pp 35–54.

[17] M.H.A. DAVIS AND A. NORMAN (1990): "Portfolio selection with transaction costs", *Mathematics of O.R.*, **15**, pp 676–713.

[18] B. DUPIRE (2003): "Résultats de dominance", seminar at University of Nice, may 14.

[19] H. FÖLLMER (1981): "Calcul d'Ito sans probabilités", in J. Azéma and M. Yor ed.: *Séminaire de probabilités XV*, Sringer Lecture Notes in Mathematics 850, Springer, Berlin, pp 143–151

[20] N.C. FRAMSTAD, B. OKSENDAL, A. SULEM (2001): "Optimal Consumption and Portfolio in a Jump Diffusion Market with Proportional Transaction Costs", *Journal of Mathematical Economics* **35**, pp. 233–257.

[21] I. GILBOA AND D. SCHMEIDLER (1989): "Maxmin Expected U tility with Non Unique Prior", *Jal of Mathematical Economics*, **18**, pp 141–153.

[22] W.M. MCENEANEY (1997): "A Robust Control Framework for Option Pricing", *Maths of Operations Research*, **22**, pp 22–221.

[23] A. MELIKYAN AND P. BERNHARD (2003): "Geometry of Optimal Paths around a Focal Singular Surface in Differential Games", *Applied Mathematics and Optimization*, to appear, 2005.

[24] G-J. OLSDER (2000): "Control-theoretic Thoughts on Option Pricing", *International Game Theory Review*, **2**, pp 209–228.

[25] D. PUJAL (2000): *Évaluation et gestion dynamiques de portefeuilles*, Thesis, Paris Dauphine University, 2000.

[26] B. ROORDA, J. ENGWERDA, AND H. SCHUMACHER (2000): "Performance of Hedging Strategies in Interval Models", preprint.

[27] B. ROORDA, J. ENGWERDA AND H. SCHUMACHER (2003): "Coherent acceptability measures in multiperiod models", preprint. (http://center.uvt.nl/staff/schumach/pub/arch/coh_rev.pdf).

[28] P. SAINT-PIERRE (2004): "Viable Capture Basin for Studying Differential and Hybrid Games", *International Game Theory Review*, **6**, pp 109–136.

[29] S.E. SCHREVE AND H.M. SONER (1994): "Optimal Investment and Consumption with Transaction Costs", *Annals of Applied probability*, **4**, pp 609–692.

[30] H.M. SONER, S.E. SHREVE, AND J. CVITANIC (1995): "There Is No Nontrivial Hedging Portfolio for Option Pricing with Transaction Costs", *The Annals of Applied Probability*, **5**, pp 327–355.

[31] P. VARAIYA (1967): "On the Existence of Solutions to a Differential Game", *SIAM Jal on Control* **5**.

A Appendix : proof of theorems 3.2 and 4.3

A.1 Theorem 4.3

We make the proof in the case of a call. The argument for a put is completely similar.

It is useful to notice an alternate, "two-stage" form of the recursion (27) :

$$W^h_{k+\frac{1}{2}}(u,v) = \max_{\tau\in[\tau_h^-,\tau_h^+]}[W^h_{k+1}((1+\tau)u,(1+\tau)v)-\tau v]\,, \tag{31}$$

$$W^h_k(u,v) = \min_{\xi}[W^h_{k+\frac{1}{2}}(u,v+\xi)-C^\varepsilon\xi]\,. \tag{32}$$

This form shows that the convexity of N is preserved, and the W^h_k are convex.[5]

Note that the formula of the theorem is correct at final time, $k=K$. Assume it is correct at time $k+1$. Consider the step (31). Because W^h_{k+1} is convex, the maximum is reached either at τ_h^- or at τ_h^+. For each u, the function to be maximized in τ is piecewise affine in v, its graph as a function of v can be represented as two wedges with one branch sloping downwards (see picture), one for each τ^ε. These can be written as

$$W^+_{k+\frac{1}{2}} := \check{w}^+_{k+\frac{1}{2}} + q^\varepsilon(\check{v}^+_{k+\frac{1}{2}} - v)\,,$$
$$W^-_{k+\frac{1}{2}} := \check{w}^-_{k+\frac{1}{2}} + q^\varepsilon(\check{v}^-_{k+\frac{1}{2}} - v)\,,$$

where $\check{v}^+_{k+\frac{1}{2}}$, $\check{v}^-_{k+\frac{1}{2}}$, $\check{w}^+_{k+\frac{1}{2}}$, and $\check{w}^-_{k+\frac{1}{2}}$ are easily written in terms of $\check{v}^h_{k+1}$ and $\check{w}^h_{k+1}$ evaluated at $(1+\tau^+)u$ and $(1+\tau^-)u$.

As a result, $\check{v}_k$ is obtained as the abscissa of the intersection of the two wedges in this graph. (In the figure, $\check{v}^\varepsilon$ stands for $\check{v}^\varepsilon_{k+\frac{1}{2}} = \check{v}_{k+1}((1+\tau^\varepsilon)u)/(1+\tau^\varepsilon)$, $\varepsilon=\pm$.)

Now, we claim the following fact:

[5] Hence, from the convergence theorem, so is $W(t,\cdot,\cdot)$.

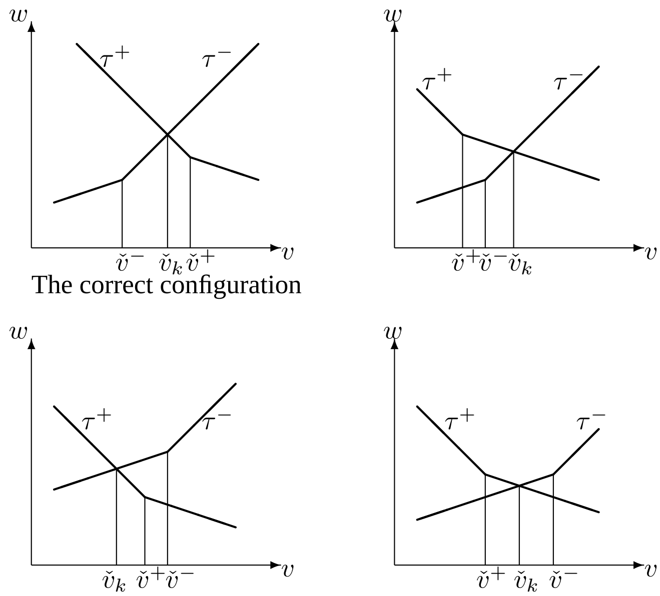

Figure 1: Four possible configurations

Proposition A.1 *We have for all* (k, u)

$$\frac{1}{1+\tau_h^-}\check{v}_{k+1}^h((1+\tau_h^-)u) \le \check{v}_k^h(u) \le \frac{1}{1+\tau_h^+}\check{v}_{k+1}^h((1+\tau_h^+)u)\,.$$

Proof Assume that the left inequality does not hold. Then a *decrease* of the price of the underlying stock (by a factor $1+\tau^-$) would result in the cheapest hedging portfolio having a *larger* content (in number of shares) in this stock than the previous one. A contradiction for a call (and for any option with an increasing payment fuction). ∎

Only the first possibility in the figure is consistent with the proposition, and it results in the max being again a simple wedge. Its minimum is achieved at the intersection of the right branch of the graph with τ^- and the left branch of the graph with τ^+. Which gives the formulas (30). (One needs to notice that the q_k^ε as given by (28) coincide with $q^\varepsilon(kh)$ as defined by (22).)

Their remains to carry out (32). It is an inf convolution with a wedge function acting on the v variable only. It leaves unchanged branches with a slope between $-C^+$ and $-C^-$ (and the min is then reached at $\xi = 0$), and replaces steeper slopes by these two limit ones. Hence the $\min$ or $\max$ operations in (22).

A.2 Theorem 3.2

We have to show that formula (25), where $\varepsilon = \operatorname{sign}(\check{v} - v)$, q^ε is given by (22), and $\mathcal{V}(t,u)$ is the solution of the PDE (21) is the (regular) viscosity solution of (20). Let

$$\begin{aligned} H(t,u,v,\mathrm{D}W,\tau) &:= W_t + \tau[W_u u + (W_v - 1)v]\,, \\ \bar{H}(t,u,v,\mathrm{D}W) &:= \max_{\tau\in[\tau^-,\tau^+]} H(t,u,v,\mathrm{D}W,\tau)\,. \end{aligned} \tag{33}$$

Then (20) reads

$$\min\{\bar{H}(t,u,v,\mathrm{D}W)\,, W_v + C^+\,, -W_v - C^-\} = 0\,. \tag{34}$$

Define $Q^\varepsilon = (q^\varepsilon \;\; 1)$ and $\mathbb{1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

A.2.1 Preliminary propositons

The proof of the theorem is by checking that formula (25) indeed provides a (sufficiently regular) viscosity solution of (34). However, the complete proof is a bit lengthy, as several manifolds of possible gradient discontinuity must be checked. We give here the important arguments and summarize the less important details.

We stress a first simple fact, as a consequence of the definition (22):

Proposition A.2 *For $\varepsilon = 1$ and $\varepsilon = -1$,*

- *if $t \le t_\varepsilon$, $q^\varepsilon = C^\varepsilon$,*
- *if $t > t_\varepsilon$, $q^\varepsilon \in [c^\varepsilon, C^\varepsilon]$ and*

$$\dot{q}^\varepsilon = -\tau^\varepsilon(q^\varepsilon + 1) \tag{35}$$

We also claim the following important fact:

Proposition A.3 *Forall $(t,u,v) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$,*

$$Q^\varepsilon \mathcal{V}_t \le 0\,, \quad \textit{or equivalently} \quad \operatorname{sign}[Q^\varepsilon(\mathcal{V}_u u - \mathbb{1}\check{v})] = \varepsilon \tag{36}$$

Proof The equivalence of the two forms of the claim comes from the fundamental PDE (21) and the the fact that

$$Q^\varepsilon \mathcal{T} = \tau^\varepsilon Q^\varepsilon\,. \tag{37}$$

Simple geometry shows that the proposition A.1 implies

$$\check{w}^+_{k+\frac12} + q^-_{k+\frac12}(\check{v}^+_{k+\frac12} - \check{v}^-_{k+\frac12}) \le \check{w}^-_{k+\frac12} \le \check{w}^+_{k+\frac12} + q^+_{k+\frac12}(\check{v}^+_{k+\frac12} - \check{v}^-_{k+\frac12})\,. \tag{38}$$

In the limit as $h \to 0$, $W^h \to W$, but also $\mathcal{V}^h \to \mathcal{V}$ that satisfies the PDE (21). And since the defining recursion (30) is a consistent discretization scheme for (21), the differentials converge, and, as a tedious but simple calculation shows, (38) converges to (36). ∎

For a given (t,u,v), let $\varepsilon = \operatorname{sign}(\check{v}(t,u) - v)$. As a consequence of (36), and keeping in mind that $q^\varepsilon + 1 > 0$,

$$\operatorname{sign}[(\check{w}_u + q^\varepsilon \check{v}_u)u - (q^\varepsilon + 1)v] = \varepsilon\,, \tag{39}$$

so that the max in $\bar{H}$ is reached at $\tau = \tau^\varepsilon$.

A.2.2 Differentiable case

We investigate first regions of (t,u,v) space where our formula (25) gives a differentiable function.

Assume first that $t > t_\varepsilon$. Using Proposition (A.2), (37) and (39), it readily comes that with the definition (25) of W, $\bar{H} = 0$, while the other two terms in (34) are positive because of (A.2).

If $t < t_\varepsilon$, (21), (37), and (39) show that $\bar{H}(t,u,\check{v},\mathrm{D}W) = 0$ and therefore $\bar{H}(t,u,v,\mathrm{D}W) \geq 0$. And according to (A.2), one of the other two terms in (34) is zero and the other one positive.

A.2.3 The singular manifold $v = \check{v}$

On the manifold $v = \check{v}(t,u)$, formula (25) for W is non differentiable. It has a non void sub-differential, obtained by replacing q^ε by $q = \lambda q^+ + (1-\lambda)q^-$ in the formulas for the patial derivatives in either of the regions $\varepsilon = -1$ or $\varepsilon = 1$. This is so because these partials are affine in q^ε.

Now, for each ϵ, the maximum in τ in $\bar{H}$, reached at τ^ε, is 0. Therefore, for $\tau^{-\varepsilon}$, $H \leq 0$. Hence, as an affine function of q (for fixed τ) which ranges from 0 to a negative number, H is non positive for all possible q's. Hence so is its max in τ, $\bar{H}$. The other two terms in (34) are trivially non positive for all λ. Therefore the minimum of the three terms is non positive, and this is the viscosity condition.

A.2.4 Boundaries of the nontrivial region

It takes some analysis of the fundamental PDE to show that along the manifolds $u = K\exp(-\tau^\varepsilon(T-t))$, the gradients of $\mathcal{V}$ may be (in fact are) discontinuous, with discontinuities $\delta\mathcal{V}_t$ and $\delta\mathcal{V}_u$ satisfying $\delta\mathcal{V}_t = -\tau^\varepsilon\delta\mathcal{V}_u$ and $Q^{-\varepsilon}\delta\mathcal{V}_u = 0$. (See Bernhard et al. 2002). Assume we are hedging a call, with thus $0 \leq v \leq u$. On the left boundary $u = K\exp(-\tau^+(T-t))$, we have $\varepsilon = -1$, and the discontinuities of the gradient of our function W are given by $(\delta W_t, \delta W_u, \delta W_v) = (Q^-\delta\mathcal{V}_t, Q^-\delta\mathcal{V}_u, 0) = 0$. Therefore the fuction (25) is smooth. A similar argument applies along the boundary $u = K\exp(-\tau^-(T-t))$. And symmetric arguments hold for a put.

A.2.5 Boudaries of the jump regions

Finally, one has to check the two manifolds $t = t_\varepsilon$, where $\mathcal{V}_t$ is discontinuous, because q_t^ε is. It can be seen that the super-differential of W is non empty there, and is made of all the vectors $(Q^\varepsilon\mathcal{V}_t + \delta, W_u, -C^\varepsilon)$ with $\delta \in [-\tau^\varepsilon(1+C^\varepsilon)(\check{v}-v), 0]$, and notice that $-\tau^\varepsilon(1+C^\varepsilon)(\check{v}-v) < 0$ (Which shows that it is the superdifferential wihch is non empty.) As a consequence, the viscosity condition reads

$$\forall\delta \in [-\tau^\varepsilon(1+C^\varepsilon)(\check{v}-v), 0]\,, \quad Q^\varepsilon\mathcal{V}_t + \delta + \tau^\varepsilon[Q^\varepsilon\mathcal{V}_u u - (C^\varepsilon+1)v] \geq 0\,.$$

However, we have already seen that this quantity is zero for δ at the lower end of the interval. And thus the inequality does hold, ending the proof.