A robust control approach to option pricing: the uniqueness theorem

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February, 02, 2012

Abstract

We prove the missing uniqueness theorem which makes our probability-free theory of option pricing in the interval market model, essentially complete.

Keywords Option pricing, viscosity solutions, robust control.
AMS classification 35D40, 91G20, 91G80, 91A23, 49N90, 49N25.

1 Introduction

In a series of papers starting with [4], and culminating, so far, with [6, 5, 7] we have developped a probability-free theory of option pricing, both for vanilla options and digital options. The most comprehensive account of this theory is in the unpublished doctoral dissertation of Stéphane Thiery [13]. A rather complete account is to appear in the volume [8].

The main claims of that new approach are, on the one hand, the possibility of constructing a consistent theory of hedging portfolios with either continuous or discrete time trading paradigms, the former being the limit of the latter for vanishing time steps, with one and the same (continuous time) market model, and, on the other hand, to accommodate transaction costs and closing costs in a natural way, with a nontrivial hedging portfolio.

It may also be argued that although it seems somewhat un-natural, still our market model implies much less knowledge about the future market

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prices than the classical probabilistic Samuelson model, used in the Black and Scholes theory. A discussion of the strengths and weaknesses of the new approach, as well as of related contributions in the literature, mostly [1] and [10], can be found in [7].

The reference [13] stresses that the last missing item is a uniqueness theorem for the viscosity solution of a particular, highly degenerate, Isaacs Differential Quasi Variational Inequality (DQVI). In the article [7], we got around that difficulty by resorting to a refined form of Isaacs'verification theorem. However, on the one hand, this relies on the true, but unpublished, fact that the viscosity condition implies satisfaction of our old "corner conditions" [3], and on the other hand, it is much less satisfactory than directly proving that uniqueness.

In this article, we sketch the overall context and prove the uniqueness sought. Notice, however, that the present proof does not account for the discontinuous payment digital option, while that of [3] can be extended to that case, thanks to the concept of barrier.

2 Modelization

2.1 Option pricing

Our problem relates to an economy with a fixed, known, riskless interest rate ρ. In a classical fashion, all monetary values will be assumed to be expressed in end-time value computed at that fixed riskless rate, so that, without loss of generality, the riskless rate can be taken as (seemingly) zero.

We consider a financial derivative called *option* characterized by

- an exercize time, or initial maturity, $T > 0$,
- an underlying security, such as a stock or currency, whose *price* on the market is always well defined. This price at time t is usually called $S(t)$. As indicated above, we shall use instead its end-time price $u(t) = \mathrm{e}^{\rho(T-t)}S(t)$,
- a closure payment $M(u(T))$. Typical instances are $M(u) = \max\{u - K, 0\}$ (for a given *exercize price* K) for a vanilla call, or $M(u) = \max\{K - u, 0\}$ for a vanilla put.

2.2 Market

We share with Roorda, Engwerda, and Schumacher [12, 11] the view that a market model is a set Ω of possible price trajectories, and we borrow from them the name of *interval model* for our model. It is defined by two real numbers $\tau^- < 0$ and $\tau^+ > 0$, and Ω is the set of all absolutely continuous functions $u(\cdot)$ such that for any two time instants t_1 and t_2,

$$e^{\tau^-(t_2-t_1)} \leq \frac{u(t_2)}{u(t_1)} \leq e^{\tau^+(t_2-t_1)} . \tag{1}$$

The notation τ^ε will be used to handle both τ^+ and τ^- at a time. Hence, in that notation, it is understood that $\varepsilon \in \{-,+\}$, sometimes identified to $\{-1,+1\}$. We shall also let $(\tau_\star, \tau^\star) = (\min_\varepsilon |\tau^\varepsilon|, \max_\varepsilon |\tau^\varepsilon|)$.

We shall make use the equivalent characterization

$$\dot{u} = \tau u\,, \quad u(0) = u_0\,, \quad \tau \in [\tau^-, \tau^+]\,. \tag{2}$$

In that formulation, $\tau(\cdot)$ is a measurable function, which plays the role of the "control" of the market. We shall let Ψ denote the set of measurable functions from $[0,T]$ into $[\tau^-,\tau^+]$. It is equivalent to specify a $u(\cdot)\in\Omega$ or a $(u(0),\tau(\cdot))\in\mathbb{R}^+\times\Psi$. This is an a priori unknown time function. The concept of nonanticipative strategies embodies that fact.

2.3 Portfolio

A (hedging) portfolio will be composed of an amount v (in end-time value) of underlying stock, and an amount y of riskless *bonds*, for a total worth of $w = v + y$. In the normalized (or end-value) representation, the bonds are seemingly with zero interest.

2.3.1 Buying and selling

We let $\xi(t)$ be the buying rate (a sale if $\xi(t) < 0$), which is the trader's control. Therefore we have, in continuous time,

$$\dot{v} = \tau v + \xi\,. \tag{3}$$

However, there is no reason to restrict the buying/selling rate, so that there is no bound on ξ. To avoid mathematical ill-posedness, we explicitly admit "infinite" buying/selling rate in the form of instantaneous block buy or sale of a finite amount of stock at time instants chosen by the trader together with the amount. Thus the control of the trader also involves the choice of finitely many time instants t_k and trading amounts ξ_k, and the model must be augmented with

$$v(t_k^+) = v(t_k) + \xi_k\,, \tag{4}$$

meaning that $v(\cdot)$ has a jump discontinuity of size ξ_k at time t_k. Equivalently, we may keep formula (3) but allow for impulses $\xi_k\delta(t-t_k)$ in $\xi(\cdot)$.

We shall therefore let $\xi(\cdot)\in\Xi$, the set of real time functions (or rather distributions) defined over $[0,T]$ which are the sum of a measurable function $\xi^c(\cdot)$ and a finite number of weighted translated Dirac impulses $\xi_k\delta(t-t_k)$.

2.3.2 Transaction costs

We assume that there are transaction costs. In this paper, we assume that they are proportional to the transaction amount. But we allow for different proportionality coefficients for a buy or a sale of underlying stock. Hence let C^+ be the cost coefficient for a buy, and $-C^-$ for a sale, so that the cost of a transaction of amount ξ is $C^\varepsilon\xi$ with $\varepsilon=\text{sign}(\xi)$. We have chosen C^- negative, so that, as it should, that formula always gives a positive cost.

We shall use the convention that when we write $C^\varepsilon(expression)$, and except if otherwise specified, the symbol ε in C^ε stands for the sign of the *expression*. We shall also let $(C_\star,C^\star)=(\min_\varepsilon|C^\varepsilon|,\max_\varepsilon|C^\varepsilon|)$.

Our portfolio will always be assumed *self-financed*; i.e., the sale of one of the commodities, underlying stock or riskless bonds, must exactly pay for the buy of the other one *and* the transaction costs. It is a simple matter to see that the worth w of the portfolio then obeys

$$\forall t\in(t_{k-1},t_k)\,,\quad \dot w=\tau v-C^\varepsilon\xi^c\,,\quad w(t_{k-1})=w(t_{k-1}^+)\,, \tag{5}$$

between two jump instants, and at jump instants,

$$w(t_k^+)=w(t_k)-C^{\varepsilon_k}\xi_k. \tag{6}$$

This is equivalent to

$$\begin{aligned} w(t)&=w(0)+\int_0^t(\tau(s)v(s)-C^\varepsilon\xi(s))\mathrm{d}s\\ &=w(0)+\int_0^t(\tau(s)v(s)-C^\varepsilon\xi^c(s))\mathrm{d}s-\sum_{k|t_k\le t}C^{\varepsilon_k}\xi_k\,. \end{aligned} \tag{7}$$

2.4 Hedging

2.4.1 Strategies

The initial portfolio is to be created at step 0. As a consequence the seller's price is obtained taking $v(0)=0$. Then, formally, admissible hedging strategies will be functions $\varphi : \Omega \to \Xi$ which enjoy the property of being nonanticipative:

$$\forall (u_1(\cdot), u_2(\cdot)) \in \Omega \times \Omega, \quad [u_1|_{[0,t)} = u_2|_{[0,t)}] \Rightarrow [\varphi(u_1(\cdot))|_{[0,t]} = \varphi(u_2(\cdot))|_{[0,t]}].$$

(It is understood here that the restriction of $\delta(t - t_k)$ to a closed interval not containing t_k is 0, and its restriction to a closed interval containing t_k is an impulse.)

In practice, we shall find optimal hedging strategies made of a jump at initial time, followed by a state feedback law $\xi(t) = \phi(t, u(t), v(t))$.

We shall call Φ the set of admissible trading strategies.

2.4.2 Closing costs

The idea of a hedging portfolio is that at exercise time, the writer is going to close off its position after abiding by its contract, buying or selling some of the underlying stock according to the necessity. We assume that it sustains proportional costs on this final transaction. We allow for the case where these costs would be different from the running transaction costs because compensation effects might lower them and also allow for the case without closing costs just by making their rate 0. Let therefore $c^+ \leq C^+$ and $-c^- \leq -C^-$ be these rates.

It is a simple matter to see that, in order to cover both cases where the buyer does or does not exercise its option, the portfolio worth at final time should be $N(u, v)$, given for a call and a closure in kind by

$$N(u, v) = \max\{c^\varepsilon(-v)\,,\; u - K + c^\varepsilon(u - v)\}\,,$$

where the notation convention for $c^\varepsilon(\mathit{expression})$ holds. We expect that on a typical optimum hedging portfolio for a call, $0 \leq v(T) \leq u(T)$. Hence

$$N(u, v) = \max\{-c^- v\,,\; u - K + c^+(u - v)\}\;. \tag{8}$$

In the case of a put, where $-u(T) \leq v(T) \leq 0$, we need to replace the above expression by

$$N(u, v) = \max\{-c^+ v\,,\; K - u - c^-(u + v)\}. \tag{9}$$

The case of a closure in cash is similar but leads to less appealing mathematical formulas in later developments. The details can be found in [5].

2.4.3 Hedging portfolio

An initial portfolio $(v(0), w(0))$ and an admissible trading strategy φ, together with a price history $u(\cdot)$, generate a dynamic portfolio. We set the following.

Definition 2.1 *An initial portfolio* $(v(0) = 0, w(0) = w_0)$ *and a trading strategy* φ *constitute a* hedge *at* u_0 *if for any* $u(\cdot) \in \Omega$ *such that* $u(0) = u_0$ *(equivalently, for any admissible* $\tau(\cdot)$*), the dynamic portfolio thus generated satisfies*

$$w(T) \geq N(u(T), v(T)) \,. \tag{10}$$

Now, we may use (7) at time T to rewrite this:

$$\forall \tau(\cdot) \in \Psi \,, \quad N(u(T), v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \xi(t) \Big) \mathrm{d}t - w_0 \leq 0 \,.$$

This in turn is clearly equivalent to

$$w_0 \geq \sup_{\tau(\cdot) \in \Psi} \left[N(u(T), v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \xi(t) \Big) \mathrm{d}t \right] .$$

We further set the following.

Definition 2.2 *The* seller's price *of the option at* u_0 *is the worth of the cheapest hedging portfolio at* u_0.

The seller's price at u_0 is therefore

$$P(u_0) = \inf_{\varphi \in \Phi} \sup_{\tau(\cdot) \in \Psi} \left[N(u(T), v(T)) + \int_0^T \Big(-\tau(t)v(t) + C^\varepsilon \xi(t) \Big) \mathrm{d}t \right] , \tag{11}$$

where it is understood that $v(0) = 0$ and that $\xi(\cdot) = \varphi(u_0, \tau(\cdot))$.

3 Solving the minimax impulse control problem

3.1 The related DQVI

We are therefore led to the investigation of the impulse control differential game whose dynamics are given by (2), (3), and (4) and the criterion by (11). In a classical fashion we introduce its Isaacs value function:

$$W(t, u, v) = \inf_{\varphi \in \Phi} \sup_{\tau(\cdot) \in \Psi} \left[N(u(T), v(T)) + \int_t^T \Big(-\tau(s)v(s) + C^\varepsilon \xi(s) \Big) \mathrm{d}s \right] \tag{12}$$

where the dynamics are integrated from $u(t) = u$, $v(t) = v$. Hence the seller's price is $P(u_0) = W(0, u_0, 0)$.

There are new features in that game, in that, on the one hand, impulse controls are allowed, and hence an Isaacs quasi-variational inequality (or QVI; see Bensoussan and Lions [2]) should be at work, but, on the other hand, impulse costs have a zero infimum. As a consequence, that QVI is degenerate, and no general result is available. In [6], we introduce the so-called Joshua transformation that lets us show the following fact.

Theorem 3.1 *The function W defined by* (12) *is a continuous viscosity solution of the following "differential QVI" (DQVI):*

$$\begin{aligned} 0 = \min\Bigg\{ & \frac{\partial W}{\partial t} + \max_{\tau\in[\tau^-,\tau^+]} \tau \left[\frac{\partial W}{\partial u}u + \left(\frac{\partial W}{\partial v} - 1\right) v\right], \\ & \frac{\partial W}{\partial v} + C^+ , \quad -\left(\frac{\partial W}{\partial v} + C^-\right)\Bigg\}, \\ W(T,u,v) = {} & N(u,v)\,. \end{aligned} \tag{13}$$

This PDE in turn lends itself to an analysis, either along the lines of the Isaacs–Breakwell theory through the construction of a field of characteristics for a transformed game as in [6], or using the theory of viscosity solutions and the representation theorem as outlined in [7]. The solution we seek may further be characterized by its behavior at infinity. Yet its uniqueness does not derive from the classical results on viscosity solutions.

3.2 Representation

We introduce two functions $\check{v}(t,u)$, a representation of the singular manifold, and $\check{w}(t,u)$, the restriction of W to that manifold, handled jointly as

$$\mathcal{V}(t,u) = \begin{pmatrix} \check{v}(t,u) \\ \check{w}(t,u) \end{pmatrix}.$$

That pair of functions is entirely defined by a linear PDE that involves the following two matrices (q^- and q^+ are defined hereafter in (15)):

$$\mathcal{S} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{T} = \frac{1}{q^+ - q^-}\begin{pmatrix} \tau^+q^+ - \tau^-q^- & \tau^+ - \tau^- \\ -(\tau^+ - \tau^-)q^+q^- & \tau^-q^+ - \tau^+q^- \end{pmatrix},$$

and it seems to play a very important role in the overall theory. Namely,

$$\mathcal{V}_t + \mathcal{T}(\mathcal{V}_u u - \mathcal{S}\mathcal{V}) = 0\,. \tag{14}$$

The definitions of q^+ and q^-, as well as the terminal conditions at T for (14), depend on the type of option considered. For a simple call or put, and a closure in kind, we have

$$\begin{aligned} q^-(t) &= \max\{(1+c^-)\exp[\tau^-(T-t)]-1\,,\ C^-\}\,,\\ q^+(t) &= \min\{(1+c^+)\exp[\tau^+(T-t)]-1\,,\ C^+\}\,. \end{aligned} \tag{15}$$

Notice that $q^\varepsilon = C^\varepsilon$ for $t \le t_\varepsilon$, with

$$T - t_\varepsilon = \frac{1}{\tau^\varepsilon}\ln\frac{1+C^\varepsilon}{1+c^\varepsilon}\ . \tag{16}$$

The terminal conditions are given, for a call, by

$$\mathcal{V}^t(T,u) = \begin{cases} (\,0 \quad 0\,) & \text{if } u < \dfrac{K}{1+c^+}\,,\\ \dfrac{(1+c^+)u-K}{c^+-c^-}(\,1 \quad -c^-\,) & \text{if } \dfrac{K}{1+c^+} \le u < \dfrac{K}{1+c^-}\,,\\ (\,u \quad u-K\,) & \text{if } u \ge \dfrac{K}{1+c^-} \end{cases} \tag{17}$$

and symmetric formulas for a put. (All combinations call/put, closure in kind/in cash, are detailed in [5]). Standard techniques of hyperbolic PDE's let us prove that that equation has a unique solution with these terminal conditions. (See [13].)

In [7], we proved the following fact:

Theorem 3.2 *The function W defined by the formula*

$$W(t,u,v) = \check{w}(t,u) + q^\varepsilon(\check{v}(t,u)-v)\,, \quad \varepsilon = \mathrm{sign}(\check{v}-v)\,, \tag{18}$$

where q^ε is given by formula (15) *(for a simple call or put), and $(\check{v}\ \check{w}) = \mathcal{V}^t$ is given by* (14) *and the terminal conditions* (17) *for a call (and symmetrical formulas for a put) is a viscosity solution of (12).*

If the uniqueness of the viscosity solution can be proved, this implies that formula (18) indeed provides the Value of the game problem, and hence solves the pricing problem via $P(u_0) = W(0,u_0,0)$. A huge computational advantage as compared to integrating (12).

3.3 Discrete trading

We consider also the case where the trader is only allowed bulk trading ("impulses" in the above setting) at predetermined instants of time $t_k = kh$

$k = 0, 1, \ldots, \mathbb{K}$, with h a given time step and $\mathbb{K}h = T$. Everything else remains unchanged, in particular the market model. A problem interesting in its own right, and, we shall see, as an approximation to the continuous trading solution.

A similar analysis leads to a discrete Isaacs equation,

$$\begin{aligned}
&\forall k < \mathbb{K}, \forall (u, v)\,,\\
&W_k^h(u, v) = \min_{\xi} \max_{\tau \in [\tau_h^-, \tau_h^+]} [W_{k+1}^h((1+\tau)u, (1+\tau)(v+\xi)) - \tau(v + \xi) + C^\varepsilon \xi],\\
&\forall (u, v)\,, \quad W_{\mathbb{K}}^h(u, v) = N(u, v)\,.
\end{aligned} \tag{19}$$

A carefull analysis shows that its solution $\{W_k^h\}_{k \in \{0, \ldots \mathbb{K}\}}$ can be obtained via the following procedure. Notice first that $q_\ell^\varepsilon := q^\varepsilon(t_\ell)$ can be computed via the recursion

$$\begin{aligned}
q_K^\varepsilon &= c^\varepsilon\,,\\
q_{k+\frac{1}{2}}^\varepsilon &= (1 + \tau_h^\varepsilon) q_{k+1}^\varepsilon + \tau_h^\varepsilon\,,\\
q_{k+1}^\varepsilon &= \varepsilon \min\{\varepsilon q_{k+\frac{1}{2}}^\varepsilon\,, \varepsilon C^\varepsilon\}\,,
\end{aligned} \tag{20}$$

Then, let, for all integer ℓ,

$$Q_\ell^\varepsilon = \begin{pmatrix} q_\ell^\varepsilon & 1 \end{pmatrix}, \quad \text{and} \quad \mathcal{V}_\ell^h(u) = \begin{pmatrix} \check{v}_\ell^h(u) \\ \check{w}_\ell^h(u) \end{pmatrix}. \tag{21}$$

Take $\check{v}_{\mathbb{K}}^h(u) = \check{v}(T, u)$, $\check{w}_{\mathbb{K}}^h(u) = \check{w}(T, u)$ as given by (17) for a call (symmetrically for a put) and

$$\mathcal{V}_k^h(u) = \frac{1}{q_{k+\frac{1}{2}}^+ - q_{k+\frac{1}{2}}^-} \begin{pmatrix} 1 & -1 \\ -q_{k+\frac{1}{2}}^- & q_{k+\frac{1}{2}}^+ \end{pmatrix} \begin{pmatrix} Q_{k+1}^+ \mathcal{V}_{k+1}^h((1+\tau^+)u) \\ Q_{k+1}^- \mathcal{V}_{k+1}^h((1+\tau^-)u) \end{pmatrix}. \tag{22}$$

We leave to the reader the tedious, but straightforward, task to check that this is indeed a consistent finite difference scheme for (14). This provides our preferred fast algorithm to compute the premiums in our theory. As a matter of fact, we claim the following:

Theorem 3.3 *The solution of* (19) *is given by* (20), (21), (22), *and* (17) *for a call, as*

$$W_k^h(u, v) = \check{w}_k^h(u) + q_k^\varepsilon(\check{v}_k^h(u) - v) = Q_k^\varepsilon \mathcal{V}_k^h(u) - q_k^\varepsilon v, \quad \varepsilon = \operatorname{sign}(\check{v}_k^h(u) - v).$$

Finally, the main theorem of [6], and a central result in that theory, is the following convergence theorem. (Which can very probably be extended to a sequence $h = T/\mathbb{K}$, $\mathbb{K} \to \infty \in \mathbb{N}$.) Let $W^h(t, u, v)$ be the Value function of the minimax problem where the minimizer is allowed to make an impulse at the initial time t, and then only at the discrete instants t_k as defined above. It is an interpolation of the sequence $\{W_k^h\}$. (This is the correct definition of $W^h(t, u, v)$. It only appears in [13]).)

Theorem 3.4 *Choose $h = T \times 2^{-n}$. As $n \to \infty$ in $\mathbb{N}$, W^h converges uniformly on any compact to a viscosity solution W of the DQVI (13).*

Since (22) can be viewed as a finite difference scheme for (14), it is clear that this limit W is the same function W as given by (18). But here again, we need a uniqueness theorem of the viscosity solution of (13) to conclude that the Value of the discrete trading problem converges towards that of the continuous trading problem.

3.4 Uniqueness

At this point, we know that if the viscosity solution of the DQVI (13) can be proved unique, we have both an interesting representation formula (18) for the value function of the continuous trading problem, and a fast algorithm (22) to approximate it via the Value function of the discrete trading problem.

In order to exploit the technical result of the next section, we need to introduce a modified problem.

Let R be a fixed positive number, and $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}$ be the region $u \in [0, R]$, $|v| \leq R$. For the time being, we consider only problems of option hedging where $(u(0), v(0)) \in \mathcal{R}$.

As a consequence, for these problems, and for all $t \in [0, T]$, we have $u(t) \leq R e^{\tau^+ T}$.

Concerning $v(\cdot)$, the control ξ might send it anywhere in $\mathbb{R}$. But we know from the analysis according to the Isaacs-Breakwell theory that the minimizing strategies never create large $v(t)$'s. As a matter of fact, let W_0 be the maximum payoff obtained by the strategy $\varphi = 0$ (after maximization in $\tau(\cdot)$) for any $(u_0, v_0) \in \mathcal{R}$. Let a be a large number, chosen satisfying $a > 2\exp(\tau^+ T)[W_0/(RC_\star) + 1]$, and $S = aR$. We claim the following:

Proposition 3.1 *Any nonanticipative strategy ϕ that causes $|v| > S$ is dominated by the strategy $\phi = 0$.*

Proof Let ζ be a positive number, $\zeta < C_\star \exp(-\tau^+T)/4\tau^+$. Any nonanticipative strategy ϕ may be challenged by the control function generated by the following rule: *If* $v(t-\zeta) > S$, *choose* $\tau(t) = \tau^-$ *if* $v(t-\zeta) < -R$ *choose* $\tau(t) = \tau^+$. Due to the small time delay ζ, it does generate an admissible control function $\tau(\cdot)$ against a nonanticipative strategy ϕ, which, in turn, cannot anticipate this control. It is easy to check that whether we reach $v = S$ or $v = -S$, from $|v_0| \leq R$, the cost $\int C^\varepsilon \xi(t)\mathrm{d}t$ is larger than $C_\star(\mathrm{e}^{-\tau^+T}S - R)$.

On the other hand, under the rule proposed to generate $\tau(\cdot)$, after at most a delay ζ, we shall have $\tau v < 0$, so that the benefit accrued to the minimizer is not more than $\int_{t-\zeta}^t v(s)\,\mathrm{d}s \leq S(\exp(\tau^+\zeta) - 1) < 2S\tau^+\zeta < C_\star \exp(-\tau^+T)S/2$. Hence, any such excursion in v costs the minimizer at least $C_\star(\exp(-\tau^+T)a/2-1)R > W_0$. Since the terminal cost $N(u(T), v(T))$ is itself non negative, that strategy ϕ does less well than $\phi = 0$. ∎

As a consequence, for initial states in $\mathcal{R}$, we may, without modifying the Value, restrict the set of admissible strategies to strategies that keep $|v| \leq S$. With these strategies, the term $\int \tau(t)v(t)\,\mathrm{d}t$ is bounded. And therefore, we can furthermore restrict the admissible strategies to be such that $\int C^\varepsilon \xi(t)\,\mathrm{d}t$ be also bounded, and therefore also $\int |\xi(t)|\,\mathrm{d}t$. (Say, by $(W_0+1+\tau^\star ST)/C_\star$.)

Let Φ_b the set of admissible non anticipative strategies thus restricted.

We now modify the original problem as follows: let $\mathcal{P}_{[a,b]}$ be the projection of $\mathbb{R}$ on $[a,b] \subset \mathbb{R}$ and,

$$N_b(u,v) = N(\mathcal{P}_{[0,\mathrm{e}^{\tau^+T}R]}(u), \mathcal{P}_{[-S,S]}(v)), \qquad L_b(v) = \mathcal{P}_{[-S,S]}(v). \tag{23}$$

We keep the same dynamics as we had, and define the pay-off as

$$W_b(0,u_0,v_0) = \inf_{\phi\in\Phi_b} \sup_{\tau\in\Psi} \left[N_b(u(T),v(T)) + \int_0^T \Big(-\tau L_b(v(t)) + C^\varepsilon \xi(t) \Big) \mathrm{d}t \right]. \tag{24}$$

We have modified the problem only for states outside $[0, \mathrm{e}^{\tau^+T}R] \times [-S,S]$, never reached from initial states (u_0,v_0) in $\mathcal{R}$. Hence in $\mathcal{R}$, the value of the modified game coincides with that of the original game: $W_b|_{\mathcal{R}} = W|_{\mathcal{R}}$.

The new point is that now, W_b is a *bounded* viscosity solution of the modified DQVI

$$\begin{aligned} 0 = \min\Bigg\{ & \frac{\partial W_b}{\partial t} + \max_{\tau\in[\tau^-,\tau^+]} \tau \left[\frac{\partial W_b}{\partial u} u + \frac{\partial W_b}{\partial v} v - L_b(v) \right], \\ & \frac{\partial W_b}{\partial v} + C^+ , \quad -\left(\frac{\partial W_b}{\partial v} + C^- \right) \Bigg\}, \\ W_b(T,u,v) = {} & N_b(u,v). \end{aligned} \tag{25}$$

Following the lines of [9], it can be shown to be uniformly continuous, i.e. in the space BUC of bounded uniformly continous functions.

We prove in the next section the following technical result:

Theorem 3.5 *The DQVI (25) admits a unique BUC viscosity solution.*

One can look at the discrete trading problem associated to the bounded pay-off (24), and define as above a function $W_b^h(t,u,v)$ solution of the related minimax problem. As soon as the step size h is smaller than $C^\star/4\tau^\star$, it holds as well that the restrictions of the original and modified discrete trading values $W^h|_{\mathcal{R}}$ and $W_b^h|_{\mathcal{R}}$ coincide. The same proof as in [7] shows that as $h \to 0$ (in a dyadic fashion), W_b^h converges to a viscosity solution of (25). But this viscosity solution being proved unique, we may conclude that $W_b^h \to W_b$, uniformly on every compact. As a consequence,

$$W^h|_{\mathcal{R}} = W_b^h|_{\mathcal{R}} \to W_b|_{\mathcal{R}} = W|_{\mathcal{R}}\,.$$

Therefore, in $\mathcal{R}$, we do have uniform convergence of the Value of the discrete trading problem to that of the continous trading problem. But R was picked arbitrarily. Therefore convergence occurs for all of $\mathbb{R}^2$, uniformly on any compact.

4 Proof of the uniqueness theorem

We now set to prove theorem (3.5). We ommit all indices b, but it should be understood all along that we are dealing with the modified problem.

4.1 The proof with three lemmas

4.1.1 Proof of the theorem

We shall consider the DQVI for $V = \mathrm{e}^t W$. It satisfies another DQVI, (35). Assume that it has two BUC viscosity solutions V and V'. Choose $\varepsilon > 0$ and $\varepsilon < \|V\|_\infty$ (it is to go to 0). Choose $\mu \in (1 - \varepsilon/\|V\|_\infty, 1)$, and let $U = \mu V$. Then,

$$\forall (t,u,v)\,, \quad |V(t,u,v) - U(t,u,v)| \le \varepsilon\,.$$

Let $\mathbb{M} = \sup_{t,u,v}(U(t,u,v) - V'(t,u,v))$. It follows that

$$\sup_{t,u,v}(V(t,u,v) - V'(t,u,v)) \le \mathbb{M} + \varepsilon\,. \tag{26}$$

We now claim the following lemma:

Lemma 4.1 *There exists $\mu^\star \in (1-\varepsilon/\|V\|_\infty, 1)$ and a constant $K > 0$, both depending only on the data of the problem, such that, if $\mu \in (\mu^\star, 1)$, then $\mathbb{M} \le K\varepsilon$.*

As a consequence,

$$\sup_{t,u,v}(V(t,u,v) - V'(t,u,v)) \le (K+1)\varepsilon\,. \tag{27}$$

Since ε was chosen arbitrary, it follows that for all (t,u,v), $V(t,u,v) \le V'(t,u,v)$. But since the argument is symmetric in V and V', necessarily $V = V'$. Q.E.D. ∎

4.1.2 Proof of lemma 4.1

Notice first that if $\mathbb{M} \le 0$, then according to (26), $\sup_{t,u,v}(V(t,u,v) - V'(t,u,v)) \le \varepsilon$, and (27) is satisfied for any positive K. We may therefore, from now on, concentrate on the case $\mathbb{M} > 0$.

Let, thus $0 < \varepsilon < \|V\|_\infty$ be given, and pick μ such that

$$1 > \mu \ge \mu^\star = 1 - \frac{\varepsilon}{\max\{\|V\|_\infty, \ell \mathrm{e}^T, m\mathrm{e}^T\}} < 1\,. \tag{28}$$

For three positive numbers α, β, γ (that we shall pick small later on), introduce the test function $\phi_{\alpha,\beta,\gamma} : [0,T] \times \mathbb{R}^2 \times [0,T] \times \mathbb{R}^2 \to \mathbb{R}$:

$$\begin{aligned}\phi_{\alpha,\beta,\gamma}(t,u,v,t',u',v') &= U(t,u,v) - V'(t',u',v') - \alpha(u^2 + u'^2 + v^2 + v'^2) \\ &\quad - \frac{(u-u')^2 + (v-v')^2}{\beta^2} - \frac{(t-t')^2}{\gamma^2}\,.\end{aligned}$$

This function reaches its maximum at

$$\max \phi_{\alpha,\beta,\gamma}(t,u,v,t',u',v') = \phi_{\alpha,\beta,\gamma}(\bar{t},\bar{u},\bar{v},\bar{t}',\bar{u}',\bar{v}') =: \mathbb{M}_{\alpha,\beta,\gamma}\,.$$

We claim the following two lemmas, both for $\mu \in (\mu^\star, 1)$ fixed, and under the hypothesis that $\mathbb{M} > 0$:

Lemma 4.2 *There exists $\alpha^\star$, $\beta^\star$, $\gamma^\star$ all positive, such that for any $\alpha \le \alpha^\star$, $\beta \le \beta^\star$, $\gamma \le \gamma^\star$,*

$$|U(\bar{t},\bar{u},\bar{v}) - V'(\bar{t}',\bar{u}',\bar{v}') - \mathbb{M}| \le \varepsilon\,, \tag{29}$$

$$\alpha(\bar{u}^2 + \bar{u}'^2 + \bar{v}^2 + \bar{v}'^2) + \frac{(\bar{u}-\bar{u}')^2 + (\bar{v}-\bar{v}')^2}{\beta^2} + \frac{(\bar{t}-\bar{t}')^2}{\gamma^2} \le 2\varepsilon\,. \tag{30}$$

Let $\tau^\star = \max\{\tau^+, -\tau^-\}$.

Lemma 4.3 *For any $\alpha \le \alpha^\star$, $\beta \le \beta^\star$, $\gamma \le \gamma^\star$,*

$$U(\bar t, \bar u, \bar v) - V'(\bar t', \bar u', \bar v') \le \max\{2, 7\tau^\star\}\varepsilon\,. \tag{31}$$

The main lemma follows clearly, with $K = \max\{3, 7\tau^\star{+}1\}$, from inequations (29) and (31). ∎

Inequation (30) is used in the proof of lemma 4.3. We split the assertions in two separate lemmas, because the first one does not make use of the DQVI while the second one does.

4.2 Proof of the lemmas 4.2 and 4.3

4.2.1 Proof of lemma 4.2

Assume that $\mathbb{M} > 0$. Choosing $(t, u, v) = (t', u', v')$, it follows that

$$\forall (t, u, v)\,, \quad \mathbb{M}_{\alpha,\beta,\gamma} \ge U(t, u, v) - V(t, u, v) - 2\alpha(u^2 + v^2)\,. \tag{32}$$

Pick a point $(t^\star, u^\star, v^\star)$ such that $\mathbb{M}$ is approached within $\varepsilon/2$:

$$U(t^\star, u^\star, v^\star) - V'(t^\star, u^\star, v^\star) \ge \mathbb{M} - \varepsilon/2\,,$$

and let $\alpha_1 = \varepsilon/[4(u^{\star 2} + v^{\star 2})]$ if $(u^{\star 2} + v^{\star 2}) \ne 0$, (and $\alpha_1 = \infty$ otherwise). It follows that for any $\alpha \le \alpha_1$,

$$U(t^\star, u^\star, v^\star) - V'(t^\star, u^\star, v^\star) - 2\alpha(u^{\star 2} + v^{\star 2}) \ge \mathbb{M} - \varepsilon\,,$$

and using (32),

$$\mathbb{M} - \varepsilon \le \mathbb{M}_{\alpha,\beta,\gamma}\,. \tag{33}$$

Hence,

$$-\varepsilon \le \mathbb{M} - \varepsilon \le \mathbb{M}_{\alpha,\beta,\gamma} \le \|U\|_\infty + \|V'\|_\infty - \alpha(\bar u^2 + \bar u'^2 + \bar v^2 + \bar v'^2) - \frac{(\bar u - \bar u')^2 + (\bar v - \bar v')^2}{\beta^2} - \frac{(\bar t - \bar t')^2}{\gamma^2}.$$

Let $r^2 := \|V\|_\infty + \|V'\|_\infty + \varepsilon$, and notice that $\|U\|_\infty < \|V\|_\infty$. It follows that

$$\alpha(\bar u^2 + \bar u'^2 + \bar v^2 + \bar v'^2) + \frac{(\bar u - \bar u')^2 + (\bar v - \bar v')^2}{\beta^2} + \frac{(\bar t - \bar t')^2}{\gamma^2} \le r^2\,,$$

and in particular that

$$\alpha(\bar u^2+\bar u'^2+\bar v^2+\bar v'^2)\le r^2\,,\ \ |\bar u-\bar u'|\le r\beta\,,\ \ |\bar v-\bar v'|\le r\beta\,,\ \ |\bar t-\bar t'|\le r\gamma\,. \tag{34}$$

Now, V' is uniformly continuous by hypothesis. Let, for u and v positive

$$n(u,v)=\sup_{\substack{|t-t'|\le v\\ |u-u'|\le u\\ |v-v'|\le u}}|V'(t,u,v)-V'(t',u',v')|$$

Clearly, n is decreasing with its arguments and decreases to 0 with $u+v$. Using (34), it follows that

$$U(\bar t,\bar u,\bar v)-V'(\bar t',\bar u',\bar v')\le U(\bar t,\bar u,\bar v)-V'(\bar t,\bar u,\bar v)+n(r\beta,r\gamma)\le \mathbb{M}+n(r\beta,r\gamma)\,.$$

Choose β_1 and γ_1 such that for $\beta\le\beta_1$ and $\gamma\le\gamma_1$, $n(r\beta,r\gamma)\le\varepsilon$. Using again (33), we get

$$\mathbb{M}-\varepsilon\le\mathbb{M}_{\alpha,\beta,\gamma}\le U(\bar t,\bar u,\bar v)-V'(\bar t',\bar u',\bar v')\le\mathbb{M}+\varepsilon\,.$$

Conclusions (29) and (30) of the lemma follow. ∎

4.2.2 Modified DQVI's

We first apply a classical transformation to DQVI (13) introducing

$$V(t,u,v):=\mathrm{e}^tW(t,u,v)\,,$$

which is BUC if and only if W is. Now, W is a viscosity solution of (13) if and only if V is a viscosity solution of the modified DQVI:

$$\begin{aligned}
&\forall(t,u,v)\in[0,T)\times\mathbb{R}^2,\\
&\qquad 0=\min\left\{\frac{\partial V}{\partial t}-V(t,u,v)+\max_{\tau\in[\tau^-,\tau^+]}\tau\left[\frac{\partial V}{\partial u}u+\frac{\partial V}{\partial v}v-\mathrm{e}^tL(v)\right],\right.\\
&\qquad\qquad\left.\frac{\partial V}{\partial v}+\mathrm{e}^tC^+\,,-\frac{\partial V}{\partial v}-\mathrm{e}^tC^-\right\}\\
&V(T,u,v)=\mathrm{e}^TM(u,v)\,,\quad\forall(u,v)\in\mathbb{R}^2\,.
\end{aligned} \tag{35}$$

We shall also make use of the following remark. For any positive μ, that we shall take smaller than one, let $U(t,u,v)=\mu V(t,u,v)$. It is a viscosity solution of the third DQVI

$$
\begin{aligned}
&\forall (t,u,v) \in [0,T) \times \mathbb{R}^2 ,\\
&\qquad 0 = \min\left\{ \frac{\partial U}{\partial t} - U(t,u,v) + \max_{\tau\in[\tau^-,\tau^+]} \tau \left[\frac{\partial U}{\partial u} u + \frac{\partial U}{\partial v} v - \mu \mathrm{e}^t L(v) \right] ,\right.\\
&\qquad\qquad \left. \frac{\partial U}{\partial v} + \mu \mathrm{e}^t C^+ \,, - \frac{\partial U}{\partial v} - \mu \mathrm{e}^t C^- \right\}\\
&U(T,u,v) = \mu \mathrm{e}^T M(u,v)\,, \quad \forall (u,v) \in \mathbb{R}^2 .
\end{aligned}
\tag{36}
$$

As a matter of fact, the DQVI (35) is a particular case of this one with $\mu = 1$. We gave it separately for reference hereafter.

4.2.3 Proof of lemma 4.3

Case $\bar t$ and $\bar t'$ smaller than T By definition of $(\bar t, \bar u, \bar v)$ and $(\bar t', \bar u', \bar v')$, we have

$$
\begin{aligned}
&U(\bar t,\bar u,\bar v) - V'(\bar t',\bar u',\bar v') - \alpha(\bar u^2 + \bar u'^2 + \bar v^2 + \bar v'^2) - \tfrac{(\bar u-\bar u')^2 + (\bar v - \bar v')^2}{\beta^2} - \tfrac{(\bar t - \bar t')^2}{\gamma^2} \geq\\
&U(t,u,v) - V'(\bar t',\bar u',\bar v') - \alpha(u^2 + \bar u'^2 + v^2 + \bar v'^2) - \tfrac{(u-\bar u')^2 + (v - \bar v')^2}{\beta^2} - \tfrac{(t - \bar t')^2}{\gamma^2},
\end{aligned}
$$

and also

$$
\begin{aligned}
&U(\bar t,\bar u,\bar v) - V'(\bar t',\bar u',\bar v') - \alpha(\bar u^2 + \bar u'^2 + \bar v^2 + \bar v'^2) - \tfrac{(\bar u-\bar u')^2 + (\bar v - \bar v')^2}{\beta^2} - \tfrac{(\bar t - \bar t')^2}{\gamma^2} \geq\\
&U(\bar t,\bar u,\bar v) - V'(t',u',v') - \alpha(\bar u^2 + u'^2 + \bar v^2 + v'^2) - \tfrac{(\bar u-u')^2 + (\bar v - v')^2}{\beta^2} - \tfrac{(\bar t - t')^2}{\gamma^2}.
\end{aligned}
$$

Define the two test functions:

$$
\begin{aligned}
&\phi(t,u,v) = V'(\bar t',\bar u',\bar v') + \alpha(u^2 + \bar u'^2 + v^2 + \bar v'^2) + \tfrac{(u-\bar u')^2 + (v - \bar v')^2}{\beta^2} + \tfrac{(t - \bar t')^2}{\gamma^2},\\
&\phi'(t',u',v') = U(\bar t,\bar u,\bar v) - \alpha(\bar u^2 + u'^2 + \bar v^2 + v'^2) - \tfrac{(\bar u-u')^2 + (\bar v - v')^2}{\beta^2} - \tfrac{(\bar t - t')^2}{\gamma^2}.
\end{aligned}
$$

The first inequality above means that $(\bar t, \bar u, \bar v)$ is a maximal point of $U - \phi$, and the second that $(\bar t', \bar u', \bar v')$ is a minimal point of $V' - \phi'$. Using the definition of a viscosity solution, it follows that[1]

$$\text{at } \bar t,\bar u,\bar v\,,\quad \min\left\{\frac{\partial\phi}{\partial t}-U+\max_{\tau\in[\tau^-,\tau^+]}\tau\left[\frac{\partial\phi}{\partial u}\bar u+\frac{\partial\phi}{\partial v}\bar v-\mu e^{\bar t}L\right],\right.$$
$$\left.\frac{\partial\phi}{\partial v}+\mu e^{\bar t}C^+\,,-\frac{\partial\phi}{\partial v}-\mu e^{\bar t}C^-\right\}\geq 0, \tag{37}$$

$$\text{at } \bar t',\bar u',\bar v'\,,\ \min\left\{\frac{\partial\phi'}{\partial t'}-V'+\max_{\tau\in[\tau^-,\tau^+]}\tau\left[\frac{\partial\phi'}{\partial u'}\bar u'+\frac{\partial\phi'}{\partial v'}\bar v'-e^{\bar t'}L\right],\right.$$
$$\left.\frac{\partial\phi'}{\partial v'}+e^{\bar t'}C^+\,,-\frac{\partial\phi'}{\partial v'}-e^{\bar t'}C^-\right\}\geq 0, \tag{38}$$

The first inequality can be decomposed into three inequalities:

$$2\frac{\bar t-\bar t'}{\gamma^2}-U(\bar t,\bar u,\bar v)$$
$$+\max_{\tau\in[\tau^-,\tau^+]}2\tau\left[\alpha\bar u^2+\frac{\bar u-\bar u'}{\beta^2}\bar u+\alpha\bar v^2+\frac{\bar v-\bar v'}{\beta^2}\bar v-\frac{\mu}{2}e^{\bar t}L(\bar v)\right]\geq 0, \tag{39}$$

$$-\mu e^{\bar t}C^+\leq 2\alpha\bar v+2\frac{\bar v-\bar v'}{\beta^2}\leq -\mu e^{\bar t}C^-\,. \tag{40}$$

The second inequality reads

$$\min\left\{2\frac{\bar t-\bar t'}{\gamma^2}-V'(\bar t',\bar u',\bar v')+\right.$$
$$\max_{\tau\in[\tau^-,\tau^+]}2\tau\left[-\alpha\bar u'^2+\frac{\bar u-\bar u'}{\beta^2}\bar u'-\alpha\bar v'^2+\frac{\bar v-\bar v'}{\beta^2}\bar v'-\frac{1}{2}e^{\bar t'}L(\bar v')\right], \tag{41}$$
$$\left.-2\alpha\bar v'+2\frac{\bar v-\bar v'}{\beta^2}+e^{\bar t'}C^+\,,\quad 2\alpha\bar v'-2\frac{\bar v-\bar v'}{\beta^2}-e^{\bar t'}C^-\right\}\leq 0.$$

We want, now, to use the inequalities (40) to show that the last two terms of (41) can be made positive, which will imply that the first one is negative. Let us therefore write the following string of inequalities, which makes use

[1] Our sign convention for the Isaacs equation follows that of control theory rather than that of the calculus of variations. It follows that the roles of maximum and minimum are reversed in the definition of viscosity super- and sub-solutions.

of (40) between the second and the third line, then of (34):

$$
\begin{aligned}
&-2\alpha\bar{v}' + 2\frac{\bar{v}-\bar{v}'}{\beta^2} + \mathrm{e}^{\bar{t}'}C^+ = \\
&2\alpha\bar{v} + 2\frac{\bar{v}-\bar{v}'}{\beta^2} + \mu\mathrm{e}^{\bar{t}}C^+ - 2\alpha(\bar{v}+\bar{v}') + (\mathrm{e}^{\bar{t}'} - \mu\mathrm{e}^{\bar{t}})C^+ \geq \\
&-2\alpha(|\bar{v}| + |\bar{v}'|) + (\mathrm{e}^{\bar{t}'} - \mathrm{e}^{\bar{t}})C^+ + (1-\mu)\mathrm{e}^{\bar{t}}C^+ \geq \\
&-4r\sqrt{\alpha} - \mathrm{e}^{T}r\gamma C^+ + (1-\mu)C^+ .
\end{aligned}
$$

Hence, choose

$$
\alpha_2 = \min\left\{\alpha_1\,, \frac{(1-\mu)^2 C^{+2}}{64r^2}\right\}
$$

and

$$
\gamma_2 = \min\left\{\gamma_1\,, \mathrm{e}^{-T}\frac{1-\mu}{2r}\right\} .
$$

The choice of $\alpha \leq \alpha_2$, $\gamma \leq \gamma_2$ insures that this term is positive, without destroying the effects sought with the choice of α_1 and γ_1.

In a similar fashion, we have

$$
\begin{aligned}
&2\alpha\bar{v}' - 2\frac{\bar{v}-\bar{v}'}{\beta^2} - \mathrm{e}^{\bar{t}'}C^- = \\
&-2\alpha\bar{v} - 2\frac{\bar{v}-\bar{v}'}{\beta^2} - \mu\mathrm{e}^{\bar{t}}C^- + 2\alpha(\bar{v}+\bar{v}') - (\mathrm{e}^{\bar{t}'} - \mu\mathrm{e}^{\bar{t}})C^- \geq \\
&-4r\sqrt{\alpha} + \mathrm{e}^{T}r\gamma C^- - (1-\mu)C^- .
\end{aligned}
$$

Again, define

$$
\alpha_3 = \min\left\{\alpha_2\,, \frac{(1-\mu)^2 C^{-2}}{64r^2}\right\}
$$

and

$$
\gamma_3 = \min\left\{\gamma_2\,, \mathrm{e}^{-T}\frac{1-\mu}{2r}\right\} ,
$$

and the choice $\alpha \leq \alpha_3$, $\gamma \leq \gamma_3$ insures that both terms are positive.

Therefore, with these choices of parameters α, β, γ, we have

$$
\begin{aligned}
&2\frac{\bar{t}-\bar{t}'}{\gamma^2} - V'(\bar{t}', \bar{u}', \bar{v}') + \\
&\max_{\tau\in[\tau^-,\tau^+]} 2\tau\left[-\alpha\bar{u}'^2 + \frac{\bar{u}-\bar{u}'}{\beta^2}\bar{u}' - \alpha\bar{v}'^2 + \frac{\bar{v}-\bar{v}'}{\beta^2}\bar{v}' - \frac{1}{2}\mathrm{e}^{\bar{t}'}L(\bar{v}')\right] \leq 0 .
\end{aligned}
\tag{42}
$$

We now make the difference (42)-(39), and make use of $\tau^\star = \max\{\tau^+, -\tau^-\}$ and the fact

$$\max_{\tau\in[\tau^-,\tau^+]} \tau A - \max_{\tau\in[\tau^-,\tau^+]} \tau B \le \max_{\tau\in[\tau^-,\tau^+]} \tau(A-B) \le \tau^\star |A-B|.$$

This yields

$$U(\bar t,\bar u,\bar v) - V'(\bar t',\bar u',\bar v') \le 2\tau^\star\Big[\alpha(\bar u^2+\bar u'^2+\bar v^2+\bar v'^2) + \frac{(\bar u-\bar u')^2+(\bar v-\bar v')^2}{\beta^2} \\ +\frac{1}{2}(\mathrm{e}^{\bar t'}L(\bar v') - \mu \mathrm{e}^{\bar t}L(\bar v))\Big]$$

Using (30), the first line of the r.h.s. above is less than $4\tau^*\varepsilon$ for any $(\alpha,\beta,\gamma)\le(\alpha_1,\beta_1,\gamma_1)$, and a fortiori for $(\alpha,\beta,\gamma)\le(\alpha_3,\beta_3,\gamma_3)$. Also

$$\begin{aligned}\mathrm{e}^{\bar t'}L(\bar v') - \mu\mathrm{e}^{\bar t}L(\bar v) &= (\mathrm{e}^{\bar t'} - \mathrm{e}^{\bar t})L(\bar v') + \mathrm{e}^{\bar t}(L(\bar v') - L(\bar v)) + (1-\mu)L(\bar v)\\ &\le \mathrm{e}^T[r\gamma\ell + |L(\bar v') - L(\bar v)| + (1-\mu)\ell]\end{aligned}$$

According to our choice (28) of μ, the last term in the r.h.s. above is not larger than ε. Let β_4 be small enough so that, for any $|\bar v' - \bar v| \le r\beta_4$, $|L(\bar v') - L(\bar v)| \le \varepsilon$, which is possible since L is uniformly continuous. Picking $\beta \le \beta_4$, and $\gamma \le \gamma_4 = \min\{\gamma_3, \varepsilon/(\mathrm{e}^T r\ell)\}$, the first two terms are also not larger than ε. Therefore, with this choice of (α,β,γ), we have

$$0 < U(\bar t,\bar u,\bar v) - V'(\bar t',\bar u',\bar v') \le 7\tau^\star\varepsilon.$$

It remains to use inequality (29) to get $\mathbb{M} \le (7\tau^\star+1)\varepsilon$. This is the inequality $\mathbb{M} \le K\varepsilon$ foretold in section 4.1.1.

Case $\bar t = T$ or $\bar t' = T$ If $\bar t = T$, it follows that $U(\bar t,\bar u,\bar v) = \mu\mathrm{e}^T M(\bar u,\bar v)$. It also holds that $V'(\bar t,\bar u,\bar v) = \mathrm{e}^T M(\bar u,\bar v)$ and $|V'(\bar t',\bar u',\bar v') - V'(\bar t,\bar u,\bar v)| \le n(r\beta, r\gamma) \le \varepsilon$. (this last inequality as soon as $\beta\le\beta_1$ and $\gamma\le\gamma_1$.) Remember that $\|M\|_\infty = m$ and (28). Hence

$$U(\bar t,\bar u,\bar v) - V'(\bar t',\bar u',\bar v') \le (1-\mu)\mathrm{e}^T m + n(r\beta,r\gamma) \le 2\varepsilon\,.$$

If $\bar t' = T$, $V'(\bar t',\bar u',\bar v') = \mathrm{e}^T M(\bar u',\bar v')$. Choose $\beta_5 < \beta_4$ and $\gamma_5 \le \gamma_4$ such that for $|u-u'| \le r\beta_5$, $|v-v'| \le r\beta_5$ and $|t-t'| \le r\gamma_5$, it results $|U(t,u,v) - U(t',u',v')| \le \varepsilon$. (This is possible since, as V', U is assumed to be uniformly continuous.) It results that

$$U(\bar t,\bar u,\bar v) - V'(\bar t',\bar u',\bar v') \le \varepsilon + (\mu-1)\mathrm{e}^T M(\bar u',\bar v') \le \varepsilon + (1-\mu)\mathrm{e}^T m \le 2\varepsilon\,.$$

Finally, the case where $\bar{t} = \bar{t}' = T$ is taken care of by any of the above two.

We may now set $\alpha^\star = \alpha_3$, $\beta^\star = \beta_5$ and $\gamma^\star = \gamma_5$, and the two lemmas are proved, hence also lemma 4.1 and the theorem.

Acknowledgment The uniqueness proof owes much to discussions with Guy Barles, of University François Rabelais, Tours, France.

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