An Impulsive Differential Game Arising in Finance with Interesting Singularities*

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Abstract

We investigate a differential game motivated by a problem in mathematical finance. This game displays two interesting features. On the one hand, one of the players, **P**ursuer say, may, and will, use infinitely large controls, i.e., impulses, producing "jumps" in the state variables. Standard optimal trajectories are made of such a jump followed by a "coasting period" where **P** exerts no control. This leads to barriers of a somewhat new type. But because the cost of jumps is only proportional to their amplitude, some singular optimal trajectories arise where **P** uses an intermediary control, nonzero but finite. (In classical impulse control, there is a minimum positive cost to any use of the control, forbidding such a mixed situation.)

On the other hand, the complete solution of the game exhibits a type of singularity, the existence of which had long been conjectured (noticeably by Arik Melikyan in discussions with the first author) but, as far as we know, never shown in actual examples: a two-dimensional focal manifold traversed by noncollinear optimal fields depending on the control used by **E**vader. It is on this manifold that intermediary controls for **P** arise.

Finally, we show that the Isaacs equation of a discrete-time version of the problem provides a discretization scheme that converges to the value function of the differential game. This is done through the investigation of a (degenerate) quasi-variational inequality and its viscosity solution, with

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the help of an equivalent, but nonimpulsive, differential game—a method of interest per se that we credit to Joshua—to which we apply essentially the classical method of Capuzzo Dolcetta extended to differential games by Pourtallier and Tidball, with some technical adaptations.

1 The Differential Game Considered

We consider a differential game arising in finance, specifically in the theory of option pricing with an “interval model.” (We refer to [4,16] for the context in finance.) This is a game in two dimensions plus time with an integral payoff, or three dimensions plus time with a terminal payoff, and two scalar controls (pursuer **P** and evader **E**), with the peculiarity that the pursuer may, and will, use arbitrarily large control values, up to the point of producing “impulses.” Thus, this player may cause discontinuities in some state variables, incurring a related cost.

1.1 Dynamics

The (3-D) dynamics are as follows. We call (x, y, z) the state variables, and u and v the controls of pursuer and evader respectively. The continuous (nonimpulsive) part of the dynamics is given in terms of $\varepsilon = \operatorname{sign}(u)$ and two numbers C_{+1} and C_{-1}, also written C^+ and C^- respectively, with $C^+ > 0$, $C^- < 0$, as follows:

$$\dot{x} = vx\,, \tag{1}$$
$$\dot{y} = vy + u\,, \tag{2}$$
$$\dot{z} = vy - C_\varepsilon u\,, \tag{3}$$

with the control constraints on v specified by two positive numbers α and β as

$$-\alpha \le v \le \beta\,. \tag{4}$$

Since u is not bounded, we allow the pursuer to cause discontinuities in the state variables at isolated time instants t_k according to the rule

$$y(t_k^+) = y(t_k^-) + u_k\,, \tag{5}$$
$$z(t_k^+) = z(t_k^-) - C_{\varepsilon_k} u_k\,. \tag{6}$$

Of course, we have set $y(t_k^-) = \lim_{t\uparrow t_k} y(t)$ and $y(t_k^+) = \lim_{t\downarrow t_k} y(t)$ and likewise for z. The jump amplitude in y is $u_k \in \mathbb{R}$, and $\varepsilon_k = \operatorname{sign}(u_k)$.

To avoid an unessential discussion later on, we shall further assume that

$$\alpha \le \beta\,, \qquad \text{and} \qquad 0 < (1 + C^+)(1 + C^-) \le 1\,. \tag{7}$$

1.2 Payoff

The game is played over a fixed time interval $[0, T]$, and is a capture-evasion *game of kind*, with capture defined in terms of a given positive number Z as

$$z(T) \geq \max\{0\,,\, x(T) - Z\} =: M(x(T))\,. \tag{8}$$

Again this rather strange setup is motivated by its finance application in [4].

We may notice that, since z does not appear in the right-hand side of its dynamics, it integrates so that (8) is equivalent to

$$z(0) \geq \int_0^T (-vy + C_\varepsilon u)\,\mathrm{d}t + \sum_k C_{\varepsilon_k} u_k + M(x(T))\,.$$

As a consequence, we may consider the *game of degree* in dimension 2 plus time with state variables (x, y), the same dynamics (1) (2) and (5)(6), and payoff $\min_u \max_v G$ with

$$G = \int_0^T (-vy + C_\varepsilon u)\,\mathrm{d}t + \sum_k C_{\varepsilon_k} u_k + M(x(T))\,. \tag{9}$$

Let $W(t, x, y)$ be its value function, an initial state is capturable iff $z \geq W(0, x, y)$, so that the graph of the value function W is the barrier of the *game of kind*.

1.3 Strategies

In this game, the pursuer chooses the function $u(t)$, the jump instants t_k, and the jump amplitudes u_k. It does so knowing past values of the state. It is a classical fact that it will only use an (instantaneous) *state feedback* which we write symbolically $u = \varphi(t, X(t))$, where X stands for the whole state. Admissible strategies are those such that the dynamical equations have for any initial state a unique solution with $y(\cdot)$ uniformly bounded over admissible $v(\cdot)$'s.

We are looking for capturable states of the *game of kind*. It is known that this is equivalent to looking for the upper value of the game of degree, and that then, whether the evader plays open loop or closed loop is irrelevant. Thus we may always assume that v is chosen open loop, as a measurable time function from $[0, T]$ into $[-\alpha, \beta]$. (This remark will play an important role in the investigation of the convergence.)

2 A Geometric Analysis: The Isaacs–Breakwell Theory

2.1 Jumps as Ordinary Trajectories

In [4], we introduced a quasi-variational inequality (QVI) naturally related to the game of degree with impulse controls. However, due to its very degenerate nature, it is not accounted for by the literature on viscosity solutions of first order QVI such as [3,2]. We prefer to use the 3-D plus time representation (1), (2), (3), (5), (6), and the formulation as a *game of kind*, and apply to it the geometrical tools of the semipermeability.

In that representation, jumps are just trajectories orthogonal to the t axis. As a matter of fact, Equations (5) and (6) show that these trajectories are also orthogonal to the x axis and have a slope either $-C^+$ or $-C^-$ in the (y, z) plane. We stress the following fact.

Proposition 2.1. *Given a smooth two-dimensional manifold $\mathcal{M}$ transverse to the jump trajectories, the hypersurface made of jump trajectories of the same slope through each point of $\mathcal{M}$ is a "safe hypersurface" for* **P***, (i.e.,* **E** *cannot force the state to cross it against* **P***'s will).*

Proof. Indeed by choosing a jump, **P** causes the state to traverse these trajectories in no time, so that **E**'s control v has no time to act. (**P** has chosen to be in the dynamics (5), (6) where v does not enter.) □

We shall in effect construct manifolds $y = \check{y}(t, x)$, $z = \check{z}(t, x)$ for some functions $\check{y}$ and $\check{z}$, construct barriers made of jump trajectories reaching that manifold, and show that upon reaching it, **P** still has a means of preventing a crossing of the composite surface.

2.2 The Natural Barrier

We proceed with the classical construction of the natural barrier through the boundary of the capture set, which here is $t = T$, $z = M(x)$, y arbitrary. This has been published in [4]. We summarize it here.

The natural barrier is made up of two sheets, one towards $x \le Z$ and one towards $x \ge Z$. They are given below, together with a corresponding *inward* semipermeable normal as the vector $(n, p, q, 1)$ (corresponding to the state variables (t, x, y, z)), leading to Isaacs' "main equation"

$$0 = \max_u \inf_{v \in [-\alpha, \beta]} \left[n + v\Big(px + (q+1)y\Big) + u(q - C_\varepsilon) \right] ,$$

and the adjoint equations

$$\dot{p} = -vp\,, \tag{10}$$

$$\dot{q} = -v(q+1). \tag{11}$$

The analysis depends on the fact that the maximum in u of $(q - C_\varepsilon)u$ is reached at $u = 0$ provided that $C^- \le q \le C^+$. (Remember that $\varepsilon = \mathrm{sign}(u)$.) When q leaves that range, there is no maximum anymore. (Or u should be infinite: we shall have a jump.)

Sheet α towards $x \le Z$. We set the parameters $x(T) = s \le Z$, $y(T) = r$. It yields $v^* = -\alpha$ and

$$
\begin{array}{rcl}
\multicolumn{3}{c}{\text{sheet } (\alpha)} \\
t &=& t \\
x(t) &=& s\mathrm{e}^{\alpha(T-t)}\,, \\
y(t) &=& r\mathrm{e}^{\alpha(T-t)}\,, \\
z(t) &=& r(\mathrm{e}^{\alpha(T-t)} - 1)\,,
\end{array}
\qquad
\begin{array}{rcl}
\multicolumn{3}{c}{\text{semipermeable normal } \nu_\alpha} \\
n(t) &=& \alpha r\,, \\
p(t) &=& 0\,, \\
q(t) &=& \mathrm{e}^{-\alpha(T-t)} - 1\,, \\
1 &=& 1\,.
\end{array}
$$

This is a valid solution as long as $q \ge C^-$, i.e., for $t \ge t_\alpha$ with

$$
\mathrm{e}^{-\alpha(T-t_\alpha)} = 1 + C^-\,, \quad \text{i.e.,} \quad T - t_\alpha = \frac{1}{\alpha}\ln\left(\frac{1}{1+C^-}\right). \tag{12}
$$

Sheet β towards $x \ge Z$. On this sheet, $x(T) = s \ge Z$, $y(T) = r$. We find that $v^* = \beta$, and

$$
\begin{array}{rcl}
\multicolumn{3}{c}{\text{sheet } (\beta)} \\
t &=& t \\
x(t) &=& s\mathrm{e}^{-\beta(T-t)}\,, \\
y(t) &=& r\mathrm{e}^{-\beta(T-t)}\,, \\
z(t) &=& r(\mathrm{e}^{-\beta(T-t)} - 1) + s - Z\,,
\end{array}
\qquad
\begin{array}{rcl}
\multicolumn{3}{c}{\text{semipermeable normal } \nu_\beta} \\
n(t) &=& \beta(s-r)\,, \\
p(t) &=& -\mathrm{e}^{\beta(T-t)}\,, \\
q(t) &=& \mathrm{e}^{\beta(T-t)} - 1\,, \\
1 &=& 1\,.
\end{array}
$$

This is a valid solution as long as $q \le C^+$, i.e., for $t \ge t_\beta$ with

$$
\mathrm{e}^{\beta(T-t_\beta)} = 1 + C^+\,, \quad \text{i.e.,} \quad T - t_\beta = \frac{1}{\beta}\ln(1+C^+)\,. \tag{13}
$$

From the hypothesis (7), we have $t_\alpha < t_\beta$.

Moreover, from final states on the boundary $z = x - Z \ge 0$ of the admissible set, a 2-D singular sheet can be constructed with $r = s$, v arbitrary, leading to

$$
x = y = z - Z = s\exp\left(-\int_t^T v(\tau)\,\mathrm{d}\tau\right),\ -p = q + 1 = \exp\left(\int_t^T v(\tau)\,\mathrm{d}\tau\right).
$$

Intersection and Composite Barrier. The two main sheets (α) and (β) intersect along a two-dimensional edge $\mathcal{D}$ that spans the domain $t \ge t_\beta$, $Z\mathrm{e}^{-\beta(T-t)} \le x \le Z\mathrm{e}^{\alpha(T-t)}$, and that can be parametrized by (t, x) as $y = \check{y}(t, x)$, $z = \check{z}(t, x)$ given by

$$
\check{y}(t,x) = \frac{(x\mathrm{e}^{\beta(T-t)} - Z)}{\mathrm{e}^{\beta(T-t)} - \mathrm{e}^{-\alpha(T-t)}}, \quad \check{z}(t,x) = (1 - \mathrm{e}^{-\alpha(T-t)})\check{y}(t,x). \tag{14}
$$

Notice that for $x = Z\exp(-\beta(T-t))$, we have $\check{y} = \check{z} = 0$, which corresponds to the sheet (α) with $r = 0$. For smaller x's, only the sheet (α) plays a role.

We find it convenient to extend the definition of $\check{y}$ and $\check{z}$ by 0 for both. For $x = Z\exp(\alpha(T-t))$, we have $\check{y} = x$, $\check{z} = x - Z$, which corresponds to the sheet (β) with $r = s = Z\exp((\alpha+\beta)(T-t_\beta))$. For larger x's, only the sheet (β) plays a role. Again, we extend the definitions of $\check{y} = x$ and $\check{z} = x - Z$ to larger x's.

We easily check that $\mathcal{D}$ is an **E**-dispersal line. States "above" (with larger z's) are indeed capturable, and the edge does not "leak" since **P**'s control on both sheets is the same: $u = 0$. Therefore, this same control prevents crossing of both barrier sheets.

The singular sheet $x = y = z + Z$ is imbedded in the sheet (β). But it can be used against $v = -\alpha$ until time t_α. In the region $x \le Z\exp(\alpha(T-t))$ it plays no role. However, it will be seen to play a prominent role in the region $x \ge Z\exp(\alpha(T-t))$ for $t \le t_\beta$, when the sheet (β) does not exist. There it behaves as a manifold drawn on an extension of the sheet (α) for $s \ge Z$.

2.3 Junction of a Jump Manifold and the Natural Barrier

For $t \le t_\beta$, the sheet (β) of the natural barrier does not exist, since it would entail a $q \ge C^+$, leading to $u = +\infty$. We therefore expect a positive jump manifold, i.e., trajectories in the (y,z) plane with slope $-C^+$. They must join on a two-dimensional manifold $\mathcal{E}$ drawn on the sheet (α), and such that, whatever v, **P** can maintain the state on or above both that sheet and the jump manifold. The manifold $\mathcal{E}$ will indeed be an "equivocal" one (in Isaacs' parlance), constructed according to the technique of a "safe contact" on a barrier, as originally discovered by Breakwell and Merz [9,12].

We first determine a control $u(v)$ that maintains the state on the barrier sheet (α). Let ν_α be the normal to that sheet; we have

$$\langle \nu_\alpha, \dot{X} \rangle = \mathrm{e}^{-\alpha(T-t)}(v+\alpha)y - u(1 + C_\varepsilon - \mathrm{e}^{-\alpha(T-t)})$$

so that we keep the state on the sheet (α) by choosing

$$u = \frac{\mathrm{e}^{-\alpha(T-t)}(v+\alpha)y}{1 + C^+ - \mathrm{e}^{-\alpha(T-t)}}\,.$$

With that control, keeping in mind that the normal to the jump manifold, say ν_j, has to be of the form $\nu_j = (n_j, p_j, C^+, 1)$, we get on $\mathcal{E}$:

$$\langle \nu_j, \dot{X} \rangle = n_j + v(p_j x + (1 + C^+)y)\,.$$

Furthermore, we want $\mathcal{E}$ to join on the boundary of $\mathcal{D}$ at $t = t_\beta$. Therefore, ν_j there should be normal to that boundary. This gives $p_j = -(1 + C^+)$, i.e., the same as in ν_β as it should be, and hence at $t = t_\beta$,

$$\langle \nu_j, \dot{X} \rangle = n - v(1 + C^+)(x - y)\,.$$

The domain considered thus far, the boundary of $\mathcal{D}$, is such that $x \ge y$. As a consequence, the minimizing v is $v = \beta$. Furthermore, if we construct the manifold $\mathcal{E}$ using $v = \beta$ in the above construction, we can check[1] that n_j remains positive, hence $(p_j x + (1+C^+)y)$ is negative. Hence $v = \beta$ is indeed minimizing, or, equivalently, we check that the strategy $u(v)$ above guarantees that the state lies on the sheet (α) *and* on the desired side of the jump manifold.

We therefore obtain the following.

Theorem 2.1. *The equations of the equivocal manifold $\mathcal{E}$ are*

$$\begin{aligned} \dot{x} &= \beta x\,, & x(t_\beta) &= \tfrac{s}{1+C^+}\,,\\ \dot{y} &= \tfrac{\beta(1+C^+)+\alpha \mathrm{e}^{-\alpha(T-t)}}{1+C^+-\mathrm{e}^{-\alpha(T-t)}}y\,, & y(t_\beta) &= \tfrac{s-Z}{1+C^+-\mathrm{e}^{-\alpha(T-t_\beta)}}\,,\\ \dot{z} &= \tfrac{\beta(1+C^+)(1-\mathrm{e}^{-\alpha(T-t)})-C^+\alpha\mathrm{e}^{-\alpha(T-t)}}{1+C^+-\mathrm{e}^{-\alpha(T-t)}}y\,, & z(t_\beta) &= (1-\mathrm{e}^{-\alpha(T-t_\beta)})y(t_\beta). \end{aligned} \tag{15}$$

We can integrate these backwards as long as the sheet (α) exists, i.e., down to $t = t_\alpha$. However, due to our restricted set of initial conditions, this will only take care of the domain $s \in [Z, Z\exp((\alpha+\beta)(T-t_\beta))]$, i.e., $Z\exp(-\beta(T-t)) \le x \le Z\exp(\alpha(T-t_\beta)-\beta(t_\beta-t))$. We need to find the extension of the manifold $\mathcal{E}$ to all values of (t,x) for $t \in [t_\alpha, t_\beta]$.

In the region $x \le Z\exp(-\beta(T-t))$, the above equations are to be taken with terminal conditions $y = z = 0$, and thus yield $y = z = 0$ down to $t = t_\alpha$.

In the region $x \ge Z\exp(\alpha(T-t))$, we do not have the sheet (α) to perform the above construction, but we do it with the singular sheet $y = x$, $z = x - Z$. A completely similar analysis yields a u proportional to $y - x$, i.e., zero on the singular sheet, which turns out itself to be the manifold $\mathcal{E}$.

This joins smoothly with $\mathcal{D}$ in the region $t \ge t_\beta$. We shall use it as terminal conditions for the equations of $\mathcal{E}$ along the boundary $x = Z\exp(\alpha(T-t))$, $t_\alpha \le t \le t_\beta$. That way, we have defined the manifold $\mathcal{E}$ in all the required domain. Again, for $t \in [t_\alpha, t_\beta]$, let $y = \check{y}(t,x)$, $z = \check{z}(t,x)$ describe this manifold. The functions $\check{y}$ and $\check{z}$ thus defined extend continuously those for $t \ge t_\beta$ defined on $\mathcal{D}$.

It turns out that the equations for y integrate analytically. See Appendix A.

2.4 The Focal Manifold

2.4.1 Principle

For $t \le t_\alpha$, neither of the two sheets of the natural barrier exist. We must therefore replace them both by jump manifolds that will join on a new manifold, which is thus a focal surface, (but with adjoining trajectories that are jump trajectories). Let us call it $\mathcal{F}$.

To explain how to construct $\mathcal{F}$, we need to introduce some notation. We shall have two jump manifolds, one with negative jump and one with positive jump.

[1] We did it numerically. There should exist an analytical proof.

Let ν^- and ν^+ be the corresponding normals. They are of the form

$$\nu^- = \begin{pmatrix} n^- \\ p^- \\ C^- \\ 1 \end{pmatrix}, \qquad \nu^+ = \begin{pmatrix} n^+ \\ p^+ \\ C^+ \\ 1 \end{pmatrix}.$$

Let also $\dot{X} = f(X, u, v)$ denote the dynamics. Upon reaching $\mathcal{F}$, player **P** will have to choose a control $u(v)$ that will maintain the state on $\mathcal{F}$ or above the composite barrier thus constructed. Assume that for the extreme values of v, i.e., $-\alpha$ and β, the state can just be maintained on $\mathcal{F}$. Let u_α and u_β be the corresponding controls. Now, we must have the following equalities:

$$\begin{array}{ll} 0 = \langle \nu^-, f(X, u_\alpha, -\alpha)\rangle, & 0 = \langle \nu^+, f(X, u_\alpha, -\alpha)\rangle, \\ 0 = \langle \nu^-, f(X, u_\beta, \beta)\rangle, & 0 = \langle \nu^+, f(X, u_\beta, \beta)\rangle. \end{array}$$

We have six unknowns, $n^-, p^-, n^+, p^+, u_\alpha, u_\beta$. We need two more equations to determine them.

2.4.2 Trajectories $v = \beta$

We choose to describe $\mathcal{F}$ as the set of trajectories obtained for $v = \beta$. Later we shall discuss this arbitrary choice. In this description, let $X^\beta(s,t)$ be our state,[2] depending on the parameter s characterizing the trajectory (say reaching the boundary of $\mathcal{E}$ at t_α at the point $u = s\exp(\beta(T - t_\alpha))$ and on t. Thus

$$\frac{\partial X^\beta}{\partial t} = X^\beta_t = \dot{X}^\beta = f(X, u_\beta, \beta).$$

We need further express that all trajectories lie in the same manifold $\mathcal{F}$. Hence, let $X^\beta_s := \partial X^\beta / \partial s$; we must further have

$$0 = \langle \nu^-, X^\beta_s\rangle, \qquad 0 = \langle \nu^+, X^\beta_s\rangle.$$

We now have six equations in six unknowns at each X. We want to use them to recover u_β and put it in the equations of the dynamics. Surprisingly, this is rather easy to do. The first four equations yield

$$u_\alpha = \frac{\alpha+\beta}{C^+ - C^-}[p^+ x + (1 + C^+) y], \quad u_\beta = \frac{\alpha+\beta}{C^+ - C^-}[p^- x + (1 + C^-) y].$$

The equation $0 = \langle \nu^-, X^\beta_s\rangle$ yields

$$p^- x^\beta_s = -(C^- y^\beta_s + z^\beta_s).$$

[2]Obviously, β here is a superindex, not a power!

Now, we still have $\dot{x^\beta} = \beta x$, i.e., $x^\beta = s\exp(-\beta(T-t))$. Thus $x^\beta(s,t) = sx_s^\beta$. Hence, the above equation reads

$$p^- x = -s(C^- y_s^\beta + z_s^\beta)\,.$$

Put this back in u_β; this finally yields a pair of coupled partial differential equations (PDEs). We use the notation

$$\gamma = \frac{\alpha+\beta}{C^+ - C^-}\,, \tag{16}$$

to get the following fact (we have dropped the superindices β).

Theorem 2.2. *The focal manifold satisfies the following system of partial differential equations:*

$$\begin{pmatrix} y_t \\ z_t \end{pmatrix} = s\gamma \begin{pmatrix} -C^- & -1 \\ C^+C^- & C^+ \end{pmatrix} \begin{pmatrix} y_s \\ z_s \end{pmatrix} + \begin{pmatrix} \beta + \gamma(1+C^-) \\ \beta - C^+\gamma(1+C^-) \end{pmatrix} y\,. \tag{17}$$

Domain and Boundary Conditions. We need to know $\mathcal{F}$ for all $t \le t_\alpha$ and all $x \ge 0$. However, for $x \le Z\exp(-\beta(T-t))$, we have previously argued that we expect the optimal (y,z) to be $(0,0)$. Also, for $x \ge Z\exp(\alpha(T-t))$, we expect the optimal (y,z) to be $(x, x-Z)$. Notice first that each of these two pairs, with $x = s\exp(-\beta(T-t))$, satisfies the PDE (17). It remains to fill the domain $\Omega := \{t \le t_\alpha,\ Z\exp(-\beta(T-t)) \le x \le Z\exp(\alpha(T-t))\}$, using the above known values at the boundaries in x, and the previously computed values on $\mathcal{E}$ at $t = t_\alpha$ for $Z\exp(-\beta(T-t_\alpha)) \le x \le Z\exp(\alpha(T-t))$.

This may entail discontinuities of the gradients of y and z along the "lateral" boundaries of Ω. Appendix B provides a proof that these two lines are precisely the possible support of such discontinuities. It also provides a further mathematical and numerical investigation of this PDE.

We therefore have a manifold $\mathcal{F}$ defined for all $t \le t_\beta$, all positive x's. We still call $\check{y}(t,x)$, $\check{z}(t,x)$ this manifold, and observe that the functions $\check{y}$ and $\check{z}$ are continuous.

2.4.3 Trajectories $v = -\alpha$

As stressed above, the choice to analyze $\mathcal{F}$ through the trajectories generated by $v = \beta$ was arbitrary. The same analysis could have been made using the trajectories $v = -\alpha$. Let them be parametrized by $u = r\exp(\alpha(T-t))$, and $X^\alpha(r,t)$ be the resulting manifold. One obtains the PDE

$$\begin{pmatrix} y_t \\ z_t \end{pmatrix} = r\gamma \begin{pmatrix} -C^+ & -1 \\ C^-C^+ & C^- \end{pmatrix} \begin{pmatrix} y_r \\ z_r \end{pmatrix} + \begin{pmatrix} -\alpha + \gamma(1+C^+) \\ -\alpha - C^-\gamma(1+C^+) \end{pmatrix} y\,. \tag{18}$$

(Notice that one gets these equations upon interchanging in (17) $-\alpha$ with β on the one hand, and C^- with C^+ on the other hand.)

Proposition 2.2. *The PDEs (17) and (18) (with the same boundary conditions) describe the same manifold in the* (t, x, y, z) *space.*

Proof. Let us pick

$$s = r \exp((\alpha + \beta)(T - t)) \tag{19}$$

so that the coordinates x coincide. Let $Y = (y\ z)^t$. We want to show (with transparent notation) that $Y^\alpha(r, t) = Y^\beta(r \exp((\alpha + \beta)(T - t)), t)$. Therefore,

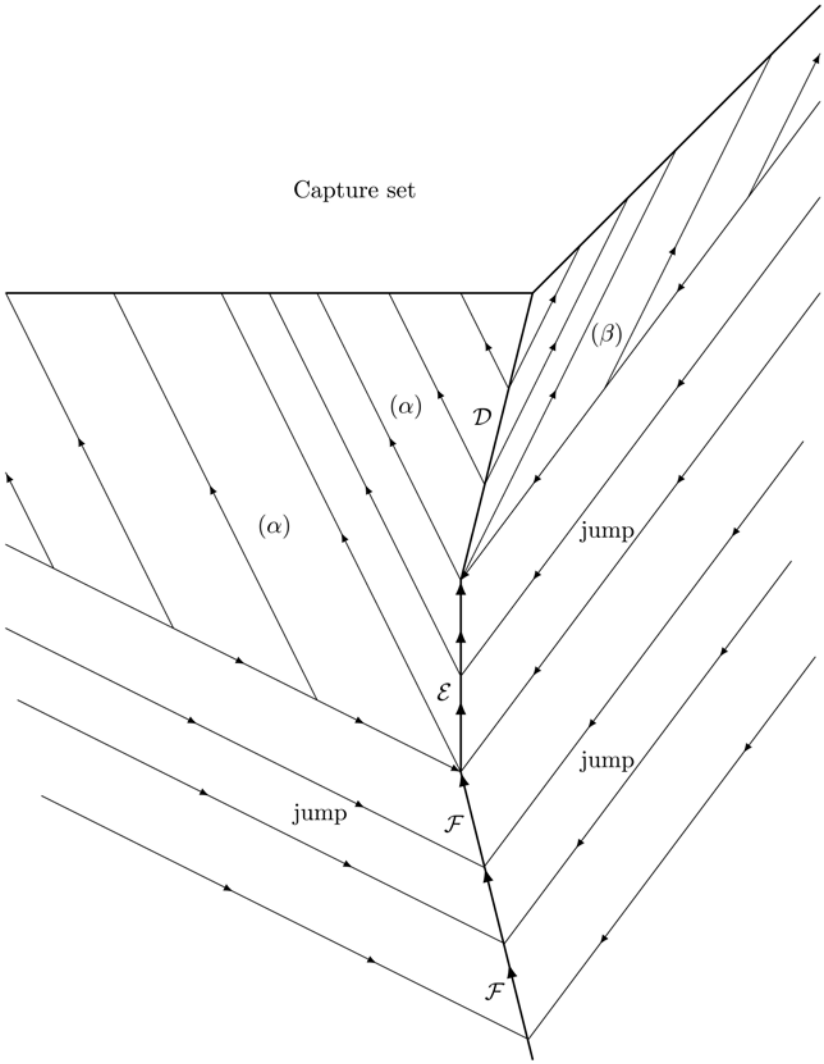


Figure 1: A 2-D sketch of the 4-D geometry of the barrier.

we should have

$$\begin{aligned} Y_r^\alpha &= \mathrm{e}^{(\alpha+\beta)(T-t)} Y_s^\beta\,, &(20)\\ Y_t^\alpha &= -(\alpha+\beta) r \mathrm{e}^{(\alpha+\beta)(T-t)} Y_s^\beta + Y_t^\beta\,. &(21)\end{aligned}$$

Write (17) and (18) respectively as

$$Y_t^\beta = sA^\beta Y_s^\beta + B^\beta Y^\beta\,, \quad Y_t^\alpha = rA^\alpha Y_r^\alpha + B^\alpha Y^\alpha\,.$$

We also need the notation $D := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. We therefore have

$$\begin{aligned} A^\beta &= \gamma \begin{pmatrix} -1 \\ C^+ \end{pmatrix} \begin{pmatrix} C^- & 1 \end{pmatrix}, \;\; B^\beta = (\beta I - A^\beta) D\,,\\ A^\alpha &= \gamma \begin{pmatrix} -1 \\ C^- \end{pmatrix} \begin{pmatrix} C^+ & 1 \end{pmatrix}, \;\; B^\alpha = (-\alpha I - A^\alpha) D\,. \end{aligned}$$

Substituting both (17) and (20) into (21) and also using (19), we get

$$Y_t^\alpha = r[-(\alpha+\beta)I + A^\beta] Y_r^\alpha + B^\beta Y^\alpha\,. \tag{22}$$

The proposition then results from the easy fact that (remembering (16))

$$A^\alpha = A^\beta - (\alpha+\beta)I\,, \quad \text{and therefore} \quad B^\alpha = B^\beta\,,$$

so that (22) coincides with (18). □

2.5 Synthesis

The boundary of the set of capturable states is given by $z = W(t,x,y)$ defined, in the domain $t \in [0,T]$, $x \geq 0$, $y \in [0,x]$, by

$$W(t,x,y) = \check{z}(t,x) + D_\eta(y - \check{y}(t,x))\,,$$

where

- The functions $\check{y}$ and $\check{z}$ are given by the requirement that they be continuous (which specifies the boundary values of the differential equations) and
  - (i) $\check{y} = \check{z} = 0$ if $x \leq Z\exp(-\beta(T-t))$,
  - (ii) $\check{y} = x$, $\check{z} = x - Z$ if $x \geq Z\exp(\alpha(T-t))$,
  - (iii) if $x \in [Z\exp(-\beta(T-t)), Z\exp(\alpha(T-t))]$,
    - · if $t \geq t_\beta$, equations (14)
    - · if $t \in [t_\alpha, t_\beta]$, differential equations (15) with terminal conditions as in (15) $t = t_\beta$, and and by continuity with region (ii) above on the boundary in x,
    - · if $t \leq t_\alpha$, equations (17) with terminal conditions by continuity with the above at $t = t_\alpha$, and by continuity with (i) and (ii) on the boundaries in x,

- $\eta = \mathrm{sign}(y - \check{y})$, and $D_{+1} = D^+$ and $D_{-1} = D^-$ are given by

$$D^+ = \begin{cases} -C^- & \text{if } t \le t_\alpha\,, \\ 1 - \mathrm{e}^{-\alpha(T-t)} & \text{if } t \ge t_\alpha\,, \end{cases}$$

$$D^- = \begin{cases} -C^+ & \text{if } t \le t_\beta\,, \\ 1 - \mathrm{e}^{\beta(T-t)} & \text{if } t \ge t_\beta\,. \end{cases}$$

This function W is therefore also the upper value function of the *game of degree* in (x, y) with payoff given by (9). Figure 1 shows a sketch of this compound manifold.

3 Discretization

3.1 The Multistage Game

In [4], we investigated a discrete-time version of the same problem. In discrete time, there are no such things as impulse controls (or there are *only* such things!), so that this is now a classical multistage game. Let $h = T/N$, with N an integer, be our time step. We shall often use a dyadic division, i.e., $N = 2^d$, with d an integer. Write $x(kh^-) = x_k$, and likewise for y, z and $W(kh, x, y) = W_k(x, y)$.

The following system is the natural discretization of our game (and is of interest per se in the finance application):

$$x_{k+1} = (1 + v_k)x_k\,, \tag{23}$$
$$y_{k+1} = (1 + v_k)(y_k + u_k)\,, \tag{24}$$
$$z_{k+1} = z_k + v_k(y_k + u_k) - C_\varepsilon u_k\,, \tag{25}$$

$$\alpha_h = 1 - \exp(-\alpha h)\,, \quad \beta_h = \exp(\beta h) - 1\,, \quad v_k \in [-\alpha_h,\, \beta_h]\,. \tag{26}$$

It is also convenient to separate the effect of the two controls via the two-step description:

$$\begin{array}{ll} x_k^+ = x_k\,, & x_{k+1} = (1 + v_k)x_k^+\,, \\ y_k^+ = y_k + u_k\,, & y_{k+1} = (1 + v_k)y_k^+\,, \\ z_k^+ = z_k - C_\varepsilon u_k\,, & z_{k+1} = z_k^+ + v_k y_k^+\,. \end{array}$$

The 3-D plus time *game of kind* is the same pursuit-evasion game as in the continuous theory, and capturable states are here again defined by $z_k \ge W_k(x_k, y_k)$ where the sequence of functions $\{W_k\}_{k\in\mathbb{N}}$ is the uppervalue function of the 2-D plus time game of degree (23), (24) with payoff

$$G = M(x_N) + \sum_{k=0}^{N-1} \Big(-v_k(y_k + u_k) + C_{\varepsilon_k} u_k \Big)\,. \tag{27}$$

Straightforward application of Isaacs' equation (see [4]) yields

Proposition 3.1. *The value function of the above discrete-time game is the only solution of the recursion*

$$W_k(x,y) = \min_u \max_{v\in[-\alpha_h,\beta_h]} [W_{k+1}((1+v)x,(1+v)(y+u)) - v(y+u) + C_\varepsilon u] \tag{28}$$

with

$$\forall x, y\,, \quad W_N(x,y) = M(x)\,. \tag{29}$$

Equation (28) is equivalent to the two-step procedure

$$\begin{aligned} W_k^+(x,y^+) &= \max_{v\in[-\alpha_h,\beta_h]} [W_{k+1}((1+v)x,(1+v)y^+) - vy^+], \\ W_k(x,y) &= \min_u [W_k^+(x,y+u) + C_\varepsilon u]. \end{aligned}$$

The two-step formulation separates the maximization and minimization operations. It proves useful in the numerical implementation.

We also recall the following theorem from [4].

Theorem 3.1. *The functions* $(x,y)\mapsto W_k(x,y)$ *are all convex.*

Proof. Notice that $(x,y)\mapsto M(x)$ is convex. Assume that W_{k+1} is convex. Then $(x,y)\mapsto W_{k+1}((1+v)x,(1+v)y) - vy$ is convex, so that W_k^+ is the maximum of a family of convex functions, and hence is convex. Now, changing u in $-u'$, W_k appears as the inf-convolution of W_k^+ and the convex extended function $\Gamma(x,y)$ equal to $+\infty$ if $x\neq 0$ and to $C_\varepsilon(-y)$ (with $\varepsilon = \mathrm{sign}(-y)$) if $x=0$. Hence it is convex. □

This, in turn, helps us in devising an efficient numerical procedure to compute that value. Because the functions $v\mapsto W_k((1+v)x,(1+v)y)$ are convex, the maximum in v is reached at either $v=-\alpha$ or $v=\beta$. As for the inf-convolution, it is easy to see that, for each fixed (k,x), one should look for

$$\begin{aligned} y_k^-(x) &= \max\{y \mid -C^+ \in \partial_y W_k(x,y)\}\,, \\ y_k^+(x) &= \min\{y \mid -C^- \in \partial_y W_k(x,y)\}\,. \end{aligned} \tag{30}$$

Then, for $y\in[y^-,y^+]$, W_k and W_k^+ coincide. For $y\le y^-$, W_k must be extended continuously with a slope in y equal to $-C^+$. For $y\ge y^+$, W_k must be extended continuously with a slope equal to $-C^-$:

$$W_k(x,y) = \begin{cases} W_k^+(x,y_k^-(x)) - C^+(y-y_k^-(x)) & \text{if } y\le y_k^-(x)\,, \\ W_k^+(x,y) & \text{if } y_k^-(x)\le y\le y_k^+(x), \\ W_k^+(x,y_k^+(x)) - C^-(y-y_k^+(x)) & \text{if } y\ge y_k^+(x)\,. \end{cases} \tag{31}$$

Implementing that procedure is much faster than computing a min via a standard search procedure.

Understanding the shape of the functions W_k will be useful in the sequel. We emphasize it in the following remark.

Remark 3.1.

- For $y < y^-$, for $0 < h < y^- - v$, $W_k(x,y) = W_k(x,y+h) + C^+h \le W_k^+(x,y)$, and $W_k(x,y) \le W_k(x,y-h) - C^-h$,
- for $y \in [y^-, y^+]$, for all $h > 0$, $W_k(x,y) = W_k^+(x,y) \le W_k(x,y-h) - C^-h$, and $W_k(x,y) \le W_k(x,y+h) + C^+h$,
- for $y > y^+$, for $0 < h < v - y^+$, $W_k(x,y) = W_k(x,y-h) - C^-h \le W_k^+(x,y)$, and $W_k(x,y) \le W_k(x,y+h) + C^+h$.

3.2 Convergence

3.2.1 Main Theorem

We introduce the function $W^h(t,x,y)$ defined as the linear interpolation in time of the functions $W_k(x,y)$ and $W_{k+1}(x,y)$ where $kh \le t < (k+1)h$, and where the functions W_k are given by Equations (28) and (29) for a time step h (in (26)). The objective of this section is to prove the following theorem.

Theorem 3.2. *Let $h = 2^{-d}T$. As d goes to infinity, the sequence of functions $\{W^h\}$ converges uniformly on every compact (and monotonously decreasing) to the value function W of the continuous-time, impulse control game of degree.*

To prove this theorem, we need to introduce another way of looking at the impulse control problem, via yet another game. Thus we name our games. Let $\mathcal{G}$ be the original, continuous-time game, with controls u either finite or impulsive. Its (upper) value function is W. We shall also use the game $\mathcal{G}'$ which is the same as $\mathcal{G}$, but where $\mathbf{P}$ may only use impulses. Let $\mathcal{G}^h$ be the discretized game of this section, and its upper value the sequence $\{W_k^h\}_k$ (the W_k's above). Let also $\mathcal{G}^{h,\ell}$ be the discrete-time game with time step h where, in addition, the variable u has been discretized with a step ℓ, i.e., $u_k \in \ell\mathbb{Z}$. Its value function is a sequence $\{W_k^{h,\ell}\}_{k\in\mathbb{N}}$, which we interpolate in a function $W^{h,\ell}(t,x,y)$ as we did for W^h.

3.2.2 Joshua's Transformation

Finally, we introduce a game $\mathcal{J}$ according to an idea initially due to Joshua [11]. The players are still $\mathbf{P}$ and $\mathbf{E}$ as previously, but $\mathbf{P}$ has a control j which can take only the values -1, 0 or +1. We shall for convenience let $\bar{\jmath} = 1 - |j|$. The game happens in an artificial time that we call τ. We denote with a prime the derivatives with respect to τ. The natural time is now a state variable, and the final τ is defined as the first instant $\tau = \mathcal{T}$ such that $t(\mathcal{T}) = T$. The dynamics of the game are

$$\begin{aligned} t' &= \bar{\jmath}, \\ x' &= \bar{\jmath}vx, \\ y' &= \bar{\jmath}vy + j, \end{aligned}$$

and the payoff is

$$J = M(x(\mathcal{T})) + \int_0^{\mathcal{T}} (-\bar{\jmath}vy + C_j j)\,\mathrm{d}\tau,$$

(with C_0 arbitrary, 0 for instance).

Observe that this is now a standard differential game, which no longer has impulse controls. Its Isaacs equation can be written in the following way:

$$0 = \min\left\{ \frac{\partial W}{\partial t} + \max_{v\in[-\alpha,\beta]} v\left[\frac{\partial W}{\partial x}x + \left(\frac{\partial W}{\partial y} - 1\right)y\right], \right.$$
$$\left. \frac{\partial W}{\partial y} + C^+, -\left(\frac{\partial W}{\partial y} + C^-\right)\right\}.$$

This is a less degenerate form of the quasi-variational inequality of [4]:

$$0 = \min\left\{ \frac{\partial W}{\partial t} + \max_{v\in[-\alpha,\beta]} v\left[\frac{\partial W}{\partial x}x + \left(\frac{\partial W}{\partial y} - 1\right)y\right], \right.$$
$$\left. \min_u [W(t,x,y+u) - W(t,x,y) + C_\varepsilon u]\right\}$$

(where we required $\partial W/\partial y \in [-C^+, -C^-]$ everywhere).

We claim the important following fact.

Proposition 3.2. *The game $\mathcal{J}$ has the same value as the game $\mathcal{G}$.*

Proof. The game $\mathcal{J}$ is in fact completely equivalent to the game $\mathcal{G}'$. When **P** chooses a control $j = 0$, the game proceeds exactly as the game $\mathcal{G}'$ between two impulses. When **P** chooses $j = +1$ or -1, the time stops (hence the reference to Joshua), and y evolves in no real time of a quantity equal to j times the duration, in artificial time, of that control, at a cost C_ε times the variation of y.

The rest of the proof depends on the following easy lemma.

Lemma 3.1. *For any **P**'s control strategy φ in the game $\mathcal{G}$, and any positive δ, there exists an admissible (causal) strategy in the game $\mathcal{G}'$ that yields against any admissible $v(\cdot)$ a payoff within δ of the payoff obtained with φ in the game $\mathcal{G}$.*

The proof of the lemma is given in Appendix C. We immediately have the following:

Corollary 3.1. *The games $\mathcal{G}$ and $\mathcal{G}'$ have the same value.*

And this, together with the fact that $\mathcal{J}$ and $\mathcal{G}'$ have the same value, proves the proposition. □

To complete the proof of the theorem, we need two more lemmas.

Lemma 3.2. *For every positive h, ℓ and every $(t = kh, x, y)$, $N \geq k \in \mathbb{N}$, one has*

$$W(t, x, y) \leq W_k^h(x, y) \leq W_k^{h,\ell}(x, y)\,. \tag{32}$$

Proof. We notice that due to our choice of α_h and β_h in (26), the quantity

$$\exp\left(\int_{t-h}^{t} v(\tau)\,\mathrm{d}\tau\right)$$

exactly spans the interval $[-\alpha_h,\ \beta_h]$. As a consequence, due to the linearity of the dynamics, the game $\mathcal{G}^h$ is an exact time sampling of the game $\mathcal{G}'$ where **P** is further constrained to placing its impulses at time instants $t_k = kh$, $k \in \mathbb{N}$. Since constraints have been placed on the admissible strategies of the minimizer, but not on the controls of the maximizer, we have the first inequality in (32). (Here and in the next lemma, the fact that $v(\cdot)$ can be taken open loop in defining the upper value plays a crucial role.)

In the game $\mathcal{G}^{h,\ell}$, further constraints are placed on the admissible strategies of **P**. Hence the second inequality follows. □

Lemma 3.3. *The functions W^h and $W^{h,h}$ with $h = 2^{-d}T$ decrease as $d \to \infty$ and converge, uniformly on any compact, to functions $\widehat{W}$ and $\widetilde{W}$ respectively.*

Proof. We have noticed that the various games $\mathcal{G}^h$ are variants of the game $\mathcal{G}'$. They differ by the frequency at which player **P** is allowed to play. The game with $h = 2^{-d}T$ can be considered itself as a variant of the game with $h = 2^{-(d+1)}T$ but where **P** is constrained to play $u = 0$ at every odd-numbered stage. Since **P** is minimizing, this constraint increases the value of the game. Hence $W^h(t, x, y)$ is decreasing for every fixed (t, x, y). Being bounded from below by zero, it converges to some $\widehat{W}(t, x, y)$. Now, the W^h are convex, thus continuous, in (x, y), and continuous in time by construction. We therefore have a monotonous convergence of continuous functions, hence it is uniform on every compact.

Concerning the functions $W_k^{h,h}$, they correspond to games where a further constraint has been imposed on u. And again, for $h = 2^{-d}T$, the admissible u's for $d + 1$ are a superset of those admissible for d. Hence the value function decreases also. The rest follows. □

The main theorem is now a consequence of a last lemma.

Lemma 3.4. *Let W be the value function of the game $\mathcal{G}$ and $\widetilde{W} = \lim_{h\to 0} W^{h,h}$. Then*

$$\widetilde{W} = W\,. \tag{33}$$

Proof. The detailed proof is given in Appendix C. It uses the method of [10] for the game $\mathcal{J}$, whose value is W according to Corollary 3.1, and uses the fact that it follows from our analysis of (28), and specifically from Remark 3.1, that the sequence $\{W_k^{h,h}\}_k$ can be identified with the value function of the discretized version of the game $\mathcal{J}$. Hence (33) follows. □

Proof of the Main Theorem. of the main theorem: It follows from Lemma 3.3 that W^h converges to some $\widehat{W}$ as $h \to 0$ in a dyadic way. It follows from (32) that $W \le \widehat{W} \le \widetilde{W}$, and from (33) that $\widehat{W} = W$. □

4 Numerical results

We have implemented the recursion (28). We have used the two-step formulation and the procedures of Section 3.1 for the maximization and minimization. We are, of course, obliged to discretize x and y. To evaluate W and W^+ between discretization points, we have used a piecewise affine interpolation on triangles, and to evaluate them beyond the domain of discretization (the evaluation at $((1+\beta)x, (1+\beta)y)$ may require that), a linear extrapolation. Notice that this affine interpolation is essentially equivalent to the space discretization procedure analyzed in [13,14]. Hence, we may expect it to converge to the desired function as the discretization step goes to zero.

We have found that in some very narrow ranges of discretization steps, depending on the parameters, one may get wide numerical instabilities. Yet, being carefull to validate the results as "reasonable," we have a very efficient program. With a 600×600 grid in the (x, y) domain, it runs in about .22 second per time step on a 1.7 GHz PC.

The numerical results corroborate our continuous-time theory. The results we discuss here correspond to the following set of parameters: a time step of $h = 0.02$, $\alpha = .10$, $\beta = .15$, $c_0 = .02$, $c_1 = .05$, and a discretization step of .005 in x and y.

For large t's (the first time steps) the program finds y^- and y^+ at both ends of the domain of y. Then for $T-t$ larger than .52, it finds y^- within the range of discretization and y^+ at the boundary. The theoretical value is $T-t_\beta = .46$. For $T-t$ larger than .70, it finds y^- and y^+ either equal or within one discretization step, the latter being a normal discretization effect. The theoretical value is $T - t_\alpha = .71$. Thus t_α has been recovered with a good accuracy (within one time step) while t_β is recovered with an error of three time steps. When y^- and y^+ differ within one discretization step, we have taken $(1/2)(y^- + y^+)$ as the approximation of $\check{y}$, and the smallest W as the approximation of $\check{z}$.

We have also implemented a numerical integration of the differential equations for $\mathcal{E}$ (or used the closed form found later. It makes no observable difference) and of the PDE for $\mathcal{F}$. The latter can exhibit numerical instabilities with bad choices of method. We got good results with a second-order centered finite difference scheme in "space" and a Runge–Kutta method of order two in time. Our computer code (in MATLAB) is still far from being optimized in terms of computing time. Thus this aspect will not be discussed here.

We have made the comparisons with a short maturity of $T = 5$ to save computation time in the computation of the focal surface. Both methods gave the same graphs for $\check{y}$ within one or two discretization steps ($.005Z$), except close to the boundary of the discretized domain, and almost the same graphs for $\check{z}$ to within two discretization steps, the discrete time $\check{z}$ being slightly larger, as expected. Both graphs are plotted in Figure 2.

5 Variants and Related Works

5.1 Another Terminal Target

Another game, maybe more significant for the finance application, but less rich in terms of game theory, is obtained by replacing $M(x)$ by $N(x,y) := M(x) + C_\varepsilon(-y)$, with $\varepsilon = \operatorname{sign}(-y)$. (See [4] for a motivation.) Then the sheet (α) of the barrier does not exist any longer, and thus neither does the dispersal manifold. The first singularity met (rearward) is an equivocal junction on the sheet (β), and before (in forward time) a focal surface. The theory is essentially the same.

5.2 The Viability Approach

In a series of papers [1,15] and in private communications, Aubin, Saint-Pierre, Pujal, and collaborators have considered, with the same motivation, essentially the same continuous-time problem, slightly more general in some aspects (they allow for constraints that were not considered in our work). They put a bound on the magnitude of our u to avoid impulses. But this is mainly for theoretical reasons, to get existence results for the viscosity solution of Isaacs' equation, an issue we did not tackle. They use a capture basin type of approach (similar to our *game of kind* approach) and discretize the corresponding PDE, leading to the same recursion (28) as ours, or a slightly different one depending on whether they use an explicit or implicit scheme. There, if taken large enough, the bounds on u are inactive.

A noteworthy feature is Saint-Pierre's "decoupling algorithm," which, for the above variant (Section 5.1), let him compute the locus $\check{y}$ and $\check{z}$, as the locus of

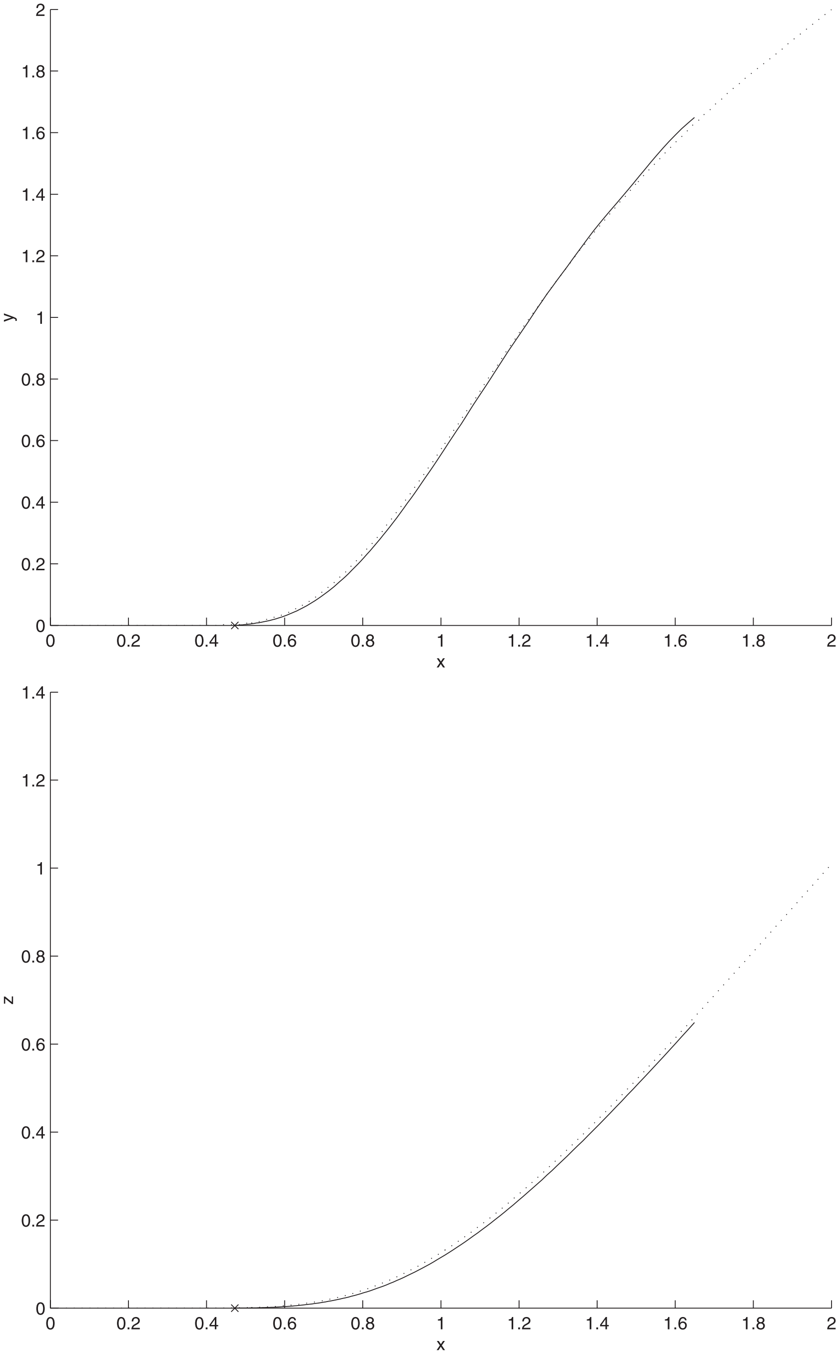


Figure 2: Cut of the focal manifold $\mathcal{F}$: $\check{y}(t,x)$ and $\check{z}(t,x)$ for $T-t=5$. Dotted line: discrete time. Solid line: continuous time.

the minimum in y of the solution, with a computing effort comparable to two dynamic programming algorithms in dimension one instead of one in dimension two. Coupled with our theory as synthesized above, this is the fastest known way to compute the value of that game.

6 Conclusion

We have provided two closely related ways of investigating impulse controls in a differential game, both linked to the fact that "jump trajectories" can be regarded as ordinary trajectories. In this game, the optimal strategy of the pursuer contains both an impulse at initial time, and finite controls later on as the state traverses the singular manifolds. Admittedly, here the optimal strategy has the weakness that it needs to sense instantaneously the opponent's control, i.e., here the time derivative of the first state variable. Breakwell has discussed this feature and approximate implementation in other papers [7,8]. Here, our discrete-time theory points to a practical solution of that problem.

This approach is feasible only because the cost of jumps was supposed to be proportional to the amplitude of the jump. It would be interesting to consider a cost affine in the amplitude, with a positive infimum. This would probably entail an investigation of the QVI according to the theory of [3].

More significantly perhaps, this analysis proves correct an old conjecture by Arik Melikyan that in higher dimensions, focal surfaces would be traversed by noncollinear optimal fields of trajectories. We have shown in detail that this is indeed the case here.

There remains to derive from the above analysis a general construction of higher-dimensional focal surfaces, which was missing in our constructive theory of singularities of co-dimension one in the Isaacs equation of (deterministic) two-person zero-sum differential games [5,6].

We have also proved and checked numerically that the continuous-time solution can be approached by the natural discrete-time game associated to our differential game. Yet, while that approach lets one numerically compute the value function, it does not give the more explicit form of Section 2.5, nor our detailed description of the optimal continuous-time strategies.

Appendix A: Equations of the Manifold $\mathcal{E}$

We recall the equations of the manifold $\mathcal{E}$:

$$\begin{aligned}
\dot{x} &= \beta x\,,\\
\dot{y} &= \frac{\beta(1+C^+)+\alpha \mathrm{e}^{-\alpha(T-t)}}{1+C^+-\mathrm{e}^{-\alpha(T-t)}}y\,,\\
\dot{z} &= \frac{\beta(1+C^+)(1-\mathrm{e}^{-\alpha(T-t)})-C^+\alpha \mathrm{e}^{-\alpha(T-t)}}{1+C^+-\mathrm{e}^{-\alpha(T-t)}}y\,,
\end{aligned}$$

to be integrated backwards from the terminal states $(t_0, x(t_0), y(t_0), z(t_0))$ either on the boundary of the manifold $\mathcal{D}$ at $t = t_\beta$ or on the boundary parametrized by $x = Z\mathrm{e}^{(\alpha(T-t))}$, $t_\alpha \le t \le t_\beta$. These equations admit a closed form solution as follows:

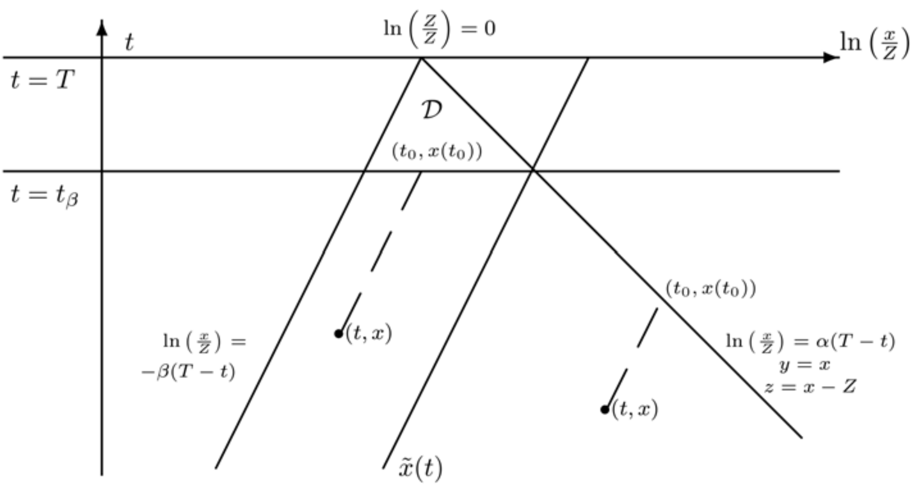
$$\begin{aligned}
x(t) &= x(t_0)e^{\beta(t-t_0)} , \\
y(t) &= y(t_0)e^{\beta(t-t_0)} \left(\frac{1 + C^+ - e^{\alpha(t_0-T)}}{1 + C^+ - e^{\alpha(t-T)}}\right)^{\frac{\alpha+\beta}{\alpha}} , \\
z(t) &= (1 - e^{-\alpha(T-t)})y(t) + z(t_0) - (1 - e^{-\alpha(T-t_0)})y(t_0),
\end{aligned}$$

as can be checked by direct differentiation. The expressions for y and z can be rewritten in terms of x and t, upon substituting for t_0, $y(t_0)$, and $z(t_0)$, to yield $\check{y}(t, x)$ and $\check{z}(t, x)$.

Let $\tilde{x}(t) = Ze^{(\alpha+\beta)(T-t_\beta)}e^{-\beta(T-t)}$:

$$\begin{cases}
\text{if } x \le \tilde{x}(t): & \begin{cases} t_0 = t_\beta \ , \\ y(t_0) = \frac{xe^{\beta(T-t)} - Z}{1 + C^+ - e^{-\alpha(T-t_\beta)}} \ , \\ z(t_0) = (1 - e^{-\alpha(T-t_\beta)})y(t_0) \ , \end{cases} \\
\\
\text{if } x \ge \tilde{x}(t) : & \begin{cases} t_0 = \frac{1}{\alpha+\beta}(\alpha T + \beta t) \ln\left(\frac{x}{Z}\right) \ , \\ y(t_0) = xe^{\beta(T-t_0)} , \\ z(t_0) = y(t_0) - Z \ . \end{cases}
\end{cases}$$

We can remark that in the region $x \le \tilde{x}(t)$ we still have $w(t) = (1 - e^{-\alpha(T-t)})y(t)$ as on the manifold $\mathcal{D}$.



Appendix B: The PDE for the Focal Manifold $\mathcal{F}$

B.1 Analysis

As the trajectories $v = \beta$ and $v = -\alpha$ describe the same focal manifold $\mathcal{F}$ in (s, x, y, z) space, we only solve the PDE system (17), which we rewrite as

$$Y_t^\beta = sA^\beta Y_s^\beta + B^\beta Y^\beta.$$

Let us pick

$$\sigma = \ln\left(\frac{s}{Z}\right),$$

which transforms the PDE system in a linear PDE system of first order with constant coefficients in (t, σ):

$$Y_t^\beta = A^\beta Y_\sigma^\beta + B^\beta Y^\beta. \tag{34}$$

Moreover, the domain of interest Ω simplifies into the new domain in (t, σ): $\Omega_\sigma := \{t \le t_\alpha,\, 0 \le \sigma \le (\alpha + \beta)(T - t)\}$.

We notice that the known solutions $\check{y}$ and $\check{z}$ outside Ω, namely $(0, 0)$ to the "left" of Ω and $(x, x - Z)$ to the right, satisfy the PDE for $\mathcal{F}$ (34). Moreover, we have the following fact.

Proposition B.1. *If (34) admits a continuous solution on* $[0, T] \times [-\infty, \infty]$ *(in the domain* (t, σ)*) with simple discontiuities in* $(\nabla y, \nabla z)$*, these discontinuities are born by lines of slope 0 or* $-(\alpha + \beta)$ *in the plane* t, σ*.*

Hence, if such discontinuities follow from the discontinuity in $\sigma = 0$ at terminal time, they will precisely be born by the boundaries of Ω.

Proof. Let Δy_t, Δy_σ, Δz_t, and Δz_σ be the discontinuities. Let (p, q) be the direction of a smooth curve bearing the discontinuity in the (t, σ) domain. The continuity of both y and z implies that

$$p \begin{pmatrix} \Delta y_t \\ \Delta z_t \end{pmatrix} + q \begin{pmatrix} \Delta y_\sigma \\ \Delta z_\sigma \end{pmatrix} = 0\,.$$

Moreover, because at the discontinuity both sides satisfy the PDE (34), it follows that

$$\begin{pmatrix} \Delta y_t \\ \Delta z_t \end{pmatrix} = A^\beta \begin{pmatrix} \Delta y_\sigma \\ \Delta z_\sigma \end{pmatrix}\,.$$

Hence, combining these two equations, we obtain

$$(pA^\beta + qI) \begin{pmatrix} \Delta y_\sigma \\ \Delta z_\sigma \end{pmatrix} = 0.$$

Since, by hypothesis, the vector is nonzero, p cannot be 0, and $-q/p$ is an eigenvalue of A^β. These are 0 and $\alpha + \beta$. $\square$

B.2 Numerical Integration

We decided to use the fact that $\check{y}$ and $\check{z}$ are known outside of the domain of interest Ω in the numerical procedure. We compared this approach with a global integration relying on the preceding analysis. But the latter gave, not surprisingly, less precise results close to the boundary of Ω.

Hence, the boundary conditions in (t, σ) are also affine but the domain is not rectangular, the range in σ is a function of t. Let $\sigma_\ell = \sigma_0 + \ell\delta_\sigma$ where $\ell = 0, \dots, N-1$ are the values of the discretization of the variable σ with a step of δ_σ on the domain $(t, \sigma) = [0, t_\alpha] \times [0, (\alpha+\beta)T + \delta_\sigma]$ including Ω_σ. We shall explain this choice hereafter.

At any time $t \leq t_\alpha$, we consider the vector of fixed dimension $2N \times 1$:

$$Y^\beta(t) = \begin{pmatrix} Y^\beta(t, \sigma_0) \\ \vdots \\ Y^\beta(t, \sigma_\ell) \\ \vdots \\ Y^\beta(t, \sigma_{N-1}) \end{pmatrix} \text{ with } Y^\beta(t, \sigma_\ell) = \begin{pmatrix} y(t, \sigma_\ell) \\ z(t, \sigma_\ell) \end{pmatrix} .$$

We denote by $Y^\beta_\sigma(t)$ the vector of the derivatives in σ of the vector $Y^\beta(t)$. We will approach it by finite differences in σ. Thus the PDE system leads to an ODE system of $2N$ equations of the form

$$Y^\beta_t(t) = M(t)Y(t). \tag{35}$$

The interest of the new domain is that we will work with a matrix $M(t)$ of constant dimension.

In the domain $\sigma \geq (\alpha+\beta)(T-t)$, we replace the PDE system (34) by the system satisfied by $x = y$, $z = x - Z$ with $\dot{x} = \beta x$, i.e.:

$$\begin{cases} \dot{y} &= \beta y \ , \\ \dot{z} &= \beta y \ . \end{cases} \tag{36}$$

This leads to a matrix $M(t)$ whose lines corresponding to $\sigma \leq (\alpha+\beta)(T-t)$ implement Equation (34) while those corresponding to $\sigma \geq (\alpha+\beta)(T-t)$ implement Equation (36). Hence, M is time varying.

To solve the ODE system (35), we have tried different numerical methods of lower order (1 or 2). Some methods exhibit numerical instabilities, but we got good results with a second-order centered finite difference scheme in “space” and a Runge–Kutta method of order two in time.

Appendix C: Proofs of the Lemmas

C.1 Proof of Lemma 3.1

Lemma 3.1. *For any* $\mathbf{P}$*'s control srategy* φ *in the game* $\mathcal{G}$*, and any positive* δ*, there exists an admissible (causal) strategy in the game* $\mathcal{G}'$ *that yields against any admissible* $v(\cdot)$ *a payoff within* δ *of the payoff obtained with* φ *in the game* $\mathcal{G}$*.*

Proof. We shall only prove that y can be approximated uniformly arbitrarily well. The proof for its integral follows. In fact, integrating (2) yields

$$y(t) = y(0)\exp\left(\int_0^t v(\tau)\,\mathrm{d}\tau\right) + \bar{y}(t),$$

where only $\bar{y}$ depends on u. Thus it suffices to approximate $\bar{y}$. Let δ be a given positive number; we shall show how to approximate $\bar{y}(t)$ within δ uniformly in t and $v(\cdot)$.

Pick a strategy φ. For a disturbance $v(\cdot)$ given, it generates a time function (or distribution) $u(\cdot)$ that may contain impulses. One has

$$\bar{y}(t) = \int_0^t \exp\left(\int_s^t v(\tau)\,\mathrm{d}\tau\right) u(s)\,\mathrm{d}s\,.$$

We decompose $u(\cdot)$ as $u(t) = u^+(t) - u^-(t)$, its positive and negative parts (including the positive and negative impulses). In an obvious way, this induces a decomposition $\bar{y} = \bar{y}^+ - \bar{y}^-$.

Proposition C.1. *Under our hypotheses, we may assume that both* $\bar{y}^+$ *and* $\bar{y}^-$ *are uniformly bounded over all admissible* $v(\cdot)$*'s for any initial state.*

Proof of the Proposition. In investigating the value of the game $\mathcal{G}$, we may restrict our attention to strategies φ that do better than a given strategy φ_0. Choose, for instance, φ_0 as the strategy made of an initial jump to $y = 0$ at time $t = 0$ (i.e., $t_0 = 0$ and $u_0 = -y(0)$), and $u = 0$ from then on. It yields $z(T) = z(0) - C_\varepsilon(-y(0))$. Thus we restrict our attention to strategies φ that yield a larger $z(T)$ for all admissible $v(\cdot)$'s. Now, since $y(t)$ is by hypothesis uniformly bounded, so is $\int_0^T v(t)y(t)\,\mathrm{d}t$. According to (3), $z(T) = z(0) + \int_0^T (vy - C_\varepsilon u)\,\mathrm{d}t$. Therefore, $\int_0^T C_\varepsilon u(t)\,\mathrm{d}t$ is also uniformly bounded. But if we let $C = \max\{C^+, -C^-\}$ (C is positive), we have $C_\varepsilon u \geq C|u|$. Hence the integral of $|u|$ is uniformly bounded, and a fortiori those of u^+ and u^-, and finally also $\bar{y}^+$ and $\bar{y}^-$. □

We shall do the approximation for each of these two parts separately. Hence, from now on, we may assume that $u(t) \geq 0$, or more precisely that $\int |u(s)|\,\mathrm{d}s = \int u(s)\,\mathrm{d}s$.

Let therefore $y_{\max}$ be an (uniform) upper bound of $\bar{y}(t)$, pick ϵ (< 1 and) such that $\epsilon y_{\max} \le \delta/2$, and let h be a positive number such that, for every admissible $v(\cdot)$, and $\forall t \in [h, T]$,

$$\left|\exp\left(\int_{t-h}^{t} v(\tau)\,\mathrm{d}\tau\right) - 1\right| \le \frac{\epsilon}{2} \le \frac{1}{2}$$

(this is possible uniformly in $v(\cdot)$ because $|v(t)|$ is bounded), and thus a fortiori, $\forall s \in [t-h, t]$,

$$\left|\exp\left(\int_{s}^{t} v(\tau)\,\mathrm{d}\tau\right) - 1\right| \le \frac{\epsilon}{2} \le \epsilon \exp\left(\int_{s}^{t} v(\tau)\,\mathrm{d}\tau\right). \tag{37}$$

We advocate the impulses-only strategy using impulses of amplitude u_k at the instants $t_k = kh$, $k \in \mathbb{N}$ as follows:

$$u_k = \int_{t_k-h}^{t_k} u(t)\,\mathrm{d}t\,.$$

This yields for $\bar{y}$ a time function that we denote $\tilde{y}$:

$$\tilde{y}(t) = \sum_{k|t_k<t} u_k \exp\left(\int_{t_k}^{t} v(\tau)\,\mathrm{d}\tau\right).$$

The difference $\Delta(t) = |\bar{y}(t) - \tilde{y}(t)|$ can be written as

$$\Delta(t) = \left|\sum_{k|t_k<t} \int_{t_k-h}^{t_k} \left[\exp\left(\int_{s}^{t_k} v(\tau)\,\mathrm{d}\tau\right) - 1\right] u(s)\,\mathrm{d}s \exp\left(\int_{t_k}^{t} v(\tau)\,\mathrm{d}\tau\right)\right|,$$

hence

$$\Delta(t) \le \sum_{k|t_k<t} \int_{t_k-h}^{t_k} \left|\exp\left(\int_{s}^{t_k} v(\tau)\,\mathrm{d}\tau\right) - 1\right| u(s)\,\mathrm{d}s \exp\left(\int_{t_k}^{t} v(\tau)\,\mathrm{d}\tau\right).$$

According to (37),

$$\Delta(t) \le \epsilon \sum_{k|t_k<t} \int_{t_k-h}^{t_k} \exp\left(\int_{s}^{t_k} v(\tau)\,\mathrm{d}\tau\right) u(s)\,\mathrm{d}s \exp\left(\int_{t_k}^{t} v(\tau)\,\mathrm{d}\tau\right) = \epsilon\bar{y}(t)\,.$$

Hence, for each of the positive and negative parts of $\bar{y}$ we have

$$\tilde{y}(t) \in [(1-\epsilon)\bar{y}(t)\,,\ (1+\epsilon)\bar{y}(t)] \subset \left[\bar{y}(t) - \frac{\delta}{2}\,,\ \bar{y}(t) + \frac{\delta}{2}\right].$$

C.2 Proof of Lemma 3.4

We consider the following discrete scheme associated to Joshua's transform:

$$\begin{cases} t((k+1)h) &= t(kh) + h\bar{\jmath}, \\ x((k+1)h) &= x(kh) + h\bar{\jmath}vx(kh), \\ y((k+1)h) &= y(kh) + h(\bar{\jmath}vy(kh) + j), \end{cases}$$

with the payoff defined with $t(\mathcal{T}) = T$ and

$$J = M(u(\mathcal{T})) + \sum_{k=0}^{N-1} h(-\bar{\jmath}vy(kh) + C_j j)\,,$$

and the controls $j \in \{-1, 0, 1\}$ and $v \in [-\alpha^h, \beta^h]$, where (see (26)):

$$\alpha^h = \frac{\alpha_h}{h} = \frac{1}{h}(1 - \mathrm{e}^{-\alpha h})\,, \qquad \beta^h = \frac{\beta_h}{h} = \frac{1}{h}(\mathrm{e}^{\beta h} - 1)\,.$$

We notice that $\alpha^h \to \alpha$ and $\beta^h \to \beta$ as $h \to 0$.

The Isaacs equation of the above multistage game concerns a function V^h and reads:

$$\begin{aligned} &\forall (t,x,y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}\,, \\ &0 = \min_{j \in \{-1,0,1\}} \max_{v \in [-\alpha^h, \beta^h]} \Big[V^h(t + h\bar{\jmath}, x + h\bar{\jmath}vx, y + h(\bar{\jmath}vy + j)) \\ &\qquad\qquad\qquad\qquad - V^h(t,x,y) + h(-\bar{\jmath}vy + C_j j) \Big], \\ &\forall t \geq T\,, \quad V^h(t,x,y) = M(x). \end{aligned} \tag{38}$$

Now, we want to prove that V^h converges towards V, where V is the viscosity solution of the following Isaacs equation, associated to the continuous Joshua form:

$$0 = \min_j \max_{v \in [-\alpha, \beta]} \left[\frac{\partial V}{\partial t}\bar{\jmath} + \frac{\partial V}{\partial x}\bar{\jmath}vx + \frac{\partial V}{\partial y}(\bar{\jmath}vy + j) + (-\bar{\jmath}vy + C_j j) \right]$$

with the same boundary condition.

We recall the definition of a viscosity solution of the last Isaacs equation. A bounded uniformly continuous function V is called a viscosity solution of the Isaacs equation above if for each $\phi \in C^1(\mathbb{R}^3)$, the following hold:

(1) if $V - \phi$ attains a strict local maximum at $a_0 = (t_0, x_0, y_0)$, then

$$\min_j \max_v \left[\frac{\partial \phi}{\partial t}(a_0)\bar{\jmath} + \frac{\partial \phi}{\partial x}(a_0)\bar{\jmath}vx_0 + \frac{\partial \phi}{\partial y}(a_0)(\bar{\jmath}vy_0 + j) - \bar{\jmath}vy_0 + C_j j \right] \geq 0\,,$$

(2) if $V - \phi$ attains a strict local minimum at $a_1 = (t_1, x_1, y_1)$, then

$$\min_j \max_v \left[\frac{\partial \phi}{\partial t}(a_1)\bar{\jmath} + \frac{\partial \phi}{\partial x}(a_1)\bar{\jmath}vx_1 + \frac{\partial \phi}{\partial y}(a_1)(\bar{\jmath}vy_1 + j) - \bar{\jmath}vy_1 + C_j j \right] \leq 0\,.$$

Proof. Notice first that, expanding the $\min_j$ according to the three possible values of j, and replacing $hv \in [-\alpha^h, \beta^h]$ by
$v \in [-\alpha_h, \beta_h]$, (38) also reads

$$\min\Big\{ \max_{v \in [-\alpha_h, \beta_h]} \left[V^h(t+h, (1+v)x, (1+v)y) - V^h(t,x,y) - vy \right] ,\\ V^h(t,x,y-h) - C^- h\,, \quad V^h(t,x,y+h) + C^+ h \Big\}\,,$$

so that, using Remark 3.1 we may conclude that V^h coincides with $W^{h,h}$. Thus, we know from Lemma 3.3 that there exists a V (called W in the body of the paper) such that

$$V^h \to V \qquad \text{uniformly on any compact of } \mathbb{R}^3 \text{when } \; h \to 0\,. \tag{39}$$

Let $\phi \in C^1(\mathbb{R}^3)$ and a_0 be a strict local maximum for $V - \phi$. Then there exists a closed ball B centered at a_0 such that

$$(V - \phi)(a_0) > (V - \phi)(a), \qquad \forall a \in B. \tag{40}$$

Let now a_0^h be a maximum point for $V^h - \phi$ over B.

Lemma C.1.

$$a_0^h \to a_0, \qquad \text{when } \; h \to 0. \tag{41}$$

Proof. a_0^h remains in the compact B. Let $\bar{a}$ be a cluster point of the sequence $\{a_0^h\}$ and $\{a_0^{h_i}\}$ be a subsequence converging to $\bar{a}$. By definition we have that $(V^{h_i} - \phi)(a_0^{h_i}) \geq (V^{h_i} - \phi)(a)$, for all $a \in B$, and then, by continuity of V^{h_i} and ϕ and using (39), we get $(V - \phi)(\bar{a}) \geq (V - \phi)(a)$, $\forall a \in B$. By unicity of the maximum, we have that $\bar{a} = a_0$. The cluster point $\bar{a}$ is then unique, so the whole sequence a_0^h converges towards a_0. $\square$

Now since $h \to 0$, we have that $(t_0^h + h\bar{\jmath}, x_0^h + h\bar{\jmath}vx_0^h, y_0^h + h(\bar{\jmath}vy_0^h + j))$ remains in B. Since a_0^h is a maximum point for $V^h - \phi$ over B, we have

$$\begin{aligned} V^h(t_0^h, x_0^h, y_0^h) &- \phi(t_0^h, x_0^h, y_0^h) \\ &\geq V^h\Big(t_0^h + h\bar{\jmath}, x_0^h + h\bar{\jmath}vx_0^h, y_0^h + h(\bar{\jmath}vy_0^h + j)\Big) \\ &\quad - \phi\Big(t_0^h + h\bar{\jmath}, x_0^h + h\bar{\jmath}vx_0^h, y_0^h + h(\bar{\jmath}vy_0^h + j)\Big), \end{aligned}$$

Using the last inequality together with Equation (38) and also using the monotonicity of the "minmax" operator, we get the following, where v is always understood to lie in $[-\alpha^h, \beta^h]$:

$$\begin{aligned} 0 &= \min_j \max_v \Big[V^h(t_0^h + h\bar{\jmath}, x_0^h + h\bar{\jmath} v x_0^h, y_0^h + h(\bar{\jmath} v y_0^h + j)) \\ &\qquad\qquad - V^h(t_0^h, x_0^h, y_0^h) + h(-\bar{\jmath} v y_0^h + C_j j) \Big] \\ &\le \min_j \max_v \Big[\phi(t_0^h + h\bar{\jmath}, x_0^h + h\bar{\jmath} v x_0^h, y_0^h + h(\bar{\jmath} v y_0^h + j)) \\ &\qquad\qquad - \phi(t_0^h, x_0^h, y_0^h) + h(-\bar{\jmath} v y_0^h + C_j j) \Big]. \end{aligned}$$

Since $\phi \in C^1(\mathbb{R}^n)$, from the last inequality, we get

$$0 \le \min_j \max_v h \left[\frac{\partial \phi}{\partial t}(b^h)\bar{\jmath} + \frac{\partial \phi}{\partial x}(b^h)\bar{\jmath} v x_0^h + \frac{\partial \phi}{\partial y}(b^h)(\bar{\jmath} v y_0^h + j) - \bar{\jmath} v y_0^h + C_j j \right],$$

where b^h is in the segment $[(t_0^h, x_0^h, y_0^h), (t_0^h + h\bar{\jmath}, x_0^h + h\bar{\jmath} v x_0^h, y_0^h + (\bar{\jmath} v y_0^h + j))]$. Since $h > 0$, we may divide through by h; then it follows that

$$0 \le \min_j \max_v \left[\frac{\partial \phi}{\partial t}(b^h)\bar{\jmath} + \frac{\partial \phi}{\partial x}(b^h)\bar{\jmath} v x_0^h + \frac{\partial \phi}{\partial y}(b^h)(\bar{\jmath} v y_0^h + j) - \bar{\jmath} v y_0^h + C_j j \right].$$

Since a_0^h converges towards a_0 and since h converges towards zero, it follows that b^h also converges towards a_0. Moreover, the bracket is continuous in $(v, (t, x, y))$, therefore in (v, h) and therefore uniformly continuous in (v, h) in a (closed) neighborhood of $[-\alpha, \beta] \times \{0\}$. Thus we may pass to the limit for each value of j to conclude that

$$\min_j \max_{v \in [-\alpha, \beta]} \left[\frac{\partial \phi}{\partial t}(a_0)\bar{\jmath} + \frac{\partial \phi}{\partial x}(a_0)\bar{\jmath} v x_0 + \frac{\partial \phi}{\partial y}(a_0)(\bar{\jmath} v y_0 + j) - \bar{\jmath} v y_0 + C_j j \right] \ge 0.$$

The proof is the same, mutatis mutandis, for point (2) of the definition of the viscosity solution of the Isaacs equation considered. □

REFERENCES

[1] Aubin J.-P., Pujal D. and Saint-Pierre P., *Dynamic Management of Portfolios with Transaction Costs under Tychastic Uncertainty*, preprint, 2001.

[2] Barles G., Deterministic impulse control problems, *SIAM Journal on Control and Optimization*, 23, 419–432, 1985.

[3] Bensoussan A. and Lions J-L., *Contrôle impulsionnel et inéquations quasivariationnelles*, Gauthier-Villars, Paris, 1982.

[4] Bernhard P., A robust control approach to option pricing including transaction costs, in A. Nowak (ed.), *Annals of the ISDG*, 7, Birkhauser, 2002.

[5] Bernhard P., Singular surfaces in differential games: an introduction, in P. Hagedorn, H.W. Knobloch, and G.J. Olsder (eds.), *Differential Games and Applications*, Proceedings of a workshop held in Twente University of Technology, 1977, Springer Verlag Lecture Notes in Control and Information Sciences, 3, 1–33, 1977.

[6] Bernhard P., *Contribution à l'étude des jeux différentiels à deux joueurs et somme nulle*, Ph.D. Dissertation, Université Paris 6, 1989.

[7] Breakwell J.V., Time optimal pursuit inside a circle, in T. Başar and P. Bernhard (eds.), *Differential Games and Applications*, Springer Lecture Notes in Control and Information Sciences, 119, Springer, Berlin, 1989.

[8] Breakwell J.V. and Bernhard P., A simple game with a singular focal line, *JOTA*, 64, 1990.

[9] Breakwell J.V. and Merz A.W., Toward a Complete Solution of the Hommicidal Chauffeur Game, *First International Conference on the Theory and Applications of Differential Games*, Amherst, 1969.

[10] Capuzzo Dolcetta I., On a discrete approximation of the Hamilton–Jacobi equation of dynamic programming, *Applied Mathematics and Optimization*, 10, 367–377, 1983.

[11] Joshua: The Book of Joshua, 10:12–13, in *The Bible* (circa 1000 to 500 AD).

[12] Merz A.W., *The Homicidal Chauffeur: A Differential Game*, Ph.D. Dissertation, SUDAAR Report, Stanford University, 1971.

[13] Pourtallier O. and Tidball M., Approximation of the value function for a class of differential games with target, *Sixth International Symposium on Dynamic Games and Applications*, St. Jovite, Canada, July 1994.

[14] Pourtallier O., Tidball M. and Altman E., Approximations in dynamic zero-sum games, II, *SIAM J. Control and Optimization*, 35, 2101–2117, 1997.

[15] Pujal D., *Évaluation et gestion dynamiques de portefeuilles*, Ph.D. Dissertation, Université Paris Dauphine, 2000.

[16] Roorda B., Engwerda J. and Schumacher H., *Performance of Hedging Strategies in Interval Models*, to appear.